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by

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Competitive Problem Solving and the Optimal Prize Schemes*

Toru Suzuki†

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Abstract

Agents compete to solve a problem. Each agent knows own computational capacity as private information and simultaneously chooses either a risky or a safe problem solving method. This paper analyzes the optimal prize schemes from the perspective of the prize designer who wishes to find a solution as quick as possible. It is shown that (i) the winner-take-all scheme can induce excessive risk taking and make problem solving slower (ii) prize schemes with milder competitive pressure induce the optimal risk taking and quicker problem solving.

Keywords: Optimal prize scheme, Risk taking, Problem solving

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1 Introduction

The X-prize foundation, one of leading innovation prize organizers, claims that risk taking is the key factor for quicker breakthroughs and innovation prizes effectively induce the risk taking. On the other hand, when all agents take a risky method and all methods are failed, the risk taking can make innovation slower. This paper analyzes the condition in which a winner-take-all competition makes problem solving slower because of "excessive" risk taking. Then, it is shown that some prize schemes with milder competitive pressure can induce the optimal risk taking and quicker problem solving.

This paper focuses on the specific aspect of problem solving: risk taking. The riskiness of problem solving methods is one of the key choice variables in innovation races. For example, in the Human Genome Project, National Institute of Health (NIH) and Celera genomics compete for determining the sequence of chemical based pairs for human DNA. NIH employed a well known safe method and Celera genomics employed a new method whose effectiveness was unknown. In this paper, there are a safe method and a risky method for problem solving. In the safe method, the agent knows when he finds out a solution given his computational capacity or budget. Thus, the safe method guarantees the time to get a solution as long as he spends a sufficient amount of time. The safe method can be interpreted as an established method in which the time to find a solution is approximately known given his ability or budget. On the other hand, in the risky method, the performance depends on his luck and the agent only knows the probability distribution of the time to find a solution. Thus, in the risky method, the agent can find a solution very quickly if he is lucky but it can take longer time than the safe method. One interpretation of the risky method is a trial-and-error process with new ideas.

In the first part of this paper, I focus on problem solving in a popular prize scheme, winner-take-all competition. The basic setting is introduced in Section 2. The model consists of two agents who face a common problem and only the first agent who solves the problem gets a prize. Each agent simultaneously chooses a problem solving method in the beginning of the game. In Section 3, it is shown that the game has a unique symmetric equilibrium which has a cutoff property.

Section 4 evaluates the equilibrium problem solving from the perspective of the prize designer. The prize designer is impatient and wishes to find a solution as early as possible. Two kinds of prize designers, an expected utility maximizer and a worst case maximizer, are considered. Then, the optimal problem solving is defined as a strategy profile which induces the optimal risk taking, i.e., the strategy profile which maximizes the designer's interest. It is shown that, in the winner-take-all competition, the optimal problem solving cannot be the equilibrium for "difficult" problems because of excessive risk taking.

In Section 5, prize schemes which induce the optimal risk taking are analyzed. A prize scheme is optimal if the optimal problem solving is an equilibrium under the prize scheme. Two classes of prize schemes are introduced. The first is a multi-prize scheme in which prizes are awarded not only to the first but also to the second winner. Second is a probabilistic prize scheme in which the prize is determined by the winner-take-all basis with some probability and an absolute performance basis with some probability. The optimal prize scheme is analyzed for each class and it is shown that the key to induce the optimal risk taking is the choice of "right" competitive pressure.

Section 6 extends our analysis to a "costly problem solving" environment where each agent has another task in addition to problem solving and the prize designer has to take into account the participation constraint. Each agent has a safe task which guarantees a small payoff but if he focuses on the safe task, it deteriorates the quality of the problem solving. In this setting, since the prize designer cannot observe the choice of the task, there is moral hazard problem. On the other hand, this multi-task environment makes the benefit of competitive schemes clear. That is, the competitive pressure provides the incentive to focus on problem solving. Then, the question is whether we can provide a reasonable level of competitive pressure which does not induce the excessive risk taking behavior yet still provides incentive to participate the competition. It is shown that when the payoff from the safe task is sufficiently low and the lack of the effort deteriorates the speed of his problem solving significantly, there is a prize scheme which induces the optimal problem solving. The prize scheme is the same as the optimal prize scheme in the non multi-task environment except for one aspect: there is a deadline for the problem solving in which the agent cannot get any prize from the competition after the deadline.

Related literature. Since agents compete to find a solution, this model is related to strategic search models. In most of strategic search models, the intensity of search is a key choice variable, e.g., Fershtman and Rubinstein (1997), R&D race models such as Dasgupta and Stiglitz (1980). However, this paper focuses on the different aspect "how to solve it." I restrict our attention to the situation where "how hard we think" is much less important than "how to solve." For example, when the budget is already determined, most of scientists try to utilize their time and budget as much as possible. Then, the key choice variable is their approach to solve the problem.

The winner-take-all competition induces an excessive risk taking behavior for some problems in our model. Risk taking in winner-take-all environments is well known, e.g., Tirole (1988), Dekel and Scotchmer (1999), Hvide (2002). Unlike their models, risk taking itself is not main interest of our paper. The focus of our paper is “excessive" risk taking and prize schemes which eliminate the excessive risk taking and induce the optimal risk taking.

This paper is also related to contest and tournament literature. In most of contest and tournament models, each agent chooses the effort or investment level, e.g., Lazear and Rosen (1981). On the other hand, in my model, the main interest is in the risk taking in problem solving. There are not many models which take into account risk taking in contest. Hvide (2002), Hvide and Kristiansen (2003), and Krakel (2008) analyzed risk taking in contest and provide some interesting economic implications, e.g., explanation of the relative performance evaluation puzzle. Unlike their papers, the main interest of our paper is in the design of the prize scheme. Thus, this paper is also related to contest/tournament design literatures, e.g., Taylor (1995), Moldovanu and Sela (2001, 2006), Che and Gale (2003). The main difference is that they design contest/tournament to induce the optimal effort or investment level. On the other hand, our prize scheme is designed to induce the optimal risk taking.

2 Basic Model

Basics. Time is continuous and indexed by $t \in [0, \infty) = \mathcal{T}$. There are two agents, $i = 1, 2$ who compete to solve a problem under a prize scheme. Let t_i^* be the time at which agent i finds a solution and (t_1^*, t_2^*) be an outcome of problem solving. Let $Z = \mathbb{R}_+$ be the set of prizes. Then, the prize function for agent i is a mapping $b_i : \mathcal{T}^2 \to (\mathcal{T} \times Z) \cup \{\varnothing\}$ which specifies the amount of prize $z \in Z$ and its delivery timing given (t_1^*, t_2^*) where $\{\varnothing\}$ denotes "no prize," i.e., his payoff is 0 for all $t \in \mathcal{T}$. A *prize scheme* is, then, $b = (b_1(t_1^*, t_2^*), b_2(t_2^*, t_1^*))$. Each agent has the same time preference. Let $u_i(z, t)$ be the utility of agent i from prize z at time t. Then, I assume that $u_i(z, t) = v(z)\delta(t)$ where $v(z)$ is strictly increasing in z and $v(0) = 0$. $\delta(t)$ is a time discount function which is continuous, strictly decreasing in t, $\lim_{t\to\infty} \delta(t) = 0$ and $\lim_{t\to 0} \delta(t) = 1$. In short, agents are impatient and the payoff from any size of prize goes to zero as the delivery timing goes to infinity.

Winner-take-all competition. In the first part of this paper, I focus on the winner-take-all prize scheme. Concretely, the winner-take-all prize scheme $b(t_i^*, t_j^*)$ is

$$b_i(t_i^*, t_j^*) = \begin{cases} x > 0 \text{ at } t = t_i^* \text{ if } t_i^* < t_j^* \\ \varnothing \text{ if } t_i^* \geq t_j^* \end{cases} .$$

Obviously, we can define a variation of the winner-take-all prize scheme by changing the prize rule at $t_i^* = t_j^*$. However, the probability of such outcome is zero and thus it has no effect on the equilibrium analysis.

Problem solving strategy. When finding a solution is not immediate but time-consuming process, "how to solve it" becomes another decision problem. There are two qualitatively different approaches to solving problems. One is a *safe method* and the other is a *risky method*. Each agent i has private information about his computational capacity $\theta_i \in \Theta = [\theta_{\min}, \theta_{\max}] \subset (0, \infty)$ and θ_i is independently drawn from an absolutely continuous distribution function $G(\theta)$ with $\text{supp}(g) = \Theta$. Then, if agent i chooses the safe method, he finds a solution for sure at $t = \theta_i \in \Theta$ but he has no chance to find the solution before θ_i. On the other hand, if agent i chooses the risky method, he does not know how much time he needs to find a solution. More concretely, the time at which agent i finds a solution t_i is independently drawn from an absolutely continuous distribution $\Psi(t)$ with $\text{supp}(\psi) = [0, T]$ but t_i is not observable for either agent. Notice that the risky method can take time to find a solution at most T.[1] I assume that $\Psi(t)$ is common knowledge and $T > \theta_{\max}$.

Timing of the game. The game consists of the following two steps.

Stage 1. Nature independently draws the type of each agent using $G(\theta)$. Each agent observes own type θ_i and chooses a problem solving method.

Stage 2. Each agent applies his problem solving method. Then, based on the prize scheme, the prize is awarded to each agent.

Equilibrium. A *problem solving strategy* of agent i is a mapping $\sigma_i : \Theta \to \{S, R\}$ where S denotes the safe method and R denotes the risky method[2]. Let $U_i(\sigma_1, \sigma_2; \theta_1, \theta_2)$ be the expected payoff to player i given a strategy profile (σ_1, σ_2), a type profile (θ_1, θ_2). Then, the strategy profile σ^* is a *Bayes Nash equilibrium* if, for all $i \in \{1, 2\}$,

$$E_{\theta_j}[U_i(\sigma^*; \theta_i, \theta_j))|\theta_i] \geq E_{\theta_j}[U_i(\sigma_i, \sigma_j^*; \theta_i, \theta_j)|\theta_i]$$

for all $\sigma_i \in \Sigma_i$.

3 Equilibrium analysis

In this section, I show that there exists a unique symmetric equilibrium. First, I introduce a class of strategies which plays an important role in the analysis.

Definition 1. *A strategy $\sigma_i(\theta)$ is cut off strategy if there is a cut off type $\hat{\theta}_i \in \Theta$ such that*

$$\begin{cases} \sigma_i(\theta_i) = S \text{ if } \theta_i \in [\theta_{\min}, \hat{\theta}_i) \\ \sigma_i(\theta_i) = R \text{ if } \theta_i \in (\hat{\theta}_i, \theta_{\max}] \end{cases}.$$

An equilibrium is cut off equilibrium if it consists of cut off strategies.

In another words, in a cut off strategy, all types which play the safe method have larger capacity than the cutoff level.

[1] The result of this paper is preserved even if we allow to have $T = \infty$.

[2] I focus on pure strategies since the set of types who play a mixed strategy in equilibrium is always measure zero.

Lemma 1. *In every equilibrium, both players use cut off strategies. However, there is no symmetric equilibrium in which* $\hat{\theta}_i = \hat{\theta}_j = \theta_{\max}$.

Proof. See appendix.

Remark 1. Observe that if the risky method tends to take a long time, type $\theta_{\max}$ prefers to play the safe method in individual problem solving. However, type $\theta_{\max}$ never plays the safe method in the equilibrium of the winner-take-all competition.

The intuition of lemma 1 is the following. First, there is no equilibrium where all agents plays the safe method for all θ. This is because, if all agents play the safe method, type $\theta_{\max}$ knows that he has no chance to solve the problem earlier than the other agent and his payoff is zero for sure. Secondly, whenever an agent plays the safe method in equilibrium, his strategy is a cut off strategy. The idea is simple. If type θ' finds that there is no profitable deviation from the safe method, then any $\theta < \theta'$ finds the safe method more profitable than the risky method.

Proposition 1. *There exists a unique symmetric equilibrium.*

The idea of Proposition 1 is the following. First, since the expected payoff of agent i from the safe method is decreasing in θ_i and the expected payoff from the risky method is independent of θ_i, whenever there exists a common cut off type for each player who is indifferent between the safe method and the risky method, the common cutoff type is an equilibrium cutoff. Second, we can show that if there exists one cut off type which satisfies the indifference condition for each player, there is no other cut off type which also does so. Roughly speaking, this is because the difference between the expected payoff to the risky method and that to the safe method for the cut off type becomes higher if the cut off level is higher. Hence, if there are two equilibrium cut off types and the indifference condition is satisfied for the lower cut off type, the condition cannot be satisfied for the larger cut off type.

Proof. By Lemma 1, every equilibrium is a cut off equilibrium. Hence, all I need to do is the following two steps. First, I establish the existence of a symmetric cut off equilibrium.

Second, I show that such symmetric cut off type is unique. Before proving the results, I introduce the following notation.

The expected payoff of agent i of type θ_i when agent i plays the safe method and agent j plays the risky method:

$$V_{i,SR}(\theta_i) = (1 - \Psi(\theta_i))u(x, \theta_i).$$

The expected payoff of agent i given his opponent's type is θ_j when agent i plays the risky method and agent j plays the safe method:

$$V_{i,RS}(\theta_j) = \int_0^{\theta_j} \psi(t)u(x,t)dt.$$

The expected payoff of agent i when both agents play the risky method:

$$V_{i,RR} = \int_0^{T} \psi(t)(1 - \Psi(t))u(x,t)dt.$$

The expected payoff of agent i given type profile (θ_i, θ_j) when both agent i and j play the safe method:

$$V_{i,SS}(\theta_i, \theta_j) = \begin{cases} u(x, \theta_i) \text{ if } \theta_i < \theta_j \\ 0 \text{ if } \theta_i > \theta_j \end{cases}.$$

Step 1. The construction of the equilibrium cutoff.

Suppose both agents follows the same cut off strategy and let $\hat{\theta}$ be the symmetric cut off type. Then, the expected payoff of agent i of type $\hat{\theta}$ from the risky method is

$$U_{iR} = (1 - G(\hat{\theta}))V_{i,RR} + \int_{\theta_j < \hat{\theta}} V_{i,RS}(\theta_j)dG(\theta_j)$$

The expected payoff of agent i of type $\hat{\theta}$ from the safe method is

$$U_{iS}(\hat{\theta}) = (1 - G(\hat{\theta}))V_{i,SR}(\hat{\theta}).$$

Then, let $z(\hat{\theta})$ be the difference between the expected payoff to the risky method and that to the safe method for cut off type $\hat{\theta}$, that is,

$$z(\hat{\theta}) = (1 - G(\hat{\theta}))[V_{i,RR} - V_{i,SR}(\hat{\theta})] + \int_{\theta_j < \hat{\theta}} V_{i,RS}(\theta_j)dG(\theta_j).$$

Notice that $\int_{\theta_j<\hat{\theta}} V_{i,RS}(\theta_j)dG(\theta_j) > 0$. Hence, if $\forall\theta \in \Theta, V_{i,RR} \geq V_{i,SR}(\theta_i)$, then $z(\hat{\theta}) > 0$ for all θ. In this case, the equilibrium cut off type is $\theta_{\min}$. That is, all agents play the risky method. There is no profitable deviation since $\forall\theta \in \Theta, V_{i,RR} \geq V_{i,SR}(\theta)$. So assume that there exists some $\theta \in \Theta$ with $V_{i,RR} < V_{i,SR}(\theta)$.

Notice that $V_{i,RR} - V_{i,SR}(\theta_i)$ has the smallest value when $\theta_i = \theta_{\min}$. Thus, if $V_{i,RR} < V_{i,SR}(\theta_i)$ for some θ, then $z(\theta_{\min}) < 0$. On the other hand, $z(\theta_{\max}) = \int_{\theta_j} V_{i,RS}(\theta_j)dG(\theta_j) > 0$. Since $V_{i,SR}(\hat{\theta})$ is continuous in $\hat{\theta}$, $z(\hat{\theta})$ is also continuous. Hence, there exists at least one $\hat{\theta}^* \in int(\Theta)$ such that $z(\hat{\theta}^*) = 0$. I claim that a symmetric cut off strategy profile with $\hat{\theta}^*$ constitutes an equilibrium. To verify the claim, suppose each agent plays the safe method for $\theta \in [\theta_{\min}, \hat{\theta}^*)$ and plays the risky method for $\theta \in (\hat{\theta}^*, \theta_{\max}]$. Since $V_{i,SR}(\theta_i)$ is decreasing in θ_i, by construction, for any $\theta_i > \hat{\theta}^*$,

$$\begin{aligned} & (1 - G(\hat{\theta}^*))V_{i,RR} + \int_{\theta_j<\hat{\theta}^*} V_{i,RS}(\theta_j)dG(\theta_j) \\ \geq \; & (1 - G(\hat{\theta}^*))V_{i,SR}(\theta_i). \end{aligned}$$

For any $\theta_i \in [\theta_{\min}, \hat{\theta}^*)$,

$$\begin{aligned} & (1 - G(\hat{\theta}^*))V_{i,SR}(\theta_i) + G(\hat{\theta}^*)u(x, \theta_i) \\ \geq \; & (1 - G(\hat{\theta}^*))V_{i,RR} + \int_{\theta_j<\hat{\theta}^*} V_{i,RS}(\theta_j)dG(\theta_j). \end{aligned}$$

Step 2. There exists unique equilibrium cutoff $\hat{\theta}^*$.

Recall that $\hat{\theta}^*$ is chosen so that $(1 - G(\hat{\theta}^*))[V_{i,RR} - V_{i,SR}(\hat{\theta})] + \int_{\theta_j<\hat{\theta}^*} V_{i,RS}(\theta_j)dG(\theta_j) = 0$ if such $\hat{\theta}$ exists and $\hat{\theta}^* = \theta_{\min}$ otherwise. Thus, I need to show that $z(\hat{\theta}) = 0$ has unique solution whenever it exists. Notice that $V_{i,RR} - V_{i,SR}(\hat{\theta})$ is increasing in $\hat{\theta}$ since $V_{i,SR}(\hat{\theta})$ is decreasing in $\hat{\theta}$. Thus, once $V_{i,RR} - V_{i,SR}(\hat{\theta}) > 0$, then $z(\hat{\theta}) > 0$ for larger $\hat{\theta}$. On the other hand, whenever there exists a cut off equilibrium with $\hat{\theta}^* \neq \theta_{\min}$, $V_{i,RR} - V_{i,SR}(\theta_{\min}) < 0$. Obviously, since $V_{i,RS}(\theta_j) > 0$, $z(\hat{\theta})$ is increasing in $\hat{\theta}$ as long as $V_{i,RR} - V_{i,SR}(\hat{\theta}) < 0$. Then, since $V_{i,RR} - V_{i,SR}(\hat{\theta}^*) < 0$, there exists unique $\hat{\theta}^*$. Q.E.D.

Remark 2. If a problem is sufficiently "difficult" for the risky method in the sense that T is large and $\Psi(T/2)$ is small, there is no asymmetric equilibrium and thus the symmetric cut off is the unique equilibrium.

Proof. See appendix.

The following is the intuition. Suppose we have an asymmetric equilibrium. Then, the agent with the lower cut off type has a extra chance to have larger capacity than the other agent when both play the safe method. Hence, the agent with the lower cut off type has a higher expected payoff from the safe method and this is not affected by $\Psi(t)$. On the other hand, if $\Psi(t)$ is sufficiently small for a sufficiently large set of t's, the expected payoff from the risky method is small and the difference of expected payoffs from the risky method is small between agents. As a result, under asymmetric cut offs, we cannot make the expected payoffs from the safe method and the risky method indifferent for both cut off types. Moreover, if a problem is sufficiently easy in the sense that $\Psi(\theta_{\min})$ is large, there is no asymmetric equilibrium. The reason is simple. The risky method is the dominant strategy if $\Psi(\theta_{\min})$ is large.

Before moving to the next section, I compare competitive problem solving with individual problem solving. Given type θ_i, agent i plays the safe method (the risky method) in individual problem solving if and only if

$$\int_0^T \psi(t)u(x,t)dt < (>) u(x,\theta_i).$$

Then, let θ^{Ind} be the cut off type which characterizes the individual decision rule. That is, if $\theta_i > (<) \theta^{Ind}$, the optimal choice is the risky (safe) method. First, it is easy to see that if $\Psi(\theta_{\min})$ is large, all agents play the risky method in both competitive problem solving and individual problem solving, i.e., $\theta_{\min} = \hat{\theta}^* = \theta^{Ind}$. Second, if $\Psi(\theta_{\max})$ is small, then all types play the safe method in individual problem solving, i.e., $\theta^{Ind} = \theta_{\max}$. On the other hand, by Lemma 1, $\theta^{Ind} > \hat{\theta}^*$.

The important point is that the meaning of "capacity" depends on whether there is a competitor or not. The agent takes into account his capacity *relative to* the opponent to choose the problem solving strategy when there is a competitor. On the other hand, in individual problem solving, relative capacity is meaningless. That is, even if the agent has $\theta_{\max}$, he plays the safe method as long as the expected time to find a solution is shorter than that in the risky method. Thus, if a problem is sufficiently "difficult", all types play the safe method in individual problem solving.

4 Optimal problem solving

In this section, I define optimal problem solving from the perspective of the prize designer. Suppose the prize designer's interest is in obtaining a solution as early as possible. Formally, let t_i^* be the time at which agent i finds a solution and let $t^* = \min\{t_1^*, t_2^*\}$. By obtaining a solution of the problem, the prize designer gets profit $B > 0$ at t^*. The payoff of the prize designer from B at t^* is $\widetilde{u}(B, t^*) = \widetilde{v}(B)\delta(t^*)$ where $\widetilde{v}(B) > 0$. I assume that the prize designer shares the same time discount function $\delta(t^*)$ with agents. Let μ be the probability distribution of t^* induced by a problem solving profile. Then, $W(\mu)$ denotes the utility of the prize designer from μ. I assume that the total budget of the prize is $x > 0$ and spending less than the budget does not improve the prize designer's payoff. Hence, the objective function of the prize designer is $W(\mu)$.

Now, consider a pair of mapping $(y_1(\theta_1, \theta_2), y_2(\theta_1, \theta_2))$ where $y_i : \Theta^2 \to \{S, R\}$. Then, let $\mu_{y_1(\theta_1,\theta_2),y_2(\theta_1,\theta_2)}$ be the distribution of t^* given $(y_1(\theta_1, \theta_2), y_2(\theta_1, \theta_2))$. Then, I define *optimal problem solving assignment (OPSA)* as $(y_1^*(\theta_1, \theta_2), y_2^*(\theta_1, \theta_2))$ which maximizes

$$W(\mu_{y_1(\theta_1,\theta_2),y_2(\theta_1,\theta_2)}),$$

given any (θ_1, θ_2). In short, this is the pair of problem solving methods the prize designer prefers to assign when the type profile is observable.

Since the type is private information in this model, I also introduce a criterion which takes into account the informational constraint. Let $\mu_{\sigma_1(\theta_1),\sigma_2(\theta_2)}$ be the distribution of t^* given strategy profile $(\sigma_1(\theta_1), \sigma_2(\theta_2))$. I define *optimal problem solving (OPS)* as a strategy profile $(\sigma_1(\theta_1), \sigma_2(\theta_2))$ which maximizes

$$\int_{\theta_1}\int_{\theta_2} W(\mu_{\sigma_1(\theta_1),\sigma_2(\theta_2)})dG(\theta_1)dG(\theta_2).$$

In this paper, I consider two kinds specifications of $W(\mu)$ depending on how the prize designer aggregates the utility index $\widetilde{u}(B, t^*)$ across t^*. First, the prize designer is an *expected utility maximizer* if his objective function is the following.

$$W(\mu) = \int_{t^*} \widetilde{u}(B, t^*)\mu(t^*)dt^*.$$

Second, it is often the case that the main concern of the prize designer is to find out a solution before the deadline[3]. In this case, the prize designer's concern may be how much time might be needed in the worst case to find out a solution. The prize designer is a *worst case maximizer* if his objective function is the following.

$$W(\mu) = \min_{t^* \in \mathrm{supp}(\mu)} \widetilde{u}(B, t^*).$$

Now, I turn to the analysis of the equilibrium problem solving in the winner-take-all scheme. For the first part of the analysis, I focus on the case where the prize designer is the expected utility maximizer. I start from the analysis of "difficult" problems.

Observation 1.

Suppose the prize designer is the expected utility maximizer.

(i) *If T is large and $\Psi(T/2)$ is sufficiently small, then the OPSA is $y_i^*(\theta_i, \theta_j) = R$ and* $y_j^*(\theta_i, \theta_j) = S$ for $\theta_i > \theta_j$.

(ii) *If T is large and $\Psi(T/2)$ is sufficiently small, the OPS consists of a symmetric cut off strategy profile.*

Proof. See appendix.

Since all equilibria are cutoff equilibrium, the winner-take-all game induces the OPSA for sufficiently "difficult" problems only if $\theta_i < \hat{\theta} < \theta_j$ given equilibrium cutoff $\hat{\theta}$. Otherwise, the equilibrium problem solving is not the OPSA.

On the other hand, the following proposition states that the winner-take-all game induces the excessive risk taking and cannot induce the OPS for "difficult" problems.

Proposition 2. *Suppose the prize designer is the expected utility maximizer. If T is large and $\Psi(T/2)$ is sufficiently small, then the symmetric equilibrium cut off type is smaller than the cut off type in the OPS.*

Proof. See appendix.

Let $\hat{\theta}_{OPS}$ be the cutoff type for the OPS. Proposition 2 says that if a problem is sufficiently "difficult" for the risky method, competition forces the agent with moderate capacity, i.e.,

[3] In computer science, average-case and worst-case performance are popular criteria to evaluate algorithms.

$\theta \in (\hat{\theta}^*, \hat{\theta}_{OPS})$, to employ the risky method, i.e., excessive risk taking. The intuition is the following. In the winner-take-all game, when an agent plays the safe method, the expected payoff depends on the type of the agent. On the other hand, when he employs the risky method, the expected payoff is independent of his type. Thus, when an agent plays the safe method in equilibrium, his capacity has to be large enough to win the game with a reasonably high probability. As a result, an agent with moderate capacity employs the risky method in equilibrium. On the other hand, notice that if both agents play the risky method, the time to find a solution can be long. Then, to reduce the chance of having such situation, the agent with moderate capacity is assigned to the safe method in the OPS.

Now, I turn to the analysis for "easy" problems, i.e., $\Psi(\theta_{\min})$ is high.

Observation 2. *Suppose the prize designer is the expected utility maximizer. If $\Psi(\theta_{\min})$ is sufficiently large, both agents play the risky method in both the OPSA and the OPS.*

Proof. The expected utility of the prize designer in which both agents employ the risky method is at least

$$\int_0^{\theta_{\min}} [1 - (1 - \psi(t))^2]\widetilde{u}(B, t)dt.$$

On the other hand, if an agent with higher capacity plays the safe method and the other employs the risky method, the expected utility for the prize designer is at most

$$\int_0^{\theta_{\min}} \psi(t)\widetilde{u}(B, t)dt + (1 - \Psi(\theta_{\min}))\widetilde{u}(B, \theta_{\min})$$

Notice that $[1 - (1 - \psi(t))^2] = 2\psi(t) - \psi(t)^2 > \psi(t)$. Hence, the expected utility of the prize designer in which both agents play the risky method is higher if

$$\Psi(\theta_{\min}) \geq 1 - \frac{\int_0^{\theta_{\min}} [(\psi(t) - \psi(t)^2]\widetilde{u}(B, t)dt}{\widetilde{u}(B, \theta_{\min})}.$$

Q.E.D.

The following proposition shows that the equilibrium can be the OPSA and the OPS for sufficiently "easy" problems.

Proposition 3. *Suppose the prize designer is the expected utility maximizer. If $\Psi(\theta_{\min})$ is sufficiently large, the symmetric equilibrium is the OPSA.*

Proof. Immediate from Observation 2 and the fact the symmetric equilibrium cutoff type is $\theta_{\min}$ for large $\Psi(\theta_{\min})$. Q.E.D.

Now, I turn to the analysis in which the prize designer is the worst case maximizer.

Observation 3. *Suppose the prize designer is the worst case maximizer.*

(i) $\theta_i \geq \theta_j$*, then* $(y_1(\theta_1, \theta_2), y_2(\theta_1, \theta_2))$ *is the OPSA whenever* $y_j^*(\theta_i, \theta_j) = S$*.*

(ii) *The OPS consists of a set of strategy profiles such that there exists agent* i *who plays the safe method for any* θ_i*.*

Note that the OPSA and the OPS do not depend on the "difficulty" of the problem. For the OPSA, since the prize designer knows that the worst case of the safe method is better than that of the risky method, he can maximize his payoff by assigning the safe method to the agent with larger capacity. For the OPS, observe that whenever each of agent has a set of types in which he plays the risky method, the prize designer's payoff is $\widetilde{u}(B, T)$. On the other hand, when one of agents employ the safe method independent of his type, the prize designer's payoff is $\widetilde{u}(B, \theta_{\max}) > \widetilde{u}(B, T)$.

The next proposition says when the prize designer is the worst case maximizer, he can be better off by asking one agent to solve the problem without competition.

Proposition 4. *Suppose the prize designer is the worst case maximizer. The symmetric equilibrium cannot be the OPS for any problem. Moreover, if* T *is large and* $\Psi(T/2)$ *is sufficiently small, then the prize designer gets higher payoff by asking one agent to solve the problem rather than setting 2-person winner-take-all competition.*

Proof. The first part is immediate from Lemma 1, and observation 3. For the second part, consider individual problem solving with large T and small $\Psi(T/2)$. Then, the optimal problem solving in the individual decision problem is to choose the safe method for all types. Note that the prize designer's payoff is at least $\widetilde{u}(B, \theta_{\max})$ if he asks one agent to solve the problem. On the other hand, by Lemma1, the prize designer's payoff from the winner-take-all competition setting is always $\widetilde{u}(B, T)$. Q.E.D.

5 Optimal prize scheme

When a problem is "difficult" for the risky method, the winner-take-all competition does not induce the OPS. This section introduces prize schemes which induce the OPS in an equilibrium. Recall that, for "difficult" problems, the competitive pressure induces the excessive risk taking in the equilibrium. Hence, the key to induce the OPS is reduce the competitive pressure of the winner-take-all scheme. I provide two kinds of such prize schemes. The first is a multi-prize scheme. In this prize scheme, unlike the winner-take-all game, not only the winner but also the loser gets some reward. The second is a probabilistic prize scheme in which both agents do not know the exact prize structure when they choose the problem solving method. The prize structure is drawn from a set of prize structures and the distribution is known to both agents.

If the focus of the analysis is "costless problem solving," i.e., participating problem solving is not costly, there is a noncompetitive scheme which can induce the OPS. A reward scheme is *cooperative* if both agents share prize x when the first agent finds a solution.

Observation 4. *Suppose the prize designer is the expected utility maximizer. The OPS is an equilibrium under the cooperative scheme.*

A drawback of the cooperative scheme is that the implementation of the OPS relies on the "costless problem solving" setting. When it is costly to participate problem solving, e.g., there is an outside option, the cooperative scheme faces free rider problem. Then, this section focuses on competitive schemes so that we can extend the analysis for a "costly problem solving" environment in the later section.

Now, I introduce the key concept of this section.

Definition 2. *An optimal prize scheme is a prize scheme which induces the OPS in an equilibrium.*

That is, in the optimal prize scheme, the strategy profile which maximizes the prize designer's interest can be supported as an equilibrium. In this section, without loss of generality, I focus on the class of prize schemes in which the total amount of prizes is no more than $x > 0$.

5.1 Multi-prize scheme

In the winner-take-all competition, only the winner receives a prize. Multi-prize scheme is a prize scheme in which the loser also receives a prize. Concretely, the prize designer gives $\beta \in [0, x]$ to the loser and $x-\beta$ to the winner at the time the agent finds a solution. Formally, *the multi-prize scheme* is $(b_1(t_1^*, t_2^*|\beta), b_2(t_2^*, t_1^*|\beta))$ where

$$b_i(t_i^*, t_j^*|\beta) = \begin{cases} x - \beta \text{ at } t = t_i^* \text{ if } t_i^* < t_j^* \\ \beta \text{ at } t = t_i^* \text{ if } t_i^* \geq t_j^* \end{cases}$$

Notice that since the agent can be rewarded even if he does not win the game, the competitive pressure of this game is lower than that of the winner-take-all game. Now, the question is whether we can find the second prize level β which makes the prize scheme optimal.

The following proposition says that, for sufficiently "difficult" problems, there exists an optimal multi-prize scheme.

Proposition 5.

(i) *Suppose the prize designer is the expected utility maximizer. If T is sufficiently large and $\Psi(T/2)$ is sufficiently small, then there exists $\beta^* \in (0, x/2)$ such that $b(t_1^*, t_2^*|\beta^*)$ is the optimal prize scheme.*

(ii) *Suppose the prize designer is the worst case maximizer. If T is sufficiently large and $\Psi(T/2)$ is sufficiently small, then $b(t_1^*, t_2^*|\beta)$ is the optimal prize scheme whenever β is sufficiently close to $x/2$.*

To provide an intuition, recall that, for both criteria, the optimal prize scheme has to eliminate the excessive risk taking behavior for "difficult" problems. Observe that when the loser can get a prize based on the time he finds a solution, the risk taking is not attractive choice whenever (i) the chance of being the loser is high and (ii) the problem is "difficult." Thus, when the loser can get a prize, it reduces the benefit from the risk taking for larger capacity types. Then, when the prize designer provides a right amount of prize to the loser, the prize scheme can eliminate excessive risk taking and maximize the prize designer's expected payoff.

Proof. Given β, the expected payoff of the cutoff type from the risky method is

$$\begin{aligned}\hat{U}_R(\hat{\theta}|\beta) &= (1-G(\hat{\theta}))\left[\int_0^T \psi(t)(1-\Psi(t))u(x-\beta,t)dt + \int_0^T \psi(t)\Psi(t)u(\beta,t)dt\right] \\ &+ \int_{\theta_j<\hat{\theta}} \int_{t=0}^{\theta_j} \psi(t)u(x-\beta,t)dtdG(\theta_j) \\ &+ \int_{\theta_j<\hat{\theta}} \int_{t=\theta_j}^{T} \psi(t)u(\beta,t)dtdG(\theta_j).\end{aligned}$$

Given β, the expected payoff of the cutoff type from the safe method is

$$\begin{aligned}\hat{U}_S(\hat{\theta}|\beta) &= (1-G(\hat{\theta}))\left[(1-\Psi(\hat{\theta}))u(x-\beta,\hat{\theta}) + \Psi(\hat{\theta})u(\beta,\hat{\theta})\right] \\ &+ G(\hat{\theta})u(\beta,\hat{\theta}).\end{aligned}$$

Now, observe that

$$\begin{aligned}\hat{U}_R(\hat{\theta}|x/2) &= \int_0^T \psi(t)u(x/2,t)dt. \\ \hat{U}_S(\hat{\theta}|x/2) &= u(x/2,\hat{\theta}).\end{aligned}$$

Moreover, recall that, when the game is the winner-take-all game, the expected payoff from each action is

$$\begin{aligned}U_R(\hat{\theta}) &= (1-G(\hat{\theta}))\int_0^T \psi(t)(1-\Psi(t))u(x,t)dt \\ &+ \int_{\theta_j<\hat{\theta}} \int_0^{\theta_j} \psi(t)u(x,t)dtdG(\theta_j), \\ U_S(\hat{\theta}) &= (1-G(\hat{\theta}))(1-\Psi(\hat{\theta}))u(x,\hat{\theta}).\end{aligned}$$

Then, let $z_{multi}(\hat{\theta}|\beta) = \hat{U}_R(\hat{\theta}|\beta) - \hat{U}_S(\hat{\theta}|\beta)$ and note that

$$\begin{aligned}z_{multi}(\hat{\theta}|0) &= z(\hat{\theta}) = U_R(\hat{\theta}) - U_S(\hat{\theta}), \\ z_{multi}(\hat{\theta}|x/2) &= \hat{U}_R(\hat{\theta}|x/2) - \hat{U}_S(\hat{\theta}|x/2).\end{aligned}$$

Suppose the prize designer is the expected utility maximizer. Recall that, when the problem has sufficiently large T and sufficiently small $\Psi(T/2)$, we have $z(\hat{\theta}_{OPS}) > 0$. Moreover, it is easy to see that $z_{multi}(\hat{\theta}_{OPS}|x/2) < 0$ if the problem has sufficiently large T and sufficiently small $\Psi(T/2)$. Then, since $z_{multi}(\hat{\theta}_{OPS}|\beta)$ is continuous in β, there always exists $\beta^* \in (0, x/2)$ such that $z_{multi}(\hat{\theta}_{OPS}|\beta^*) = 0$.

Suppose the prize designer is the worst case maximizer. If T is sufficiently large and $\Psi(T/2)$ is sufficiently small, at least one of agents play the safe method in the OPS. Obviously, if T is sufficiently large and $\Psi(T/2)$ is sufficiently small, we have $z_{multi}(\hat{\theta}|x/2) < 0$ for all $\hat{\theta}$. Then, given large T and sufficiently small $\Psi(T/2)$, if β is sufficiently close to $x/2$, then, by continuity, $z_{multi}(\hat{\theta}|\beta) < 0$ for all $\hat{\theta}$. Then, all agents choose the safe method in the equilibrium. This is the OPS. Q.E.D.

5.2 Probabilistic prize scheme

Suppose the prize structure is not known among agents when each agent chooses a problem solving method. Concretely, suppose there are two possible prize structures, i.e., the winner-take-all and an absolute performance basis prize. In the winner-take-all competition, the agent is rewarded only if he finds a solution earlier than the other, i.e., being winner. The absolute performance basis prize means that the reward is based only on the time the agent finds a solution. Then, a probabilistic prize scheme is characterized by the probability that the prize structure is the winner-take-all competition. Concretely, the probabilistic prize scheme $b(t_1^*, t_2^*|q)$ is as follows.

$$b_i(t_i^*, t_j^*|q) = \begin{cases} b_{i,wta}(t_i^*, t_j^*) \text{ with probability } q \\ b_{i,abs}(t_i^*, t_j^*) \text{ with probability } 1-q \end{cases}$$

where

$$b_{i,wta}(t_i^*, t_j^*) = \begin{cases} x \text{ at } t_i^* \text{ if } t_i^* < t_j^* \\ \varnothing \text{ if } t_i^* \geq t_j^* \end{cases}$$

and

$$b_{i,abs}(t_i^*, t_j^*) = x/2 \text{ at } t_i^*.$$

Notice that since the agent faces no competition with probability $1-q$, the competitive pressure of this game is lower than that of the winner-take-all game.

The time line of the game is as follows.

Step 1. The prize designer chooses q which is observable for agents. Then, based on q, one of prize structures is chosen by the prize designer but this is not observable for agents.

Step 2. Given q, each agent chooses his problem solving strategy. When the agent finds a solution, he receives the prize based on the prize structure.

The next proposition shows that, for each criterion, there exists probability q which induces the OPS in an equilibrium.

Proposition 6.

(i) *Suppose the prize designer is the expected utility maximizer. If T is sufficiently large and $\Psi(T/2)$ is sufficiently small, there exists a unique $q^* \in (0,1)$ in which $b(t_1^*, t_2^*|q^*)$ is the optimal prize scheme.*

(ii) *Suppose the prize designer is the worst case maximizer. If T is sufficiently large and $\Psi(T/2)$ is sufficiently small, $b(t_1^*, t_2^*|q)$ is the optimal prize scheme for sufficiently small q.*

The reason that this scheme eliminates the risk taking behavior is similar to that of the multi-prize scheme. When agents face the absolute performance basis prize structure with some probability, the incentive of risk taking is lower for "difficult" problems. Then, we can always find a probability which eliminates the excessive risk taking.

Proof. Given q, the expected payoff of $\hat{\theta}$ from the risky method is

$$\begin{aligned} U_{iR}(\hat{\theta}|q) &= q\left[(1-G(\hat{\theta}))V_{i,RR} + \int_{\theta_j<\hat{\theta}} V_{i,RS}(\theta_j)dG(\theta_j)\right] \\ &\quad +(1-q)\int_0^T \psi(t)u(x/2,t)dt. \end{aligned}$$

On the other hand, given q, the expected payoff of $\hat{\theta}$ from the safe method is

$$U_{iS}(\hat{\theta}|q) = q(1-G(\hat{\theta}))V_{i,SR}(\hat{\theta}) + (1-q)u(x/2,\hat{\theta}).$$

Recall that $z(\hat{\theta})$ is the difference of expected payoffs between the risky method and the safe method in the winner-take-all scheme. Then, let $z_{\text{rand}}(\hat{\theta}|q)$ be the difference of expected payoffs between the risky method and the safe method in which the type $\hat{\theta}$ faces the winner-take-all game with probability q. That is,

$$z_{\text{rand}}(\hat{\theta}|q) = qz(\hat{\theta}) + (1-q)\left[\int_0^T \psi(t)u(x/2,t)dt - u(x/2,\hat{\theta})\right].$$

Now, suppose the prize designer is the expected utility maximizer. To show that there exists q which induces the OPS, recall that, if T is sufficiently large and $\Psi(T/2)$ is sufficiently small,

$$\begin{aligned} z_{\text{rand}}(\theta^*_{OPS}|1) &= (1-G(\theta^*_{OPS}))[V_{i,RR}-V_{i,SR}(\theta^*_{OPS})] \\ &\quad + \int_{\theta_j<\theta^*_{OPS}} V_{i,RS}(\theta_j)dG(\theta_j) > 0, \\ z_{\text{rand}}(\theta^*_{OPS}|0) &= \left[\int_0^T \psi(t)u(x/2,t)dt - u(x/2,\hat{\theta})\right] < 0. \end{aligned}$$

Then, since $z_{\text{rand}}(\theta^*_{OPS}|q)$ is continuous and strictly increasing in q, we always find a unique $q^* \in (0,1)$ such that $z(\theta^*_{OPS}|q^*) = 0$.

Suppose the prize designer is the worst case maximizer, observe that $z_{\text{rand}}(\hat{\theta}|0) < 0$ for all $\hat{\theta}$ if T is sufficiently large and $\Psi(T/2)$ is sufficiently small. Then, by continuity, whenever q is sufficiently close to 0, $z_{\text{rand}}(\hat{\theta}|q) < 0$ for all $\hat{\theta}$. Then, the symmetric equilibrium is the OPS. Q.E.D.

Remark 3. The multi-prize scheme and the probabilistic prize scheme are similar in the sense that it reduces the competitive pressure to induce the OPS. However, one clear merit of the probabilistic prize scheme over the multi-prize scheme is the following. Consider the case where there are I agents. Moreover, suppose the OPS is characterized by a symmetric cutoff type. Then, it is easy to find the optimal probabilistic prize scheme since all we need to find is the probability q. On the other hand, it is more difficult to find the optimal multi-prize structure for I prizes.

6 A multi-task environment and participation constraint

When agents have another task in addition to problem solving, the agent has to decide which task he focuses on, i.e., a costly problem solving. Then, one natural question is whether the optimal prize scheme in the last section still induces the OPS in a multi-task environment. To investigate the question, suppose there is a safe task in which the agent gets $v > 0$ for sure whenever the agent focuses on the safe task. Then, the agent has to choose whether he focuses on problem solving or not. Concretely, we add a new action "focusing on the safe task," denoted by N, to the set of feasible actions. Hence, a strategy of agent i is

$\sigma_i : \Theta \to \{S, R, N\}$. I assume that when an agent does not focus on problem solving, he can find a solution but it tends to take longer time to find a solution. Concretely, without effort, the time at which the agent finds a solution follows probability distribution $\Psi^N(t)$ with $\text{supp}(\psi^N) = [0, T_N]$ where $\Psi^N(t) < \Psi(t)$ for all $t \in [0, T]$. Obviously, it implies $T_N > T$. I assume that the prize designer gets no benefit from the safe task and his objective function is the same as that of the basic setting. An important assumption here is that the prize designer of the competition cannot observe whether the agent focuses on problem solving or not, that is, the choice of "which to focus" is a hidden action. Thus, the prize structure cannot be conditional on the hidden action.

When we take into account the choice of "which to focus," the benefit of competition becomes clear. The following observation clarifies the benefit of competitive schemes in the multi-task environment.

Observation 5.

(i) *If $u(x, 0) - u(x, T_N)$ is sufficiently small, then the optimal decision in the individual problem solving is N for all θ.*

(ii) *If $u(x, 0) - u(x, T_N)$ is sufficiently small, then there is no equilibrium where both agents focus on problem solving in the cooperative scheme (Free rider problem).*

(iii) *In the winner-take-all competition, both agents focus on problem solving in an equilibrium if v is sufficiently small given x.*

To see Observation 5 (i), in the individual problem, the expected payoff from focusing on problem solving is at most

$$\max\left\{\int_{t=0}^{T} \psi(t)u(x,t)dt, u(x,\theta_{\min})\right\}.$$

Observe that, given $\varepsilon > 0$, we can always find small $u(x, 0) - u(x, T_N)$ such that

$$\begin{aligned} u(x,0) - \max\left\{\int_{t=0}^{T} \psi(t)u(x,t)dt, u(x,\theta_{\min})\right\} &< \varepsilon, \\ u(x,0) - \int_{t=0}^{T_N} \psi^N(t)u(x,t)dt &< \varepsilon. \end{aligned}$$

Hence, if $u(x, 0) - u(x, T_N)$ is sufficiently small,

$$\max\left\{\int_{t=0}^{T} \psi(t)u(x,t)dt, u(x,\theta_{\min})\right\} < \int_{t=0}^{T_N} \psi^N(t)u(x,t)dt + v.$$

Thus, the agent prefers to choose N. We can explain Observation 5 (ii) with a similar argument.

To understand Observation 5 (iii), suppose the prize is given by the winner-take-all fashion and both agents focus on problem solving. Then, if the agent deviates and focuses on the safe task, he never wins the competition and his payoff is only v. Hence, if v is sufficiently small, there is no incentive to deviate.

The winner-take-all scheme can provide the incentive to participate problem solving but it also induces the excessive risk taking behavior. Then, our question is whether optimal prize schemes in the last section can provide the incentive to participate problem solving. Let b_{mlt} be the optimal multi-prize scheme in the single task environment and b_{rnd} be the optimal probabilistic prize scheme in the single task environment. The next observation says that, for "difficult" problems, optimal incentive schemes in the single task environment cannot induce the OPS when agents are sufficiently patient.

Observation 6. *Suppose T is large and $\Psi(T/2)$ is sufficiently small so that the OPS is characterized by symmetric cutoff $\hat{\theta}_{OPS}$. Then, if $u(x,0)-u(x,T_N)$ is sufficiently small, type $\theta_{\max}$ always has incentive to deviate from the OPS under b_{mlt} and b_{rnd}.*

Observation 6 says that the optimal prize schemes in the last section do not satisfy the "participation condition" when agents are sufficiently patient. To see the claim, recall that prize scheme b_{mlt} rewards the loser. When the agent is $\theta_{\max}$ and the problem is "difficult", the probability that he loses the competition is very low. Then, since the safe task guarantees the payoff v, there is no reason to make an effort to get the second prize with cost v as long as he is sufficiently patient. The analogous argument can be applied to the case of the probabilistic prize scheme.

Now, in order to provide the incentive of focusing on problem solving, I introduce the following class of prize scheme.

Definition 3. *Given prize scheme $b=(b_1(t_1^*,t_2^*),b_2(t_2^*,t_1^*))$, a prize scheme with deadline is $b^D=(b_1^D(t_1^*,t_2^*),b_2^D(t_2^*,t_1^*))$ such that*

$$b_i^D(t_i^*,t_j^*)=\begin{cases} b_i(t_i^*,t_j^*) \text{ if } t_i^*<T \\ \varnothing \text{ if } t_i^*\geq T \end{cases}.$$

In short, whenever agent i finds a solution after deadline T, the expected payoff from "not focusing on problem" for agent i is only $v > 0$. Let b_{opt} be an optimal prize scheme when $v = 0$. The next proposition claims that if (i) the payoff from the safe task is small and (ii) the lack of focus makes problem solving sufficiently slower, the prize scheme with deadline b_{opt}^D induces the OPS in an equilibrium.

Proposition 7. *Suppose $\hat{\theta}_{OPS} \in int(\Theta)$. Given x, if $\Psi^N(T)$ is sufficiently small, there exists $\bar{v}_x$ such that b_{opt}^D induces the OPS in an equilibrium if and only if $v \in [0, \bar{v}_x]$.*

Proof. "If" part. Given b_{opt}, let $\hat{U}_R(\hat{\theta}_{OPS})$ be the expected payoff of the equilibrium cutoff type from the risky method and $\hat{U}_S(\hat{\theta}_{OPS})$ be the expected payoff of the equilibrium cutoff type from the safe method. Then, since $\hat{\theta}_{OPS} \in int(\Theta)$, we have $\hat{U}_R(\hat{\theta}_{OPS}) = \hat{U}_S(\hat{\theta}_{OPS})$ under b_{opt}. Note that $\hat{U}_S(\hat{\theta}_{OPS})$ and $\hat{U}_R(\hat{\theta}_{OPS})$ are the same under b_{opt}^D.

Now, suppose one of agents deviate and play N. Then, let $U_N^{dev}(\Psi^N) + v$ be the expected payoff from the deviation given Ψ^N where $U_N^{dev}(\Psi^N)$ is the expected payoff from the competition. Observe that, whenever $\Psi^N(T)$ is small, $\hat{U}_S(\hat{\theta}_{OPS}) > U_N^{dev}(\Psi^N)$. Then, we can find $v' > 0$ such that $\hat{U}_S(\hat{\theta}_{OPS}) = U_N^{dev}(\Psi^N) + v'$ and let $\bar{v}_x = v'$. Obviously, if $v \leq \bar{v}_x$, the cutoff type has no incentive to deviate to N. Thus, the OPS can be induced in an equilibrium whenever $v \leq \bar{v}_x$.

"Only if" part. Suppose $v > \bar{v}_x$. Then, by construction of $\bar{v}_x$, the expected payoff of the cutoff type from N is always strictly higher than that from the OPS. Q.E.D.

To get an intuition, suppose one agent deviates from the OPS. Then, he gets payoff v for sure and may get the prize from the competition if he can solve the problem before the deadline. However, if $\Psi^N(T)$ is smaller, the probability that the agent gets the prize becomes lower because of the deadline. Hence, whenever both $\Psi^N(T)$ and v are sufficiently low, there is no incentive to deviate from the OPS.

The prize scheme with deadline does not work well if the lack of the focus does not deteriorate his performance of problem solving very much. Consider the case that $\Psi^N(T)$ is large. Then, the expected payoff from N can be high under deadline T. Then, the difference between the expected payoff from N and that from the risky method can be small. As a result, when v is sufficiently large, the prize scheme with deadline cannot induce the OPS.

Remark 4. Given a prize scheme, $\bar{v}_x$ is increasing in the level of prize x. Thus, given $v > 0$, we can always find large x which makes $\bar{v}_x > v$.

7 Discussion

This paper shows that the key to induce the optimal risk taking is the choice of competitive pressure. For instance, it is shown that the second prize can eliminate the excessive risk taking. The second prize is observed in innovation prizes in the real world. For example, the Google Lunar X PRIZE has \$5 million second prize (\$20 million for the first prize).[4] The X-prize foundation claims innovation prizes are the effective incentive to induce risk taking which brings breakthroughs. Thus, if their aim is to encourage the optimal risk taking, their second prize is consistent with the implication of this paper.

On the other hand, it is important to notice that “optimal risk taking” in this paper depends on the setting where there is a safe method. For some cases, e.g., sequencing DNA, this is a reasonable assumption. However, if there is no promising/ safe approach for a problem, the excessive risk taking is not well defined and the rationale of the second prize is not clear. This paper implies that whether the problem has relatively "safe" approach or not can be a key factor for the design of the optimal prize scheme.

8 Summary

In the first part of this paper, I analyzed the situation where agents compete to solve a problem in a winner-take-all fashion. Each agent chooses either a risky or a safe method.

- The winner-take-all competition induces the problem solving which maximizes the prize designer's objective when the problem is "easy" for the risky method. However, the winner-take-all competition induces the excessive risk taking behavior when a problem is "difficult" for the risky method.

In the second part of this paper, I introduced prize schemes which induce the optimal risk taking in an equilibrium for those "difficult" problems. The following classes of prize

[4] http://www.googlelunarxprize.org/lunar/about-the-prize/rules-and-guidelines

schemes can induce the optimal problem solving.

- Multi-prize scheme: In this scheme, the designer gives a prize not only to the winner but also loser. The prize designer chooses the first and second prizes.
- Probabilistic prize scheme: In this scheme, the prize structure is determined by a lottery. One of prize structures is the winner-take-all and the other is an absolute performance basis prize. Each agent does not know the prize structure when he chooses his problem solving method but the probability of the prize structure is common knowledge among agents. The prize designer chooses the probability of each prize structure.

In the extension, I investigated a multi-task environment. Each agent has another "safe" task and chooses whether he focuses on problem solving or not.

- Optimal prize schemes in the basic setting cannot provide the incentive to focus on problem solving when agents are sufficiently patient: Moral hazard problem.

To resolve the problem, I introduced a deadline to the prize schemes. That is, the agent gets a prize from competition only if he finds a solution before the deadline. Then, I found the following.

- The competition with deadline can provide the incentive to focus on problem solving and induces the optimal problem solving if the lack of focus deteriorates the performance of problem solving significantly and the payoff from the safe task is sufficiently small.

9 Appendix

9.1 Proof of Lemma 1

Claim 1. There is no equilibrium where all agents play the safe method for all θ.

Proof of claim 1. Suppose all types play the safe method in equilibrium. Then, the expected payoff of agent i is $(1-G(\theta_i))u(x,\theta_i)$ which is decreasing in θ_i and goes to 0 as $\theta_i \to \theta_{\max}$. On the other hand, if agent i deviates to the risky method, his expected payoff is $\int_{\theta_j} V_{i,RS}(\theta_j)dG(\theta_j) > 0$ and constant in θ_i. Hence, for large θ_i, the deviation is always profitable. Q.E.D.

Claim 2. Suppose there exists a pure strategy equilibrium in which $\sigma_i(\theta) = S$ for some θ. Then, agent i plays a cut off strategy in the equilibrium.

Proof of claim 2. Suppose claim 2 is not true. Then, there exists an equilibrium such that $\sigma_i(\theta_i') = R$ for θ' and $\sigma_i(\theta_i'') = S$ for $\theta_i'' > \theta_i'$. Let $\Theta_i^R \subset \Theta$ be the set of types in which the equilibrium strategy for agent i is the risky method and $\Theta_i^S \subset \Theta$ be the set of types in which the equilibrium strategy for agent j is the safe method. For type $\theta_i' \in \Theta_i^R$,

$$\begin{aligned} \text{C1} \quad &: \int_{\theta_j \in \Theta_j^R} V_{i,RR} dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} V_{i,RS}(\theta_j) dG(\theta_j) \\ &\geq \int_{\theta_j \in \Theta_j^R} V_{i,SR}(\theta_i') dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} V_{i,SS}(\theta_i', \theta_j) dG(\theta_j) \end{aligned}$$

On the other hand, the equilibrium expected payoff for $\theta_i'' \in \Theta_i^S$ is

$$\int_{\theta_j \in \Theta_j^R} V_{i,SR}(\theta_i'') dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} V_{i,SS}(\theta_i'', \theta_j) dG(\theta_j).$$

Then, for $\theta_i'' \in \Theta_i^S$, the expected payoff from deviation is

$$\int_{\theta_j \in \Theta_j^R} V_{i,RR} dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} V_{i,RS}(\theta_j) dG(\theta_j),$$

which is independent of θ_i.

Then, since $\int_{\theta_j \in \Theta_j^R} V_{i,SR}(\theta_i) dG(\theta_j)$ and $\int_{\theta_j \in \Theta_j^S} V_{i,SS}(\theta_i, \theta_j) dG(\theta_j)$ are both strictly decreasing in θ_i, this deviation is always profitable as long as C1 holds. A contradiction. Q.E.D.

9.2 Proof of Remark 2

Claim. There is no asymmetric equilibrium if T is large and $\Psi(T/2)$ is sufficiently small.

Proof. Suppose there exists an equilibrium which consists of cut off strategies with a pair of cut off $(\hat{\theta}_i, \hat{\theta}_j)$ such that $\hat{\theta}_i > \hat{\theta}_j$. Then, the expected payoffs of agent i and j from the risky method are

$$\begin{aligned} U_{iR} &= (1 - G(\hat{\theta}_j))V_{i,SS} + \int_{\theta_j < \hat{\theta}_j} V_{i,RS}(\theta_j) dG(\theta_j) \\ U_{jR} &= (1 - G(\hat{\theta}_i))V_{j,RR} + \int_{\theta_i < \hat{\theta}_i} V_{j,RS}(\theta_i) dG(\theta_i). \end{aligned}$$

The expected payoffs of agent i and j from the safe method are

$$\begin{aligned} U_{iS}(\hat{\theta}_i|\hat{\theta}_j) &= (1 - G(\hat{\theta}_j))V_{i,SR}(\hat{\theta}_i) \\ U_{jS}(\hat{\theta}_j|\hat{\theta}_i) &= (1 - G(\hat{\theta}_i))V_{j,SR}(\hat{\theta}_j) + \int_{\theta_i \in [\hat{\theta}_j, \hat{\theta}_i]} V_{j,RS}(\theta_i) dG(\theta_i). \end{aligned}$$

Notice that, in equilibrium, we need to have $U_{iR} = U_{iS}(\hat{\theta}_i|\hat{\theta}_j)$ and $U_{jR} = U_{jS}(\hat{\theta}_j|\hat{\theta}_i)$. I will show that this is not possible if T is large and $\Psi(T/2)$ is sufficiently small. Let $\hat{\theta}^*$ be the cut off type in a symmetric cut off equilibrium.

Case 1. $\hat{\theta}_i, \hat{\theta}_j < \hat{\theta}^*$.

Let $\hat{\theta}_j' = \hat{\theta}_i$. Notice that if $\hat{\theta}_i < \hat{\theta}^*$, then $z(\hat{\theta}_i) < 0$ or $U_{jR} < U_{jS}(\hat{\theta}_j'|\hat{\theta}_i)$. On the other hand, if the cut off type of agent j is $\hat{\theta}_j$, U_{jR} stays in the same level but $U_{jS}(\hat{\theta}_j|\hat{\theta}_i) > U_{jS}(\hat{\theta}_j'|\hat{\theta}_i)$. Thus, $U_{jR} < U_{jS}(\hat{\theta}_j|\hat{\theta}_i)$.

Case 2. $\hat{\theta}_i, \hat{\theta}_j > \hat{\theta}^*$.

Let $\hat{\theta}_i' = \hat{\theta}_j$. Notice that if $\hat{\theta}_j > \hat{\theta}^*$, then $z(\hat{\theta}_j) > 0$ or $U_{iR} > U_{iS}(\hat{\theta}_i'|\hat{\theta}_j)$. On the other hand, if the cut off type of agent i is $\hat{\theta}_i > \hat{\theta}_i'$, U_{iR} stays in the same level but $U_{iS}(\hat{\theta}_i|\hat{\theta}_j) < U_{iS}(\hat{\theta}_i'|\hat{\theta}_j)$. Thus, $U_{iR} > U_{iS}(\hat{\theta}_j|\hat{\theta}_j)$.

Case 3. $\hat{\theta}_i > \hat{\theta}^* > \hat{\theta}_j$.

Let $\hat{\theta}_j' = \hat{\theta}_i' = \hat{\theta}^*$. $U_{jS}(\hat{\theta}_j'|\hat{\theta}_i') < U_{jS}(\hat{\theta}_j|\hat{\theta}_i')$ because $\hat{\theta}_j < \hat{\theta}_j'$ and the probability that agent j wins for sure becomes higher. Moreover, $U_{jR}(\hat{\theta}_i') < U_{jS}(\hat{\theta}_j|\hat{\theta}_i')$ since $U_{jR}(\hat{\theta}_i') = U_{jS}(\hat{\theta}_j'|\hat{\theta}_i')$ by definition of $\hat{\theta}^*$. Now, suppose the cut off type of agent i is $\hat{\theta}_i > \hat{\theta}_i'$. Then, $U_{jS}(\hat{\theta}_j|\hat{\theta}_i') < U_{jS}(\hat{\theta}_j|\hat{\theta}_i)$. On the other hand, $|U_{jR}(\hat{\theta}_i) - U_{jR}(\hat{\theta}_i')|$ can be arbitrarily small by choosing large T and small $\Psi(T/2)$. Thus, $U_{jR}(\hat{\theta}_i) < U_{jS}(\hat{\theta}_j|\hat{\theta}_i)$. Q.E.D.

9.3 Proof of Observation 1

Proof of (i). Immediate.

Proof of (ii). First, I show that the OPS is always cutoff strategy profile. Let $\sigma_i^*(\theta_i)$ and $\sigma_j^*(\theta_j)$ be the OPS. Then, for each θ_i, a problem solving method $\sigma_i^*(\theta_i)$ maximizes the expected utility of the prize designer given $\sigma_j^*(\theta_j)$. Now, suppose $\sigma_i^*(\theta_i') = R$ for θ' and $\sigma_i^*(\theta_i'') = S$ for $\theta_i'' > \theta_i'$. Let $\Theta_i^a = \{\theta_i | \sigma_i^*(\theta_i) = a\}$ and, analogously, I define $\Theta_j^a \subset \Theta$. Moreover, let $W_{i,a_i,a_j}(\theta_i, \theta_j)$ be the expected payoff of the prize designer given (a_i, a_j) and (θ_i, θ_j). Then, whenever the strategy profile is the OPS, given $\theta_i' \in \Theta_i^R$,

$$
\begin{aligned}
&\int_{\theta_j \in \Theta_j^R} W_{i,RR}(\theta_i', \theta_j) dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} W_{i,RS}(\theta_i', \theta_j) dG(\theta_j) \\
\geq &\int_{\theta_j \in \Theta_j^R} W_{i,SR}(\theta_i', \theta_j) dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} W_{i,SS}(\theta_i', \theta_j) dG(\theta_j)
\end{aligned}
$$

The expected payoff of the prize designer, given $\theta_i'' \in \Theta_i^S$, is

$$\int_{\theta_j \in \Theta_j^R} W_{i,SR}(\theta_i'', \theta_j) dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} W_{i,SS}(\theta_i'', \theta_j) dG(\theta_j).$$

It is easy to see that, given θ_j, $W_{i,SR}(\theta_i', \theta_j) > W_{i,SR}(\theta_i'', \theta_j)$ and $W_{i,SS}(\theta_i', \theta_j) \geq W_{i,SS}(\theta_i'', \theta_j)$. On the other hand, note that

$$
\begin{aligned}
&\int_{\theta_j \in \Theta_j^R} W_{i,RR}(\theta_i'', \theta_j) dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} W_{i,RS}(\theta_i'', \theta_j) dG(\theta_j) \\
= &\int_{\theta_j \in \Theta_j^R} W_{i,RR}(\theta_i', \theta_j) dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} W_{i,RS}(\theta_i', \theta_j) dG(\theta_j) \\
> &\int_{\theta_j \in \Theta_j^R} W_{i,SR}(\theta_i'', \theta_j) dG(\theta_j) + \int_{\theta_j \in \Theta_j^S} W_{i,SS}(\theta_i'', \theta_j) dG(\theta_j)..
\end{aligned}
$$

A contradiction.

Second, I show that if T is large and $\Psi(T/2)$ is sufficiently small, the OPS is a symmetric cut off strategy. Given $\hat{\theta}_j$, the expected payoffs for the prize designer in which agent i plays the risky method is

$$W_{iR}(\hat{\theta}_i | \hat{\theta}_j) = \int_{\theta_j > \hat{\theta}_j} W_{i,RR}(\hat{\theta}_i, \theta_j) dG(\theta_j) + \int_{\theta_j < \hat{\theta}_j} W_{i,RS}(\hat{\theta}_i, \theta_j) dG(\theta_j)$$

Note that $W_{iR}(\hat{\theta}_i | \hat{\theta}_j)$ is constant in $\hat{\theta}_i$. On the other hand, given $\hat{\theta}_j$ and $\hat{\theta}_i$, the expected payoff for the prize designer in which agent i plays the safe method is

$$W_{iS}(\hat{\theta}_i | \hat{\theta}_j) = \int_{\theta_j > \hat{\theta}_j} W_{i,SR}(\hat{\theta}_i, \theta_j) dG(\hat{\theta}_i) + \int_{\theta_j < \hat{\theta}_j} W_{i,SS}(\hat{\theta}_i, \theta_j) dG(\theta_j).$$

Let $\hat{\theta}_i(\hat{\theta}_j)$ be

$$\begin{cases} \hat{\theta}_i^{**} \text{ if there exists } \hat{\theta}_i^{**} \text{ s.t. } W_{iR}(\hat{\theta}_i^{**}|\hat{\theta}_j) = W_{iS}(\hat{\theta}_i^{**}|\hat{\theta}_j), \\ \theta_{\min} \text{ if } W_{iR}(\hat{\theta}_i|\hat{\theta}_j) > W_{iS}(\hat{\theta}_i|\hat{\theta}_j) \text{ for any } \hat{\theta}_i \in \Theta, \\ \theta_{\max} \text{ if } W_{iR}(\hat{\theta}_i|\hat{\theta}_j) < W_{iS}(\hat{\theta}_i|\hat{\theta}_j) \text{ for any } \hat{\theta}_i \in \Theta. \end{cases}$$

Moreover, let $\widetilde{W}_{ia}(\hat{\theta}_i|\hat{\theta}_j)$ be $W_{ia}(\hat{\theta}_i|\hat{\theta}_j)$ with $\Psi(\theta_{\max}) = 0$, that is,

$$\begin{aligned} \widetilde{W}_{iR}(\hat{\theta}_i|\hat{\theta}_j) &= \int_{\theta_j<\hat{\theta}_j} \widetilde{u}(B,\theta_j)dG(\theta_j), \\ \widetilde{W}_{iS}(\hat{\theta}_i|\hat{\theta}_j) &= \begin{cases} \int_{\theta_j<\hat{\theta}_j} \widetilde{u}(B,\theta_j)dG(\theta_j) + \int_{\theta_j>\hat{\theta}_j} \widetilde{u}(B,\hat{\theta}_i)dG(\theta_j) \text{ if } \hat{\theta}_i \geq \hat{\theta}_j \\ \int_{\theta_j\in[\theta_{\min},\hat{\theta}_i)} \widetilde{u}(B,\theta_j)dG(\theta_j) + \int_{\theta_j\in[\hat{\theta}_i,\hat{\theta}_j)} \widetilde{u}(B,\hat{\theta}_i)dG(\theta_j) \\ \quad + \int_{\theta_j>\hat{\theta}_j} \widetilde{u}(B,\hat{\theta}_i)dG(\theta_j) \text{ if } \hat{\theta}_i < \hat{\theta}_j \end{cases} \end{aligned}$$

Note that, by choosing large T and small $\Psi(T/2)$, we can make $W_{ia}(\hat{\theta}_i|\hat{\theta}_j)$ arbitrarily close to $\widetilde{W}_{ia}(\hat{\theta}_i|\hat{\theta}_j)$.

First, I claim that, if T is large and $\Psi(T/2)$ is sufficiently small, $\hat{\theta}_i(\hat{\theta}_j)$ is non increasing in $\hat{\theta}_j$. To see the claim, observe that

$$\frac{d}{d\hat{\theta}_j}[\widetilde{W}_{iR}(\hat{\theta}_i|\hat{\theta}_j) - \widetilde{W}_{iS}(\hat{\theta}_i|\hat{\theta}_j)] = \begin{cases} \widetilde{u}(B,\hat{\theta}_i)g(\hat{\theta}_j) \text{ if } \hat{\theta}_i \geq \hat{\theta}_j \\ \widetilde{u}(B,\hat{\theta}_j)g(\hat{\theta}_j) \text{ if } \hat{\theta}_i < \hat{\theta}_j \end{cases}.$$

Thus, $\frac{d}{d\hat{\theta}_j}[\widetilde{W}_{iR}(\hat{\theta}_i|\hat{\theta}_j) - \widetilde{W}_{iS}(\hat{\theta}_i|\hat{\theta}_j)] > 0$. Then, since $W_{iR}(\hat{\theta}_i|\hat{\theta}_j) - W_{iS}(\hat{\theta}_i|\hat{\theta}_j)$ is continuous in $\hat{\theta}_i$, if T is large and $\Psi(T/2)$ is sufficiently small, $\hat{\theta}_i(\hat{\theta}_j)$ is non increasing in $\hat{\theta}_j$.

Second, I claim that if T is large and $\Psi(T/2)$ is sufficiently small, $\hat{\theta}_i(\theta_{\min}) = \theta_{\max}$. To see the claim, note that $\widetilde{W}_{iS}(\hat{\theta}_i|\theta_{\min}) > \widetilde{W}_{iR}(\hat{\theta}_i|\theta_{\min}) = 0$. Hence, if T is large and $\Psi(T/2)$ is sufficiently small, $W_{iR}(\hat{\theta}_i|\theta_{\min}) < W_{iS}(\hat{\theta}_i|\theta_{\min})$ for all $\hat{\theta}_i$.

Third, I claim that if T is large and $\Psi(T/2)$ is sufficiently small $\hat{\theta}_i(\theta_{\max}) < \theta_{\max}$. To see the claim, note that $\widetilde{W}_{iR}(\theta_{\max}|\theta_{\max}) = \int_{\theta_j} \widetilde{u}(B,\theta_j)dG(\theta_j) = \widetilde{W}_{iS}(\theta_{\max}|\theta_{\max}) = \int_{\theta_j} \widetilde{u}(B,\hat{\theta}_i)dG(\theta_j)$. Note that $\widetilde{W}_{iR}(\theta_{\max}|\theta_{\max}) < W_{iR}(\theta_{\max}|\theta_{\max})$ and $\widetilde{W}_{iS}(\theta_{\max}|\theta_{\max}) = W_{iS}(\theta_{\max}|\theta_{\max})$. Hence, if T is large and $\Psi(T/2)$ is sufficiently small, $W_{iR}(\theta_{\max}|\theta_{\max}) > W_{iS}(\theta_{\max}|\theta_{\max})$ and thus $\hat{\theta}_i(\theta_{\max}) \neq \theta_{\max}$.

Finally, observe that $\hat{\theta}_i(\hat{\theta}_j)$ is continuous in $\hat{\theta}_j$ and, by symmetry we know $\hat{\theta}_j(\hat{\theta}_i)$ has the same property as $\hat{\theta}_i(\hat{\theta}_j)$. Then, the above three claims imply that if T is large and

$\Psi(T/2)$ is sufficiently small, there exists unique $\hat{\theta}_i^{**}, \hat{\theta}_j^{**} < \theta_{\max}$ such that $\hat{\theta}_i(\hat{\theta}_j(\hat{\theta}_i^{**})) = \hat{\theta}_i^{**}$, $\hat{\theta}_j(\hat{\theta}_i(\hat{\theta}_j^{**})) = \hat{\theta}_j^{**}$. By symmetry, $\hat{\theta}_i^{**} = \hat{\theta}_j^{**} = \hat{\theta}^{OPS}$. Q.E.D.

9.4 Proof of Proposition 2

Suppose both agents follow the same cut off strategy and let $\hat{\theta}$ be the symmetric cut off type. Then, the expected payoffs of the prize designer given $a_i = R$ is

$$W_{iR}(\hat{\theta}) = \int_{\theta_j > \hat{\theta}} W_{i,RR} dG(\theta_j) + \int_{\theta_j < \hat{\theta}} W_{i,RS}(\theta_j) dG(\theta_j).$$

The expected payoffs of the prize designer given $a_i = S$ is

$$W_{iS}(\hat{\theta}) = \int_{\theta_j > \hat{\theta}} W_{i,SR}(\hat{\theta}) dG(\hat{\theta}_i) + \int_{\theta_j < \hat{\theta}} W_{i,SS}(\hat{\theta}, \theta_j) dG(\theta_j).$$

Let $\widetilde{z}(\hat{\theta}) = W_{iR}(\hat{\theta}) - W_{iS}(\hat{\theta})$.

$$\begin{aligned} \widetilde{z}(\hat{\theta}) &= \int_{\theta_j > \hat{\theta}} [W_{i,RR} - W_{i,SR}(\hat{\theta})] dG(\theta_j) \\ &\quad + \int_{\theta_j < \hat{\theta}} [W_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}, \theta_j)] dG(\theta_j). \end{aligned}$$

First, I show that whenever there exists $\hat{\theta}^{**}$ such that $\widetilde{z}(\hat{\theta}^{**}) = 0$, it has to be unique and $\hat{\theta}^{**} = \hat{\theta}_{OPS}$. Observe that, if $\theta_j < \hat{\theta}$, $W_{i,SS}(\hat{\theta}, \theta_j)$ is the same as the expected payoff of the prize designer in which only agent j solves the problem with the safe method. Thus, for any $\theta_j < \hat{\theta}$,

$$W_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}, \theta_j) > 0.$$

Then,

$$\widetilde{z}(\theta_{\max}) = W_{i,RS}(\theta_j) - W_{i,SS}(\theta_{\max}, \theta_j) > 0.$$

Since $W_{i,SR}(\hat{\theta})$ is decreasing in $\hat{\theta}$, $W_{iRR} - W_{i,SR}(\hat{\theta})$ is increasing in $\hat{\theta}$. Hence, whenever there exists $\hat{\theta}$ such that $\widetilde{z}(\hat{\theta}) = 0$,

$$\widetilde{z}(\theta_{\min}) = W_{i,RR} - W_{i,SR}(\theta_{\min}) < 0.$$

Then, it is easy to see that, for small $\hat{\theta}$ such that $W_{i,RR} < W_{i,SR}(\hat{\theta})$, $\widetilde{z}(\hat{\theta})$ is increasing in $\hat{\theta}$. On the other hand, once $\hat{\theta}$ becomes large enough so that $W_{i,RR} > W_{i,SR}(\hat{\theta})$, then $\widetilde{z}(\hat{\theta}) > 0$

for larger $\hat{\theta}$. Therefore, whenever there exists $\hat{\theta}^{**}$ such that $\widetilde{z}(\hat{\theta}^{**}) = 0$, it has to be unique. Then, by observation 1-(ii), we know that $\hat{\theta}^{**} = \hat{\theta}_{OPS}$.

Now, let $\hat{\theta}^{*}$ be the equilibrium cutoff type. I show that if T is large and $\Psi(T/2)$ is sufficiently small, then $\hat{\theta}^{*} < \hat{\theta}_{OPS}$. The proof consists of 4 steps.

First, let us rewrite $\widetilde{z}(\hat{\theta}^{*})$ in the following way.$\widetilde{v}(B)$

$$
\begin{aligned}
\widetilde{z}(\hat{\theta}^{*}) = {} & \frac{v(B)}{v(x)} z(\hat{\theta}^{*}) + (1 - G(\hat{\theta}^{*}))(W_{i,RR} - W_{i,SR}(\hat{\theta}^{*}) - \frac{\widetilde{v}(B)}{v(x)}(V_{i,SS} - V_{i,SR}(\hat{\theta}^{*}))) \\
& + \int_{\theta_j < \hat{\theta}^{*}} [W_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}^{*}, \theta_j) - \frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j)] dG(\theta_j) \\
= {} & (1 - G(\hat{\theta}^{*}))(W_{i,RR} - \frac{\widetilde{v}(B)}{v(x)} V_{i,RR} + V_{i,SR}(\hat{\theta}^{*}) - W_{i,SR}(\hat{\theta}^{*})) \\
& + \int_{\theta_j < \hat{\theta}^{*}} [W_{i,RS}(\theta_j) - \frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}^{*}, \theta_j)] dG(\theta_j).
\end{aligned}
$$

Step 1. If T is large and $\Psi(T/2)$ is sufficiently small, then $W_{i,RS}(\theta_j) - \frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}^{*}, \theta_j) < 0$ for any $\theta_j < \hat{\theta}^{*}$.

To see the claim, let $\mu(t_i^{*}|a_i a_j, i)$ be the probability that agent i finds a solution at t_i^{*} conditional on his winning and action profile (a_i, a_j). Then,

$$
\begin{aligned}
W_{i,RS}(\theta_j) &= \int_{t_i^{*}} \widetilde{u}(B, t_i^{*}) \mu(t_i^{*}|RS, i) dt_i^{*} + \int_{t_j^{*}} \widetilde{u}(B, t_j^{*}) \mu(t_j^{*}|RS, j) dt_j^{*}, \\
\frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j) &= \int_{t_i^{*}} \widetilde{u}(B, t_i^{*}) \mu(t_i^{*}|RS, i) dt_i^{*}, \\
W_{i,SS}(\hat{\theta}^{*}, \theta_j) &= \int_{t_j^{*}} \widetilde{u}(B, t_j^{*}) \mu(t_j^{*}|SS, j) dt_j^{*} \text{ for any } \theta_j < \hat{\theta}^{*}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& W_{i,RS}(\theta_j) - \frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}^{*}, \theta_j) \\
= {} & \int_{t_i^{*}} \widetilde{u}(B, t_i^{*}) \mu(t_i^{*}|RS, i) dt_i^{*} - \int_{t_i^{*}} \widetilde{u}(B, t_i^{*}) \mu(t_i^{*}|RS, i) dt_i^{*} \\
& + \int_{t_j^{*}} \widetilde{u}(B, t_j^{*}) \mu(t_j^{*}|RS, j) dt_j^{*} - \int_{t_j^{*}} \widetilde{u}(B, t_j^{*}) \mu(t_j^{*}|SS, j) dt_j^{*} \\
= {} & \int_{t_j^{*}} \widetilde{u}(B, t_j^{*}) [\mu(t_j^{*}|RS, j) - \mu(t_j^{*}|SS, j)] dt_j^{*}
\end{aligned}
$$

Note that, if T is large and $\Psi(T/2)$ is sufficiently small, then $\int_{t_j^*} \widetilde{u}(B, t_j^*)[\mu(t_j^*|RS, j) - \mu(t_j^*|SS, j)]dt_j^* < 0$. Thus, $W_{i,RS}(\theta_j) - \frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}^*, \theta_j) < 0$ for any $\theta_j < \hat{\theta}^*$.

Step 2. By choosing large T and small $\Psi(T/2)$, we can make $1 - G(\hat{\theta}^*)$ arbitrarily close to 0.

To see the claim, let $z(\hat{\theta}; \Psi)$ be $z(\hat{\theta})$ given Ψ. Suppose $z(\hat{\theta}; \Psi) = k$. Observe that, by choosing large T and small $\Psi(T/2)$, we can make $V_{i,RR}$ and $V_{i,RS}(\theta_j)$ arbitrarily close to 0. Hence, by choosing Ψ' with larger T and small $\Psi(T/2)$, we have $z(\hat{\theta}; \Psi') < k$. Thus, given any $\theta < \theta_{\max}$, by choosing large T and small $\Psi(T/2)$, we can make $z(\theta; \Psi) < 0$. Then, since $z(\hat{\theta}; \Psi) > 0$ for sufficiently large $\hat{\theta}$, by choosing large T and small $\Psi(T/2)$, we can make $\hat{\theta}^*$ arbitrarily close to $\theta_{\max}$ and $G(\hat{\theta}^*)$ arbitrarily close to 1.

Step 3. If T is large and $\Psi(T/2)$ is sufficiently small, $\widetilde{z}(\hat{\theta}^*) < 0$.

By step 1, if $\theta_j < \hat{\theta}^*$, $W_{i,RS}(\theta_j) - \frac{\widetilde{v}(B)}{v(x)} V_{i,RS}(\theta_j) - W_{i,SS}(\hat{\theta}^*, \theta_j) < 0$ for large T and small $\Psi(T/2)$. On the other hand, step 2 says that, by choosing large T and small $\Psi(T/2)$, we can make $1 - G(\hat{\theta}^*)$ arbitrarily close to 0. Hence, $\widetilde{z}(\hat{\theta}^*) < 0$ for large T and small $\Psi(T/2)$.

Step 4. $\hat{\theta}_{OPS} > \hat{\theta}^*$ if T is large and $\Psi(T/2)$ is sufficiently small.

To see the claim, recall that $\widetilde{z}(\hat{\theta}) = 0$ has unique solution $\hat{\theta}_{OPS}$ and $\widetilde{z}(\hat{\theta}) > 0$ for $\theta > \hat{\theta}_{OPS}$ and $\widetilde{z}(\hat{\theta}) < 0$ for $\theta < \hat{\theta}_{OPS}$. Therefore, if $\widetilde{z}(\hat{\theta}^*) < 0$, then $\hat{\theta}_{OPS} > \hat{\theta}^*$. Q.E.D.

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