Admissibility and Event-Rationality*

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Abstract

We develop an alternative approach to providing epistemic conditions for admissible behavior in games. Instead of using lexicographic beliefs to capture infinitely less likely conjectures, we postulate that players use tie-breaking sets to help decide among strategies that are outcome-equivalent given their conjectures. A player is event-rational if she best responds to a conjecture and uses a list of subsets of the other players' strategies to break-ties among outcome-equivalent strategies. Using type spaces to capture interactive beliefs, we show that common belief of event-rationality (RCBER) implies that players play strategies in $S^\infty W$, that is, admissible strategies that also survive iterated elimination of dominated strategies (Dekel and Fudenberg (1990)). We strengthen standard belief to consistent belief and we show that event-rationality and common consistent belief of event-rationality (RCcBER) implies that players play iterated admissible strategies (IA). We show that in complete, continuous and compact type structures, RCBER and RCcBER are nonempty, and hence we obtain epistemic criteria for $S^\infty W$ and IA.

Keywords: Epistemic game theory; Admissibility; Iterated weak dominance; Common Knowledge; Rationality; Completeness.

1 Introduction

As noted by Samuelson (1992) and many others, there is an intrinsic impossibility in dealing with common knowledge of admissibility in games, which is known as the inclusion-exclusion problem. The reason is that a strategy is admissible if and only if it is a best response to a conjecture with full support. If we capture knowledge by the support of the agent's belief and assume that she is rational, that is, she optimizes given her belief, then playing an admissible strategy implies that she must necessarily consider all strategies of the other players as possible, including the strategies

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that are not admissible. So she cannot know that her opponents play admissible strategies because she cannot exclude from consideration their inadmissible strategies.

The most appealing approach to dealing with this issue is provided by Brandenburger et al. (2008), henceforth BFK, by using lexicographic probability systems (LPS) and the notion of assumption in the place of certainty. Roughly speaking, a player with conjectures that form an LPS can have a fully supported conjecture while "assuming" certain events that are not equal to the whole state space. BFK show that common assumption of admissibility (RCAR) characterizes iterated admissibility (IA), but RCAR may be empty in complete type structures. Yang (2009), Keisler (2009) and Lee (2009) have recently extended BFK's analysis and obtained non emptiness of RCAR in complete type structures.

We propose an alternative route. Instead of an LPS-based analysis, we use event-rationality to allow for players to break ties with lists of subsets of opponents' strategies. That is, we use a different notion of rationality: the LPS-based approaches assume that players are lexicographic expected utility maximizers. We assume that players are event-rational. The two notions of rationality equally describe admissible behavior. The difference comes into play in the analysis of interactive beliefs. Interactive beliefs are described by type spaces. In our framework, a type of a player determines her beliefs over the strategies and types of the other players (as in the standard framework) and in addition it determines the tie-breaking list that the (event-rational) type uses. As a result, common belief of event-rationality does not run into the tension of having to exclude and include the same event. In contrast, in an LPS-based analysis a type of a player determines her lexicographic beliefs over the strategies and types of the other players, and the inclusion-exclusion tension is avoided by the use of "assumption" in the place of certainty. Under our approach, we provide epistemic foundations for both the solution concept proposed by Dekel and Fudenberg (1990) ($S^{\infty}W$) and iterated admissibility (IA).

We consider finite two player games in strategic form. The two players are Ann and Bob, denoted by superscripts "a" and "b". In order to provide some intuition about event-rationality, note that if a strategy s^a of Ann's is rational then it is a best response to some conjecture, $v \in \Delta(S^b)$, where S^b is the set of Bob's strategies. If s^a is inadmissible and therefore weakly dominated by some (mixed) strategy σ^a, then s^a and σ^a give the same payoff for all strategies of Bob on the support of v while σ^a is strictly better than s^a for all conjectures with support on the complement of the support of v. Hence, whenever Ann chooses an admissible strategy, it is as if she optimizes given her conjecture, as usual, but when she is totally indifferent between two strategies she compares them using a measure with support on the difference between S^b and the support of her conjecture. We say that she "breaks ties" using the event that is the complement of her support (with respect to S^b). In other words, Ann is confident in trusting her belief, just like any other rational agent. But if two of her strategies are outcome-equivalent under her belief, she chooses the one that is also optimal under a measure with support being the complement of the support of her belief.

There is nothing particular about breaking ties with respect to the complement of her support when defining event-rationality. Ann can conceivably break ties using any other set, as long as it is outside her current frame, that is, disjoint from the support of her belief. Furthermore, Ann need not use a single such tie-breaking set. She may well have many such sets, each providing extra validation for her chosen strategy.

The principle behind event-rationality is, therefore, the following: if two strategies are outcome-equivalent given Ann's conjecture, then Ann has no way of deciding among them within her frame of mind: the two strategies yield the same outcome for whichever strategy of Bob she considers possible. Ann must, therefore, resort to information beyond her frame to make a decision. She could, for instance, flip coins, that is, resort to fully external means. But in doing so Ann would be missing information about her two strategies, contained in how they fare against strategies of Bob that are considered impossible by Ann's conjecture. Event-rationality postulates that Ann goes beyond her frame, without changing what she thinks about Bob's choices.

Turn now to interactive beliefs, captured by type structures. Let T^a and T^b be the sets of types of Ann and Bob. A type $t^a \in T^a$ determines Ann's conjectures over Bob's choices, Ann's beliefs over Bob's types and so on, together with the tie-breaking list used by Ann. A state for Ann is a strategy-type pair (s^a, t^a) and her beliefs over Bob are given by her beliefs over $S^b \times T^b$. A strategy-type pair (s^a, t^a) of Ann's is called event-rational if s^a is optimal given t^a's conjecture and breaks ties for all sets in t^a's tie-breaking list. Event-rationality and common belief of event-rationality is then captured as the intersection of infinitely many events: Ann is event-rational, and so is Bob; Ann is certain that Bob is event-rational and Bob is certain that Ann is event-rational. And so on. This yields our RCBER set of states.

Event-rationality captures the idea of choosing a strategy cautiously, in the sense that a strategy has to be optimal under one's conjecture, but also pass a series of tie-breaking tests. We also introduce the idea of cautiously believing an event. Suppose that we intepret each tie-breaking set of Ann's as the support of a "secondary" conjecture. Then, we say that Ann cautiously believes an event if it is the smallest event that is believed by her primary and one of her secondary conjectures. In other words, Ann cautiously believes an event if it is the sharpest description of how the others play on which both her primary and one of her secondary conjectures agree that it is true.[1] We say that Ann consistenly believes an event if she believes it (assigns probability 1) and cautiously believes it. Event-rationality and common consistent belief of event-rationality is again captured as the intersection of infinitely many events: Ann and Bob are event-rational. Ann consistently believes that Bob is event-rational and Bob consistently believes that Ann is event-rational. And so on. This yields our RCcBER set of states.

[1] We essentially apply cautious belief only on events describing the other player's strategies.

Our results are as follows. We characterize the strategies that are compatible with RCBER by a solution concept, hypo-admissible sets (HAS), which is related to the self-admissible sets (SAS) of BFK but it is neither weaker or stronger. In a complete structure, RCBER produces the set of strategies that survive one round of elimination of non admissible strategies followed by iterated elimination of strongly dominated strategies ($S^{\infty}W$). We characterize RCcBER with a solution concept we call hypo-iteratively admissible sets (HIA). In a complete type structure, the resulting set of strategies is precisely the set of iterated admissible strategies (IA). We then show that strategies played under RCcBER constitute an SAS, but the converse is not necessarily true, meaning that the RCcBER construction is more restrictive than the RCAR construction of BFK. Nevertheless, we show that the RCBER and the RCcBER are nonempty whenever the type structure is complete, continuous and compact, therefore providing epistemic criteria for $S^{\infty}W$ and IA.

Our approach provides an alternative, effective and simple perspective in dealing with common "knowledge" of admissibility in games. The solution to the inclusion-exclusion problem lies in separating what a player knows from the strategies that she includes in her conjectures. This separation can also be obtained with LPS-based approaches as in BFK, Brandenburger (1992), Stahl (1995) and Yang (2009). But LPS-based approaches may add technical elements that are not necessarily relevant for the issue.[2] For instance, BFK's impossibility result suggests that IA is a solution concept that requires that the players are experienced enough with each other so that the type structure used to describe their beliefs is not complete (Brandenburger and Friedenberg (forthcoming)). In other words, it suggests that IA is to be viewed as a strong solution concept, that is not at the same level as iterated elimination of dominated strategies (IEDS) but rather closer to Nash equilibria, whose epistemic conditions require incomplete type structures (see Aumann and Brandenburger (1995) and Barelli (2009)). But this suggestion is an artifact of the technical details of an LPS-based approach. In fact, RCcBER is more restrictive than RCAR, and it is nonempty in a complete, continuous and compact type structure.

1.1 Related Literature

Bernheim (1984) and Pearce (1984) provide epistemic foundations for the iteratively undominated strategies via the concept of rationality and common belief in rationality. Admissibility, or the avoidance of weakly dominated strategies, has a long history in decision and game theory (see Kohlberg and Mertens (1986)). However, Samuelson (1992) shows that common knowledge of admissibility is not equivalent to iterated admissibility and does not always exist. Foundations for the $S^{\infty}W$ strategies (Dekel and Fudenberg (1990)) are provided by Börgers (1994) (using

[2] A simple example is that lexicographic type structures typically fail to be compact, whereas the universal type structure, without lexicographic beliefs, is compact.

approximate common knowledge), Brandenburger (1992) (using lexicographic probability systems (Blume et al. (1991)) and 0-level belief) and Ben-Porath (1997) (in extensive form games). Stahl (1995) defines the notion of lexicographic rationalizability and shows that it is equivalent to iterated admissibility.

BFK use lexicographic probability systems and characterize rationality and common assumption of rationality (RCAR) by the solution concept of self-admissible sets. They show that rationality and m-th order assumption of rationality is characterized by the strategies that survive $m+1$ rounds of elimination of inadmissible strategies. Finally, RCAR is empty in a complete and continuous lexicographic type structure when the agent is not indifferent. Hence, although the IA set can be captured by RmAR, for big enough m (note that games are finite), BFK do not provide an epistemic criterion for IA. Yang (2009) provides an epistemic criterion for IA, with an analogous version of BFK's RCAR, that makes use of a weaker notion of "assumption". Keisler (2009) and Lee (2009) independently show that the emptiness of RCAR can be overcome if one drops continuity. The message from Yang (2009), Keisler (2009) and Lee (2009) is that continuity strengthens the notion of caution implied by fully supported LPS. The notion of caution implied by event-rationality is independent of continuity.

The paper is organized as follows. In the following section we illustrate the differences between the various notions of rationality and belief through examples. In Sections 3 and 4 we set up the framework and provide the relevant definitions, including event-rationality, RCBER and RCcBER. In Section 5 we characterize RCBER and show that RmBER (m rounds of mutual belief) generates $S^{\infty}W$, for big enough m. In Section 6 we characterize RCcBER, show that it is more restrictive than RCAR of BFK and show that RmcBER generates the IA set, for big enough m. In Section 7 we show that RCBER and RCcBER are always nonempty in compact, complete and continuous type structures, therefore providing epistemic criteria for $S^{\infty}W$ and IA. Finally, the Appendix provides decision theoretic foundations for event-rationality.

2 Examples

In order to illustrate the differences between the BFK approach and that of the present paper, consider the following game from Samuelson (1992) and BFK. There are two players, Ann and Bob.

		1	[1]
		L	R
1	U	$1,1$	$0,1$
[1]	D	$0,2$	$1,0$

From Bernheim (1984) and Pearce (1984) we know that rationality and common belief of rationality (RCBR) is characterized by the best response sets (BRS) and, in a complete structure, the strategies that survive iterated deletion of strongly dominated strategies.[3] Can we get a similar result for the admissible strategies and the iteratively admissible strategies if we modify the notions of belief and of rationality? Recall that a strategy is admissible if and only if it is a best response to a full support measure (no action of the other player is excluded). Then, the obvious solution is to specify that rationality incorporates full support beliefs.

But such a specification does not always work. In the game above, if Ann is rational, she assigns positive probability to Bob playing L and R. If Bob is rational, he assigns positive probability to Ann playing U and D. Hence, Bob plays L. If Ann knows that Bob is rational, she assigns positive probability only on Bob playing L. But then, Ann is not rational! In other words, the modified RCBR set is empty for this game.

One solution is obtained using lexigographic beliefs. Suppose Ann's primary hypothesis assigns probability 1 to Bob playing L, and her secondary hypothesis assigns probability 1 to Bob playing R. Bob's primary hypothesis assigns 1 on U and his secondary hypothesis assigns 1 on D. Then, Bob playing L is rational because he is indifferent between L and R given his primary measure, but strictly prefers L given his secondary measure.[4] Ann playing U is rational because U is the best response given her primary measure. She assumes that Bob is rational, because she considers Bob playing L infinitely more likely than Bob playing R. Similarly, Bob assumes that Ann is rational. As a result, rationality and common assumption of rationality (RCAR) is nonempty.

A similar result can be obtained if we use the definition of event-rationality in the context of standard type structures. Suppose Ann's belief assigns probability 1 to Bob playing L and Bob's belief μ assigns probability 1 to Ann playing U. Moreover, Bob has set $S^b \backslash \text{supp } \mu$ in his tie-breaking list. Bob playing L is event-rational because he plays best response given his beliefs and, although L and R are outcome-equivalent under his support, L is better under a conjecture with support $S^b \backslash \text{supp } \mu$. Similarly, Ann is event-rational since, under her conjecture, she does not need to break ties. Finally, Ann believes that Bob is event-rational and Bob believes that Ann is event-rational. Hence, rationality and common belief of event-rationality (RCBER) is nonempty.

In the game above RCAR and RCBER produce the same strategies because the IA and the $S^\infty W$ sets are equal. However, this is not always true. Consider the following game which illustrates the difference between RCBER (which yields the $S^\infty W$ set) and RCcBER (which yields the IA set).

[3] $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

[4] That is, the associated sequence of payoffs under L is lexicographically greater than the sequence under R.

	L	R
U	$1,0$	$1,3$
M	$0,2$	$2,2$
D	$0,4$	$1,1$

Since D is strongly dominated, event-rational Ann will not play that strategy. In a complete structure though, event-rational Ann will play U or M, while event-rational Bob will play L or R. For example, Ann's type playing U is event-rational if she assigns probability 1 to Bob playing L. Ann's type playing M is also event-rational if she assigns probability 1 to Bob playing R. Note that Ann never needs to break ties. Moreover, for both U and M there are event-rational types of Ann's who assign positive probability to event-rational types of Bob playing L or R. And similarly for Bob. In other words, these types of Ann believe the event "Bob is event-rational", Bob's types believe the event "Ann is event-rational", and so on for any finite order of beliefs about beliefs. Hence, event-rationality and common belief of event-rationality (RCBER) yields the $S^{\infty}W$ set, $\{\text{U}, \text{M}\} \times \{\text{L}, \text{R}\}$.

Suppose we repeat the same procedure but now impose a stronger form of belief. Take an event $E \subseteq S^b \times T^b$, where S^b, T^b is the set of Bob's strategies and types, respectively. We say that Ann cautiously believes E if the strategies of Bob described by E is the smallest event that is believed by her conjecture and one of her secondary conjectures. Say that Ann consistenly believes E if she believes (assigns probability 1) and cautiously believes it. Imposing event-rationality and common consistent belief of event-rationality gives us RCcBER.

Which strategies are generated by RCcBER? The first round of RCcBER yields the set of event-rational types for Ann and event-rational types for Bob, just like RCBER. But the second round of RCcBER requires that each of Ann's types consistenly believes the event "Bob is rational", and similarly for Bob. Then, all types playing L are excluded. To see this, note that if Bob is event-rational and consistenly believes event "Ann is event-rational", then he must cautiously believe the strategies played by event-rational types of Ann's, namely $\{U, M\}$. The only event-rational types of Bob playing L (and consistenly believing Ann is event-rational) are the ones that assign probability 1 on Ann playing M. In order to be able to cautiously believe $\{U, M\} \times T_0^a$, where T_0^a is Ann's event-rational types, Bob must have U as a tie-breaking set in his list. Moreover, he assigns probability 1 to M and therefore needs to break ties, because L and R are outcome equivalent given his support. But L is never a best response for any conjecture with support on U. Hence, Bob, assigning probability one on M, cannot consistenly believe that Ann is event-rational.

In the third round of RCcBER, Ann consistenly believes that Bob consistenly believes that Ann is event-rational. This means that types of Ann's playing U are excluded, because those types assign positive probability to Bob's types playing L, and none of them consistently believes that Ann is event-rational. The only event-rational types of Ann playing M and of Bob playing R survive all rounds of RCcBER and generate the IA set, $\{M\} \times \{R\}$.

3 Set Up

Let (S^a, S^b, π^a, π^b) be a two player finite strategic form game, with $\pi^a : S^a \times S^b \to \mathbb{R}$, and similarly for b (as usual, a stands for Ann, and b stands for Bob). For any given topological space X, let $\Delta(X)$ denote the space of probability measures defined on the Borel subsets of X, endowed with the weak* topology. We extend π^a to $\Delta(S^a) \times \Delta(S^b)$ in the usual way: $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b)\in S^a\times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$. Similarly for π^b. A strategy $s^a \in S^a$ is a best response to a conjecture $v \in \Delta(S^b)$ if $\pi^a(s^a, v) \geq \pi^a(\hat{s}^a, v)$ for every $\hat{s}^a \in S^a$. It is denoted by $s^a \in BR^a(v)$. Similarly for b.

3.1 Admissibility and Event-Rationality

The following definition and Lemma are taken from BFK.

Definition 1. *Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is* **weakly dominated** *with respect to $X \times Y$ if there exists $\sigma^a \in \Delta(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$ and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is* **admissible** *with respect to $X \times Y$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is admissible.*

Lemma 1. *A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there exists $\sigma^b \in \Delta(S^b)$, with supp $\sigma^b = Y$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

Lexicographic beliefs have been used in dealing with the inclusion-exclusion issue identified by Samuelson (1992) (see BFK, Brandenburger (1992), Stahl (1995) and Yang (2009)). We follow an alternative approach, based on "tie-breaking lists." By a **list of subsets** of S^b we mean a collection $l^b = \{F_1, ..., F_k\}$, with $F_i \subset S^b$ for every $i = 1, ..., k$, for some $k \geq 1$, with the property that $F_i \neq F_j$ for every distinct pair $i, j \in \{1, ..., k\}$. Let L^b be the set that contains all such lists. Because S^b is a finite set, L^b is also a finite set, and we endow it with the discrete topology. Similarly for a.

For a given conjecture $v \in \Delta(S^b)$, let $\sigma^a \sim_{\text{supp } v} s^a$ denote that the mixed strategy $\sigma^a \in \Delta(S^a)$ satisfies $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for every $s^b \in$ supp v. That is, $\sigma^a \sim_{\text{supp } v} s^a$ means that σ^a is outcome equivalent to s^a in supp v.

Definition 2. *A strategy $s^a \in S^a$ is event-rational if there exists a conjecture $v \in \Delta(S^b)$ and a list $l^b \in L^b$ such that:*

- $s^a \in BR^a(v)$,

- *for each $F \in l^b$ with $F \setminus$ supp $v \neq \emptyset$ and mixed strategy $\sigma^a \in \Delta(S^a)$ with $\sigma^a \sim_{supp\ v} s^a$, there exists a conjecture $v' \in \Delta(S^b)$ with supp $v' = F \setminus$ supp v such that $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$,*

- *$S^b \in l^b$.*

Likewise for b.

The idea is that Ann uses each of the sets in the list l^b to break ties: whenever she has a conjecture $v \in \Delta(S^b)$ over Bob's choices under which s^a is optimal and s^a is outcome-equivalent to a (mixed) strategy σ^a in supp v, Ann uses each $F \in l^b$ as the "tie-breaking hypotheses": there has to exist a conjecture v' with support on $F \setminus$ supp v that justifies the choice of s^a. Ann is fully confident in her assessment v and in her best response s^a to v as long as there is no σ^a that is outcome equivalent to s^a in supp v. In that case, her probabilistic assessments are irrelevant, for whichever other conjecture $\hat{v}$ with supp $\hat{v} =$ supp v would not help Ann breaking ties between s^a and σ^a. In that case, Ann uses the tie breaking list l^b.

It is important to note that, although the "tie-breaking hypotheses" are additional measures that Ann uses to guide her choices, they do not play the role of additional hypotheses in a lexicographic framework. If s^a is indifferent to σ^a according to v, but not outcome equivalent in supp v, then there is no need to break ties. Moreover, the tie-breaking sets are not mutually disjoint, as it is the case with the supports of the measures in a lexicographic probability system. The following lemma shows the connection between admissibility and event-rationality.

Lemma 2. *For each $F \in l^b$, if s^a is event-rational under l^b and v such that supp $v \subseteq F$, then s^a is admissible with respect to $S^a \times F$. Conversely, if s^a is admissible with respect to $S^a \times F$, for each $F \in l^b$ and $S^b \in l^b$, then s^a is event-rational under l^b.*

Proof. Suppose that s^a is event-rational for v such that supp $v \subseteq F$. If supp $v = F$ then the result is immediate so suppose supp $v \subset F$ and $F \setminus$ supp $v \neq \emptyset$. Suppose there exists $\sigma^a \in \Delta(S^a)$ with $\pi(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in F$, with strict inequality for some $s^b \in F$. Because $s^a \in BR^a(v)$, we have $s^a \sim_{\text{supp } v} \sigma^a$, which implies that there exists v' with supp $v' = F \setminus$ supp v and $\pi(s^a, v') \geq \pi(\sigma^a, v')$, a contradiction. Conversely, because s^a is admissible with respect to $S^a \times S^b$, there exists v with supp $v = S^b$ such that $s^a \in BR(v)$. Moreover, for each $F \in l^b$ we have $F \setminus$ supp v $= \emptyset$.

□

3.1.1 Preference Basis

Turn now to decision theoretic considerations. Instead of using lexicographic beliefs to deal with counter-factuals, we postulate that a decision maker (Ann) has several theories. Her primary theory is captured by her preference relation $\succsim$ and the resulting probability measure μ. Let $F_0 =$ supp μ

and write $\succsim$ as $\succsim_0$. Moreover, Ann thinks outside her support, by contemplating counter-factual scenarios. This is captured by a list of conditional preferences, where the conditioning event is outside F_0. Let F_i be the support of the measure derived from each such conditional preference, written as $\succsim_i$. Hence, Ann's list l^b contains all F_i, $i \neq 0$.

Summarizing, Ann's thought is captured by a list of preferences $(\succsim_0, \succsim_1, \ldots, \succsim_k)$ and the resulting supports $(F_0, ..., F_k)$. The primary hypothesis is F_0, while all other events are the secondary hypotheses. F_0 describes Ann's frame of mind, as it contains the states that Ann considers possible. The secondary hypotheses $F_1, ..., F_k$ describe zero probability counter-factuals, since $F_0 \cap F_i = \emptyset$ for each $i = 1, \ldots, k$. Ann resorts to the secondary hypotheses only to help her decide between outcome-equivalent acts. Formally, act x is preferred to act y if $x \succsim_0 y$ and if x is outcome-equivalent to y in F_0, then we must also have $x \succsim_i y$ for all $i = 1, ..., k$. The Appendix provides a more detailed exposition and shows that the notion just defined is equivalent to event-rationality.

3.1.2 Counter-Factuals

It is important to stress that each F_i, $i > 0$, is considered impossible by Ann, as it is the support of a preference conditional on an event which is disjoint from her support, F_0. Resorting to an alternative theory to break-ties does not entail considering the alternative theory possible. For instance, one may wonder what would have happened if Germany had won World War II, and use it to help deciding whether to move to Germany or not. But one knows that Germany did not win. The following example illustrates this point further.[5]

	L	C	R
U	$4,6$	$0,0$	$4,3$
M	$0,0$	$4,6$	$0,3$
D	$2,3$	$2,3$	$0,0$

Suppose that Ann is represented by a measure with support $F_0 = \{\text{L,C}\}$ and she contemplates one counter-factual scenario, that produces a measure with support $F_1 = \{\text{R}\}$. Ann's subjective belief assigns 50% probability to L and C respectively. Conditional on F_1, Ann's subjetive belief assigns 100% probability to R. D is outcome equivalent to a coin-flip between U and M under $\succsim_0$, so Ann cannot decide between D and this coin-flip, and resorts to F_1 for help. Under $\succsim_1$, D is strictly dominated by the coin flip, so the coin flip is preferred to D (equivalently, Ann's tie-breaking list consists of the set R, and there's no conjecture supported in R that makes D better than the coin flip, so D is not event-rational). Note that R is weakly dominated by a coin flip between L and C. So Ann resorts to a secondary theory whereby Bob plays an inadmissible strategy. But, as we

[5]We thank an anonymous referee for suggesting this example.

indicated above, this does not mean that Ann does not believe that Bob plays admissible: her primary theory only considers possible Bob playing either L or C, which are admissible. So Ann believes that Bob plays admissible, and at the same time Ann uses an alternative theory to help break ties.

Moreover, the alternative theories are not restricted to be measurable with respect to Bob's rationality (or lack of it). The same referee suggested the following modification of the example:

	L	C	R	E
U	$4,6$	$0,0$	$4,3$	$4,6$
M	$0,0$	$4,6$	$0,3$	$0,0$
D	$2,3$	$2,3$	$0,0$	$0,3$

Imagine again that Ann is represented by two preferences, with $F_0 = \{\text{L,C}\}$ and $F_1 = \{\text{R,E}\}$. Conditional on F_1 Ann's subjective belief assigns 50% probability to R and E respectively, and $\succsim_0$ is as above. Ann again decides for the coin flip between U and M over D by resorting to $\succsim_1$, which is a theory that envisages Bob playing an admissible strategy E and an inadmissible strategy R. Yet again, Ann knows that Bob plays admissible (either L or C).

What is at stake here is our perspective over counter-factuals. Instead of having "infinitely less likely events" represent what Ann believes is impossible and yet possible, we fix that Ann only considers F_0 possible. The counter-factuals are the events $\{F_1, ..., F_k\}$, which Ann believes are impossible. Yet, Ann uses the information about counter-factuals on these events to help break ties.

Furthermore, in our analysis below we also consider a stronger notion of belief, that of consistent beliefs. Roughly, Ann consistently believes an event if the event is believed by her primary preference $\succsim_0$ and by one of her secondary preferences $\succsim_i$. In the second example above, say that Ann is represented by the same $\succsim_0$ as before, $F_1 = \{\text{R}\}$ and $F_2 = \{\text{E}\}$. The coin flip between U and M is again preferred to D, as it is at least as good as D for each of the secondary measures (strictly preferred under $\succsim_1$). And Ann consistently believes that Bob plays admissible, because under F_0 and F_2 Bob plays admissible.

Note that because Ann is not indifferent between two strategies that are outcome equivalent under her support, she "considers everything to be possible" in terms of how she acts. However, when reasoning about Bob, she uses her measure μ (and perhaps also an additional secondary measure, when we consider consistent beliefs) and therefore believes that Bob is event-rational. The combination of considering everything possible and believing that Bob is event-rational resolves the inclusion/exclusion tension. In the two examples above, Ann's primary theory only considers Bob playing admissible strategies, so Ann includes only admissible strategies. At the same time, event-rational Ann breaks ties with counter-factual theories that envisage Bob playing either admissible or inadmissible strategies, so Ann does not have to include all of Bob's strategies in her frame of mind.

3.2 Type Structures and Beliefs

Type structures are used to describe interactive beliefs. Because our notion of rationality has players using tie-breaking sets, a type of a player must determine a conjecture and a list of tie-breaking sets. We will use the following notation: $\bar{\Delta}(X \times Y \times Z)$ denotes the space of Borel probablity measures on the topological space $X \times Y \times Z$, endowed with the weak* topology, with marginals on Z as mass points. That is, if $\mu \in \bar{\Delta}(X \times Y \times Z)$ then the cardinality of supp $\text{marg}_Z \mu$ is equal to one. Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

Definition 3. *An (S^a, S^b)-based type structure with tie-breaking lists is a structure*

$$\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle,$$

where $\lambda^a : T^a \to \bar{\Delta}(S^b \times T^b \times L^b)$, and similarly for b. Members of T^a, T^b are called types, members of L^a, L^b are called lists and members of $S^a \times T^a \times S^b \times T^b$ are called states.

We refer to an (S^a, S^b)-based type structure with tie-breaking lists as simply a type structure. The types spaces T^a and T^b are assumed topological. The sets S^a, S^b, L^a, L^b are finite, and we endow each with the discrete topology so that they are compact spaces. The belief mappings λ^a and λ^b are assumed Borel measurable. A type structure is **complete** when λ^a and λ^b are surjective and it is **continuous** when these belief mappings are continuous. It is straightforward to verify that the standard construction of all coherent hierarchies of beliefs (c.f. Mertens and Zamir (1985) and also the Appendix) yields a type structure with continuous and surjective belief mappings and compact type spaces. Such type structures are called complete, continuous and compact type structures.

Fix an event $E \subseteq S^b \times T^b$ and write

$$B^a(E) = \{t^a \in T^a : \text{marg}_{S^b \times T^b} \lambda(t^a)(E) = 1\}$$

as the set of types that are certain of the event E. This is the standard definition of certainty (as 1-belief): the states of Bob are the strategy type pairs in $S^b \times T^b$, and Ann's beliefs are over Bob's states. The belief mapping λ^a determines such beliefs and also the tie-breaking list used by Ann, so in determining her beliefs over Bob's states what matters is the marginal on $S^b \times T^b$. Note that B^a satisfies monotonicity: if Ann is certain of E and $E \subset F$ then Ann is also certain of F.

We say that a type of Ann's **cautiously believes** event E if $\text{proj}_{S^b} E$ is the smallest event that is believed by her primary and one of her secondary measures. Fix $E \subseteq S^b \times T^b$ and define the cautious belief operator

$$B^a_c(E) = \{t^a \in T^a : \text{proj}_{S^b} E \in \text{marg}_{L^b} \lambda^a(t^a)\}.$$

A type of Ann's consistenly believes an event E if she believes and cautiously believes it. That is, the set of types of Ann that **consistently believe** an event $E \subseteq S^b \times T^b$ is given by

$$B^a_*(E) = B^a(E) \cap B^a_c(E).$$

The appendix provides a preference based characterization of the notions of beliefs defined above.

3.3 RCBER - Rationality and Common Belief of Event-Rationality

With type structures, a state for Ann is a pair (s^a, t^a) determining what she plays (s^a) and her state of mind (t^a). We extend the definition of event-rationality to strategy-type pairs as follows:

Definition 4. *Strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is event-rational if*

- $s^a \in BR^a(v)$, *for* $v = marg_{S^b} \lambda^a(t^a)$,
- *for each $F \in marg_{L^b} \lambda^a(t^a)$ with $F \setminus supp\ v \neq \emptyset$ and mixed strategy $\sigma^a \in \Delta(S^a)$ with $\sigma^a \sim_{supp\ v} s^a$, there exists a conjecture $v' \in \Delta(S^b)$ with $supp\ v' = F \setminus supp\ v$ such that $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$,*
- $S^b \in marg_{L^b} \lambda^a(t^a)$.

Likewise for b.

Let R^a_1 be the set of event-rational strategy-type pairs (s^a, t^a). For finite m, define R^a_m inductively by

$$R^a_{m+1} = R^a_m \cap [S^a \times B^a(R^b_m)].$$

Similarly for b.

Definition 5. *If $(s^a, t^a, s^b, t^b) \in R^a_{m+1} \times R^b_{m+1}$, say there is event-rationality and mth-order belief of event-rationality (RmBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} R^a_m \times \bigcap_{m=1}^{\infty} R^b_m$ say there is event-rationality and common belief of event-rationality (RCBER) at this state.*

In words, there is RCBER at a state if Ann is event-rational, Ann believes that Bob is event-rational, Ann believes that Bob believes that Ann is event-rational, and so on. Similarly for Bob. Believing that Bob is event-rational means that Ann is certain that Bob only chooses strategies that are best responses to Bob's conjectures that Ann considers possible, and that Bob breaks ties using the sets of strategies in his list.

Note that for a strategy-type pair (s^a, t^a) to belong to R^a_m the following conditions are satisfied. Strategy s^a is a best response to $v = \text{marg}_{S^b}\lambda^a(t^a)$, $\text{marg}_{S^b \times T^b}\lambda^a(t^a)(R^b_{m-1}) = 1$ and whenever $\sigma^a \sim_{\text{supp } v} s^a$, for each $E^b \in \text{marg}_{L^b}\lambda^a(t^a)$, there exists a conjecture v' in $E^b \setminus \text{supp } v$ for which $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. Notice that Ann is certain that the conjectures of Bob are of the form $v = \text{marg}_{S^a}\lambda^b(t^b)$, for $t^b \in \text{proj}_{T^b} R^b_{m-1}$, and knows that, for each such conjecture, Bob breaks each tie using some v' in $E^b \setminus \text{supp } v$. We show below that this flexibility implies that the set of strategies compatible with RCBER are the ones that survive one round of elimination of inadmissible strategies, followed by iterated elimination of strongly dominated strategies.

3.4 RCcBER - Rationality and Common consistent Belief of Event-Rationality

Let $\overline{R}^a_1$ be the set of event-rational strategy-type pairs (s^a, t^a). For finite m, define $\overline{R}^a_m$ inductively by

$$\overline{R}^a_{m+1} = \overline{R}^a_m \cap [S^a \times B^a_*(\overline{R}^b_m)].$$

Similarly for b.

The only difference with RCBER is that we use the consistent belief operator instead of the standard one.

Definition 6. *If $(s^a, t^a, s^b, t^b) \in \overline{R}^a_{m+1} \times \overline{R}^b_{m+1}$, say there is event-rationality and mth-order consistent belief of event-rationality (RmcBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \bigcap_{m=1}^{\infty} \overline{R}^b_m$ say there is event-rationality and common consistent belief of event-rationality (RCcBER) at this state.*

Because consistent beliefs are stronger than standard beliefs, RCcBER $\subseteq$ RCBER.

Note again that RCBER and RCcBER avoid the inclusion-exclusion tension. What a type t^a of Ann believes about Bob's choices is given by the marginal of $\lambda^a(t^a)$ over S^b. And a type that knows that Bob's strategy-type pairs are in $\overline{R}^b_m$ is a type that assigns positive probability only to the strategies that are consistent with $\overline{R}^b_m$. So many of Bob's strategies can be excluded from t^a's consideration, without causing any contradiction in the construction. The event-rational (s^a, t^a) resorts to the tie-breaking list $\text{marg}_{L^b}\lambda^a(t^a)$ to handle counter-factuals, without having to believe that the counter-factuals are a real possibility.

4 Solution Concepts

4.1 Self-Admissible and Hypo-Admissible Sets

By construction, event-rationality implies playing admissible strategies. If we add common belief of event-rationality, then the solution concept is that of a hypo-admissible set (HAS) that we define below. We compare the HAS with several solution concepts that have been proposed in the literature. But first a definition.

Definition 7. *Say that r^a supports s^a given Q^b if there exists some $\sigma^a \in \Delta(S^a)$ with $r^a \in supp\ \sigma^a$ and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in Q^b$. Write $su_{Q^b}(s^a)$ for the set of $r^a \in S^a$ that supports s^a given Q^b. Likewise for b.*

This is a generalization of the definition in BFK of the support of a strategy s^a, which they denote $\mathrm{su}(s^a)$. In particular, $\mathrm{su}_{S^b}(s^a) = \mathrm{su}(s^a)$.

BFK characterize rationality and common assumption of rationality (RCAR) by the solution concept of a self-admissible set (SAS).

Definition 8. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an SAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{S^b}(s^a)$, then $r^a \in Q^a$.*

Likewise for b.

In particular, BFK show that the projection of the RCAR into $S^a \times S^b$ is an SAS. Conversely, given an SAS $Q^a \times Q^b$, there is a type structure such that the projection of RCAR into $S^a \times S^b$ is equal to $Q^a \times Q^b$. BFK discuss the need for the third requirement in the definition of an SAS. In particular, consider the weak best response sets (WBRS), which does not include a restriction on convex combinations.

Definition 9. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is a WBRS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is not strongly dominated with respect to $S^a \times Q^b$.*

Likewise for b.

An "almost" characterization of the WBRS is obtained if, as in Brandenburger (1992) and Börgers (1994), common assumption of rationality is relaxed to common belief at level 0 of rationality (RCB0R) (that is, believing E means $\mu_0(E) = 1$, where μ_0 is the first measure of the agent's LPS). More specifically, on the one hand the projection of RCB0R into $S^a \times S^b$ is a WBRS. On the other hand, given a WBRS $Q^a \times Q^b$, there is a type structure such that $Q^a \times Q^b$ is contained in (but not necessarily equal to) the projection of RCB0R into $S^a \times S^b$.[6]

We are now ready to introduce the solution concept of hypo-admissible sets (HAS).

[6] See Section 11 in BFK.

Definition 10. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an HAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$.*

  For each $s^a \in Q^a$ there is nonempty $Q_0 \subseteq Q^b$ such that

- *s^a is admissible with respect to $S^a \times Q_0$,*

- *for any $s^a \in Q^a$, if $r^a \in su_{Q_0}(s^a)$ and r^a is admissible with respect to $S^a \times S^b$ then $r^a \in Q^a$.*

Likewise for b.

Note that the first two properties for a WBRS are equivalent to the first two properties for an HAS and they are implied by the first two properties for an SAS. Hence, the SAS and the HAS are always WBRS but the opposite does not hold. Moreover, an SAS is not necessarily an HAS and an HAS is not necessarily an SAS. The differences between the HAS and the SAS can be further illustrated by the following two solution concepts. The first is $S^\infty W$, the set of strategies that survive one round of deletion of inadmissible strategies followed by iterated deletion of strongly dominated strategies (Dekel and Fudenberg (1990)).

Definition 11. *Set $SW_1^i = S_1^i$, for $i = a, b$ be the set admissible strategies and define inductively for $m \geq 1$,*

$$SW_{m+1}^i = \{s^i \in SW_m^i : s^i \text{ is not strongly dominated with respect to } SW_m^a \times SW_m^b\}.$$

Let $S^\infty W = \bigcap_{m=1}^\infty SW_m^a \times \bigcap_{m=1}^\infty SW_m^a$.

The second is the set of strategies that survive iterated deletion of weakly dominated strategies, the IA set.

Definition 12. *Set $S_0^i = S^i$ for $i = a, b$ and define inductively*

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is admissible with respect to } S_m^a \times S_m^b\}.$$

A strategy $s^i \in S_m^i$ is called m-admissible. A strategy $s^i \in \bigcap_{m=0}^\infty S_m^i$ is called iteratively admissible (IA).

We then have that the $S^\infty W$ set is both an HAS and a WBRS (but not an SAS) and the IA set is an SAS and a WBRS (but not a HAS). The following game from Section 2 illustrates the various definitions.

	L	R
U	$1, 0$	$1, 3$
M	$0, 2$	$2, 2$
D	$0, 4$	$1, 1$

The IA set is $\{M\} \times \{R\}$. It is an SAS but not an HAS, because although $L \in \mathrm{su}_{\{M\}}(R)$ and L is admissible, it does not belong to the IA set. Moreover, $S^\infty W = \{U, M\} \times \{L, R\}$ is an HAS but not an SAS, because L is not admissible with respect to $\{U, M\}$. That is, in a sense the SAS captures IA whereas the HAS captures $S^\infty W$.

4.2 Generalized Self-Admissible and Hypo-Iteratively Admissible Sets

In Section 5 we show that HAS characterizes RCBER with $E = S$. With a view to obtain a characterization of RCcBER and to relate it to the concepts presented above, we introduce the following two solution concepts.

Definition 13. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an $SAS_{P^a \times P^b}$ if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{P^b}(s^a)$ and r^a is admissible with respect to $S^a \times S^b$, then $r^a \in Q^a$.*

Likewise for b.

This is a generalization of the SAS, since the only difference is that the support $\mathrm{su}_{P^b}(s^a)$ is with respect to an abstract set P^b, not S^b. This means that the SAS is equivalent to the $\mathrm{SAS}_{S^a \times S^b}$.[7] Moreover, if $Q^a \times Q^b \subseteq P^a \times P^b$ then an $\mathrm{SAS}_{Q^a \times Q^b}$ is also an $\mathrm{SAS}_{P^a \times P^b}$, but the reverse may not hold. This means that for any $P^a \times P^b$, an $\mathrm{SAS}_{P^a \times P^b}$ is also an SAS. Moreover, an $\mathrm{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is also an HAS.

Definition 14. *A set $Q^a \times Q^b$ is a hypo-iteratively admissible (HIA) set if there exist sequences of sets $\{W_i^a\}_{i=0}^\infty$, $\{W_i^b\}_{i=0}^\infty$, with $W_0^a = S^a$, $W_0^b = S^b$, such that for each $m \geq 0$,*

- *each $s^a \in W_{m+1}^a$ is admissible with respect to $S^a \times W_m^b$ and belongs to W_m^a,*
- *for any k, m, where $k \geq m$, if $s^a \in W_{k+1}^a$, $r^a \in su_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$, then $r^a \in W_{m+1}^a$,*
- *there is k such that for all $m \geq k$, $W_m^a = Q^a$.*

Likewise for b.

[7] Note that if $r^a \in \mathrm{su}_{S^b}(s^a)$ and s^a is admissible, then r^a is also admissible. Hence, the third condition for a $\mathrm{SAS}_{S^a \times S^b}$ is identical to the third condition for a SAS.

The HIA sets resemble the IA set, with the only difference that one starts with a subset of admissible strategies and always includes the strategies that are equivalent (in the sense of su_Q) to strategies that survive subsequent rounds. Moreover, the HIA can be thought of as an analogue of the best response set (BRS).[8] If we replace admissible with strongly undominated in the definition of HIA then we get a BRS. Conversely, each BRS $Q^a \times Q^b$ can be written as a modified HIA (just set $W_i^a = Q^a$ and $W_i^b = Q^b$ for all $i \geq 1$).

5 Characterization of RCBER

Our first result shows that HAS characterizes RCBER. We say that a type structure is **rich** if for each type t^a that cautiously believes events $E_i^b \times T^b$, $i = 1, \ldots, n$, where $E_1^b \supsetneq E_2^b \supsetneq \ldots \supsetneq E_n^b$, there exists type t_0^a that caustiously believes events $E_i^b \times T^b$, $i = 1, \ldots, n-1$, but not $E_n^b \times T^b$, and $\text{marg}_{S^b \times T^b} \lambda^a(t^a) = \text{marg}_{S^b \times T^b} \lambda^a(t_0^a)$. In words, for each type there is another type that differs only in that it cautiously believes fewer events.

Recall our notation: RCBER is given by $\bigcap_{m=1}^{\infty} R_m^a \times \bigcap_{m=1}^{\infty} R_m^b$.

Proposition 1. *(i) Fix a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $\text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is an HAS.*

(ii) Fix an HAS $Q^a \times Q^b$. Then there is a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$.

Proof. Throughout we keep the convention that for any two sets, E and F, $E \times F = \emptyset$ implies $E = \emptyset$ and $F = \emptyset$. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$ is empty, then the conditions for HAS are satisfied, so suppose that it is nonempty and fix $s^a \in Q^a = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a$. Then, for some t^a, (s^a, t^a) is consistent with RCBER and s^a is admissible, by Lemma 2. Since t^a believes each R_m^b, for all m, it also believes $\bigcap_{m=1}^{\infty} R_m^b$. From the conjuction and marginalization properties of belief there is $v = \text{marg}_{S^b} \lambda^a(t^a)$, with support contained in $\text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$, such that s^a is optimal under v.

Let $Q_0 = \text{supp } v$. We have that s^a is admissible with respect to $Q_0 = \text{supp } v$, which is a subset of $Q^b = \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. Suppose $s^a \in Q^a$, $r^a \in \text{su}_{\text{supp } v}(s^a)$ and r^a is admissible. From Lemma D.2 in BFK, r^a is optimal under v whenever $(s^a, t^a) \in R_1^a$.[9] Because the type structure is rich, there exists type t_0^a that is identical to t^a, except that it only cautiously believes events that consider all strategies in S^b to be possible. Since r^a is admissible, we have that $(r^a, t_0^a) \in R_1^a$. The same is true for all R_m^a, hence the third property for an HAS is satisfied.

[8]Recall that $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

[9]Lemma D.2 specifies that if F is a face of a polytope P and $x \in F$, then $su(x) \subseteq F$, where $su(x)$ is the set of points that support x. The geometry of polytopes is presented in Appendix D in BFK.

For part (ii) fix an HAS $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to $Q_{s^a} \subseteq Q^b$, there is a v with supp $v = Q_{s^a}$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q_{s^a}}(s^a)$ (Lemma D.4 in BFK).[10] Define list space L^b containing element $l^b = \{S^b\}$. Similarly for b. Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that supp $\lambda^a(s^a) = \{(s^b, l^b, s^b) : s^b \in Q_{s^a}\}$, $l^b = \{S^b\}$ and $v = \text{marg}_{S^b}\lambda^a(s^a)$ for the v found above. Similarly for b. Note that the type structure is rich, because each type t^a does not cautiously believe $E^b \times T^b$, if $E^b \neq S^b$.

First, we show that for each $s^a \in Q^a$, (s^a, s^a) is event-rational. By construction, s^a is optimal under $v = \text{marg}_{S^b}\lambda^a(s^a)$ and admissible. Hence, (s^a, s^a) is event-rational and $Q^a \subseteq \text{proj}_{S^a} R_1^a$. Suppose $(r^a, t^a) \in R_1^a$, where $t^a = s^a$. Then, $r^a \in \text{su}_{Q_{s^a}}(s^a)$ and r^a is admissible with respect to Q_{s^a}. From Lemma 2, r^a is admissible. From the definition of an HAS this implies that $r^a \in Q^a$ and $Q^a = \text{proj}_{S^a} R_1^a$. Applying similar arguments we have that $Q^b = \text{proj}_{S^b} R_1^b$.

By construction, each $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b) which consists of event-rational strategy-type pairs, hence t^a believes R_1^b and $(s^a, s^a) \in R_2^a$. This implies that $R_2^a = R_1^a$ and likewise for b. Thus, $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m, by induction. Since $\text{proj}_{S^a} R_1^a \times \text{proj}_{S^b} R_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. $\square$

That is, the strategies consistent with RCBER are the hypo-admissible strategies according to the definition of an HAS. In a complete structure, m rounds of mutual belief generate the $SW_m^a \times SW_m^b$ strategies.

Proposition 2. *Fix a complete structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m,*

$$\mathit{proj}_{S^a} R_m^a \times \mathit{proj}_{S^b} R_m^b = SW_m^a \times SW_m^b.$$

Proof. Let T_0^a be the set of types t^a such that $\text{marg}_{L^b}\lambda^a(t^a) = \{S^b\}$. From Lemma 2 we have that $(s^a, t^a) \in R_1^a$ implies s^a is admissible. Conversely, since we have a complete structure, if s^a is admissible then there exists $t^a \in T_0^a$ such that $(s^a, t^a) \in R_1^a$. Hence, $\text{proj}_{S^a} R_1^a = S_1^a = SW_1^a$ and $\text{proj}_{S^b} R_1^b = S_1^b = SW_1^b$. Suppose that for up to m we have that $\text{proj}_{S^a} R_m^a = SW_m^a$ and $\text{proj}_{S^b} R_m^b = SW_m^b$. Suppose $s^a \in SW_{m+1}^a$. Then, $s^a \in SW_m^a = \text{proj}_{S^a} R_m^a$. Because s^a is not strongly dominated with respect to $SW_m^a \times SW_m^b$, it is also not strongly dominated with respect to $S^a \times SW_m^b$. Hence, there is a v with supp $v \subseteq SW_m^b$ under which s^a is optimal. We take (s^a, t^a), $t^a \in T_0^a$, with supp $\text{marg}_{S^b \times T^b}\lambda^a(t^a) \subseteq R_m^b$ and $\text{marg}_{S^b}\lambda^a(t^a) = v$. Because s^a is admissible with respect to S^b, (s^a, t^a) is event-rational. Because $t^a \in B^a(R_m^b)$ and $R_m^b \subseteq R_k^b$, $1 \leq k \leq m$, we have that $(s^a, t^a) \in R_{m+1}^a$ and $s^a \in \text{proj}_{S^a} R_{m+1}^a$.

[10]Lemma D.4 specifies that if x belongs to a strictly positive face of a polytope P, then $su(x)$ is a strictly positive face of P.

Suppose $s^a \in \mathrm{proj}_{S^a} R^a_{m+1}$. Then, $s^a \in SW^a_m = \mathrm{proj}_{S^a} R^a_m$ and supp $\mathrm{marg}_{S^b}\lambda^a(t^a) \subseteq SW^b_m = \mathrm{proj}_{S^b} R^b_m$. Because s^a is optimal under v, where supp $v \subseteq SW^b_m$, s^a is not strongly dominated with respect to SW^b_m and therefore $s^a \in SW^a_{m+1}$. □

6 Characterization of RCcBER

The following two Propositions show that RCcBER is characterized by the HIA set and RmcBER generates the IA set in a complete type structure, for big enough m.

Recall our notation: RCcBER is given by $\bigcap_{m=1}^{\infty} \overline{R}^a_m \times \bigcap_{m=1}^{\infty} \overline{R}^b_m$.

Proposition 3.

(i) Fix a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. *Then* $\mathit{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \mathit{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$ *is an HIA set.*

(ii) Fix an HIA set $Q^a \times Q^b$. *Then there is a rich type structure* $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ *with* $Q^a \times Q^b = \mathit{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \mathit{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$.

Proof. For part (i), if $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$ is empty, then the conditions for an HIA set are satisfied, so suppose that it is nonempty.

Set $W^a_m = \mathrm{proj}_{S^a} \overline{R}^a_m$ for $m \geq 1$ and likewise for b. From Lemma 2, all strategies in $\mathrm{proj}_{S^b} \overline{R}^a_{m+1}$ are admissible with respect to $S^a \times W^b_m$ and, by construction, belong to $\mathrm{proj}_{S^b} \overline{R}^a_m$.

Suppose that for some k, m, where $k \geq m$, we have that $s^a \in W^a_{k+1} = \mathrm{proj}_{S^b} \overline{R}^a_{k+1}$, $r^a \in su_{W^b_k}(s^a) \cap W^a_m$ and r^a is admissible with respect to $S^a \times W^b_m$. This implies that for some t^a, $(s^a, t^a) \in \overline{R}^a_{k+1}$, where supp $\mathrm{marg}_{S^b}\lambda^a(t^a) \subseteq W^b_k$ and list $\mathrm{marg}_{L^b}\lambda^a(t^a)$ contains at least all sets W^b_l, for $l \leq m$. Because the type structure is rich, there exists type t^a_0, with list $\mathrm{marg}_{L^b}\lambda^a(t^a_0)$ that contains all sets W^b_l, for $l \leq m$, and nothing else. Moreover, t^a_0 is identical to t^a in all other respects. Since $r^a \in su_{W^b_k}(s^a)$, r^a is optimal given $\mathrm{marg}_{S^b}\lambda^a(t^a_0)$. Moreover, r^a is admissible with respect to $S^a \times W^b_l$, for $l \leq m$.

All these imply that $(r^a, t^a_0) \in \overline{R}^a_{m+1}$. The third condition is satisfied because $\mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$ is nonempty and the strategies are finite.

For part (ii), fix an HIA set $Q^a \times Q^b$, with sequences of sets $\{W^a_m\}_{m=0}^{m=n'}$, $\{W^b_m\}_{m=0}^{m=n}$, where $W^a_{n'} = Q^a$ and $W^b_n = Q^b$. Construct the following type structure. Similarly for a. For each $m \geq 1$, for each $s^a \in W^a_m$, find the measure $v(s^a, m)$ with support on W^b_{m-1} such that r^a is a best response to $v(s^a, m)$ if and only if $r^a \in \mathrm{su}_{W^b_{m-1}}(s^a)$. This is possible because of Lemma D.4 in BFK. Type $t^a(s^a, m)$ has a marginal $v(s^a, m)$ on S^b, marginal $l^b = \{W^b_0, \ldots, W^b_{m-1}\}$ on L^b (ommitting W^b_{m-j} if it is equal to W^b_{m-j-1}) and assigns positive probability only to strategy-types $(s^b, t^b(s^b, m-1))$, for $s^b \in W^b_{m-1}$. Finally, assign to each $s^a \in S^a$ type $t^a(r^a, 0)$ which is equal to $t^a(r^a, k)$, for some $r^a \in W^a_k$, $k > 0$.

We now show that RCcBER generates the HIA set. For $m = 1$, we show that $\text{proj}_{S^a}\overline{R}_1^a = W_1^a$. Suppose that $s^a \in W_1^a$. Because s^a is admissible and a best response to $v(s^a, 1)$, we have $(s^a, t^a(s^a, 1)) \in \overline{R}_1^a$ and $s^a \in \text{proj}_{S^a}\overline{R}_1^a$. Suppose $r^a \in \text{proj}_{S^a}\overline{R}_1^a$. Then, r^a is a best response to some measure $v(s^a, k+1)$, $k \geq 0$, for $s^a \in W_{k+1}^a$ and $r^a \in \text{su}_{W_k^b}(s^a) \cap W_0^a$. Because $(r^a, t^a(s^a, k+1))$ is event-rational, r^a is admissible. Therefore, by the second property for an HIA set, $r^a \in W_1^a$. Moreover, by construction, for each $s^a \in W_1^a$, $(s^a, t^a(s^a, 1)) \in \overline{R}_1^a$, and similarly for b.

Assume that for up to m, $\text{proj}_{S^a}\overline{R}_m^a = W_m^a$ and for each $s^a \in W_m^a$, $(s^a, t^a(s^a, m)) \in \overline{R}_m^a$. Similarly for b. Suppose that $s^a \in W_{m+1}^a$. By construction, s^a is a best response to $v(s^a, m+1)$, which has a support of $W_m^b = \text{proj}_{S^b}\overline{R}_m^b$, and it is admissible with respect to $S^a \times W_m^b$. Moreover, $\text{marg}_{L^b} t^a(s^a, m+1) = \{W_0^b, \ldots, W_m^b\}$ and type $t^a(s^a, m+1)$ assigns positive probability only to types $(s^b, t^b(s^b, m)) \in \overline{R}_m^b$, for $s^b \in W_m^b$. This implies that $(s^a, t^a(s^a, m+1)) \in \overline{R}_{m+1}^a$ and $s^a \in \text{proj}_{S^a}\overline{R}_{m+1}^a$. Suppose $r^a \in \text{proj}_{S^a}\overline{R}_{m+1}^a$. By construction, the only measures that have support which is a subset of W_m^b are measures that are associated with strategies s^a that belong to W_{k+1}^a, where $k+1 > m$. Hence, $(r^a, t^a(s^a, k+1)) \in \overline{R}_{m+1}^a$ and r^a is a best response to some measure $v(s^a, k+1)$. By construction, $r^a \in \text{su}_{W_k^b}(s^a)$. Moreover, r^a is admissible with respect to $S^a \times W_m^b$. Hence, by the second property for an HIA set we have that $r^a \in W_{m+1}^a$.

□

Proposition 4. *Fix a complete type structure* $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. *Then, for each* m,

$$\textit{proj}_{S^a}\overline{R}_m^a \times \textit{proj}_{S^b}\overline{R}_m^b = S_m^a \times S_m^b.$$

Proof. For $m = 1$, Lemma 2 and a complete structure imply $\text{proj}_{S^a}\overline{R}_1^a = S_1^a$. Suppose that for up to m we have that $\text{proj}_{S^a}\overline{R}_m^a = S_m^a$ and $\text{proj}_{S^b}\overline{R}_m^b = S_m^b$. Suppose $s^a \in S_{m+1}^a$. Then, $s^a \in S_m^a = \text{proj}_{S^a}\overline{R}_m^a$. Because s^a is admissible with respect to $S_m^a \times S_m^b$, it is also admissible with respect to $S^a \times S_m^b$ and we can take (s^a, t^a) such that $\text{supp marg}_{S^b \times T^b} \lambda^a(t^a) = \overline{R}_m^b$, $\text{marg}_{S^b} \lambda^a(t^a) = v$, $\text{marg}_{L^b} \lambda^a(t^a) = \{S^b, S_1^b, \ldots, S_m^b\}$. Then, (s^a, t^a) is event-rational and $t^a \in B_*^a(\overline{R}_k^b)$ for all $k \leq m$, which implies that $(s^a, t^a) \in \overline{R}_{m+1}^a$ and $s^a \in \text{proj}_{S^a}\overline{R}_{m+1}^a$.

Suppose $s^a \in \text{proj}_{S^a}\overline{R}_{m+1}^a$. Then, $s^a \in S_m^a = \text{proj}_{S^a}\overline{R}_m^a$ and there exists t^a such that $(s^a, t^a) \in \overline{R}_{m+1}^a$ and $\text{supp marg}_{S^b} \lambda^a(t^a) \subseteq S_m^b = \text{proj}_{S^b}\overline{R}_m^b$. Because $t^a \in B_*^a(\overline{R}_m^b)$, $S_m^b \in \text{marg}_{L^b} \lambda^a(t^a)$. Hence, we have that s^a is admissible with respect to $S_m^a \times S_m^b$ and $s^a \in S_{m+1}^a$.

□

6.1 Comparison with BFK

BFK's LPS-based approach uses the following construction. Let $\mathcal{L}^+(X)$ be the space of fully supported LPS's over X, that is, the space of finite sequences $\sigma = (\mu_0, \ldots, \mu_{n-1})$, for some integer n, where $\mu_i \in \Delta(X)$ and $\bigcup_{i=0}^{n-1} \text{supp } \mu_i = X$. In addition, the measures μ_i in σ are required to be non-overlapping, that is, mutually singular. A lexicographic type structure is a type structure where $\lambda^a : T^a \to \mathcal{L}^+(S^b \times T^b)$, and similarly for b. An event E is **assumed** if and only if the closure of the event is equal to the union of the supports of the first j levels of the player's LPS. That is, there is a level j such that the player assigns probability one to the event E for all of his/her hypothesis up to level j, and assigns probability zero to the event for all of his/her hypothesis of levels higher than j. Yang (2009) uses a weaker notion that allows the levels higher than j to assign positive (and strictly smaller than 1) weights to the event. The use of lexicographic beliefs is to be contrasted with our use of standard beliefs.

RCAR in BFK is characterized by the SAS and RmAR (m levels of mutual assumption) produces the IA set in a complete structure, for big enough m. Since RmcBER generates the IA set as well, it is important to know what is the relationship between RCAR and RCcBER in terms of the solution concepts they generate. The following Proposition and examples show that RCcBER generates a strict subclass of SAS, hence it is a more restrictive notion than RCAR. However, as we show in the following section, RCcBER and RCBER are always nonempty in a complete, continuous and compact structure, unlike RCAR. Let A^a and A^b be the set of Ann's and Bob's admissible strategies, respectively.

Proposition 5.

(i) Fix a type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is an $SAS_{A^a \times A^b}$.

(ii) Fix an $SAS_{Q^a \times Q^b}$ $Q^a \times Q^b$. Then there is a type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proof. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is empty, then the conditions for $\text{SAS}_{A^a \times A^b}$ are satisfied, so suppose that it is nonempty. By definition of event-rationality and Lemma 2, each $s^a \in Q^a = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a$ is admissible with respect to $S^a \times S^b$ and $S^a \times Q^b$.

Suppose $s^a \in Q^a$, $r^a \in \text{su}_{A^b}(s^a)$ and r^a is admissible. This implies that for any t^a, $(s^a, t^a) \in \bigcap_{m=1}^{\infty} \overline{R}_m^a$ implies that supp $\text{marg}_{S^b} \lambda^a(t^a) \subseteq A^b$ and r^a is optimal under $v = \text{marg}_{S^b} \lambda^a(t^a)$ (Lemma D.2 in BFK). Because r^a is admissible we have that $(r^a, t^a) \in \overline{R}_1^a$. For each $m \geq 2$, $(s^a, t^a) \in \overline{R}_m^a$ implies that t^a believes and cautiously believes R_{m-1}^b. Because $\text{proj}_{S^b} R_{m-1}^b \subseteq A^b$ and $r^a \in \text{su}_{A^b}(s^a)$, we have that $(r^a, t^a) \in \overline{R}_m^a$ and $r^a \in Q^a$.

For part (ii) fix an $\text{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to Q^b, there is a v with supp $v = Q^b$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q^b}(s^a)$ (Lemma D.4 in BFK). Define type spaces $T^a = Q^a$, $T^b = Q^b$, $L^a = l^a$, $L^b = l^b$, where $l^a = \{S^a\}$, $l^b = \{S^b\}$, with λ^a and λ^b chosen so that supp $\lambda^a(s^a) = \{(s^b, l^b, s^b) : s^b \in Q^b\}$ and supp $\lambda^b(s^b) = \{(s^a, l^a, s^a) : s^a \in Q^a\}$.

By construction and applying similar arguments as in the proof of Proposition 1, we have that $Q^a = \mathrm{proj}_{S^a}\overline{R}_1^a$ and $Q^b = \mathrm{proj}_{S^b}\overline{R}_1^b$. Moreover, each type $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, l^b, s^b), which consists of event-rational strategy-type pairs, hence t^a consistently believes $\overline{R}_1^b$. Since $l^b = \{S^b\}$, we have that $\overline{R}_m^a = \overline{R}_1^a$ and $\overline{R}_m^b = \overline{R}_1^b$ for all m, by induction. Since $\mathrm{proj}_{S^a}\overline{R}_1^a \times \mathrm{proj}_{S^b}\overline{R}_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$. □

In words, for a given type structure, the strategies compatible with RCcBER form a subclass of all of the SAS, and there is a class of SAS (the $Q^a \times Q^b$ sets that are $\mathrm{SAS}_{Q^a \times Q^b}$) whose strategies are compatible with RCcBER for some type structure. Because an $\mathrm{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is an $\mathrm{SAS}_{A^a \times A^b}$ but the converse is not true, Proposition 5 does not provide a characterization of RCcBER. It does show, however, that RCAR, which is characterized by SAS (BFK, Proposition 8.1), is less restrictive than RCcBER.

In fact, the following game provides an example of an SAS that is not an $\mathrm{SAS}_{A^a \times A^b}$ and cannot be generated by RCcBER for any type structure. Hence, RCcBER generates a strict subclass of SAS.

	L	C	R
U	$1,1$	$2,1$	$1,1$
M	$2,2$	$0,1$	$1,0$
D	$0,1$	$4,2$	$0,0$

Note that all strategies except for R are admissible and that $\{\mathrm{U}\} \times \{\mathrm{L}, \mathrm{C}\}$ is an SAS but not an $\mathrm{SAS}_{A^a \times A^b}$. The reason is that D and M are in the support of a mixed strategy (assigning weight 1/2 to each) that is equivalent to U given that Bob plays his admissible strategies L and C, but not given the set of all strategies S^b. Since D and M are not included in $\{\mathrm{U}\} \times \{\mathrm{L}, \mathrm{C}\}$, this is not an $\mathrm{SAS}_{A^a \times A^b}$.

We now argue that $\{\mathrm{U}\} \times \{\mathrm{L}, \mathrm{C}\}$ cannot be the outcome of RCcBER. First, note that if this were the case, the types of Ann included in RCcBER should assign zero probability to Bob playing R. Note also that U is a best response only when $Pr(\mathrm{L}) = \frac{2}{3}$ and $Pr(\mathrm{C}) = \frac{1}{3}$ and, for these conjectures, also M and D are best responses. Is it possible that M and D are excluded because types playing these strategies are not $\{\mathrm{L}, \mathrm{C}\}$-rational or S^b-rational? No, because M and D are admissible with respect to both $\{\mathrm{L}, \mathrm{C}\}$ and S^b. Hence, under RCcBER, for any type structure, whenever U is included, M and D are included as well.

In the following game all strategies are admissible, hence an SAS is equivalent to an $\mathrm{SAS}_{A^a \times A^b}$.

	L	C	R
U	$1,1$	$2,1$	$1,1$
M	$2,2$	$0,1$	$1,5$
D	$0,1$	$4,2$	$0,0$

The same arguments show that RCcBER cannot produce $\{\mathrm{U}\} \times \{\mathrm{L}, \mathrm{C}\}$ which is both an SAS and an $\mathrm{SAS}_{A^a \times A^b}$ but not an $\mathrm{SAS}_{Q^a \times Q^b}$. Hence, we cannot have a tighter characterization in terms of Proposition 5.

7 Possibility Results for RCBER and RCcBER

Since the games are assumed to be finite, Propositions 2 and 4 suggest that RmBER and RmcBER generate the $S^\infty W$ and IA sets, respectively, for m large enough. However, an epistemic criterion for $S^\infty W$ and IA has to be the same across all games and therefore independent of m. Below we show that RCBER and RCcBER are nonempty whenever the type structure is complete, continuous and compact (and recall that the universal type structure (Mertens and Zamir (1985) and Appendix) satisfies these properties), hence providing an epistemic criterion for $S^\infty W$ and IA.

Proposition 6. *Fix a complete, continuous and compact type structure* $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. *Then RCBER and RCcBER are nonempty.*

Proof. First note that from Propositions 2 and 4, the sets $R^a_m \times R^b_m$ and $\overline{R}^a_m \times \overline{R}^b_m$ are non-empty for each $m \geq 1$.

We first show that R^a_1 is closed. Note that T^a is compact. For any sequence (s^a_n, t^a_n) in R^a_1, we have $s^a_n \in BR(v^a_n)$, where $v^a_n = \mathrm{marg}_{S^b} \lambda^a(t^a_n)$. If $(s^a_n, t^a_n) \to (s^a, t^a)$, then $v^a_n \to v^a = \mathrm{marg}_{S^b} \lambda^a(t^a)$, implying that $s^a \in BR(v^a)$. Also, because S^a is finite, we have $s^a = s^a_n$ for large n, so $s^a \in BR^a(v^a_n)$. Further, because S^b is finite, we can choose a subsequence with $\mathrm{supp}\ v^a_n = \mathrm{supp}\ v^a_k$ for all indices n, k and a fortiori $\mathrm{supp}\ v^a \subset \mathrm{supp}\ v^a_n$. Let σ^a satisfy $\sigma^a \sim_{\mathrm{supp}\ v^a} s^a$. If $\mathrm{supp}\ v^a = \mathrm{supp}\ v^a_n$ we have $\sigma^a \sim_{\mathrm{supp}\ v^a_n} s^a$. Hence, for each $F_i \in \mathrm{marg}_{L^b} \lambda^a(t^a)$, there exists v_i with support equal to $F_i \setminus \mathrm{supp}\ v^a$, such that $\pi^a(s^a, v_i) \geq \pi^a(\sigma^a, v_i)$. If $\mathrm{supp}\ v^a \neq \mathrm{supp}\ v^a_n$, then because $s^a \in BR^a(v^a_n)$ and $\sigma^a \sim_{\mathrm{supp}\ v^a} s^a$, it must be that there exists $\mu \in \Delta(S^b)$ with $\pi^a(s^a, \mu) \geq \pi^a(\sigma^a, \mu)$ and $\mathrm{supp}\ \mu = \mathrm{supp}\ v^a_n \setminus \mathrm{supp}\ v^a$ (μ can be taken as the conditional of v^a_n on $\mathrm{supp}\ v^a_n \setminus \mathrm{supp}\ v^a$). Now put $\mu' = \alpha\mu + (1-\alpha)v_i$ for some $\alpha \in (0,1)$, note that $\mathrm{supp}\ \mu' = F_i \setminus \mathrm{supp}\ v^a$ and that $\pi^a(s^a, \mu') \geq \pi^a(\sigma^a, \mu')$. That is, $(s^a, t^a) \in R^a_1$, so it is a closed subset of the compact space $S^a \times T^a$.

Consider $R^a_2 = R^a_1 \cap [S^a \times B^a(R^b_1)]$, and pick a convergent sequence (s^a_n, t^a_n) therein, with limit (s^a, t^a). Because R^b_1 is closed and λ^a is continuous, we have $\limsup_{t^a_n \to t^a} \lambda^a(t^a_n)(R^b_1) \leq \lambda^a(t^a)(R^b_1)$.

Hence $\text{marg}_{S^b \times T^b} \lambda^a(t^a)(R_1^b) = 1$ because $\text{marg}_{S^b \times T^b} \lambda^a(t_n^a)(R_1^b) = 1$ for every n. Also, event-rationality follows from an argument similar to the argument above, and we conclude that R_2^a is compact. Inductively, R_m^a is compact for all m. It follows that $\bigcap_{m \geq 1} R_m^a \neq \emptyset$ because the family $\{R_m^a\}_{m \geq 1}$ has the finite intersection property: for any finite list $\{m_1, \ldots, m_K\}$ of positive numbers, let $m_{\overline{k}}$ be the largest. Then we know that $R^a_{m_{\overline{k}}} \neq \emptyset$ and it is included in $\bigcap_{k=1}^K R^a_{m_k}$.

We also have compactness of the sets $\overline{R}_m^a$. Pick a sequence (s_n^a, t_n^a) in $\overline{R}_m^a$ converging to (s^a, t^a), and without loss of generality focus on a subsequence with $\text{supp marg}_{L^b} \lambda^a(t_n^a) = \text{supp marg}_{L^b} \lambda^a(t_k^a)$ for all n, k. Because those marginals are mass points, and L^b is finite, it must be that case that $\text{supp marg}_{L^b} \lambda^a(t_n^a) = \text{supp marg}_{L^b} \lambda^a(t^a)$. Repeat the argument in the first paragraph of the proof to conclude that (s^a, t^a) is event-rational because (s_n^a, t_n^a) is event-rational for each n, and $\text{proj}_{S^b} \overline{R}_{m-1}^b \in \text{marg}_{L^b} \lambda^a(t^a)$, so $(s^a, t^a) \in \overline{R}_m^a$. Hence we have a nested sequence of non-empty compact spaces, so by the finite intersection property, we have $\bigcap_{m \geq 1} R_m^a \neq \emptyset$.

The same arguments apply to b.

□

8 Conclusion

We showed that event-rationality can be used to analyze common belief of admissibility in games. In particular, epistemic criteria for $S^\infty W$ and for IA are obtained. Moreover, IA is placed at the same level as IEDS as a solution concept. IA does require that players know more about each other than IEDS does (i.e. common consistent belief of event-rationality instead of common belief of rationality), but it certainly does not require that players know each other's conjectures. The fact that each player can perform the IA procedure on her own by considering that the other players only play admissible strategies, much as each player can perform the IEDS procedure on her own (by considering that the other players only play rational strategies), suggests that the epistemic requirements for IA ought not be much more restrictive than those for IEDS, as we indeed show using RCcBER.

Finally, because we adopt a perspective different from LPS-based approaches, our analysis is a straightforward extension of the standard analysis of common knowledge of rationality. That is, by noting that admissibility can be captured by breaking ties outside of one's conjectures, we are able to separate beliefs from conjectures and work with standard type spaces.

A Preference Basis

Let Ω be a state space and $\mathcal{A}$ the set of all measurable functions from Ω to $[0, 1]$. For simplicity, assume that Ω is finite (modulo technical details, the considerations below carry through in a more general state space). A decision maker has preferences over elements of $\mathcal{A}$. We assume that the outcome space $[0,1]$ is in utils. That is, all preferences considered below agree on constant acts over an outcome space, so the Bernoulli indices are uniquely defined and omitted from the analysis that follows. For $x, y \in \mathcal{A}$, $0 \leq \alpha \leq 1$, $\alpha x + (1-\alpha)y$ is the act that at ω gives payoff $\alpha x(\omega) + (1-\alpha)y(\omega)$. Unless otherwise noted, we assume that a preference relation $\succsim$ satisfies completeness, transitivity, independence and has an expected utility representation.

Definition 15. *$x \succsim_E y$ if for some $z \in \mathcal{A}$, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$.*

Note that for preferences satisfying the aforementioned axioms, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$ holds for all z if it holds for some z. An event E is **Savage null** if $x \sim_E y$ for all $x, y \in \mathcal{A}$. For a given $\succsim$, the set $N(\succsim) \subset \Omega$ denotes the union of all non Savage null events according to $\succsim$.

Fix a game and the resulting set of available acts $\mathcal{B}$. An act $x \in \mathcal{B}$ is **event-rational** if there exist a preference $\succsim$ and a list $l = \{F_1, ..., F_k\}$, with $F_i \subset \Omega$ for $i = 1, ..., k$ such that

- $x \succsim y$ for every $y \in \mathcal{B}$,
- for each $F_i \in l$ with $F_i \setminus N(\succsim) \neq \emptyset$ and act $y \in \mathcal{B}$ with $x(\omega) = y(\omega)$ for all $\omega \in N(\succsim)$, there exists a preference $\succsim'$ with $N(\succsim') = F_i \setminus N(\succsim)$ such that $x \succsim' y$,
- $\Omega \in l$.

Therefore, the definition of event-rationality is identical to that of the main text.

Consider a decision maker represented by a list of preferences $\{\succsim_i\}_{i=0}^k$ with $N(\succsim_i) \cap N(\succsim_0) = \varnothing$ for $i = 1, ..., k$ and $N(\succsim_i) = \Omega \setminus N(\succsim_0)$ for some i.[11] The interpretation is that $N(\succsim_0)$ is the primary hypothesis of the decision maker, and the secondary hypotheses $\{N(\succsim_i)\}_{i=1}^k$ are probability-zero counter-factuals. The preference $\succsim_0$ is the decision maker's "primary preference", and she resorts to the "secondary preferences" in $\{\succsim_i\}_{i=1}^k$ to break ties. Formally, given a list of preferences $\{\succsim_i\}_{i=0}^k$ satisfying the aforementioned two properties we define an induced preference relation over acts, $\succsim^c$, as follows:

Definition 16. *$x \succsim^c y$ if and only if either*

- *$x \succsim_0 y$ and $x \neq y$ on $N(\succsim_0)$ or*
- *$x = y$ on $N(\succsim_0)$ and $x \succsim_i y$ for $i = 1, ..., k$.*

Note that $\succsim^c$ is incomplete but transitive. An act x is $\succsim^c$-rational if $x \succsim^c y$ for every $y \in \mathcal{B}$.

Proposition 7. *An act x is $\succsim^c$-rational if and only if it is event-rational.*

[11]One can think of conditional preferences, as in Luce and Krantz (1971), Fishburn (1973) and Ghirardato (2002).

Proof. By definition, if x is $\succsim^c$-rational, then it is event-rational under $\succsim = \succsim_0$ and $l = \{F_1, ..., F_k\}$, with $F_i = N(\succsim_i) \cup N(\succsim_0)$ for $i = 1, ..., k$.

Conversely, let x be event-rational under $\hat{\succsim}$ and $l = \{F_1, ..., F_k\}$. If $x \neq y$ on $N(\hat{\succsim})$, then $x \succsim^c y$ using $\succsim_0 = \hat{\succsim}$. So let us focus on acts in $C = \{y \in \mathcal{B} : y = x \text{ on } N(\hat{\succsim})\}$. Let $m = \#\Omega \setminus N(\hat{\succsim})$, and note that the set C can be identified as a convex in $[0,1]^m$, with $x \in C$. For each $i = 1, ..., k$ where $E_i = F_i \setminus N(\hat{\succsim}) \neq \emptyset$, let $B_i = \{r \in \mathbb{R}^m_+ : r|_{E_i} \gg x|_{E_i}\}$, where $x|_{E_i}$ denotes the vector x restricted to states in E_i. Note that $B_i \cap C = \emptyset$, because otherwise there would exist an act y that is outcome-equivalent to x and strictly preferred to x for any preference $\succsim'$ with $N(\succsim') = E_i$, contradicting event-rationality of x. Because B_i is also convex, by the separating hyperplane theorem there exists $\alpha_i \in \mathbb{R}^m$ with $\alpha_i \cdot r > \alpha_i \cdot y$ for all $r \in B_i$ and $y \in C$. Take $r^\varepsilon \in \mathbb{R}^m_+$ with $r^\varepsilon(\omega) = x(\omega)$ for $\omega \notin E_i$ and $r^\varepsilon(\omega) = x(\omega) + \varepsilon$ for $\omega \in E_i$ and $\varepsilon > 0$. Then $r^\varepsilon \in B_i$. Letting $\varepsilon \to 0$, we have $r^\varepsilon \to x$ and we obtain $\alpha_i \cdot x \geq \alpha_i \cdot y$ for every $y \in C$.

Also, α_i can be chosen to satisfy $\alpha_i(\omega) > 0$ only if $\omega \in E_i$. Otherwise, say that $\alpha_i(\omega') > 0$ and $\omega' \notin E_i$. If $y(\omega') = 0$ for every act in $\mathcal{B}$, then $\alpha_i(\omega')$ can be set equal to zero without loss. If $x(\omega') = 0$ and there exists $y \in C$ with $y(\omega') > 0$, then it cannot be the case that $F_i = \{\omega'\}$ for any $i = 1, ..., k$. So set $y(\omega) = x(\omega)$ for every $\omega \neq \omega'$ and $y(\omega') > x(\omega')$, with $y \in C$. Such a y exists because $E_i \neq \Omega \setminus N(\hat{\succsim})$ (if it was equal, then ω' would not exist) and there is no F_i equal to $\{\omega'\}$. Then $\alpha_i \cdot r^\varepsilon > \alpha_i \cdot y$, for the r^ε constructed above. But as $\varepsilon \to 0$, $r^\varepsilon \to x$ and $\alpha_i \cdot x < \alpha_i \cdot y$ by construction. This contradicts $\alpha_i \cdot r^\varepsilon > \alpha_i \cdot y$ for all ε. In the case that $x(\omega') > 0$, change the r^ε above by having $r^\varepsilon(\omega') = 0$, while keeping the other values. Then as $\varepsilon \to 0$, we must get $\alpha_i \cdot r^\varepsilon < \alpha_i \cdot x$, another contradiction. So the support of α_i is contained in E_i.

Moreover, because for each $y \in C$ there exists $\succsim'$ with $N(\succsim') = E_i$ and $x \succsim' y$, it must be that $\alpha(\omega) > 0$ if $\omega \in E_i$. If not, then there is $\omega' \in E_i$ with $\alpha_i(\omega') = 0$, and there is no other α'_i with $\alpha'_i(\omega') > 0$ that would separate B_i and C. Now take the original r^ε and $y \in C$ with $y(\omega') > x(\omega')$. Such a y must exist, for otherwise there would exist the required α'_i. But there is no $\succsim'$ with $N(\succsim') = E_i$ and $x \succsim' y$, a contradiction. So it must be that $\alpha_i(\omega) > 0$ if and only if $\omega \in E_i$.

Normalizing α_i yields a probability distribution ν_i with $\operatorname{supp} \nu_i = E_i$ for which x is a better response than any $y \in C$. Let $\succsim_i$ be the preference relation represented by the underlying Bernoulli index and ν_i. The construction above is true for every $i = 1, ..., k$. Setting $\succsim_0 = \hat{\succsim}$ and collecting the list $\{\succsim_0, \succsim_1, ..., \succsim_k\}$ it follows that x is $\succsim^c$-rational. $\square$

In what follows, for ease of notation, we use $N_i = N(\succsim_i)$ for $i = 0, ..., k$, $x \succ_{iE} y$ to denote that x is preferred to y according to $\succsim_i$ conditional on E (according to Definition 15), and $x =_{0E} y$ to denote that $x(\omega) = y(\omega)$ for all $\omega \in N_0 \cap E \neq \emptyset$. The notions of beliefs we use in the main text are as follows.

Definition 17. *Event E is believed under $\succsim^c$ if $N_0 \subset E$.*

Definition 18. *Event E is cautiously believed under $\succsim^c$ and i if $E = N_0 \cup N_i$.*

In words, the decision maker believes an event E if she believes it according to her primary preference; and she cautiously believes it if it is the smallest event that is believed according to her primary and one other of her preference relations. Note that it may well be that $i = 0$, so $E = N_0$ is cautiously believed. Ann consistenly believes E if she believes and cautiously believes it. In the text, cautious belief is restricted to events that describe strategies only. Here we deal with the general case, so that cautious belief is equivalent to consistent belief. It is straightforward to consider a product state space $\Omega = \Omega_1 \times \Omega_2$ and define belief for events on Ω and cautious belief for events on Ω_1.

We now define a notion of conditional $\succsim^c$-preference that is consistent with tie-breaking ideas.

Definition 19. *Say that $x \succ_E^c y$ under i if*

- *$x \succ_{0E} y$ or*
- *$x =_{0E} y$, $x \succ_{iE} y$ and $x \succsim_j y$ for every $j \neq i$.*

Say that $x \succ_E^c y$ if $x \succ_E^c y$ for some i. Note that $x \succ_E^c y$ under i and $x =_{0E} y$ necessarily mean that $i > 0$.

Definition 20. *An event E is **non trivial under** $\succsim^c$ and i if*

- *there is a pair x, y with $x \succ_E^c y$ under i, and*
- *if $\omega \in E$ is such that there is no pair x, y with $x \succ_\omega^c y$, then there is a pair x, y with $x = y$ on N_0 such that $x \succ_{E(\omega)}^c y$ under i, where $E(\omega) = E \cap (N_0 \cup \{\omega\})$.*

Definition 21. *An event E satisfies **strict determination under** $\succsim^c$ and i if for all x, y, $x \succ_E^c y$ under i implies $x \succ^c y$.*

The following Lemma characterizes cautious belief with respect to non-triviality and strict determination.

Lemma 3. *There exists i such that E is cautiously believed under $\succsim^c$ and i if and only if it is non trivial and satisfies strict determination under $\succsim^c$ and i.*

Proof. By non triviality, $E \cap N_0 \neq \varnothing$, for otherwise there would exist no pair x, y with $x \succ_E^c y$. Assume by way of contradiction that there exists $\hat{\omega} \in N_0 \setminus E$. Also, let $\omega' \in E \cap N_0$. Set $x(\omega') = 1$ and zero otherwise, and set

$$y(\omega) = \begin{cases} a & \text{if } \omega = \hat{\omega} \\ b & \text{if } \omega = \omega' \\ 0 & \text{otherwise} \end{cases}$$

where $a > \frac{v_0(\omega')(1-b)}{v_0(\hat{\omega})}$, $0 < b < 1$, and v_0 is the conjecture associated with $\succsim_0$. Then, conditional on E, the payoff of x is equal to 1 whereas the payoff of y is $b < 1$, so $x \succ^c_E y$; But the unconditional payoff of x is equal to $v_0(\omega')$ whereas the payoff of y is $av_0(\hat{\omega}) + bv_0(\omega')$, so $y \succ^c x$, contradicting strict determination. Hence $N_0 \subset E$. Therefore, if for all $\omega \in E$ there exists a pair x, y with $x \succ^c_\omega y$, then $E \subset N_0$, and we conclude that $E = N_0 \cup N_i$, with $i = 0$.

If there is $\omega \in E$ for which there is no pair x, y with $x \succ^c_\omega y$, then $\omega \notin N_0$. By non triviality, there is a pair x, y with $x = y$ on N_0 with $x \succ^c_{E(\omega)} y$ under i, meaning that $x \succ_{iE(\omega)} y$, which in turn means that $\omega \in N_i$ and $i \neq 0$. Hence we must have $E \subset N_0 \cup N_i$. Similarly to above, assume by way of contradiction that there exists $\hat{\omega} \in N_i \setminus E$. Also, let $\omega' \in E \cap N_i$. Construct x and y as follows: $x = y$ on N_0, and on $\Omega \setminus N_0$ x and y are as above, with $a > \frac{v_i(\omega')(1-b)}{v_i(\hat{\omega})}$. Strict determination is again violated, so we must have $N_0 \cup N_i \subset E$, and we conclude that $E = N_0 \cup N_i$ with $i > 0$.

Conversely, assume that $E = N_0 \cup N_i$ for some i. Let $x = 1$ on N_0, 0 otherwise and $y(\omega) = 0$ for every ω. Then $x \succ^c_0 y$ and $x \succ^c_E y$ under i. For the second condition, if $i = 0$, then $E = N_0$ and there does not exist $\omega \in E$ such that there is no pair x, y with $x \succ^c_\omega y$. If $i \neq 0$, pick $\omega \in N_i$ (so $\omega \notin N_0$). Set $x = y$ on N_0, $x(\omega) = 1$, $y(\omega) = 0$ and $x = y = 0$ elsewhere. Then $x \succ^c_{E(\omega)} y$, so non triviality is satisfied.

Finally, let $x \succ^c_E y$ under i. If $x \succ_{0E} y$ then $x \succ_0 y$, implying that $x \succ^c y$. If $x =_{0E} y$, $x \succ_{iE} y$ and $x \succsim_j y$ for every $j \neq i$, then $x = y$ on N_0, $x \succ_i y$ and $x \succsim_j y$ for every $j \neq i$, which again means that $x \succ^c y$. So strict determination is satisfied.

$\square$

Corollary 1. *An event E is believed under $\succsim^c$ if and only if it satisfies strict determination under $\succsim^c$ and $i = 0$ and there exists a pair x, y with $x \succ^c_E y$ under $i = 0$.*

B Sequentially rationalizable choice

The preference basis provided above postulates that the decision maker is represented by a list of conditional preferences, as in Luce and Krantz (1971), Fishburn (1973) and Ghirardato (2002), to name a few. We also provide an alternative, more direct approach using choice correspondences rather than preferences, based on Manzini and Mariotti (2007). In particular, we show that event-rational strategies can be generated by a procedure of sequentially maximizing under two rationales (strict preferences). Note that although Manzini and Mariotti (2007) use their model to explain cyclical patterns of choice, we show that the particular choice modelled here does not exhibit cyclical patterns.

Let X be a finite set of alternatives with $|X| > 2$. Given an asymmetric preference relation $\succ$, denote the set of $\succ$-maximal elements of $S \subseteq X$ by

$$\max(S; \succ) = \{x \in S : \nexists\, y \in S \text{ for which } y \succ x\}.$$

Let $\mathcal{P}(X)$ be the set of all nonempty subsets of X. A choice correspondence $\gamma : \mathcal{P}(X) \rightrightarrows X$ selects a set of alternatives from each element of $\mathcal{P}(X)$.

Definition 22. *A choice correspondence is sequentially rationalized by an ordered pair $(\succ_1, \succ_2)$ of asymmetric relations, with $\succ_i \subseteq X \times X$ for $i = 1, 2$, if*

$$\gamma(S) = \max(\max(S; \succ_1); \succ_2) \text{ for all } S \in \mathcal{P}(X).$$

We call $\succ_i$ a rationale.

If γ is a function then it is called a Rational Shortlist Method (RSM) by Manzini and Mariotti (2007). It is characterized with respect to two axioms, Weak WARP and Expansion. Manzini and Mariotti (2007) also show that RSMs have empirical content, in the sense that they only allow for a specific type of irrationality and are testable.

We say that list $l = \{F_1, ..., F_k\}$ is proper under measure μ if supp $\mu \cap F = \emptyset$ for each $F \in l$ and supp $\mu \cup F = X$ for some $F \in l$. If μ is understood, we just say that list l is proper.

We assume that the agent uses a sequentially rationalizable choice, using two rationales. The first rationale is the Pareto criterion. Suppose there are two strategies σ_1, σ_2, that are outcome equivalent under the support of her primary conjecture. Then, the agent has no way of distinguishing between the two according to her primary conjecture, and she resorts to her secondary measures. Because these are not ordered (as in lexicographic preferences), she treats them equally. In particular, she strictly prefers σ_1 to σ_2 if and only if σ_1 Pareto dominates σ_2.

Definition 23. *Say that $\succ_1$ satisfies the Pareto criterion given proper list $l = \{F_1, ..., F_k\}$ and measure μ if there exist measures μ_i, where supp $\mu_i = F_i$, $i = 1, \ldots k$, such that for each pair of strategies σ_1, σ_2, we have $\sigma_1 \succ_1 \sigma_2$ if and only if $\sigma_1 \sim_{supp\ \mu} \sigma_2$, $\pi^a(\sigma_1, \mu_i) \geq \pi^a(\sigma_2, \mu_i)$ for all $i = 1, \ldots, k$, and $\pi^a(\sigma_1, \mu_i) > \pi^a(\sigma_2, \mu_i)$ for some i.*

Ann strictly prefers σ_1 to σ_2 according to $\succ_1$ if and only if they are outcome equivalent under her primary measure μ and σ_1 Pareto dominates σ_2, under her secondary measures. Note that if σ_1 and σ_2 are not outcome equivalent given supp μ, then they are not ranked according to $\succ_1$, which is therefore incomplete but transitive.

The second rationale, $\succ_2$, is derived from Ann's primary measure.

Definition 24. *Say that asymmetric and transitive preference $\succ_2$ is derived given proper list $l = \{F_1, ..., F_k\}$ and measure μ if there exist measures μ_i, where supp $\mu_i = F_i$, $i = 1, \ldots k$, such that for each pair of strategies σ_1, σ_2,*

- $\pi^a(\sigma_1, \mu) > \pi^a(\sigma_2, \mu)$ *implies* $\sigma_1 \succ_2 \sigma_2$,
- $\pi^a(\sigma_1, \mu_i) > \pi^a(\sigma_2, \mu_i)$, *for some* i, *and* $\sigma_1 \sim_{supp\ \mu} \sigma_2$, *implies* $\sigma_2 \not\succ_2 \sigma_1$.

The first condition specifies that $\succ_2$ respects the strict preferences implied by conjecture μ. To understand the second condition, note that if $\sigma_1 \sim_{\text{supp } \mu} \sigma_2$, then Ann cannot distinguish between the two strategies given the support of her primary measure and uses the Pareto criterion. Because $\pi^a(\sigma_1, \mu_i) > \pi^a(\sigma_2, \mu_i)$, for some i, there are two cases. First, σ_1 Pareto dominates σ_2 under her secondary measures. Then, having $\sigma_2 \succ_2 \sigma_1$ would contradict $\sigma_1 \succ_1 \sigma_2$. Second, $\pi^a(\sigma_2, \mu_j) > \pi^a(\sigma_1, \mu_j)$ for $j \neq i$, which means that the Pareto criterion does not rank σ_1 and σ_2, because no strategy dominates the other. This means that Ann has no way of distinguishing between the two strategies by relying to the two rationales, so we specify that neither $\sigma_2 \succ_2 \sigma_1$ nor $\sigma_1 \succ_2 \sigma_2$ is true.[12]

Lemma 4. *Suppose that $(\succ_1, \succ_2)$ sequentially rationalizes γ, where $\succ_1$ satisfies the Pareto criterion and $\succ_2$ is derived given proper list l and measure μ. If $\{s^a\} = \gamma(\Delta(S))$, then s^a is admissible with respect to $S^a \times E^b$, where $E^b =$ supp $\mu \cup F_i$, for each $F_i \in l$.*

Proof. Suppose that $\{s^a\} = \gamma(\Delta(S))$ and fix $F_i \in l$. Let $E^b = \text{supp } \mu \cup F_i$. Suppose there exists $\sigma^a \in \Delta(S^a)$ with $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in E^b$, with strict inequality for some $s^b \in E^b$. We first show that $s^a \in BR^a(\mu)$. Suppose not. Then, there exists σ such that $\pi^a(\sigma, \mu) > \pi^a(s^a, \mu)$, which implies that $\sigma \succ_2 s^a$. By the definition of sequential rationalizability, $\sigma \notin \max\ (\Delta(S); \succ_1)$. But this implies that there exists $\sigma' \in \max\ (\Delta(S); \succ_1)$, such that $\sigma' \sim_{\text{supp } \mu} \sigma$, $\sigma' \succ_1 \sigma$ and $s^a \succ_2 \sigma'$. These imply that $\pi^a(s^a, \mu) \geq \pi^a(\sigma', \mu) = \pi^a(\sigma, \mu)$, a contradiction. Because $s^a \in BR^a(\mu)$, we have $s^a \sim_{\text{supp } \mu} \sigma^a$. If s^a and σ^a are ranked by $\succ_1$, then $s^a \succ_1 \sigma^a$ and $\pi^a(s^a, \mu_i) \geq \pi^a(\sigma^a, \mu_i)$, supp $\mu_i = F_i$, a contradiction.

If s^a and σ^a are not ranked by $\succ_1$, there are two cases. First, $\sigma^a \notin \max\ (\Delta(S); \succ_1)$, which implies that there exists $\sigma' \in \max\ (\Delta(S); \succ_1)$ such that $\sigma' \succ_1 \sigma^a$. This implies that $\sigma' \sim_{\text{supp } \mu} \sigma^a$ and $\pi^a(\sigma', \mu_i) \geq \pi^a(\sigma^a, \mu_i)$, for all i, with strict inequality for some i. Hence, s^a and σ' are not ranked by $\succ_1$, which implies that either there exist i and j such that $\pi^a(s^a, \mu_i) > \pi^a(\sigma', \mu_i)$ and $\pi^a(\sigma', \mu_j) > \pi^a(s^a, \mu_j)$, or $\pi^a(\sigma', \mu_i) = \pi^a(s^a, \mu_i)$ for all i. In the latter case and because of the transitivity of the Pareto criterion, s^a Pareto dominates σ^a and $s^a \succ_1 \sigma^a$, a contradiction. In the former case, s^a and σ' are not ranked by $\succ_2$. This is impossible, because transitivity and the fact that γ picks a unique element from $\Delta(S)$ imply that $s^a \succ_2 \sigma'$.

Second, $\sigma^a \in \max\ (\Delta(S); \succ_1)$. Because s^a and σ^a are not ranked by $\succ_1$, we have that either there exist i and j such that $\pi^a(s^a, \mu_i) > \pi^a(\sigma^a, \mu_i)$ and $\pi^a(\sigma^a, \mu_j) > \pi^a(s^a, \mu_j)$, or $\pi^a(\sigma^a, \mu_i) = \pi^a(s^a, \mu_i)$ for all i. In the latter case, σ^a cannot weakly dominate s^a on $E^b = \text{supp } \mu \cup F_i$, a contradiction. In the former case, s^a and σ^a are not ranked by $\succ_2$. This is impossible, because transitivity and the fact that γ picks a unique element from $\Delta(S)$ imply that $s^a \succ_2 \sigma^a$.

□

[12] Note that there is the case where $\sigma_1 \sim_{\text{supp}\mu} \sigma_2$ and $\pi^a(\sigma_2, \mu_i) = \pi^a(\sigma_1, \mu_i)$, for all i. This is the case of total indifference, so it is plausible that Ann could have a ranking between σ_1 and σ_2, for example by flipping a coin.

Then, we can define event-rationality given a type space as follows.

Definition 25. *Strategy-type pair* $(s^a, t^a) \in S^a \times T^a$ *is event-rational if* $\{s^a\} = \gamma(\Delta(S))$ *and* $(\succ_1, \succ_2)$ *sequentially rationalizes* γ*, where* $\succ_1$ *satisfies the Pareto criterion and* $\succ_2$ *is derived given proper list* $marg_{L^b}\lambda^a(t^a)$ *and measure* $marg_{S^b}\lambda^a(t^a)$*.*

Finally, we show that sequentially rationalizable choice does not exhibit cyclical patterns. For ease of notation, write $\gamma(x_1 x_2)$ instead of $\gamma(\{x_1, x_2\})$.

Definition 26. No Binary Cycles *For all* $x_1, \ldots, x_{n+1} \in X$*:* $[\gamma(x_i x_{i+1}) = x_i,\ i = 1, \ldots, n] \implies [x_1 = \gamma(x_1 x_{n+1})]$.

We show that if γ is restricted to be a function, it satisfies no binary cycles.

Lemma 5. *Suppose that* $(\succ_1, \succ_2)$ *sequentially rationalizes function* γ*, where* $\succ_1$ *satisfies the Pareto criterion and* $\succ_2$ *is derived given proper list* l *and measure* μ*. Then* γ *satisfies no binary cycles.*

Proof. We first show for $n = 3$. The argument easily generalizes for $n > 3$. Note that both $\succ_1$ and $\succ_2$ are transitive. Suppose $x_1 = \gamma(x_1 x_2)$ and $x_2 = \gamma(x_2 x_3)$. To prove by contradiction, suppose that $x_3 = \gamma(x_1 x_3)$.

There are two cases. First, $x_3 \succ_1 x_1$. This implies that $x_3 \sim_{\text{supp } \mu} x_1$ and $\pi^a(x_3, \mu_i) > \pi^a(x_1, \mu_i)$ for some i. Hence, $x_1 \not\succ_2 x_3$. If $x_1 \succ_1 x_2$ then $x_3 \succ_1 x_2$, a contradiction. If $x_2 \succ_1 x_3$ then $x_2 \succ_1 x_1$, a contradiction. If $x_1 \succ_2 x_2$ and $x_2 \succ_2 x_3$ then $x_1 \succ_2 x_3$, a contradiction.

Second, $x_3 \succ_2 x_1$. If $x_1 \succ_1 x_2$ and $x_2 \succ_1 x_3$ then $x_1 \succ_1 x_3$, so $x_1 = \gamma(x_1 x_3)$, a contradiction. If $x_1 \succ_2 x_2$ and $x_2 \succ_2 x_3$ then $x_1 \succ_2 x_3$, a contradiction. If $x_1 \succ_2 x_2$ and $x_2 \succ_1 x_3$ then, because $x_3 \succ_2 x_1$, we have $x_3 \succ_2 x_2$. But $x_2 \succ_1 x_3$ implies $x_3 \not\succ_2 x_2$, a contradiction. Finally, if $x_1 \succ_1 x_2$ and $x_2 \succ_2 x_3$, because $x_3 \succ_2 x_1$, we have $x_2 \succ_2 x_1$. But $x_1 \succ_1 x_2$ implies $x_2 \not\succ_2 x_1$, a contradiction. $\square$

C Type Spaces

In what follows, let $\bar{\Delta}(X \times L^b)$ denote the space of Borel probability measures on $X \times L^b$ endowed with the weak* topology, and with marginals on L^b (L^a for Bob's construction) as mass points. Let $\Omega_1^a = S^b \times L^b$ and $T_1^a = \bar{\Delta}(S^b \times L^b)$. Inductively set $\Omega_{k+1}^a = S^b \times L^b \times T_k^b$ where

$$T_{k+1}^a = \{(\mu_1^a, ..., \mu_k^a, \mu_{k+1}^a) \in T_k^a \times \bar{\Delta}(\Omega_{k+1}^a) : \text{marg}_{\Omega_k^a} \mu_{k+1}^a = \mu_k^a\}$$

likewise for b. Then the standard arguments in the literature (Mertens and Zamir (1985)) show the existence of compact spaces T^a and T^b, with T^a homeomorphic to $\bar{\Delta}(S^b \times T^b \times L^b)$ and T^b homeomorphic to $\bar{\Delta}(S^a \times T^a \times L^a)$. Letting λ^a and λ^b denote the homeomorphisms, we have constructed a complete, continuous and compact type structure.

It is important to emphasize a conceptual point here. The two players form beliefs about beliefs about what is relevant for rational choices. That is, Ann has beliefs over $S^b \times L^b$, and these beliefs are given by a conjecture over S^b and a point belief over lists (that is, a single list, not a general probability distribution over L^b). What is relevant for event-rational choices is precisely the conjecture and the list. But Ann does not know what Bob's beliefs are, and the hierarchies of beliefs about beliefs constructed above yield a type structure as the one we use in the paper. Also important, given Proposition 7, it is without loss to consider hierarchies of beliefs over $S^b \times L^b$, instead of hierarchies of beliefs over lists of preferences: it suffices that players keep track of the supports of the secondary measures.

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