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# EFFICIENT PROVISION OF PUBLIC GOODS WITH ENDOGENOUS REDISTRIBUTION*

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ABSTRACT. We study a continuous and balanced mechanism that is capable of implementing in Nash equilibrium all the Pareto-efficient individually rational allocations for an economy with public goods. The Government chooses a set of weights directly related to the Lindahl prices corresponding to the Pareto-efficient allocation it wants to implement. The mechanism then guarantees that initial endowments are re-allocated so that the chosen vector of Lindahl prices is indeed a Lindahl equilibrium, and implements the corresponding Lindahl allocation.

Previously known mechanisms that implement the Lindahl correspondence do not allow the Government to choose which point on the Pareto frontier should be implemented, unless it can also redistribute initial endowments in the appropriate way. By contrast, in our case the Government directly controls the distribution of welfare in the economy.

Finally, besides being balanced and continuous, our mechanism is 'simple'. Each agent has to declare a desired increase in the amount of public good, and a vector of redistributive transfers of initial endowments (across other agents).

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# 1. Introduction

## 1.1. Motivation

Economists have long been concerned with the efficient allocation of resources in an economy with public goods. At least since Samuelson (1954) it has been an issue central to modern analytical Economics.

The problem is, of course, simple to state. Because of their non-rival and non-excludable characteristics, consumers have an incentive to under-bid for the public goods. In a competitive environment, the price of public goods will therefore be below their marginal social value. In the absence of Government intervention or other corrective mechanisms, this in turn will generate an inefficiently low, under-supply of public goods.

The study of mechanisms (games) that elicit sufficient information about agents' preferences and other relevant data in the economy so that an efficient level of provision of public goods can be ensured is not a new venture by any stretch of the imagination.[1] This paper is a contribution to the study of such implementation mechanisms.

We study a continuous[2] and balanced[3] mechanism that is capable of implementing in Nash equilibrium all the Pareto-efficient individually rational allocations in an economy with public goods. The Government has a large degree of choice in our set-up. It can choose a set of weights (Lindahl prices) that correspond to the share of public expenditure that each agent has to pay. The mechanism we present then ensures that initial endowments are re-allocated so that the chosen vector of weights is indeed a vector of (constrained) equilibrium Lindahl prices, and implements the corresponding Lindahl equilibrium allocation.

In the mechanism that we analyze, by its choice of Lindahl prices, the Government is able directly to control the distribution of welfare in the economy. We do not

[1] The relevant literature is truly vast, and we do not even try to survey it anywhere in this paper. We devote Subsection 1.2 below to an attempt to fit the present paper into the previous, directly related, literature of which we are aware.

[2] In the obvious sense that the outcome that it generates is a continuous function, except at the boundary, of the actions of the participants.

[3] In the obvious sense that the Government's budget is always balanced, both in and out of equilibrium.

model the process that leads the Government to its choice of Lindahl prices. Since all Pareto-efficient individually rational allocations are implementable in our model, the choice of Lindahl prices becomes a purely distributional problem. The use of the mechanism we propose would open the way for a benevolent Government to attempt to aggregate social preferences over the distribution of welfare to guide its choice of Lindahl prices. In particular a variety of voting mechanisms could yield outcomes that are deemed desirable according to predefined criteria. In other words, a system of 'fiscal democracy' could be used to make the distributional choice. The mechanism that we analyze below is then able to deliver the corresponding efficient allocation of resources in the economy.

### *1.2. Related Literature*

As we mentioned in Subsection 1.1, it would be foolish to even attempt a survey of the literature on the optimal provision of public goods.[4] What follows is just an attempt to fit the present paper into the literature concerned with *mechanisms* that *implement* an efficient allocation of resources in the presence of public goods. Even this particular strand of literature is too vast to make a proper review viable.[5] We simply refer to those papers of which we are aware, and that are directly related to the present one.

The first attempts to construct a mechanism that decentralizes an efficient allocation of resources in an economy with public goods can be traced back to at least Clarke (1971), Groves and Ledyard (1977), and Groves and Ledyard (1980). These papers proposed mechanisms that guarantee an efficient allocation of resources in an economy with public goods and in which 'truthful revelation' is a *dominant strategy* for every participant.

Laffont and Maskin (1980) provide a cornerstone that changed the focus of all subsequent contributions. They show that the quest for a dominant strategy mechanism as, for instance, in Groves and Ledyard (1977) is an impossible goal. In particular, they demonstrate that (even restricting attention to quasi-linear preferences) it

[4]As a starting point for an overview of the literature on public goods, we refer the reader to Green and Laffont (1979) and Laffont (1982) and the references therein.

[5]Laffont and Maskin (1982) provide an overview of the literature on incentives. Their paper surveys much of the literature we are referring to indirectly here.

is impossible to design a mechanism that will implement an efficient allocation of resources in an economy with public goods that is also *balanced* in the sense that the Government budget is always balanced, both in and out of equilibrium. Thus, if we take the issue of the (game-theoretic) 'credibility' of the mechanism seriously, we are restricted to mechanisms that implement Pareto-efficient allocations in Nash equilibrium whenever we are unwilling to restrict attention to some narrow class of preferences.[6]

Hurwicz (1979b) and Hurwicz (1979a) characterize the set of allocations that are implementable in Nash equilibrium and present a mechanism that implements in Nash equilibrium the Lindahl equilibria of an economy with public goods. Postlewaite and Wettstein (1989) and Tian (1989) shift the focus of attention to *continuous* mechanisms that respectively implement in Nash equilibrium the Walrasian and Lindahl correspondences. In particular Tian (1989) presents a continuous and balanced mechanism that implements the Lindahl correspondence in an economy with public goods. The emphasis on continuity can be traced to a dissatisfaction with previous mechanisms that were deemed too complex to be viewed as practically 'playable games'

This paper proceeds one step further. We present a continuous and balanced mechanism in which the Government is in effect able to *choose* the Lindahl prices (the shares of total public expenditure to be paid by each individual in the economy) and implements the corresponding Pareto-efficient allocation in Nash equilibrium. Thus, the Government is able to choose which point on the Pareto-efficient frontier of the economy it wants to implement. This is clearly not possible using a mechanism that simply implements the Lindahl correspondence as in Tian (1989) since in general the set of equilibrium Lindahl prices corresponding to a given vector of initial endowments is finite. The redistribution of endowments necessary to reach all (individually rational) Pareto-efficient allocations is endogenous in our mechanism.

### *1.3. Overview*

The material in the paper is organized as follows. In Section 2 we describe formally the set of economies with public goods to which our results apply. In Section 3 we set

[6] See for instance Holmström (1979).

up the machinery that allows us to define the individually rational Lindahl equilibria for the economy, and we prove some of the basic properties of Lindahl equilibria. In Section 4 we describe informally the structure of our proposed mechanism for the efficient provision of public goods. In Section 5 we describe formally the proposed mechanism, and we state our main results. For ease of exposition, the proofs of all remarks, propositions and theorems are relegated to the Appendix. In the numbering of equations, propositions, and other objects a prefix of 'A' indicates that the relevant item is to be found in the Appendix.

## 2. THE MODEL: ONE PRIVATE AND ONE PUBLIC GOOD

We consider an economy with one private consumption good ($c$) and one public consumption good ($G$). The economy consists of a finite number, $H$, of individuals who are indexed by a superscript $h = 1, \ldots, H$. The set of individuals in the economy is denoted by $\mathcal{H} = \{1, \ldots, H\}$. Each individual has preferences over the non-negative orthant $\mathbb{R}^2_+$ — the possible non-negative private good-public good pairs. Individual $h$'s preferences are represented by the utility function $U^h(\cdot, \cdot)$.

ASSUMPTION 1: *The number of agents $H$ in the economy is at least three.*

ASSUMPTION 2: *Each individual's utility function is continuous, strictly increasing in both arguments and strictly quasi-concave*[7] *over the consumption set $\mathbb{R}^2_+$.*

ASSUMPTION 3: *Each individual $h$ is endowed with a strictly positive amount, $\omega^h$, of the private good. The initial endowment of public good in the economy is zero. The vector $(\omega^1, \ldots, \omega^H)$ is denoted by $\omega$.*

ASSUMPTION 4: *The production technology that transforms the private good $c$ into the public good $G$ is linear. Without loss of generality, we can therefore take it to be the case that the available technology transforms one unit of $c$ into one unit of $G$.*

[7]Both strict monotonicity and strict quasi-concavity are stronger assumptions than needed. All our results hold in the case of preferences exhibiting quasi-concavity and local non-satiation. Obviously, the analysis is greatly simplified in the present formulation.

An economy, consisting of an assignment of private good endowments $\omega >> 0$ and utility functions, is denoted by $E(\omega)$. The total amount of resources available in the economy is denoted by $\mathcal{R} = \sum_{h=1}^{H} \omega^h$. Given a set of individuals $\mathcal{H}$ and an assignment of utility functions the set of all $E(\omega)$ with $\omega >> 0$ and given total resources $\mathcal{R}$ is denoted by $\mathcal{E}$.

In a completely standard way, we define an allocation of resources to be an array of non-negative numbers $\mathcal{C} = \{c^1, \ldots, c^h, \ldots, G\}$ specifying the amount of private consumption for each individual and the amount of public consumption in the economy. Obviously, given Assumption 4, an allocation $\mathcal{C}$ is feasible if and only if $G = \mathcal{R} - \sum_{h=1}^{H} c^h$.

The set of Pareto-efficient allocations is also completely straightforward to define. An allocation $\mathcal{C}$ is Pareto-efficient if and only if it is feasible and there exists no other feasible allocation $\mathcal{C}'$ such that $U^h(c^{h\prime}, G') \geq U^h(c^h, G)$ for every $h \in \mathcal{H}$, and $U^h(c^{h\prime}, G') > U^h(c^h, G)$ for some $h \in \mathcal{H}$. The set of Pareto-efficient allocations is denoted by $\mathcal{P}$.

We are interested in economies in which it is optimal to provide a strictly positive amount of the public good and in which it is not optimal to transform the entire aggregate endowment into public good.

ASSUMPTION 5: *Every Pareto-efficient allocation contains an amount of public good $G$ which is strictly positive and strictly below $\mathcal{R}$. In other words, for every $\mathcal{C} \in \mathcal{P}$ we have that $0 < G < \mathcal{R}$.*

Our focus will be on those allocations that are both Pareto-efficient and that do not force any individual below the utility level that he could achieve by opting out of the economy altogether.

DEFINITION 1: *Given $\omega$, an allocation $\mathcal{C}$ is said to be individually rational if and only if $U^h(c^h, G) \geq U^h(\omega^h, 0)$ for every $h \in \mathcal{H}$. The set of individually rational allocations given $\omega$ is denoted by $\mathcal{I}(\omega)$.*

## 3. Lindahl Equilibria

We start by defining the set of Lindahl equilibria for the economy. We find it convenient to work with a minor variation of the standard definition of (constrained) Lindahl equilibrium. We will comment on the differences between our way of setting up the equilibrium concept and the standard one after we have given the formal definitions below.

Throughout the paper, we denote the $H-1$-dimensional simplex by $\Delta^{H-1} = \{(x^1, \dots, x^H) \in \mathrm{IR}^H_+ \text{ such that } \sum_{h=1}^H x^h = 1\}$. The interior of $\Delta^{H-1}$ will be denoted by $\widehat{\Delta}^{H-1}$.

In essence, we consider a situation in which each individual $h$ proposes to add a certain amount $\hat{T}^h$ to the total amount of public good (and hence total taxes paid) in the economy. The total tax burden is then shared according to a vector of weights $\tau = (\tau^1, \dots, \tau^H) \in \Delta^{H-1}$. Each individual maximizes utility, given the declarations $\hat{T}^{-h} = \{\hat{T}^i\}_{i \neq h}$ of all other individuals in the economy, and the vector of weights $\tau$.

Let a vector $\hat{T} = \{\hat{T}^1, \dots, \hat{T}^H\}$ be given and, for every $h \in \mathcal{H}$, consider the following problem

$$
\begin{aligned}
\max_{c^h, T^h} \quad & U^h(c^h, G^h) \\
\text{s.t.} \quad & G^h = T^h + \textstyle\sum_{i \neq h} \hat{T}^i \\
& c^h + \tau^h G^h \leq \omega^h \\
& c^h + G^h \leq \mathcal{R} \\
& c^h \geq 0 \;\; G^h \geq 0
\end{aligned}
\tag{1}
$$

By Assumption 2, and by inspection of the constraints, for every $\tau \in \Delta^{H-1}$, and every $\hat{T} \in \mathrm{IR}^H$, a solution to Problem (1), exists unique for every $h \in \mathcal{H}$. Moreover, it is immediate to check that the values of $c^h$ and $G^h$ that solve Problem (1) do not depend on the vector $\hat{T}^{-h}$. Let $\hat{T}^h(\tau^h, \omega^h, \hat{T}^{-h})$ and $c^h(\tau^h, \omega^h)$ be the solution to Problem (1), and $G^h(\tau^h, \omega^h)$ be the corresponding value of $G^h$ obtained from the first constraint.

Definition 2: *A Lindahl equilibrium is a vector* $\tau \in \Delta^{H-1}$ *such that, for some vector* $\hat{T} = (\hat{T}^1, \ldots, \hat{T}^H)$, *we have that* $\hat{T}^h = \hat{T}^h(\tau^h, \omega^h, \hat{T}^{-h})$ *for every* $h \in \mathcal{H}$.

*Notice that if* $\tau$ *is a Lindahl equilibrium, by definition we have that for every* $h \in \mathcal{H}$, *(i)* $G^h(\tau, \omega) = \sum_{i=1}^{H} \hat{T}^i \geq 0$, *and (ii)* $c^h(\tau^h, \omega^h) = \omega^h - \tau^h \sum_{i=1}^{H} \hat{T}^i$.

Throughout the rest of the paper, the set of Lindahl equilibria of an economy $E(\omega)$ is denoted by $T[E(\omega)]$. Moreover, given a Lindahl equilibrium $\tau \in T[E(\omega)]$, the corresponding allocation will be denoted by $\mathcal{C}(\tau, \omega) = [c^1(\tau, \omega), \ldots, c^H(\tau\omega), G(\tau, \omega)]$, while the set of all Lindahl equilibrium allocations for an economy $E(\omega)$ will be denoted by $L[E(\omega)]$.

Definition 3: *The set of Lindahl equilibrium allocations for economies in* $\mathcal{E}$ *is denoted by* $\mathcal{L}$. *In other words an allocation* $\mathcal{C}$ *is in* $\mathcal{L}$ *if and only if* $\mathcal{C} \in L[E(\omega)]$ *for some* $E(\omega) \in \mathcal{E}$.

The following Proposition states that the equivalent of the First and Second Welfare Theorems hold in our model. Every Lindahl equilibrium allocation is Pareto-efficient, and every Pareto-efficient allocation is a Lindahl equilibrium allocation for some redistribution of initial endowments. Except for the fact that our definition of Lindahl equilibrium is a minor modification of the more common one, this is a standard result.[8]

Proposition 1: *Under Assumptions 1 to 5, the set of all possible Lindahl equilibrium allocations and the set of Pareto-efficient allocations coincide. In other words*

$$\mathcal{L} = \mathcal{P}$$

Our last preliminary characterization of Lindahl, equilibria is the following. We want to show that every $\tau \in \widehat{\Delta}^{H-1}$ is a Lindahl equilibrium for some redistribution of initial endowments. Formally, we have

Proposition 2: *Under Assumptions 1 to 5, for every* $\tau \in \widehat{\Delta}^{H-1}$ *there exists* $E(\omega) \in \mathcal{E}$ *such that* $\tau$ *is a Lindahl equilibrium for* $E(\omega)$.

[8]See, for instance, Tian (1988).

Throughout the rest of the paper we adopt the following notational conventions. Given any $\tau \in \Delta^{H-1}$, we let $\Omega(\tau)$ denote the set of (redistributed) endowment vectors that guarantee that $\tau$ is a Lindahl equilibrium of the economy. A typical element of $\Omega(\tau)$ will be denoted by $\omega = (\omega^1(\tau), \ldots, \omega^H(\tau))$. The corresponding set of Lindahl equilibrium allocations will be denoted by $\Upsilon(\tau)$. A typical element of $\Upsilon(\tau)$ will be denoted by $\mathcal{C}(\tau) = (c^1(\tau), \ldots, c^H(\tau), G(\tau))$.

DEFINITION 4: *Somewhat abusing terminology, for the remainder of the paper, we say that a vector of tax loadings $\tau \in \Delta^{H-1}$ is individually rational (given $\omega$) if and only if there exists an allocation $\mathcal{C}(\tau) \in \Upsilon(\tau)$ that is individually rational according to Definition 1 above.*

*Given a $\tau \in \Delta^{H-1}$ and $\omega$, we denote by $\Xi(\tau, \omega)$ the set of allocations which are in $\Upsilon(\tau)$ and which are individually rational according to Definition 1 above. In other words, we set $\Xi(\tau, \omega) = \Upsilon(\tau) \cap \mathcal{I}(\omega)$.*

REMARK 1: *Notice that the set of individually rational vectors of tax loadings may or may not coincide with the interior of the simplex $\widehat{\Delta}^{H-1}$. For instance, if every individual's preferences are such that $U^h(c^h, G) > U^h(x, 0)$ and $U^h(c^h, G) > U^h(0, x)$ for all $c^h > 0$, $G > 0$ and $x > 0$, then all vectors $\tau \in \widehat{\Delta}^{H-1}$ are individually rational according to Definition 4.*[9]

We conclude this Section with a brief discussion of the definition of Lindahl equilibrium that we use in this paper, compared to the standard text-book way of proceeding.

There are two differences between Definition 2 and the standard text-book definition of a Lindahl equilibrium. The first is entirely immaterial. We ask each individual to choose a $c^h$ and a $T^h$ and then set $G^h = T^h + \sum_{i \neq h} \hat{T}^i$. The definition of equilibrium (Definition 2) then ensures that $G^h$ is the same across all individuals. The standard way of proceeding would be to ask each individual to choose a $c^h$ and a level of overall public expenditure $G^h$ subject to the constraint that $c^h + \tau^h G^h \leq \omega^h$, and then to require that in equilibrium the choice of $G^h$ is the same across individuals.

[9]For instance, this would clearly be the case if all agents have preferences of the Cobb-Douglas type.

This is clearly immaterial. As we remarked above, the levels of private and public consumption chosen by each individual in Problem (1) do not in fact depend on the vector $\hat{T}^{-h}$.

The second difference between Definition 2 and the standard one is the presence of the third constraint in Problem (1). Notice that if we defined a Lindahl equilibrium without imposing the third constraint in Problem (1) we would clearly obtain a set that is weakly contained in the set $T[E(\omega)]$. However, from Remark 1 we know that the equilibria that may be added in this way all have Pareto-efficient allocations. Since our focus is on implementing the Pareto-efficient allocations of resources for the economy, this does not weaken our results in any way.

This second modification of the concept of Lindahl equilibrium roughly corresponds — modulo the fact that we ask each individual to declare a separate $\hat{T}^h$ — to considering what is known as a 'constrained' Lindahl equilibrium.[10]

## 4. Decentralizing Lindahl Equilibria

Consider a Lindahl equilibrium $\tau$. Clearly, if the Government could somehow work out the vector of tax loadings $\tau$, the equilibrium allocation is decentralizable in the following sense. Once $\tau$ is announced, each individual in the economy announces a tax amount $\hat{T}^h$ taking the announcements $\hat{T}^{-h}$ of other individuals as given, and so as to maximize utility, knowing that the Government will set $G = \sum_{h=1}^{H} \hat{T}^h$, and the tax payable by each individual equal to $\tau^h G$. In other words, given a vector $\tau$ which is a Lindahl equilibrium, the individuals in the economy can be viewed as playing a game in which they each choose their own $\hat{T}^h$, which, by Proposition 1, is guaranteed to yield a Pareto-efficient allocation of resources in the economy as a Nash equilibrium outcome.

As is well known (Tian 1989, Hurwicz, Maskin, and Postlewaite 1995, among others) it is indeed possible to design a mechanism that will implement the Lindahl correspondence. That is, it is possible to design a mechanism that, given a vector of endowments will have as a Nash equilibrium the corresponding Lindahl equilibrium allocation(s). Thus, in terms of our simplified description of the problem above, it

[10] See for instance Hurwicz (1986) or Tian (1988).

is possible to construct a mechanism that will induce the agents to reveal enough about their preferences and endowments so as to allow the Government to set the appropriate vector of Lindahl prices.

Our point of departure from the existing results on the implementation of the Lindahl correspondence is the following observation. In general, for given endowments, the set of Lindahl equilibrium allocations is finite. In effect, this means that using a mechanism that implements the Lindahl correspondence as above, the Government can ensure efficiency, but has no control over which point on the Pareto-frontier is actually implemented. It can ensure efficiency, but it has no control over the distribution of welfare.[11] However, in view of Propositions 1 and 2 above, it is clear that if the Government could somehow *choose* the vector of Lindahl prices it would indeed regain control of the distribution of welfare. It is also evident that, in order to be able to choose the Lindahl prices at will, the scheme proposed by the Government must endogenize the process of redistribution of initial endowments.

Suppose that the Government were to pick an arbitrary vector $\tau$ and then asked the individuals in the economy to play the game described informally above. Then, unless by fluke $\tau$ is a transformed Lindahl equilibrium, it is not hard to show that the game may *not* have a Nash equilibrium. Therefore, far from decentralizing a Pareto-efficient allocation of resources, the game that we have sketched out may have an ill-defined outcome.

The scheme for the decentralized efficient provision of public goods which we analyze below revolves on the following key observation. Proposition 2 above guarantees that *any* $\tau$ is a Lindahl equilibrium for *some* redistribution of initial endowments. Suppose then that, together with the tax declarations $\hat{T}^h$, the individuals were asked to declare a vector of lump-sum transfers which redistributes the endowments in the economy. Suppose moreover that, given any arbitrary $\tau$, we could construct the redistribution part of the game in such a way that in equilibrium, after endowments are redistributed, $\tau$ is indeed a Lindahl equilibrium. Then we would have achieved two important goals. First of all we would have a decentralized way to implement the

[11] The set $\Upsilon(\tau)$ of Lindahl equilibrium allocations corresponding to a given $\tau$ is typically a finite set. If the public good is a normal good for every individual in the economy, it is trivial to show that to any vector $\tau$ there corresponds a unique equilibrium allocation. In other words, in this case, $\Upsilon(\tau)$ contains only allocation for every possible $\tau$.

efficient allocation of resources associated with the Lindahl equilibrium $\tau$. In other words, the Government could announce an *arbitrary* (individually rational) vector of tax loadings $\tau$, and be assured that in equilibrium endowments would be redistributed so as to make $\tau$ a Lindahl equilibrium, and that the associated efficient allocation of resources would result.

The second objective which a scheme as above would fulfill is the following. Recall that by Remark 1 we know that as $\tau$ varies in $\Delta^{H-1}$ the set of Lindahl equilibrium allocations spans the entire set of Pareto-efficient allocations. Therefore, announcing different vectors of tax loadings $\tau$ the Government now has at its disposal a way to decentralize *any* efficient and individually rational allocation of resources in the economy.

In the remainder of the paper, we develop rigorously a scheme for the decentralized efficient provision of public goods along the lines we have just described intuitively. Besides the two properties we have just described, the proposed scheme has the chief virtues of being continuous and balanced. In our view, in practical terms, we believe it could be described as a *playable game*. The latter claim, although clearly vague in some way, will be substantiated in Subsection 5.1 below in where we describe the basic scheme. In short, we believe that the game we propose is simple and easy to interpret from the part of the players; each agent only has to declare an amount of desired increase in the amount of public good, and a vector of redistributive transfers of initial endowments (across other agents).

## 5. A Scheme for the Efficient Provision of Public Goods

### *5.1. The Scheme*

The description of the general scheme which we propose is as follows. Each agent $h$ is asked to call a number, $\hat{T}^h$, and a vector $\hat{b}^h$. The components of $\hat{b}^h$ are denoted by $\hat{b}^h_1, \ldots, \hat{b}^h_H$, so that, formally $\hat{b}^h$ is an $H$-dimensional vector of real numbers.[12] The square matrix $[\hat{b}^1, \ldots, \hat{b}^H]$ obtained from the $H$ (column) vectors $\hat{b}^h$ is denoted by $\hat{B}$, while the (column) vector $(\hat{T}^1, \ldots, \hat{T}^H)$ is denoted by $\hat{T}$. Each vector $\hat{b}^h$ is restricted

[12]As is clear from (2) below, each $\hat{b}^h$ in fact lives in a subset of $\mathbb{R}^{H-1}$. We consider these vectors to be elements of $\mathbb{R}^H$ with the restrictions (2) purely to simplify notation.

to satisfy

$$\hat{b}_h^h \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{H} \hat{b}_i^h = 0 \tag{2}$$

We denote by $\mathcal{A}^h$ the set of all possible declarations by $h$. In other words $\mathcal{A}^h$ is the set of all $(\hat{T}^h, \hat{b}^h) \in \mathbb{R}^{H+1}$ which satisfy (2).

The Government fixes a vector of weights $\tau = (\tau^1, \ldots, \tau^H) \in \Delta^{H-1}$. Given the pair $(\hat{T}, \hat{B})$, the taxes, transfers, and the amount of public good in the economy are then determined as follows. Let

$$T^h = \tau^h \sum_{i=1}^{H} \hat{T}^i \tag{3}$$

and[13]

$$B^h = \sum_{i=1}^{H} \hat{b}_h^i \tag{4}$$

The consumption of individual $h$ is then given by

$$c^h(\tau, \hat{T}, \hat{B}) = \omega^h - T^h - B^h \tag{5}$$

Since the Government's budget is, by assumption, always balanced, we can then compute the amount of public good as follows. Using (3), (4), (5), and the restrictions (2), we have

$$G(\tau, \hat{T}, \hat{B}) = \mathcal{R} - \sum_{h=1}^{H} c^h(\tau, \hat{T}, \hat{B}) = \sum_{h=1}^{H} T^h = \sum_{h=1}^{H} \hat{T}^h \tag{6}$$

Throughout the rest of the paper, we let $\mathcal{C}(\tau, \hat{T}, \hat{B})$ denote the allocation of resources yielded by (3), (4), (5) and (6), given $\tau$ and the array of declarations $(\hat{T}, \hat{B})$. In other words, we set,

$$\mathcal{C}(\tau, \hat{T}, \hat{B}) = (c^1(\tau, \hat{T}, \hat{B}), \ldots, c^H(\tau, \hat{T}, \hat{B}), G(\tau, \hat{T}, \hat{B})) \tag{7}$$

[13]Notice that, because of (2), we immediately have that $\sum_{h=1}^{H} B^h = 0$.

### *5.2. Feasibility of Allocations Under the Scheme*

By inspection of (3), (4), (5) and (6) it is straightforward to check that, in our proposed scheme, given the actions of others, each individual has the correct incentives to implement a Pareto-efficient allocation of resources. This is because, roughly speaking, the individuals' maximization problems in our scheme coincide with those associated with the Lindahl equilibrium $\tau$.

However, we know that an arbitrary $\tau$ is a Lindahl equilibrium only for some redistribution of private endowments in the economy. Therefore, the first thing we need to check about the proposed scheme is that in principle it is in fact able to deliver all possible re-allocations of initial endowments. This is indeed the case because of the large number of degrees of freedom we have built into the scheme via the redistribution declarations $\hat{B}$. The next remark tells us that any redistribution of initial endowments can in principle be delivered by our scheme.

REMARK 2: *Let a vector of initial endowments $\omega$ be given. Then, for any other vector of endowments $\omega^*$ such that $\sum_{i=1}^{H} \omega^{h*} = \sum_{i=1}^{H} \omega^h$, there exist declarations $\hat{B}^*$ satisfying (2) and $\hat{b}_h^{h*} = 0$ for all $h$, and such that $\omega^h - B^h = \omega^{h*}$, for every $h \in \mathcal{H}$.*

### *5.3. Bankruptcy and Negative Public Goods*

There is one difficulty with our proposed general scheme which we have not dealt with yet. It is very easy to explain what the problem is. By inspection of (3), (4), (5) and (6), it is immediate that for some vectors of declarations it is possible that some (or all) agents may be bankrupt ($c^h < 0$), and/or that the scheme yields a negative value for[14] $G$.

In order to deal with this difficulty, we 'augment' our proposed scheme as embodied in (3), (4), (5) and (6) so that even when an array of declarations $(\hat{T}, \hat{B})$, yields bankruptcy and/or negative public goods, the actual outcome of the scheme yields well defined payoffs to all agents.

There are three ways to pursue this objective. The first is to adjust the values of private consumption and of public good so that they are forced to remain non-negative. The second is to include in the scheme an extra declaration that induces the

[14]Clearly, it is *not* possible that *all* agents be bankrupt and that $G \leq 0$ at the same time.

agents truthfully to reveal their endowments. The third is to give each of the agents something like a veto vote when the proposed allocation makes them bankrupt. We pursue all three possibilities in turn.

The first of the two possibilities above corresponds to the case of a *partially informed Government*, namely a Government that knows the agents' endowments. The second and third ways to solve the bankruptcy and/or negative public goods corresponds to the case of an *uninformed Government*, namely a Government that has no information about agents' endowments.

### *5.4. Efficient Provision of Public Goods With a Partially Informed Government*

To work towards our first result, we start by defining the sets declarations $\hat{T}$ and $\hat{B}$ which guarantee that all agents consume non-negative amounts of both the private and of the public good. Given $\tau \in \Delta^{H-1}$, define the set $\mathcal{F}(\tau)$ to be as follows

$$\left\{(\hat{T},\hat{B}) \in \mathcal{A} \;\;|\;\; \mathcal{R} > \sum_{h=1}^{H} \hat{T}^h \geq 0 \;\text{ and }\; \omega^h - \sum_{i=1}^{H} \hat{b}_h^i - \tau^h \sum_{i=1}^{H} \hat{T}^i \geq 0 \;\; \forall\, h\right\} \tag{8}$$

Whenever $\hat{T}$ and $\hat{B}$ are not in the set $\mathcal{F}(\tau)$, we proceed to zero out the declarations: we set the consumption of each agent to be equal to his endowment of private good, and $G$ to be equal to 0.

More formally, consider the following outcome map $\mathcal{O} : \mathcal{A} \to \mathbb{R}_+^{H+1}$ that, for any given $\tau \in \Delta^{H-1}$, associates an allocation $\mathcal{C}$ to every possible array of declarations $\hat{T}, \hat{B} \in \mathcal{A}$. Let $\mathcal{C}^0$ be the allocation $(\omega^1, \ldots, \omega^H, 0)$. We then define

$$\mathcal{O}(\tau,\hat{T},\hat{B}) \;=\; \begin{cases} \mathcal{C}(\tau,\hat{T},\hat{B}) & \text{if } (\hat{T},\hat{B}) \in \mathcal{F}(\tau) \\ \mathcal{C}^0 & \text{otherwise} \end{cases} \tag{9}$$

Throughout the rest of the paper, for every $h \in \mathcal{H}$, we denote by $\mathcal{O}^h(\tau,\hat{T},\hat{B})$ the private and public good consumption for individual $h$ in the allocation $\mathcal{O}(\tau,\hat{T},\hat{B})$.

This gives us all the ingredients to describe formally our augmented scheme.

Remark 3: *Given a $\tau \in \Delta^{H-1}$ consider an augmented scheme defined as follows. For every array of declarations $(\hat{T},\hat{B}) \in \mathcal{A}$, the resulting allocation is given by $\mathcal{O}(\tau,\hat{T},\hat{B})$.*

*The augmented scheme which we have just described, yields a well defined $H$-player strategic-form game which we will denote by $\Gamma(\tau)$. The strategy set of each player $h$ in $\Gamma(\tau)$ is $\mathcal{A}^h$. Player $h$'s payoff given any strategy profile in $\mathcal{A}$ is given by his utility from consumption of private and public good $U^h[\mathcal{O}^h(\tau, \hat{T}, \hat{B})]$.*

*For the remainder of the paper, for a given $\Gamma(\tau)$, by the set of Nash equilibrium outcomes of $\Gamma(\tau)$ we mean the set of private consumption and public good allocations that result from any Nash equilibrium of the strategic-form game.*

We are now ready to state formally our first main result.

THEOREM 1: *Let any vector of tax loadings $\tau \in \widehat{\Delta}^{H-1}$ and $\omega$ be given. Assume that $\tau$ is individually rational according to Definition 4.*

*Then the set of Nash equilibrium outcomes of $\Gamma(\tau)$ is $\Xi(\tau, \omega)$.*

The intuition behind Theorem 1 above is not difficult to outline. It can be divided into three distinct steps. First of all, it can be shown that any Nash equilibrium of $\Gamma(\tau)$ must be in the set $\mathcal{F}(\tau)$ as defined in (8). To see that this must be the case, notice that because of (2) either the amount of redistribution is zero for every $h$, or there must be some individual $h$ that in fact gains from the redistribution part of the game ($B^h < 0$). If the strategy profile $(\hat{T}, \hat{B})$ is outside $\mathcal{F}(\tau)$, this particular individual will always want to deviate and push the system back in $\mathcal{F}(\tau)$. He will be able to do so because, using the fact that $B^h < 0$, he can make sure that all other individuals face non-degenerate budget sets, while still keeping some of the redistribution in his favor given by $B^h$.

Once we know that any Nash equilibrium of $\Gamma(\tau)$ is in $\mathcal{F}(\tau)$ we can proceed as follows. To show that every Lindahl equilibrium must be a Nash equilibrium of $\Gamma(\tau)$ we use Proposition 2. This tells us that for some redistribution of endowments, the given $\tau$ is in fact a Lindahl equilibrium of the economy. It is then clear that if we take an array $\hat{B}$ that redistributes endowments in just the right way, together with the array $\hat{T}$ that yields precisely the amount of $G$ in the Lindahl equilibrium, we must have a Nash equilibrium of $\Gamma(\tau)$.

Lastly, to show that every Nash equilibrium of $\Gamma(\tau)$ is a Lindahl equilibrium of the economy for the appropriate redistribution of endowments we make the following observation. Some fairly straightforward computations show that the opportunity set (in terms of $(c^h, G^h)$ pairs) that each individual faces in $\Gamma(\tau)$, given $\hat{B}^{-h}$ is the same as the opportunity set he faces in Problem (1) if we give him an initial endowment equal to $\omega^h - B^h$. It then follows that, if no individual has a profitable deviation in $\Gamma(\tau)$, the array $(\hat{T}^h, \hat{B}^h)$ must be a Lindahl equilibrium for the economy with redistributed endowments.

### *5.5. Uninformed Government: Endowment Declarations*

The second version of our main result applies to the case of an uninformed Government: a Government that does not know the agents' endowments.

We augment the basic scheme described in Section 5.1 with a vector of endowment declarations. In the same spirit as Postlewaite and Wettstein (1989) we ask each agent to declare his endowment, but with the restriction that he cannot declare more than his true endowment.[15] The interpretation of this restriction is that overstatement of an agent's endowment is physically impossible. The Government can ask the agent to put his money on the table and hence verify if he really has as much as is being declared. On the other hand agents can hide their endowments (report an endowment below their true one) and keep the hidden part for their own consumption.

The scheme that we describe below, in essence, is the same as the one described in Section 5.4, but in which the feasible set $\mathcal{F}$ is defined on the basis of declared endowments rather than the true ones (which the Government does not observe). In addition, a redistributive prize (penalty) is given to those agents that declare an endowment higher (lower) than the average declared endowment. It turns out that this is sufficient to give all agents the corrective incentives in their declaration of endowments.

To describe our new augmented scheme formally we need to introduce some further notation at this point. We will denote by $\hat{\omega}^h$ agent $h$'s declaration of his endowment, while $\hat{W}$ will denote the array $\{\hat{\omega}^1, \ldots, \hat{\omega}^H\}$. As we mentioned above, we take each

[15] Postlewaite and Wettstein (1989) are concerned with the implementation of the (constrained) Walrasian correspondence, not with an economy with public goods. On the issue of the manipulation of endowments in the implementation of arbitrary social choice correspondences see also Hurwicz, Maskin, and Postlewaite (1995).

agent's endowment declaration to satisfy $\hat{\omega}^h \in (0, \omega^h]$.[16] We also let $\overline{\omega}$ be the average endowment declaration across agents so that $\overline{\omega} = (\sum_{h=1}^{H} \hat{\omega}^h)/H$.

As we discussed informally above, the agents' endowment declarations trigger an amount of redistribution which is a function of the deviation of each declaration from the average declared endowment $\overline{\omega}$. Let

$$\tilde{\omega}^h \; = \; \hat{\omega}^h \; + \; \frac{1}{H} \, (\hat{\omega}^h \; - \; \overline{\omega}) \tag{10}$$

In deciding whether the outcome of the scheme is feasible or not, the Government takes the endowment declarations at face value, and of course takes into account the redistribution that they induce. Therefore, we can re-write the set of feasible allocations of Section 5.4 as follows. Let $\hat{\mathcal{R}} = \sum_{h=1}^{H} \hat{\omega}^h$ and, given $\tau$ and $\hat{W}$, define the set $\mathcal{F}^W(\tau, \hat{W})$ to be as follows

$$\left\{ (\hat{T}, \hat{B}) \in \mathcal{A} \;\; | \;\; \hat{\mathcal{R}} > \sum_{h=1}^{H} \hat{T}^h \geq 0 \;\; \text{and} \;\; \tilde{\omega}^h \, - \, \sum_{i=1}^{H} \hat{b}_h^i \, - \, \tau^h \sum_{i=1}^{H} \hat{T}^i \geq 0 \;\; \forall \, h \right\} \tag{11}$$

Analogously to Section 5.4, whenever $\hat{T}$ and $\hat{B}$ are not in the set $\mathcal{F}(\tau, \hat{W})$, we proceed to zero out the agents declarations entirely. In this case, as before, we set the consumption of each agent to be equal to his (true) endowment $\omega^h$, and $G$ to be equal to 0.

Whenever the declarations $(\hat{T}, \hat{B}, \hat{W})$ are such that $(\hat{T}, \hat{B}) \in \mathcal{F}(\tau, \hat{W})$, the scheme allocates consumption of private and public to each agent as follows. The amount of public good $G$ is exactly as before, and therefore is given by (6). In addition, each agent gets to consume any part of his endowment of private good that was hidden by his declaration of $\hat{\omega}^h$. Therefore, taking into account the redistribution term yielded by (10), the allocation of private consumption is given by,

$$c^h \; = \; \omega^h \, + \, \tilde{\omega}^h \, - \, \hat{\omega}^h \, - \, \tau^h G \, - \, B^h \tag{12}$$

Given a $\tau$ and an array $(\hat{T}, \hat{B}, \hat{W})$, let $\mathcal{C}^W(\tau, \hat{T}, \hat{B}, \hat{W}) = (\mathcal{C}^{W1}(\tau, \hat{T}, \hat{B}, \hat{W}), \ldots,$

[16]Notice that we are excluding the possibility that $\hat{\omega}^h = 0$. This can be interpreted as an assumption stipulating that the Government knows that no agent has an endowment of zero.

$\mathcal{C}^{WH}(\tau, \hat{T}, \hat{B}, \hat{W}), G)$ be the allocation yielded by setting private consumption as in (12) and $G$ as in (6).

We are now ready to complete the formal description of our augmented schemed. For each $h$, let $\tilde{\mathcal{A}}^h = \mathcal{A}^h \times (0, \omega^h]$, and set $\tilde{\mathcal{A}} = \displaystyle\mathop{\times}_{h=1}^{H} \tilde{\mathcal{A}}^h$. Next, consider the following outcome map $\mathcal{O}^W$ that, for any given $\tau$, associates an allocation $\mathcal{C}$ to every possible array of declarations $(\hat{T}, \hat{B}, \hat{W}) \in \tilde{\mathcal{A}}$.

$$\mathcal{O}^W(\tau, \hat{T}, \hat{B}, \hat{W}) \;=\; \begin{cases} \mathcal{C}^W(\tau, \hat{T}, \hat{B}) & \text{if } \; (\hat{T}, \hat{B}, \hat{W}) \in \mathcal{F}(\tau, \hat{W}) \\ \mathcal{C}^0 & \text{otherwise} \end{cases} \tag{13}$$

Throughout the rest of the paper, for every $h \in \mathcal{H}$, we denote by $\mathcal{O}^{Wh}(\tau,\, \hat{T},\, \hat{B},\, \hat{W})$ the private and public good consumption for individual $h$ in the allocation $\mathcal{O}^W(\tau,\, \hat{T},\, \hat{B},\, \hat{W})$.

This gives us all the ingredients to describe formally our second augmented scheme.

REMARK 4: *Given a $\tau \in \Delta^{H-1}$ consider an augmented scheme defined as follows. For every array of declarations $(\hat{T}, \hat{B}, \hat{W}) \in \tilde{\mathcal{A}}$, the resulting allocation is given by $\mathcal{O}^W(\tau, \hat{T}, \hat{B}, \hat{W})$.*

*The augmented scheme which we have just described, yields a well defined $H$-player strategic-form game which we will denote by $\Gamma^W(\tau)$. The strategy set of each player $h$ in $\Gamma^W(\tau)$ is $\tilde{\mathcal{A}}^h$. Player $h$'s payoff given any strategy profile in $\tilde{\mathcal{A}}$ is given by his utility from consumption of private and public good $U^h[\mathcal{O}^{Wh}(\tau, \hat{T}, \hat{B}, \hat{W})]$.*

*For the remainder of the paper, for a given $\Gamma^W(\tau)$, by the set of Nash equilibrium outcomes of $\Gamma^W(\tau)$ we mean the set of private consumption and public good allocations that result from any Nash equilibrium of the strategic-form game.*

We are now ready to state formally our second main result.

THEOREM 2: *Let any vector of tax loadings $\tau \in \widehat{\Delta}^{H-1}$ and $\omega$ be given. Assume that $\tau$ is individually rational according to Definition 4.*

*Then the set of Nash equilibrium outcomes of $\Gamma^W(\tau)$ is $\Xi(\tau, \omega)$.*

The intuition behind Theorem 2 is closely related to the one we outlined for Theorem 1. The main difference is that, here, the agents have the correct incentives to reveal their true endowments because of the redistribution terms yielded by (10). In equilibrium, this part of the redistribution of endowments is combined with the redistribution terms given by the declarations $\hat{B}$ so that, relative to our previous augmented scheme, its net effect on the equilibrium allocation is in fact zero.

### *5.6. Uninformed Government: Veto Power*

The third and last version of our main result also applies to the case of an uninformed Government that does not know the agents' endowments. This time, instead of asking the agents to declare their endowments as in Section 5.5, we give each of the agents the possibility to veto the proposed allocation.

The third version of our main result holds under mildly more stringent conditions than Theorems 1 and 2. These are embodied in Assumptions 6 and 7 below. We believe that this version of our main result is worth pursuing for two main reasons. First of all, by contrast to Theorem 2 above, in our next theorem the Government does not need to be able to verify the maximum value of any of the agents' endowments. Secondly, we believe that showing that our main result still holds when each agent is given the option to veto the proposed allocation is interesting in its own right. Theorem 3 below highlights the fact that in our model the agents can be considered free to opt out of the system entirely, and consume their endowment of private good ignoring the rest of the agents in the economy. Of course in equilibrium this will not be the case.

To state the third version of our main result, we need to extend the agents's utility functions to be defined for all possible values of consumption of private and public good. We simply assume that negative values of the consumption of either good yield utilities which are no better than the utility yielded by the consumption of *any* feasible bundle.

ASSUMPTION 6: *For each $h = 1, \ldots, H$, $\overline{U}^h$ is defined as follows. For each $h$ let $\mathcal{U}^h$ be a real number satisfying*

$$\mathcal{U}^h \;\leq\; \inf_{c^h \geq 0, G \geq 0} U^h(c^h, G) \tag{14}$$

*We then set*

$$\overline{U}^h(c^h, G) \;=\; \begin{cases} U^h(c^h, G) & \text{if} \quad c^h \geq 0 \;\text{ and }\; G \geq 0 \\ \mathcal{U}^h & \text{otherwise} \end{cases} \tag{15}$$

Before we are able to state our next theorem, we also need to strengthen Assumption 5 above by adding the following boundary condition to the utility function of each agent.[17]

Assumption 7: *The utility function of every individual $h$ satisfies the following property: for every positive private consumption level $c^h > 0$ there exists an $\epsilon > 0$ such that $U^h(c^h - \epsilon, \epsilon) > U^h(c^h, 0)$.*

The second way in which we augment the basic scheme embodied in (3), (4), (5) and (6) is designed to make the 'minmax' utility of each agent to be the level of utility that he would get by opting out of the system, not consuming any public good, and simply keeping his endowment of private good. We believe this to be an appealing feature of our next result.

Intuitively, we want to give to any agent that receives a negative transfer the possibility to veto the proposed allocation. If one or more agents veto the proposed allocation the augmented scheme yields the allocation $\mathcal{C}^0$.

Some extra notation is needed at this point. For each $h$, let $\widehat{\mathcal{A}}^h = \mathcal{A}^h \times \{0, 1\}$, with generic element $(\hat{T}^h, \hat{b}^h, v^h)$, and set $\widehat{\mathcal{A}} = \mathop{\times}\limits_{h=1}^{H} \widehat{\mathcal{A}}$. Agent's $h$ declaration of $(\hat{T}^h, \hat{b}^h)$ together with a '$v^h = 1$' is interpreted as agent $h$ putting a veto on the proposed allocation. Conversely, a declaration of '$v^h = 0$' is interpreted as $h$ not putting a veto on the proposed allocation. Recall that only the veto votes of those agents whose endowments are being reduced by the redistribution declarations $\hat{B}$ are considered effective. Any agent can set $v^h = 1$ at any time, but this will have an effect on the outcome of the game only if $\hat{B}$ is such that $B^h > 0$.

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[17]Notice that Assumption 7 in fact implies one side of Assumption 5 above. In particular, it is clear that it implies that in every Pareto-efficient allocation we must have that $G > 0$. We have chosen to state it as an *additional* condition purely for the sake of simplicity.

Let $V = \{0,1\}^H$, with generic element $v$. Lastly, given any $\hat{B}$, let $\mathcal{V}(\hat{B}) \subset \mathcal{H}$ be the subset of agents for which $B^h > 0$. We can now define the outcome function $\mathcal{O}^V$ for our augmented scheme with veto power. Let

$$\mathcal{F}^V = \left\{(\hat{T}, \hat{B}, v) \in \widehat{\mathcal{A}} \;\mid\; h \in \mathcal{V}(\hat{B}) \;\Rightarrow\; v^h = 0\right\} \tag{16}$$

and

$$\mathcal{O}^V(\tau, \hat{T}, \hat{B}, v) = \begin{cases} \mathcal{C}(\tau, \hat{T}, \hat{B}) & \text{if } (\hat{T}, \hat{B}, v) \in \mathcal{F}^V \\ \mathcal{C}^0 & \text{otherwise} \end{cases} \tag{17}$$

Throughout the rest of the paper, for every $h \in \mathcal{H}$, we denote by $\mathcal{O}^{Vh}(\tau, \hat{T}, \hat{B}, v)$ the private and public good consumption for individual $h$ in the allocation $\mathcal{O}^V(\tau, \hat{T}, \hat{B}, v)$.

This gives us all the ingredients to describe formally our next augmented scheme.

REMARK 5: *Given a $\tau \in \Delta^{H-1}$ consider an augmented scheme defined as follows. For every array of declarations $(\hat{T}, \hat{B}, v) \in \widehat{\mathcal{A}}$, the resulting allocation is given by $\mathcal{O}^V(\tau, \hat{T}, \hat{B}, v)$. Player $h$'s payoff corresponding to any vector $(c^H, G) \in \mathbb{R}^2$ is given by $\overline{U}^h(c^h, G)$, defined as in Assumption 6.*

*The augmented scheme which we have just described, yields a well defined $H$-player strategic-form game which we will denote by $\Gamma^V(\tau)$. The strategy set of each player $h$ in $\Gamma^V(\tau)$ is $\widehat{\mathcal{A}}^h$. Player $h$'s payoff given any strategy profile in $\widehat{\mathcal{A}}$ is given by his utility from consumption of private and public good $\overline{U}^h[\mathcal{O}^{Vh}(\tau, \hat{T}, \hat{B}, v)]$.*

*For the remainder of the paper, for a given $\Gamma^V(\tau)$, by the set of Nash equilibrium outcomes of $\Gamma^V(\tau)$ we mean the set of private consumption and public good allocations that result from any Nash equilibrium of the strategic-form game.*

We can now state our third main result.

THEOREM 3: *Let Assumptions 1, 2, 3, 4, 5, 6 and 7 hold.*

*Let any vector of tax loadings $\tau \in \widehat{\Delta}^{H-1}$ and $\omega$ be given, and assume that $\tau$ is individually rational according to Definition 4.*

*Then the set of Nash equilibrium outcomes of $\Gamma^V(\tau)$ is $\Xi(\tau, \omega)$.*

The intuition behind Theorem 3 is very close to the one behind Theorem 1. The main difference is connected to the need to assume that preferences satisfy Assumption 7 in addition to what we previously maintained. In particular, we need to show that any equilibrium of $\Gamma^V(\tau)$ will not involve an (effective) veto from the part of any agent. As for theorem 1, we rely on the fact that (unless $B^h = 0$ for every $h$) for some $h$ it will have to be the case that $B^h < 0$. Notice that this particular agent can always make sure that the amount of redistribution of endowments in the economy is precisely zero, thus neutralizing the effect of any veto from the part of any agent. However, we also need to show that he will have an actual incentive to do so whenever an allocation has been effectively vetoed by some agent in the economy. Assumption 7 guarantees that this will in fact be the case. Agent $h$ can simply zero out the redistribution of endowments in the economy, and then set the amount of public good equal to some (small) $\epsilon > 0$ making himself better off.

Once we know that no effective veto will take place in any Nash equilibrium of $\Gamma^V(\tau)$ the result can be established using an argument that follows the same lines as we described for the proof of Theorem 1 above.

# APPENDIX

## A.1. LINDAHL EQUILIBRIA

We begin by proving Propositions 1 and 2. Of course the proof of Proposition 1, which is equivalent to the First and Second Welfare Theorems for Lindahl equilibria, runs along well known standard lines. We report it here for the sake of completeness only. For the sake of clarity, it is convenient to break up the argument into several distinct Lemmas.

LEMMA A.1: *Every Lindahl equilibrium allocation is Pareto-efficient. In other words, $\mathcal{L} \subseteq \mathcal{P}$.*

PROOF: Let $\tau$ be a Lindahl equilibrium and $\mathcal{C} = (c^1, \ldots, c^H, G)$ be the associated Lindahl equilibrium allocation. Let $\hat{T}$ be the associated vector of tax declarations such that $\hat{T}^h = \hat{T}^h(\tau^h, \omega^h, \hat{T}^{-h})$ for every $h$.

Suppose by way of contradiction that $\mathcal{C}$ is not Pareto-efficient. By strict monotonicity of preferences (Assumption 2) this implies that there exists an allocation $\mathcal{C}^*$ that is feasible and such that $U^h(c^{h*}, G^*) > U^h(c^h, G)$ for every $h$. We now want to show that this implies that

$$c^{h*} + \tau^h G^* > c^h + \tau^h G \qquad \forall\, h = 1, \ldots, H \tag{A.1}$$

Given the constraints in (1) it is obvious that, for each $h$, the fact that $U^h(c^{h*}, G^*) > U^h(c^h, G)$ implies that either $c^{h*} + G^* > \mathcal{R}$, or $c^{h*} + \tau^h G^* > \omega^h$ must hold. If the former is the case for any $h$, then clearly $\mathcal{C}^*$ cannot be a feasible allocation. Therefore, we conclude that (A.1) must hold.

Consider now summing the inequalities (A.1) over $h$. Using $\sum_{h=1}^{H} \tau^h = 1$ we obtain that

$$G^* + \sum_{h=1}^{H} c^{h*} > G + \sum_{h=1}^{H} c^h \tag{A.2}$$

Finally, using strict monotonicity of preferences, we know that $G + \sum_{h=1}^{H} c^h = \mathcal{R}$. Therefore, (A.2) above implies that $\mathcal{C}^*$ is not feasible. This contradiction is clearly enough to prove the claim. ■

Proof of Remark 1: The claim is rather obvious given Proposition 2. Indeed it is enough to notice that for every $\omega >> 0$, the given property of $U^h(\cdot, \cdot)$ is enough to guarantee that in any Lindahl equilibrium allocation we must have that $c^h > 0$ for every $h$ and $G > 0$. ■

Lemma A.2: *Let $\mathcal{G} = (c^1, \ldots, c^H, G^1, \ldots, G^H)$ denote a generic element of $\mathbb{R}^{2H}$, and consider the following set.*

$$F = \left\{ \mathcal{G} \in \mathbb{R}^{2H} \mid \max_{h \in \mathcal{H}} G^h + \sum_{h=1}^{H} c^h \leq \mathcal{R} \right\} \tag{A.3}$$

*Let also $Q \subset F$ be defined as follows: $\mathcal{G} \in Q$ iff there does not exist a $\widehat{\mathcal{G}} \in F$ such that $U^h(\hat{c}^h, \widehat{G}^h) \geq U(c^h, G^h)$ for all $h$, and $U^h(\hat{c}^h, \widehat{G}^h) > U(c^h, G^h)$ for some $h$.*

*Then, if $\mathcal{G}^* \in Q$ it must be the case that (i) $G^{h*} = \max_{i \in \mathcal{H}} G^{i*}$ for every $h$, and (ii) $\max_{h \in \mathcal{H}} G^{h*} + \sum_{h-1}^{H} c^{h*} = \mathcal{R}$.*

Proof: Claim (ii) is immediate by monotonicity of preferences.

To establish (i) we proceed by contradiction. Without loss of generality, assume that $G^{H*} < \max_{h \in \mathcal{H}} G^{h*}$. Consider next the array $\mathcal{G}$ defined as follows. Let $c^h = c^{h*}$ and $G^h = G^{h*}$ for all $h = 1, \ldots, H-1$. Set also $c^H = c^{H*}$, and $G^H = \max_{h \in \mathcal{H}} G^{h*} > G^{H*}$. Clearly, using (A.3), we have that $\mathcal{G} \in F$. Moreover, using the monotonicity of preferences, the utility of $H$ is higher with $\mathcal{G}$ than with $\mathcal{G}^*$. Since the utility of all other agents is the same with $\mathcal{G}$ and with $\mathcal{G}^*$, this contradicts the fact that $\mathcal{G}^* \in Q$. This is sufficient to prove the claim. ■

Lemma A.3: *Let any $\mathcal{G}^* \in Q$ be given and consider the set*

$$P(\mathcal{G}^*) = \{ \mathcal{G} \in \mathbb{R}^{2H} \mid U^h(c^h, G^H) \geq U^h(c^{h*}, G^{h*}) \, \forall \, h \} \tag{A.4}$$

*Then the sets $F$ and $\mathcal{P}(\mathcal{G}^*)$ are both closed and convex. Moreover, $F \cap P(\mathcal{G}^*) = \mathcal{G}^*$.*

Proof: The fact that both $F$ and $P(\mathcal{G}^*)$ are closed and convex is a direct result of (A.3) and (A.4) together with Assumption 2 (continuity), and we omit the details.

To complete the proof, we proceed by contradiction. Assume that there exists some $\mathcal{G} \neq \mathcal{G}^*$ such that $\mathcal{G} \in F \cap P(\mathcal{G}^*)$. Let $\widehat{\mathcal{G}} = \mu\mathcal{G} + (1-\mu)\mathcal{G}^*$ with $\mu \in (0,1)$. Since $F$ is convex it must be the case that $\widehat{\mathcal{G}} \in F$. Since preferences are strictly quasi-concave (Assumption 2), $\mathcal{G} \neq \mathcal{G}^*$ and $\mathcal{G} \in P(\mathcal{G}^*)$ we have that $U^h(\hat{c}^h, \widehat{G}^h) \geq U^h(c^{h*}, G^{h*})$ for every $h$, and $U^h(\hat{c}^h, \widehat{G}^h) > U^h(c^{h*}, G^{h*})$ for some $h$. However, the latter inequalities contradict the fact that $\mathcal{G}^* \in Q$. This clearly enough to conclude the proof. ■

Lemma A.4: *Let any $\mathcal{G}^* \in Q$, $P(\mathcal{G}^*)$ and $F$ be given as in Lemma A.3. Then there exists a vector $\xi = (\xi^0, \ldots, \xi^H)$ with $\xi^h > 0$ for every $h = 0, \ldots, H$, such that*

$$\begin{array}{ll} (i) & \xi^0 \left[ \sum_{h=1}^H \left(c^h - c^{h*}\right) \right] + \sum_{h=1}^H \xi^h \left(G^h - G^{h*}\right) > 0 \quad \forall\, \mathcal{G} \in P(\mathcal{G}^*)/F \\ (ii) & \xi^0 \left[ \sum_{h=1}^H \left(c^h - c^{h*}\right) \right] + \sum_{h=1}^H \xi^h \left(G^h - G^{h*}\right) \leq 0 \quad \forall\, \mathcal{G} \in F/P(\mathcal{G}^*) \end{array} \tag{A.5}$$

Proof: The fact that a vector $\xi$ satisfying (A.5) exists is a direct consequence of Lemma A.3 and of the separating hyperplane theorem.

To show that $\xi >> 0$ we proceed by contradiction. Suppose first that $\xi^0 \leq 0$. Consider $\mathcal{G}$ defined as $c^h = c^{h*} + \epsilon$ with $\epsilon > 0$ for every $h$, and $G^h = G^{h*}$ for every $h$. Clearly $G \in P(\mathcal{G}^*)/F$. However, since $\xi^0 \leq 0$ we have that

$$\xi^0 \left[ \sum_{h=1}^H \left(c^h - c^{h*}\right) \right] + \sum_{h=1}^H \xi^h \left(G^h - G^{h*}\right) = \xi^0 \epsilon H \leq 0 \tag{A.6}$$

which contradicts (i) of (A.5).

Next, suppose that $\xi^h \leq 0$ for some $h \geq 1$, and consider $\mathcal{G}$ defined as $c^h = c^{h*}$ for every $h$, $G^i = G^{i*}$ for every $i \neq h$ and $G^h = G^{h*} + \epsilon$ with $\epsilon > 0$. Clearly, $G \in P(\mathcal{G}^*)/F$. However, since $\xi^h \leq 0$ we have that

$$\xi^0 \left[ \sum_{h=1}^H \left(c^h - c^{h*}\right) \right] + \sum_{h=1}^H \xi^h \left(G^h - G^{h*}\right) = \xi^h \epsilon \leq 0 \tag{A.7}$$

which contradicts (ii) of (A.5). This is clearly enough to conclude the proof of the claim. ■

Lemma A.5: *Let any $\mathcal{G}^* \in Q$, $P(\mathcal{G}^*)$ and $F$ be given as in Lemma A.3, and let $\xi >> 0$ be the corresponding separating hyperplane yielded by Lemma A.4. Define*

$$\lambda = \left(\xi^1/\xi^0, \ldots, \xi^H/\xi^0\right) \tag{A.8}$$

*Then,* $\sum_{h=1}^H \lambda^h = 1$

PROOF: We proceed by contradiction. Suppose first that $\sum_{h=1}^{H} \lambda^h > 1$, and consider $\mathcal{G}$ defined as $c^h = 0$ for every $h$ and $G^h = \mathcal{R}$ for every $h$. Clearly, $\mathcal{G} \in F/P(\mathcal{G}^*)$. Recall now that, by Lemma A.2 we have that $G^{h*} = \max_{i\in\mathcal{H}} G^{i*}$ for every $h$. Moreover, by Assumption 5, we must clearly have that $\max_{h\in\mathcal{H}} G^{h*} < \mathcal{R}$. Therefore

$$\sum_{h=1}^{H} \left(c^h - c^{h*}\right) + \sum_{h=1}^{H} \lambda^h \left(G^h - G^{h*}\right) = \left(\sum_{h=1}^{H} \lambda^h - 1\right) \left(\mathcal{R} - \max_{h\in\mathcal{H}} G^{h*}\right) > 0 \tag{A.9}$$

However, using (A.8), (A.9) clearly contradicts (ii) of (A.5).

Suppose next that $\sum_{h=1}^{H} \lambda^h < 1$, and consider $\mathcal{G}$ defined as $c^h = \mathcal{R}/H$ for every $h$ and $G^h = 0$ for every $h$. Clearly, $\mathcal{G} \in F/P(\mathcal{G}^*)$. Recall also that, by Assumption 5, we must have that $\max_{h\in\mathcal{H}} G^{h*} > 0$. Therefore, using Lemma A.2 again, we get

$$\sum_{h=1}^{H} \left(c^h - c^{h*}\right) + \sum_{h=1}^{H} \lambda^h \left(G^h - G^{h*}\right) = \max_{h\in\mathcal{H}} G^{h*} \left(1 - \sum_{h=1}^{H} \lambda^h\right) > 0 \tag{A.10}$$

However, using (A.8), (A.10) clearly contradicts (ii) of (A.5). Therefore, the proof of the lemma is complete. ■

LEMMA A.6: *Let any $\mathcal{G}^* \in Q$, $P(\mathcal{G}^*)$ and $F$ be given as in Lemma A.3, and let $\lambda >> 0$ be the corresponding vector of weights yielded by Lemma A.8. Let also $\omega^{h*} = c^{h*} + \lambda^h G^*$. Then, for every $h \in \mathcal{H}$, the pair $(c^{h*}, G^{h*})$ solves the following problem.*

$$\begin{aligned} \max_{c^h,\, G^h} \quad & U^h(c^h, G^h) \\ \text{s.t.} \quad & c^h + \lambda^h G^h \le \omega^{h*} \\ & G^h \ge 0, \quad c^h \ge 0 \end{aligned} \tag{A.11}$$

PROOF: We proceed by contradiction. Suppose, without loss of generality, that the statement is false for $h = H$. Let the solution to problem (A.11) when $h = H$ be denoted by $(c^{H\prime}, G^{H\prime}) \neq (c^{H*}, G^{H*})$. Evidently, we must have that

$$U^H(c^{H\prime}, G^{H\prime}) > U^H(c^{H*}, G^{H*}) \tag{A.12}$$

Consider now $\widehat{\mathcal{G}} \in \mathrm{I\!R}^{2H}$ defined as $(\hat{c}^h, \widehat{G}^h) = (c^{h*}, G^{h*})$ for every $h \le H - 1$, and $(\hat{c}^H, \widehat{G}^H) = (c^{H\prime}, G^{H\prime})$. Notice that using (A.12) have that $\widehat{\mathcal{G}} \in P(\mathcal{G}^*)$. Since $\widehat{\mathcal{G}} \neq \mathcal{G}^*$, using Lemma A.3, we can now conclude that $\widehat{\mathcal{G}} \in \mathcal{P}(\mathcal{G}^*)/F$. However, since, by assumption $(\hat{c}^H, \widehat{G}^H)$ is feasible in problem (A.11), and by construction $c^{h*} + \lambda^h G^{h*} = \omega^{h*}$, we now have that

$$\sum_{h=1}^{H} \left(c^h - c^{h*}\right) + \sum_{h=1}^{H} \lambda^h \left(G^h - G^{h*}\right) = \hat{c}^H + \lambda^H \widehat{G}^H - c^{H*} - \lambda^H G^{H*} \le 0 \tag{A.13}$$

Since $\widehat{\mathcal{G}} \in \mathcal{P}(\mathcal{G}^*)/F$, (A.13) directly contradicts (i) of (A.5). Therefore, the proof of the lemma is now complete. ■

LEMMA A.7: *Every Pareto-efficient allocation is a Lindahl equilibrium allocation for some redistribution of initial endowments. In other words, $\mathcal{P} \subseteq \mathcal{L}$.*

PROOF: Let any Pareto-efficient allocation $\widehat{\mathcal{C}} = (\hat{c}^1, \ldots, \hat{c}^H, \widehat{G}) \in \mathcal{P}$ be given. Consider now $\mathcal{G}^*$ defined as $c^{h*} = \hat{c}^h$ and $G^{h*} = \widehat{G}$ for every $h$. Clearly, since $\widehat{\mathcal{C}} \in \mathcal{P}$, it must be the case that $\mathcal{G}^* \in Q$, where $Q$ is as in Lemma A.2. Let $\lambda$ be the vector of weights associated with $\mathcal{G}^*$ yielded by (A.8), and set $\omega^{h*} = c^{h*} + \lambda^h G^{h*}$ for every $h$.

Next, let $\hat{T}^h = \widehat{G}/H$ for every $h$. Using Lemma A.6 it is now immediate to check that setting $T^h = \hat{T}^h$ and $c^h = c^{h*}$ for every $h$ yields a solution to Problem (1) for $\tau = \lambda$ and $\omega = \omega^*$.

Therefore, it is clear from Definition 2 that setting the vector of endowments $\omega^*$ as above yields a Lindahl equilibrium with prices $\tau = \lambda$ and a corresponding Lindahl equilibrium allocation equal to $\widehat{\mathcal{C}}$. This is clearly enough to prove the claim. ■

PROOF OF PROPOSITION 1: The claim is a direct consequence of Lemma A.1 and Lemma A.7. ■

## A.2. THE PROOF OF PROPOSITION 2

Let any $\tau \in \widehat{\Delta}^{H-1}$ be given, and define Let $\Omega(\mathcal{R}) = \{\omega \in R^H_+ \text{ such that } \sum_{h=1}^H \omega^h = \mathcal{R}\}$, a nonempty, convex and compact set. Let $f$ be a map $\Omega(\mathcal{R}) \to R^H$, with the $h$-th component of $f$ defined as

$$f^h(\omega) = \omega^h \; + \; \tau^h \left[ \sum_{i=1}^H \tau^i G^i(\tau^i, \omega^i) \; - \; G^h(\tau^h, \omega^h) \right] \tag{A.14}$$

where, for every $h$, $G^h(\tau^h, \omega^h)$ is given by the solution to Problem (1). Clearly, $f(\omega)$ is continuous by Assumption 2 and by inspection of Problem 1.

Notice next that (A.14) immediately implies that $\sum_{i=h}^H f^h(\omega) = \mathcal{R}$ for every $\omega \in \Omega(\mathcal{R})$. Moreover, (A.14) trivially implies that $f^h(\omega) \geq 0$ for every $h$ and for every $\omega \in \Omega(\mathcal{R})$. Therefore, $f(\omega)$ is a continuous map from $\Omega(\mathcal{R})$ into $\Omega(\mathcal{R})$. Therefore, $f$ has a fixed point $\omega^* = f(\omega^*)$.

Now set $G^* = \sum_{i=1}^H \tau^i G^i(\tau^i, \omega^{i*})$. Using the definition (A.14) of the map $f(\cdot)$, and the fact that $\tau^h > 0$ for every $h$ it is now immediate to see that we then get

$$G^h(\tau^h, \omega^{h*}) = G^* \qquad \forall\, h \tag{A.15}$$

From (A.15) it is now clear that if we set $\hat{T}^h = G^*/H$ and $\omega^h = \omega^{h*}$ for every $h$, we get that the solution to Problem (1) satisfies $\hat{T}^h = \hat{T}^h(\tau^h, \omega^h, \hat{T}^{-h})$ for every $h$. Therefore, there only remains to show that at the fixed point of $f(\cdot)$ we obtain $\omega^{h*} > 0$ as required by the statement of the Proposition.

Notice that, using an argument virtually identical to the proof of Lemma A.1, and the properties of the fixed point we have just shown to be true, it is possible to show that the allocation $\mathcal{C}^* = (\omega^{1*} - \tau^1 G^*, \ldots, \omega^{H*} - \tau^H G^*, G^*)$ is in fact Pareto-efficient. (To avoid repetition, we omit the details of this claim.) Suppose now, by way of contradiction, that for some $h$ we have that $\omega^{h*} = 0$. From (A.14) it is immediate that this would imply that $G^* = 0$. However, by Assumption 5 this implies that $\mathcal{C}^*$ cannot be Pareto-efficient, contrary to what we have already shown. This contradiction is enough to prove that we must have that $\omega^{h*} > 0$ for every $h$, and therefore the proof is now complete . ■

## A.3. Feasible Allocations Under the Scheme

PROOF OF REMARK 2: It is clearly sufficient to show that for every $\beta^* \in \mathbb{R}^H$ such that

$$\sum_{h=1}^{H} \beta^{h*} = 0 \tag{A.16}$$

there exists a solution to the following system of $3H$ linear equations in the $H^2$ unknowns $\hat{B}$.

$$\begin{gathered}
\sum_{i=1}^{H} \hat{b}_1^i = \beta^{1*} \quad \sum_{i=1}^{H} \hat{b}_2^i = \beta^{2*} \quad \ldots \quad \sum_{i=1}^{H} b_H^i = \beta^{H*} \\
\hat{b}_1^1 = 0 \quad \hat{b}_2^2 = 0 \quad \ldots \quad \hat{b}_H^H = 0 \\
\sum_{i=1}^{H} \hat{b}_i^1 = 0 \quad \sum_{i=1}^{H} \hat{b}_i^2 = 0 \quad \ldots \quad \sum_{i=1}^{H} b_i^H = 0
\end{gathered} \tag{A.17}$$

Since we are assuming throughout that $H \geq 3$ (Assumption 1), it is obvious that the system (A.17) has at least as many unknowns as it has equations.[18]

It is also easy to see that the system (A.17) must have a solution $\hat{B}^*$. This can be checked directly, for instance, by setting $\hat{B}^*$ as follows. For every $1 \leq h \leq H-2$ set $\hat{b}_h^{h+1*} = \beta^{h*} + \beta^{H*}$, $\hat{b}_1^{H*} = -\beta^{H*}$ and $\hat{b}_h^{j*} = 0$ for every $j$ other than $j = h+1$ and $j = H$. Also set $\hat{b}_{H-1}^{1*} = \beta^{H-1*} + \beta^{H*}$, $\hat{b}_{H-1}^{H*} = -\beta^{H*}$ and $\hat{b}_{H-1}^{j*} = 0$ for every $j$ other than $j = 1$ and $j = H$. Moreover, set $\hat{b}_H^{1*} = -\beta^{H-1*} - \beta^{H*}$, as well as $\hat{b}_H^{j*} = -\beta^{j*} - \beta^{H*}$, and finally $\hat{b}_H^{H*} = -(H-1)\beta^{H*}$. It is immediate to verify that these values solve (A.17) for every $\beta \in \mathbb{R}^H$ that satisfies (A.16). ■

## A.4. The Proof of Theorem 1

We begin with several Lemmas, leading up to the proof of Theorem 1.

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[18] It is worth remarking that our assumption that $H \geq 3$ does real work here. It is easy to check that when $H = 2$, the system (A.17) in fact has a solution only when $\beta^{1*} = \beta^{2*} = 0$.

LEMMA A.8: *Let any $\tau \in \Delta^H$ be given and consider the strategic-form game $\Gamma(\tau)$ described in Remark 3. Let any array $(\hat{T}, \hat{B})$ with $\min_{h \in \mathcal{H}} B^h < 0$ and $(\hat{T}, \hat{B}) \notin \mathcal{F}(\tau)$ as defined in (8) be given. Then $(\hat{T}, \hat{B})$ cannot be a Nash equilibrium of $\Gamma(\tau)$.*

PROOF: We proceed by contradiction. Assume, without loss of generality that $(\hat{T}, \hat{B}) \notin \mathcal{F}(\tau)$ is a Nash equilibrium of $\Gamma(\tau)$ and that $B^H < 0$. We want to show that $H$ can unilaterally deviate and increase his utility.

Recall that the utility of $H$ corresponding to $(\hat{T}, \hat{B})$ is

$$U^H(\omega^H, 0) \tag{A.18}$$

Now consider $H$ deviating from playing $(\hat{T}^H, \hat{b}^H)$ to playing $(\hat{T}^{H*}, \hat{b}^{H*})$, with the latter defined as follows.

$$\hat{T}^{H*} = -\sum_{h=1}^{H-1} \hat{T}^h \tag{A.19}$$

and

$$\hat{b}_h^{H*} = \frac{\epsilon}{H-1} - \sum_{i=1}^{H-1} \hat{b}_h^i \qquad \forall\, h = 1, \ldots, H-1 \tag{A.20}$$

and

$$\hat{b}_H^{H*} = -\epsilon - \sum_{h=1}^{H-1} \hat{b}_H^h \tag{A.21}$$

with $\epsilon > 0$ satisfying $\epsilon \leq (H-1)\omega^h$ for every $h = 1, \ldots, H-1$.

Notice that using (2) we immediately know that $\sum_{h=1}^{H} \hat{b}_h^{H*} = 0$ as required. Moreover, since $B^H < 0$ by assumption, we also know that $\hat{b}^{H*} \leq 0$ as required. Plugging (A.19), (A.20) and (A.21) into (3), (4), (5) and (6), we obtain that, after $H$'s deviation to $(\hat{T}^{H*}, \hat{b}^{H*})$ the outcome of the game is as follows. Every $h = 1, \ldots, H-1$ consumes $c^h = \omega^h - \epsilon/(H-1)$ of private good and $G = 0$ public good, while $H$'s consumption is given by $c^H = \omega^H + \epsilon$ of private good and $G = 0$ public good. Therefore, after the deviation $H$'s utility is given by

$$U^H(\omega^H + \epsilon, 0) \tag{A.22}$$

By Assumption 2 (monotonicity), the quantity in (A.22) is strictly greater than the quantity in (A.18). Therefore what we have described is a profitable deviation for $H$, and this clearly suffices to prove the claim. ■

Lemma A.9: *Let any $\tau \in \Delta^H$ be given and consider the strategic-form game $\Gamma(\tau)$ described in Remark 3. Let any array $(\hat{T}, \hat{B})$ with $\min_{h \in \mathcal{H}} B^h = 0$ and $(\hat{T}, \hat{B}) \notin \mathcal{F}(\tau)$ as defined in (8) be given. Then $(\hat{T}, \hat{B})$ cannot be a Nash equilibrium of $\Gamma(\tau)$.*

Proof: Notice first that, using (2) it is immediate that if $\min_{h \in \mathcal{H}} B^h = 0$ then $B^h = 0$ for every $h = 1, \ldots, H$.

The rest of the argument is by contradiction. Assume that an array$(\hat{T}, \hat{B})$ as in the statement of the Lemma is indeed a Nash equilibrium of $\Gamma(\tau)$. Recall that since $(\hat{T}, \hat{B}) \notin \mathcal{F}(\tau)$, the utility of every $h$ corresponding to $(\hat{T}, \hat{B})$ is given by $U^h(\omega^h, 0)$.

Since $(\hat{T}, \hat{B})$ is a Nash equilibrium of $\Gamma(\tau)$, of course it must be the case that no $h$ has a profitable unilateral deviation starting from $(\hat{T}, \hat{B})$. However, by inspection of problem (1), this implies that $\hat{T}^h(\tau^h, \omega^h, 0, \ldots, 0) = 0$ for every $h \in \mathcal{H}$. Therefore $\tau$ is a Lindahl equilibrium for $E(\omega)$ with corresponding Lindahl equilibrium allocation $\mathcal{C} = (\omega^1, \ldots \omega^H, 0) = \mathcal{C}^0$. By Proposition 1 this in turn implies that $\mathcal{C}^0$ is Pareto-efficient. Since by Assumption 5 we know that $\mathcal{C}^0 \notin \mathcal{P}$ this is a contradiction, and hence the proof of the claim is now complete. ■

Lemma A.10: *Let any $\tau \in \Delta^H$ be given and consider the strategic-form game $\Gamma(\tau)$ described in Remark 3. Then any array of declarations $(\hat{T}, \hat{B})$ which constitutes a Nash equilibrium point of $\Gamma(\tau)$ must have the property that $(\hat{T}, \hat{B}) \in \mathcal{F}(\tau)$ as defined in (8).*

Proof: Since by (2) it must be the case that $\sum_{h=1}^{H} B^h = 0$, the claim is a direct consequence of Lemma A.8 and Lemma A.9. ■

Lemma A.11: *Let a vector of individually rational tax loadings $\tau \in \widehat{\Delta}^{H-1}$ and an allocation $\mathcal{C}^*$ $\in \Xi(\tau, \omega)$ be given. Then $\mathcal{C}^*$ is a Nash equilibrium outcome of $\Gamma(\tau)$.*

Proof: By Proposition 2 we know that for some (redistributed) endowment vector $\omega^* \in \Omega(\tau)$, $\tau$ is a Lindahl equilibrium of $E(\omega^*)$ with corresponding equilibrium allocation $\mathcal{C}^*$. By Remark 2 we know that for some $\hat{B}^*$ satisfying (2) and $\hat{b}_h^{h*} = 0$ for every $h$, we have that

$$\omega^{h*} = \omega^h - B^{h*} \qquad \forall\, h = 1, \ldots, H \tag{A.23}$$

Consider now an array $(\hat{T}^*, \hat{B}^*)$ with $\hat{B}^*$ as required for (A.23) to hold and $\hat{T}^{h*} = G^*/H$ for every $h$. Observe that the outcome associated with $(\hat{T}^*, \hat{B}^*)$ in $\Gamma(\tau)$ is precisely $\mathcal{C}^*$. To see that $(\hat{T}^*, \hat{B}^*)$ is an equilibrium of $\Gamma(\tau)$ we proceed as follows.

Notice first of all that since $\tau$ is a Lindahl equilibrium of $E(\omega^*)$, if we set $\omega^h = \omega^{h*}$ and $\hat{T}^{-h} = \hat{T}^{-h*}$ in problem (1) for every $h$, we have that the highest possible utility for $h$ is attained by setting $\hat{T}^h = \hat{T}^{h*}$ and $c^h = c^{h*}$. Therefore, the highest possible utility that $h$ can achieve in $\Gamma(\tau)$, given $(\hat{T}^{-h}, \hat{B}^{-h})$, is

$$\max \left\{ U^h(\omega^h, 0)\,,\; U^h(c^{h*}, G^*) \right\} \tag{A.24}$$

Since $\mathcal{C}^* \in \Xi(\tau, \omega)$ is individually rational the maximum in (A.24) is equal to $U^h(c^{h*}, G^*)$. It now follows directly that $h$'s payoff in $\Gamma(\tau)$, given $(\hat{T}^{-h*}, \hat{B}^{-h*})$ is maximized by setting $\hat{T}^h = \hat{T}^{h*}$ and $c^h = c^{h*}$. This is clearly enough to prove the claim. ■

LEMMA A.12: *Let $\omega^h$, $\tau$ and an array $(\hat{T}^{-h}, \hat{B}^{-h})$ be given. Assume that $\omega^h$ and $\hat{B}^{-h}$ are such that $\omega^{h*} = \omega^h - \sum_{i \neq h} \hat{b}^i_h \geq 0$.*

*Let also a pair $(c^h, G^h)$ that is feasible in Problem (1), given $\hat{T}^{-h}$ and when $h$'s endowment is set equal to $\omega^{h*} = \omega^h - \sum_{i \neq h} \hat{b}^i_h$, be given. Then there exists a pair $(\hat{T}^h, \hat{b}^h)$ such that (i) the array $(\hat{T}^h, \hat{T}^{-h}, \hat{b}^h, \hat{B}^{-h})$ is in $\mathcal{F}(\tau)$, where $\mathcal{F}(\tau)$ is defined as in (8), and (ii) $\mathcal{O}^h(\tau, \hat{T}, \hat{B}) = (c^h, G^h)$.*

PROOF: By inspection of the constraint in Problem (1) and of the definition of $\mathcal{F}(\tau)$ it is enough to prove the claim for any pair $(c^h, G^h)$ that is on the boundary of the feasible set in Problem (1). We divide the rest of the argument into two parts, dealing with two separate possible cases.

The first case we consider is the one in which $\tau^h$, $\omega^{h*}$ and $\mathcal{R}$ are such that for every pair $(c^h, G^h)$ such that $c^h + \tau^h G^h \leq \omega^{h*}$ it is also the case that $c^h + G^h \leq \mathcal{R}$. Notice that in this case the boundary of the feasible set in Problem (1) consists entirely of pairs $(c^h, G^h)$ such that $c^h + \tau^h G^h = \omega^{h*}$. Next, let any pair $(c^h, G^h)$ satisfying the last equality be given. We now construct the corresponding $(\hat{T}^h, \hat{b}^h)$ as follows. Let

$$\hat{T}^h \;=\; G^h \;-\; \sum_{i \neq h} \hat{T}^i \tag{A.25}$$

set $\hat{b}^h_h = 0$, and for every $i \neq h$ let

$$\hat{b}^h_i \;=\; \omega^i \;-\; \sum_{j \neq h} \hat{b}^j_i \;-\; (\mathcal{R} - \omega^{h*}) \frac{\tau^i}{\displaystyle\sum_{j \neq h} \tau^j} \tag{A.26}$$

Notice that since $\sum_{h \in \mathcal{H}} \omega^h = \mathcal{R}$, taking sums over $i \neq h$ and using (2 immediately gives $\sum_{i \in \mathcal{H}} \hat{b}^h_i$ $= 0$. Therefore $\hat{b}^h$ satisfies (2) as required.

Our next step is to check that the array $(\hat{T}^h, \hat{T}^{-h}, \hat{b}^h, \hat{B}^{-h})$ is in $\mathcal{F}(\tau)$ when $\hat{T}^h$ is as in (A.25), $\hat{b}^h_h = 0$ and $\hat{b}^h_i$ is as in (A.26) for every $i \neq h$. Using (A.25) and (A.26) it is enough to check that, for every $i \neq h$ we have that

$$\tau^i G^h \;\leq\; (\mathcal{R} - \omega^{h*}) \frac{\tau^i}{\displaystyle\sum_{j \neq h} \tau^j} \tag{A.27}$$

Using the fact that $\sum_{h\in\mathcal{H}} \tau^h = 1$, it is immediate to see that (A.27) is equivalent to

$$\omega^{h*} - \tau^h G^h \leq \mathcal{R} - G^h \tag{A.28}$$

Notice now that since we are assuming that $c^h + \tau^h G^h = \omega^{h*}$, the left-hand side of (A.28) is clearly equal to $c^h$. Since, by assumption, $(c^h, G^h)$ is feasible in Problem (1), the right-hand side of (A.28) must be greater or equal to $c^h$. Therefore (A.28) must be verified. This is clearly sufficient to show that, in this case $(\hat{T}^h, \hat{T}^{-h}, \hat{b}^h, \hat{B}^{-h})$ is in $\mathcal{F}(\tau)$. Using (A.25) and $\hat{b}_h^h = 0$, it is now obvious that $\mathcal{O}^h(\tau, \hat{T}, \hat{B}) = (c^h, G^h)$. This clearly suffices to prove our claim in the first case.

The second case we consider is the one in which some pairs $(c^h, G^h)$ on the boundary of the feasible set of Problem (1) are such that $c^h + G^h = \mathcal{R}$ and $c^h + \tau^h G^h < \omega^{h*}$. (Evidently, this is only possible when $\tau^h \mathcal{R} < \omega^{h*}$). In this case, for those points on the boundary of the feasible set of Problem (1) that satisfy $c^h + \tau^h G^h = \omega^{h*}$ our claim can be proved exactly as in the first case, and we do not repeat the argument here. Therefore, there only remains to prove our claim for those pairs on the boundary of the feasible set of Problem (1) that satisfy $c^h + G^h = \mathcal{R}$ and $c^h + \tau^h G^h < \omega^{h*}$. Let any such pair be given. We now construct the corresponding $(\hat{T}^h, \hat{b}^h)$ as follows. Let $\hat{T}^h$ be as in (A.25), and set

$$\hat{b}_h^h = \omega^h - \mathcal{R} + G^h(1 - \tau^h) - \sum_{i \neq h} \hat{b}_h^i \tag{A.29}$$

Notice that since $c^h + G^h = \mathcal{R}$ and $c^h + \tau^h G^h < \omega^{h*}$, this implies that $\hat{b}_h^h \leq 0$, as required. For every $i \neq h$ set

$$\hat{b}_i^h = \omega^i - \sum_{j \neq h} \hat{b}_i^j - G^h(1 - \tau^h) \frac{\tau^i}{\sum\limits_{j \neq h} \tau^j} \tag{A.30}$$

Our next step is to check that the array $(\hat{T}^h, \hat{T}^{-h}, \hat{b}^h, \hat{B}^{-h})$ is in $\mathcal{F}(\tau)$ in this case. Using (A.25), (A.29) and (A.30) it is enough to check that, for every $i \neq h$ we have that

$$\tau^i G^h \leq G^h(1 - \tau^h) \frac{\tau^i}{\sum\limits_{j \neq h} \tau^j} \tag{A.31}$$

However, using the fact that $\sum_{h\in\mathcal{H}} \tau^h = 1$, it is immediate that (A.31) must always be true. This is clearly sufficient to show that, in this case $(\hat{T}^h, \hat{T}^{-h}, \hat{b}^h, \hat{B}^{-h})$ is in $\mathcal{F}(\tau)$. Using (A.25), and (A.29) it is now obvious that $\mathcal{O}(\tau, \hat{T}, \hat{B}) = (c^h, G^h)$ as required. Therefore, this is enough to conclude the proof of the claim. ■

Lemma A.13: *Let a vector of individually rational tax loadings $\tau \in \widehat{\Delta}^{H-1}$. Let an allocation $\mathcal{C}^* \notin \Xi(\tau, \omega)$ be given. Then $\mathcal{C}^*$ is not a Nash equilibrium outcome of $\Gamma(\tau)$.*

PROOF: Since the minmax payoff of each player in $\Gamma(\tau)$ is $U^h(\omega^h, 0)$ if $\mathcal{C}^*$ is not an individually rational allocation the claim is trivial and we omit the details.

Suppose next that $\mathcal{C}^*$ is an individually rational allocation that is a Nash equilibrium outcome of $\Gamma(\tau)$, and let $(\hat{T}^*, \hat{B}^*)$ be the corresponding equilibrium strategy profile. Recall that, by Lemma A.10 we know that $(\hat{T}^*, \hat{B}^*) \in \mathcal{F}(\tau)$. For each $h$, let $\omega^{h*} = \omega^h - \sum_{i \neq h} \hat{b}_h^{i*}$, and $\omega^* = (\omega^{1*}, \ldots, \omega^{H*})$.

Since $(\hat{T}^*, \hat{B}^*) \in \mathcal{F}(\tau)$, using Lemma A.12 it is now immediate that since $(\hat{T}^*, \hat{B}^*)$ is a Nash equilibrium of $\Gamma(\tau)$ we must have that $\hat{T}^h(\tau^h, \omega^{h*}, \hat{T}^{-h*}) = \hat{T}^{h*}$ for every $h$. Therefore $\tau$ is a Lindahl equilibrium for $E(\omega^*)$ with corresponding Lindahl equilibrium allocation $\mathcal{C}^*$. Therefore $\mathcal{C}^* \in \Xi(\tau, \omega)$. This is clearly enough to prove the claim. ■

PROOF OF THEOREM 1: The claim is a direct consequence of Lemma A.11 and Lemma A.13. ■

## A.5. THE PROOF OF THEOREM 2

The proof of Theorem 2 also begins with several Lemmas which are then used to prove the main result.

LEMMA A.14: *Let any $\tau \in \Delta^H$ be given and consider the strategic-form game $\Gamma^W(\tau)$ described in Remark 4. Let any array $(\hat{T}, \hat{B}, \hat{W})$ with $\min_{h \in \mathcal{H}}[(\hat{\omega}^h - \overline{\omega})/H] - B^h < 0$ and $(\hat{T}, \hat{B}) \notin \mathcal{F}^W(\tau, \hat{W})$ as defined in (11) be given. Then $(\hat{T}, \hat{B}, \hat{W})$ cannot be a Nash equilibrium of $\Gamma^W(\tau)$.*

PROOF: A minor adaptation of the proof of Lemma A.8 above is sufficient to prove the claim. In particular, $H$'s deviation must now be computed taking into account the overall transfers to each agent $h$ which now total $[(\hat{\omega}^h - \overline{\omega})/H] - B^h$, rather than $-B^h$ as before. The rest of the details are omitted for the sake of brevity. ■

LEMMA A.15: *Let any $\tau \in \Delta^H$ be given and consider the strategic-form game $\Gamma^W(\tau)$ described in Remark 4. Let any array $(\hat{T}, \hat{B}, \hat{W})$ with $\min_{h \in \mathcal{H}}[(\hat{\omega}^h - \overline{\omega})/H] - B^h = 0$ and $(\hat{T}, \hat{B}) \notin \mathcal{F}^W(\tau, \hat{W})$ as defined in (11) be given. Then $(\hat{T}, \hat{B}, \hat{W})$ cannot be a Nash equilibrium of $\Gamma^W(\tau)$.*

PROOF: Notice first of all that $\min_{h \in \mathcal{H}}[(\hat{\omega}^h - \overline{\omega})/H] - B^h = 0$ implies that

$$\frac{\hat{\omega}^h - \overline{\omega}}{H} - B^h = 0 \quad \forall\, h = 1, \ldots, H \tag{A.32}$$

Notice also that, using (10) and (A.32), together with the fact that we restrict $\hat{\omega}^h$ to be greater that 0, clearly implies that in this case we must have that $\tilde{\omega}^h - B^h = \hat{\omega}^h > 0$.

Consider now the maximization problem (1) for every individual $h$. Our first step is to notice that, for the given $\tau^h$ and actual endowment $\omega^h$ it must be that $G^h(\tau^h, \omega^h) > 0$ for at least one $h$.

To see why this must be the case suppose, by way of contradiction, that $G^h(\tau^h, \omega^h) = 0$ for every $h$. Then $\tau$ must be a Lindahl equilibrium for $E(\omega)$ with corresponding Lindahl equilibrium allocation $\mathcal{C}^0$. Therefore, by Proposition 1, $\mathcal{C}^0$ would have to be a Pareto-efficient allocation, which directly contradicts Assumption 5.

The rest of the argument is also by contradiction. Therefore suppose that $(\hat{T}, \hat{B}, \hat{W})$ with $[(\hat{\omega}^h - \overline{\omega})/H] - B^h = 0$ for every $h$, and $(\hat{T}, \hat{B}) \notin \mathcal{F}^W(\tau, \hat{W})$ is a Nash Equilibrium of $\Gamma^W(\tau)$. We then want to show that at least one agent has a profitable unilateral deviation available.

Using our first step, without loss of generality up to a re-labeling of agents, assume that $G^H(\tau^H, \omega^H) > 0$. From the strict quasi-concavity of $U^H$ it now follows that

$$U^H \left\{ \lambda(\omega^H, 0) + (1-\lambda) \left[ c^H(\tau^H, \omega^H), G^H(\tau^H, \omega^H) \right] \right\} \; > \; U^H(\omega^H, 0) \qquad \forall \, \lambda[0, 1) \tag{A.33}$$

Recall that, as we observed above, $\tilde{\omega}^h - B^h = \hat{\omega}^h > 0$ for every $h$. Therefore it is immediate that there exists a $\overline{\lambda} < 1$ such that

$$(1-\lambda) \, \tau^h \, G^H(\tau^H, \omega^H) \; \leq \; \tilde{\omega}^h - B^h \qquad \forall \, \lambda \geq \overline{\lambda} \qquad \forall \, h = 1, \dots, H \tag{A.34}$$

Now consider $H$ deviating from playing $(\hat{T}^H, \hat{b}^H, \hat{\omega}^H)$ to playing $(\hat{T}^{H*}, \hat{b}^{H*}, \hat{\omega}^{H*})$, with the latter defined as follows.

$$\hat{T}^{H*} \; = \; - \sum_{h=1}^{H-1} \hat{T}^h \; + \; (1-\lambda) \, G^H(\tau^H, \omega^H) \tag{A.35}$$

with $\lambda \in (\overline{\lambda}, 1)$, while $\hat{b}^{H*} = \hat{b}^H$ and $\hat{\omega}^{H*} = \hat{\omega}^H$. Plugging $(\hat{T}^{H*}, \hat{b}^{H*}, \hat{\omega}^{H*})$ into (3), (4), (5) and (6) we now obtain that after $H$'s deviation the outcome of the game gives $H$ a utility of

$$U^H \left\{ \lambda(\omega^H, 0) + (1-\lambda) \left[ c^H(\tau^H, \omega^H), G^H(\tau^H, \omega^H) \right] \right\} \tag{A.36}$$

Since before the deviation $H$ utility was $U^H(\omega^H, 0)$, (A.33) is now sufficient to show that this is a profitable deviation for $H$. This is clearly enough to conclude the proof of the Lemma. ■

Lemma A.16: *Let any $\tau \in \Delta^H$ be given and consider the strategic-form game $\Gamma^W(\tau)$ described in Remark 4. Then any array of declarations $(\hat{T}, \hat{B}, \hat{W})$ that constitutes a Nash equilibrium point of $\Gamma^W(\tau)$ must have the property that $(\hat{T}, \hat{B}) \in \mathcal{F}^W(\tau, \hat{W})$ as defined in (11).*

Proof: Since by (2) it must be the case that $\sum_{h=1}^{H} B^h = 0$, the claim is a direct consequence of Lemma A.14 and Lemma A.15. ■

Lemma A.17: *Let $\tau$ be given and consider $\Gamma^W(\tau)$. Let also any $(\hat{T}, \hat{B}, \hat{W})$ that constitutes a Nash equilibrium of $\Gamma^W(\tau)$ be given. Moreover, assume that, in this equilibrium, for some $j \in \mathcal{H}$ we have that*

$$\tilde{\omega}^j \; - \; \tau^j \sum_{h=1}^{H} \hat{T}^h \; - \; B^j \; > \; 0 \tag{A.37}$$

*then it must be the case that*

$$\hat{\omega}^h \;\; = \;\; \omega^h \qquad\qquad \forall\, h \neq j \tag{A.38}$$

Proof: Without loss of generality, up to a re-labeling of agents, assume that

$$\tilde{\omega}^1 \; - \; \tau^1 \sum_{h=1}^{H} \hat{T}^h \; - \; B^1 \; > \; 0 \tag{A.39}$$

By way of contradiction (without loss of generality again) assume also that

$$\hat{\omega}^H \;\; < \;\; \omega^H \tag{A.40}$$

Consider now the following deviation by $H$, from playing $(\hat{T}^H, \hat{b}^H, \hat{\omega}^H)$ to playing $(\hat{T}^{H*}, \hat{b}^{H*}, \hat{\omega}^{H*})$, with the latter defined as follows.

$$\hat{T}^{H*} \; = \; \hat{T}^H \tag{A.41}$$

and

$$\hat{\omega}^{H*} \;\; = \;\; \hat{\omega}^H \; + \epsilon \tag{A.42}$$

with $\epsilon > 0$, and

$$b_i^{H*} \;\; = \;\; b_i^H \;\; - \;\; \frac{1}{H^2}\,\epsilon \qquad \forall\, i = 1, \ldots, H-1 \tag{A.43}$$

and

$$b_1^{H*} \;\; = \;\; b_1^H \qquad \text{and} \qquad b_H^{H*} \;\; = \;\; b_H^H \tag{A.44}$$

Plugging (A.41), (A.42), (A.43) and (A.44) into (3), (4), (5) and (6) immediately shows that whenever

$$\epsilon \; < \; \frac{H}{H-1} \left[ \tilde{\omega}^1 \; - \; \tau^1 \sum_{h=1}^{H} \hat{T}^h \; - \; B^1 \right] \tag{A.45}$$

then it must be the case that

$$\left(\hat{T}^{-H}, \hat{T}^{H*}, \hat{B}^{-H}, \hat{b}^{H*}\right) \in \mathcal{F}^W\left(\tau, \hat{W}^{-H}, \hat{\omega}^{H*}\right) \tag{A.46}$$

It now follows that, using (A.41), (A.42), (A.43), and (A.44) together with (12), after $H$'s deviation his consumption of private good has increased by $[(H-1)/H]\epsilon$, while the amount of public good provided in the economy has not changed. This is clearly sufficient to show that this is a profitable deviation for $H$, and hence concludes the proof of the Lemma. ■

LEMMA A.18: *Let $\tau$ be given and consider $\Gamma^W(\tau)$. Let also any $(\hat{T}, \hat{B}, \hat{W})$ that constitutes a Nash equilibrium of $\Gamma^W(\tau)$ be given. Moreover, assume that, in this equilibrium*

$$\tilde{\omega}^j - \tau^j \sum_{h=1}^{H} \hat{T}^h - B^j > 0 \tag{A.47}$$

*for at least two distinct agents. Then it must be the case that*

$$\hat{\omega}^h = \omega^h \qquad \forall\, h = 1, \ldots, H \tag{A.48}$$

PROOF: The claim is an immediate consequence of Lemma A.17. ■

LEMMA A.19: *Let $\tau$ be given and consider $\Gamma^W(\tau)$. Let also any $(\hat{T}, \hat{B}, \hat{W})$ that constitutes a Nash equilibrium of $\Gamma^W(\tau)$ be given. Moreover, assume that, in this equilibrium, for some $j \in \mathcal{H}$ we have that*

$$\tilde{\omega}^j - \tau^j \sum_{h=1}^{H} \hat{T}^h - B^j > 0 \tag{A.49}$$

*while*

$$\tilde{\omega}^h - \tau^h \sum_{i=1}^{H} \hat{T}^i - B^h = 0 \qquad \forall\, h \neq j \tag{A.50}$$

*Then there exists a $(\hat{T}^*, \hat{B}^*, \hat{W}^*)$ that also constitutes a Nash equilibrium of $\Gamma^W(\tau)$ and such that*

$$\mathcal{O}^{Wh}(\hat{T}^*, \hat{B}^*, \hat{W}^*) = \mathcal{O}^{Wh}(\hat{T}, \hat{B}, \hat{W}) \qquad \forall\, h = 1, \ldots, H \tag{A.51}$$

*and*

$$\hat{\omega}^{h*} = \omega^h \qquad \forall\, h = 1, \ldots, H \tag{A.52}$$

Proof: By Lemma A.17 it is immediate that

$$\hat{\omega}^h = \omega^h \qquad \forall\, h \neq j \tag{A.53}$$

We now construct the new Nash equilibrium as in the statement of the Lemma as follows. Set

$$(\hat{T}^{h*}, \hat{b}^{h*}, \hat{\omega}^{h*}) = (\hat{T}^h, \hat{b}^h, \omega^h) \qquad \forall\, h \neq j \tag{A.54}$$

and

$$\hat{T}^{j*} = \hat{T}^j \tag{A.55}$$

and

$$\hat{\omega}^{j*} = \omega^j \tag{A.56}$$

while

$$\hat{b}_h^{j*} = \frac{1}{H^2}(\omega^j - \hat{\omega}^j) \qquad \forall\, h \neq j \tag{A.57}$$

and

$$\hat{b}_j^{j*} = \frac{H-1}{H^2}(\omega^j - \hat{\omega}^j) \tag{A.58}$$

Using (A.54), (A.55), (A.56), (A.57) and (A.58) together with (3), (4), (5) and (6) it is now immediate to check that

$$\mathcal{O}^{Wh}(\hat{T}^*, \hat{B}^*, \hat{W}^*) = \mathcal{O}^{Wh}(\hat{T}, \hat{B}, \hat{W}) \qquad \forall\, h = 1, \ldots, H \tag{A.59}$$

as required. Moreover, in the new proposed equilibrium the maximization problem faced by agent $j$ clearly has not changed. Therefore $j$'s new strategy must be a best response to the strategies of all other agents in $(\hat{T}^*, \hat{B}^*, \hat{W}^*)$.

Using (A.50) and (A.53) it is also straightforward to check that, for every agent $h \neq j$, after $j$'s change of strategy the opportunity set in terms of $c^h$ and $G$ has in fact not changed. This, using (A.59), is clearly enough to show that $(\hat{T}^*, \hat{B}^*, \hat{W}^*)$ must be a Nash equilibrium of $\Gamma^W(\tau)$. This concludes the proof of the Lemma. ■

Lemma A.20: *Let $\tau$ be given and consider $\Gamma^W(\tau)$. Let also any $(\hat{T}, \hat{B}, \hat{W})$ that constitutes a Nash equilibrium of $\Gamma^W(\tau)$ be given.*

*Then there exists a $(\hat{T}^*, \hat{B}^*, \hat{W}^*)$ that also constitutes a Nash equilibrium of $\Gamma^W(\tau)$ and such* *that*

$$\mathcal{O}^{Wh}(\hat{T}^*, \hat{B}^*, \hat{W}^*) \;=\; \mathcal{O}^{Wh}(\hat{T}, \hat{B}, \hat{W}) \qquad \forall\, h = 1, \ldots, H \tag{A.60}$$

*and*

$$\hat{\omega}^{h*} \;=\; \omega^h \qquad \forall\, h = 1, \ldots, H \tag{A.61}$$

Proof: Recall that by Lemma A.16 we know that any Nash equilibrium of $\Gamma^W(\tau)$ must satisfy $(\hat{T}^h, \hat{B}^h, \hat{W}^h) \in \mathcal{F}^W(\tau, \hat{W})$. Therefore, using (10) and (11), we know that

$$\tilde{\omega}^h \;-\; \tau^h \sum_{i=1}^{H} \hat{T}^i \;-\; B^h \;\geq\; 0 \qquad \forall\, h = 1, \ldots, H \tag{A.62}$$

and

$$\tilde{\omega}^j \;-\; \tau^h \sum_{h=1}^{H} \hat{T}^h \;-\; B^j \;>\; 0 \tag{A.63}$$

for at least one agent $j$. Therefore the Lemma is a direct consequence of Lemma A.17 and Lemma A.18. ■

Proof of Theorem 2: From Lemma A.20, we know that there is no loss of generality in restricting attention to Nash equilibria of $\Gamma^W(\tau)$ with the property that $\hat{\omega}^h = \omega^h$ for every $h = 1, \ldots, H$.

The rest of the proof is therefore a minor adaptation of the argument that we used to prove Theorem 1 (see Lemma A.11, Lemma A.12 and Lemma A.13). The details are omitted for the sake of brevity. ■

## A.6. The Proof of Theorem 3

As before, we use several preliminary Lemmas to lead up to the proof of Theorem 3.

Lemma A.21: *Consider the strategic-form game $\Gamma^V(\tau)$ described in Remark 5 and let Assumptions 1 through to 7 hold. Then any array of declarations $(\hat{T}, \hat{B}, v)$ that constitutes a Nash equilibrium point of $\Gamma^V(\tau)$ must have the property that $(\hat{T}, \hat{B}, v) \in \mathcal{F}^V(\tau)$ as defined in (16).*

Proof: Let $(\hat{T}, \hat{B}, v)$ be a Nash equilibrium of $\Gamma^V(\tau)$. Without loss of generality assume that $B^H = \min_{h \in \mathcal{H}} b^h$.

Notice next that if $B^H = 0$, then, since $\sum_{h \in \mathcal{H}} B^h = 0$, we must have that $B^h = 0$ for every $h$ and therefore it must be the case that $\mathcal{V}(\hat{B}) = \emptyset$. It follows that in this case we already know that $(\hat{T}, \hat{B}, v) \in \mathcal{F}^V(\tau)$.

The rest of the argument is by contradiction. Assume then that some $(\hat{T}, \hat{B}, v) \notin F^V(\tau)$ with $B^H < 0$ is a Nash equilibrium of $\Gamma^V(\tau)$. We now want to show that $H$ has a profitable unilateral deviation available.

Recall that $H$'s utility corresponding to the Nash equilibrium strategy profile $(\hat{T}, \hat{B}, v)$ is

$$U^H(\omega^H, 0) \tag{A.64}$$

Now consider $H$ deviating from playing $(\hat{T}^H, \hat{b}^H, v^H)$ to playing $(\hat{T}^{H*}, \hat{b}^{H*}, v^{H*})$ with the latter defined as follows.

$$v^{H*} = 0 \tag{A.65}$$

and

$$\hat{T}^{H*} = -\sum_{h=1}^{H-1} \hat{T}^h + \epsilon \tag{A.66}$$

with $\epsilon > 0$, and

$$\hat{b}_h^{H*} = -\sum_{i=1}^{H-1} \hat{b}_h^i \qquad \forall\, h = 1, \ldots, H \tag{A.67}$$

Notice that using (2) we immediately know that $\sum_{h=1}^{H} \hat{b}_h^{H*} = 0$ as required. Moreover, since $B^H$ $< 0$ by assumption, we also know that $\hat{b}^{H*} \leq 0$ as required. Plugging (A.66) and (A.67) into (3), (4), (5) and (6), we obtain that, after $H$'s deviation to $(\hat{T}^{H*}, \hat{b}^{H*})$ it must be that $B^{h*} = 0$ for every $h$ so that we immediately know that $\mathcal{V}(\hat{B}^{H*}, \hat{B}^{-H*}) = \emptyset$ and therefore that $(\hat{T}^{H*}, \hat{T}^{-H*}, \hat{B}^{H*}, \hat{B}^{-H*})$ $\in \mathcal{F}^V(\tau)$.

Moreover, the outcome of the game after $H$'s deviation to $(\hat{T}^{H*}, \hat{B}^{H*})$ is as follows. Every $h = 1, \ldots, H$ consumes $c^h = \omega^h - \tau^h \epsilon$ of private good and $G = \epsilon$ public of good. Therefore, after the deviation the utility of $H$ is given by

$$U^H(\omega^H - \tau^H \epsilon, \epsilon) \tag{A.68}$$

By Assumption 7, for some $\epsilon > 0$, the quantity in (A.68) is greater than the quantity in (A.64). Therefore, for the appropriate $\epsilon > 0$, $(\hat{T}^{H*}, \hat{b}^{H*})$ is a profitable unilateral deviation for $H$ from the proposed equilibrium. This is clearly enough to conclude the proof of the claim. ■

Lemma A.22: *Consider the strategic-form game $\Gamma^V(\tau)$ described in Remark 5 and let Assumptions 1 trough to 7 hold. Then any array of declarations $(\hat{T}, \hat{B}, v)$ that constitutes a Nash equilibrium point of $\Gamma^V(\tau)$ must have the property that $(\hat{T}, \hat{B}, v) \in \mathcal{F}(\tau)$ where $\mathcal{F}(\tau)$ is defined as in (8).*

PROOF: We start by showing that in any Nash equilibrium $(\hat{T}, \hat{B})$ of $\Gamma^V(\tau)$ we must have that the utility of each agent $h$ must be at least

$$U^h(\omega^h, 0) \tag{A.69}$$

Notice that for any $h \in \mathcal{V}(\hat{B})$ it is immediate that (A.69) must hold since given the action of the others, any $h \in \mathcal{V}(\hat{B})$ can unilaterally deviate to set $v^h = 1$, and hence achieve utility $U^h(\omega^h, 0)$.

Consider now any $h \notin \mathcal{V}(\hat{B})$. If for some $i \in \mathcal{V}(\hat{B})$ we have that $v^i = 1$, then the utility of $h$ is clearly equal to $U^h(\omega^h, 0)$. If $v^i = 0$ for every $i \in \mathcal{V}(\hat{B})$, notice that $h$ can always unilaterally deviate to setting $\hat{T}^{h*} = -\sum_{i \neq h} \hat{T}^i$. Clearly in this way he can guarantee utility $U^h(\omega^h - B^h, 0)$. Since $h \notin \mathcal{V}(\hat{B})$ we know that $B^h < 0$ in this case. Therefore $U^h(\omega^h - B^h, 0) > U^h(\omega^h, 0)$. Hence, the equilibrium utility of any agent must be at least $U^h(\omega^h, 0)$.

Observe now that, by Assumption 2 (monotonicity), we have that for every $h$

$$U^h(\omega^h, 0) > U^h(0, 0) \geq \mathcal{U}^h \tag{A.70}$$

where $\mathcal{U}^h$ is defined as in (14).

Lastly notice that, since by Lemma A.21 we know that any equilibrium $(\hat{T}, \hat{B})$ must satisfy $(\hat{T}, \hat{B}) \in \mathcal{F}^V(\tau)$, it is immediate that if $(\hat{T}, \hat{B}) \notin \mathcal{F}(\tau)$ then for some $h$ we must have that $U^h(\mathcal{O}^{Vh}(\hat{T}, \hat{B})) = \mathcal{U}^h$. This clearly contradicts the fact that the utility of every agent in any Nash equilibrium must be at least $U^h(\omega^h, 0)$. Therefore the proof of the claim is complete. ■

LEMMA A.23: *Let a vector of individually rational tax loadings $\tau \in \widehat{\Delta}^{H-1}$ be given, and let Assumptions 1, 2, 3, 4, 6 and 7 hold. Let an allocation $\mathcal{C}^* \in \Xi(\tau, \omega)$ be given. Then $\mathcal{C}^*$ is a Nash equilibrium outcome of $\overline{\Gamma}(\tau)$.*

The argument is virtually identical to the proof of Lemma A.11. All that needs to be added is that no agent will want deviate to set $v^h = 1$ since $\mathcal{C}^*$ is individually rational according to Definition 1. The rest of the details are omitted. ■

LEMMA A.24: *Let a vector of individually rational tax loadings $\tau \in \Delta^{H-1}$, and let Assumptions 1, 2, 3, 4, 5, 6 and 7 hold. Let an allocation $\mathcal{C}^* \notin \Xi(\tau, \omega)$ be given. Then $\mathcal{C}^*$ is not a Nash equilibrium outcome of $\overline{\Gamma}(\tau)$.*

Since by Lemma A.22 we know that any Nash equilibrium of $\Gamma^V(\tau)$ must be in $\mathcal{F}(\tau)$, the claim can be proved using an argument that is identical to the proof of Lemma A.13. The rest of the details are omitted. ■

PROOF OF THEOREM 3: The claim is a direct consequence of Lemma A.23 and Lemma A.24. ■

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