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EQUILIBRIA IN INCOMPLETE FINANCIAL MARKETS WITH PORTFOLIO CONSTRAINTS AND TRANSACTION COSTS

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Equilibria in incomplete financial markets with portfolio constraints and transaction costs

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Abstract : In this paper we consider a model for potentially incomplete financial markets with nominal assets where the trade of the assets is constrained by two types of constraints. The first type are called portfolio constraints. This means that there are constraints on the portfolios that an agent is allowed to purchase. So each agent has a trade set, which is the set of portfolios that he is allowed to purchase. The second type are transaction costs which have to be paid to a broker in return for his intermediation between buyer and seller. The brokerage house is owned by the agents. We show that equilibria exist on such a market. Furthermore, we show that the presence of trading constraints implies that one has to distinguish between two types of arbitrage possibilities, from which only one is incompatible with equilibrium.

Keywords : incomplete markets, nominal assets, portfolio constraints, transaction costs, arbitrage possibilities.

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1. Introduction

In this paper we consider a (potentially) incomplete financial markets economy where the trade of assets is constrained by two different types of constraints. The first type will be called ***portfolio constraints***. These are constraints on the portfolio holdings of the agents. Several examples of models with constraints on portfolio holdings can be found in the literature. In [Balasko, Cass, Siconolfi (1990)] the authors consider an incomplete markets model with real asset structure where each agent has a ***portfolio choice set***. This is the set of portfolios that the agent is allowed to purchase. In their paper, these portfolio choice sets are considered to be linear subspaces. Another example is [Younès (1988)]. Here, the author defines the concept of a ***submarket***, which is a market where only a certain number of consumption bundles can be traded against each other. Agents have restricted access to these submarkets. The access is again defined by linear subspaces. In [Siconolfi (1989)] existence of equilibria for incomplete financial markets with nominal asset structure is established under the presence of fairly general portfolio choice sets.

The second type of constraints we consider in our model are ***transaction costs***. As opposed to models where transaction costs are real inputs and intermediaries are modeled as profit maximizers over their transaction set (see for instance [Hahn (1973)] or [Starrett (1973)]), we consider the case where intermediaries get a commission for their intermediation. This commission depends on the prices of the assets and the traded quantities of each of the assets. The brokerage houses are owned by the agents, so in the end, the broker's profit returns to the agents.

The assets in our model are considered to be nominal with exogenous yield, i.e. assets pay off in units of account, and the payoff does not depend on spot prices. It is well known that, in the absence of transaction costs or portfolio constraints, equilibria exist on potentially incomplete financial markets with nominal assets (see for instance [Werner (1985)]). So, as opposed to other approaches ([Préchac (1993)]), we don't introduce transaction costs in order to prove that they can eliminate problems of nonexistence of equilibria, because there are no such problems in the case of nominal assets. We introduce portfolio constraints as well as transaction costs because very often the trade of assets is constrained by institutional rules, and by the fact that there is an intermediary person who gets a certain percentage of the price paid by the buyer. This situation typically occurs for instance in insurance markets. Insurance contracts are very often traded through insurance brokers, who get a certain percentage of the price of the contract. On the other hand, the trade of insurance contracts is constrained by institutional rules such as the prohibition of overinsurance. So both types of constraints are present on some markets. Therefore, it seems interesting to study the effects of such constraints on the trade.

The aim in this paper is to show that equilibria exist on financial markets with two types of trading constraints, namely portfolio constraints and transaction costs. Therefore, it is important to distinguish between two types of arbitrage possibilities, those that can yield an infinite profit for at least one agent (type one) and those that can "only" yield a finite profit for all agents (type two). Arbitrage possibilities of type two occur when there exists a portfolio z which yields a non-negative payoff in each state, strictly positive in at least one state, and has a negative price, such that there exists an $n \in I\!N$ such that tz cannot be traded (as a result of the portfolio constraints) as soon as $t \geq n$. Clearly, type two arbitrage possibilities are not necessarily incompatible with utility maximization. This implies that equilibrium prices are no longer a priori no-arbitrage prices. By giving an example of an incomplete market with portfolio constraints where an equilibrium price for a nominal asset with positive payoff is strictly negative, we show that type two arbitrage possibilities are indeed not necessarily incompatible with equilibrium. Only the first type of arbitrage possibilities is incompatible with equilibrium. This shows that the presence of portfolio constraints can have an important influence on the structure of the equilibrium prices. The underlying reason is that, as a result of these constraints, even under the presence of arbitrage possibilities of type two, budget sets remain compact, and therefore agents are still able to maximize their utility over their budget set.

As mentioned before, the transaction costs are characterized by a function depending on the prices of the J assets, and the quantities bought or sold of each of the J assets. The portfolio constraints are characterized by a collection of subsets of $I\!R^J$, called *trade sets*. So each agent has a trade set, which is the set of portfolios that the agent is allowed to purchase. These portfolio constraints are asymmetric in the sense that these trade sets don't have to be the same for each agent. For example in insurance markets, the prohibition of overinsurance implies that each agent has a portfolio set which consists of those insurance portfolios that the agent can trade without getting overinsured. This of course depends upon his initial risk position. So, vaguely stated, agents who bare "more risk" will be allowed to buy "more" insurance contracts than those who bare "less risk".

We proceed in the following way : in section 2, we define the model and we give an example. In section 3, we define the equilibrium concept. In section 4, we study the properties of demand and excess demand correspondences. This includes the definition of the set of asset prices for which type one arbitrage possibilities do not exist. In section 5, we prove existence of equilibria. Section 6 concludes.

2. The market model

We consider an economy where L physical goods are available. There are two periods, which we will call date zero and date one. At date one, the world can be in S different states, indexed by $s \in \{1, 2, \ldots, S\}$. Date zero will be denoted state zero. Each of the L goods can be traded at each of the $S+1$ spot markets, one at date zero and one for each possible state of the world at date one.

We will denote $p_s = (p_{s,1}, p_{s,2}, \cdots, p_{s,L})$ for the row vector containing the spot prices of each of the L goods in state s. So for each state $s \in \{0, 1, \ldots, S\}$, we have $p_s \in \mathbb{R}^L$.

The consumption set X^i of agent $i \in \{1, 2, \ldots, I\}$ equals $\mathbb{R}_+^{L\times(S+1)}$. So consumption bundles are matrices in $\mathbb{R}_+^{L\times(S+1)}$. We will denote $x = (x_0, x_1, \ldots x_S)$ for a consumption bundle where, for each state $s \in \{0, 1, \ldots, S\}$, $x_s = (x_{s,1}, x_{s,2}, \cdots, x_{s,L})^t$ is the column vector containing the amount of each of the L goods if the world is in state s. Each agent has an initial endowment vector $w^i = (w_0^i, w_1^i, \ldots, w_S^i)$. So for each agent $i \in \{1, 2, \ldots, I\}$ and for every state $s \in \{0, 1, \ldots, S\}$, $w_s^i = (w_{s,1}^i, w_{s,2}^i, \cdots, w_{s,L}^i)^t$ is the column vector containing the amount of each of the L goods owned by the agent before any trade took place if the market is in state s. For each spot price matrix $p \in \mathbb{R}^{(S+1)\times L}$ and for each consumption bundle $x \in \mathbb{R}^{L\times(S+1)}$, we will denote $(px)_1$ for the column vector $(p_1x_1, p_2x_2, \ldots, p_Sx_S)^t$.

Each agent has a preference relation $\succeq^i \subset X^i \times X^i$ to express his preferences between different consumption bundles $x, y \in X^i$.

Agents will trade on spot markets as well as on asset markets in order to maximize their utility. There are J financial assets which pay off at date one. Assets are considered to be nominal, i.e. they pay off in units of account. The payoff of the assets does not depend on spot prices, it only depends on the state of the world at date one. We will denote $A_j = (A_{1j}, A_{2j}, \cdots, A_{Sj})^t \in \mathbb{R}^S$ for vector containing the payoff of asset j in each of the S states of the world at date one. We denote A for the matrix in $\mathbb{R}^{S\times J}$ with j^{th} column equal to A_j for $j \in \{1, 2, \ldots, J\}$. Portfolios will be denoted $z = (z_1, z_2, \ldots, z_J)^t \in \mathbb{R}^J$.

The portfolio constraints are expressed by the fact that each agent has a *trade set* $Z^i \subset \mathbb{R}^J$. The interpretation is that agent i is only allowed to purchase portfolios $z \in Z^i$.

For each asset $j \in \{1, 2, \ldots, J\}$, the price of the asset (in units of account) will be denoted q_j. We will denote q for the row vector $(q_1, q_2, \cdots, q_J) \in \mathbb{R}^J$. Prices have to be paid at date zero.

Since there are transaction costs, and transaction costs have to be paid for buying assets as well as for selling assets, it is convenient to define for each vector $z \in \mathbb{R}^J$, the vector $|z| \in \mathbb{R}^J_+$ as follows :

$$|z| = (|z_1|, |z_2|, \dots, |z_J|)^t.$$

In the sequel, we will assume that the transaction cost for buying or selling z_j units of asset j equals a certain percentage of $|q_j||z_j|$. So the transaction cost for purchasing a portfolio $z \in \mathbb{R}^J$ through the intermediation of a certain broker, given that the prices of the assets are given by the vector $q \in \mathbb{R}^J$, is equal to

$$c|q||z| = c \sum_{j=1}^{J} |q_j||z_j|,$$

where $c \in [0, 1[$ is exogenous. Therefore, the total cost of a portfolio z equals $qz + c|q||z|$. We take the absolute value of q and of z in the definition of the transaction costs because transaction costs have to be positive, whatever the sign of the prices $q_j, j \in \{1, 2, \dots, J\}$. Furthermore, transaction costs have to be paid for buying assets ($z_j \geq 0$) as well as for selling assets ($z_j \leq 0$). Remark that for positive asset prices $q \geq 0$, this definition of the transaction costs is the same as in [Préchac (1993)].

These transaction costs are paid to an intermediary person, called the broker. The profit of the broker will be denoted π. We assume that the brokerage house is owned by the agents. The fraction of the brokerage house owned by agent i, i.e. the share of agent i in the profits of the brokerage house, will be denoted $\nu_i \in [0, 1]$. Furthermore, $\sum_{i=1}^{I} \nu_i = 1$.

In order to simplify notations, we will assume that there is only one broker on the market. In case of more brokers (with different percentages c) the results still hold. The reason why we present them in the case of one broker is because with more brokers, say $n > 1$, instead of having one portfolio per agent, we would need n portfolios per agent (one for each broker), in order to be able to determine transaction costs. Therefore, notations get more complex.

Furthermore, it can be seen easily that all the results in this paper remain valid if we consider the case where there are several "classes" of assets on the market and each class has a different commission c. An example of such a market would be a market where part of the assets are financial assets and the other ones are insurance contracts. Then it seems reasonable to assume that there are different commissions c for those two types of assets. In the remainder of this section, we will treat this example in detail.

Examples : 1) Consider a market where part of the agents are insurers, indexed by $i \in \mathcal{I} \subset \{1, 2, \dots, I\}$. There is a fixed number of insurance contracts on the market, say R, each corresponding to a risk bared by at least one agent $i \in \{1, 2, \dots, I\}$. There is also a number of financial assets K on the market. Let A denote the matrix in

$I\!R^{S\times(K+R)}$, where the first K columns denote the payoff of the financial assets, and the last R columns denote the payoff of the insurance contracts. We assume in this example that the only portfolio constraints are the prohibition of overinsurance and the fact that agents who are not insurers ($i \notin \mathcal{I}$) are not allowed to sell insurance contracts. Then the trade sets would be :

$$Z^i = I\!R^K \times \prod_{j=1}^{R}]-\infty, c_j^i],$$

for insurers $i \in \mathcal{I}$, and

$$Z^i = I\!R^K \times \prod_{j=1}^{R} [0, c_j^i],$$

for agents who are not insurers ($i \notin \mathcal{I}$). In both cases c_j^i denotes the fraction of the corresponding risk j bared by the agent. So $z_j \leq c_j^i$ implies that the agent is not allowed to get overinsured on that risk. For insurers, $z_j \in]-\infty, c_j^i]$ implies that they can sell insurance contracts for risk j without constraints, and they can get reinsured on risk j but they cannot get "over-reinsured". For agents $i \notin \mathcal{I}$, $0 \leq z_j \leq c_j^i$ implies that the agent is not allowed to sell the insurance contract, and is not allowed to get overinsured on that risk.

Furthermore, there is a broker who is an intermediate between the seller of an insurance contract (the insurance agents $i \in \mathcal{I}$) and the buyer of the contract. He receives a commission of $c_r 100\%$ of the price of the insured risk. On the other hand, there is a broker who is an intermediate between the seller and the buyer of financial assets. He receives a commission of $c_a 100\%$ of the price of the asset. So, if the prices of the assets are given by $q_a \in I\!R^K$, and the prices of the insurance contracts are given by $q_r \in I\!R^L$, the total cost of a portfolio $(z_a, z_r) \in Z^i$ would be :

$$q_a z_a + q_r z_r + c_a|q_a||z_a| + c_r|q_r||z_r|.$$

The effects of the trading constraints $Z^i, i \in \{1, 2, \ldots, I\}$ on such a mixed financial–(re)insurance market are studied in detail in [De Waegenaere (1994)].

2) If we take $Z^i = I\!R^J$ for all agents $i \in \{1, 2, \ldots, I\}$, then we get an incomplete markets model for nominal assets with transaction costs but without portfolio constraints. Equivalently, if we take $c = 0$, then we get an incomplete markets model for nominal assets with portfolio constraints but without transaction costs. Finally, the incomplete markets model as in [Werner (1985)] is a special case of our model if we take $c = 0$ and $Z^i = I\!R^J, i \in \{1, 2, \ldots, I\}$.

3. Equilibria on financial markets

We can now define the budget set of an agent, i.e. the set of vectors, consisting of a consumption bundle and a portfolio of assets that the agent can obtain as a result of an allowed trade, taking into account the portfolio constraints, transaction costs and the broker's profit. From now on, we will assume that initial endowment vectors $w^i, i \in \{1, 2, \ldots, I\}$, as well as shares $\nu_i, i \in \{1, 2, \ldots, I\}$, are given exogenously.

Definition 3.1 : *The budget set of agent $i \in \{1, 2, \ldots, I\}$ for given spot prices $p \in \mathbb{R}^{(S+1)\times L}$, asset prices $q \in \mathbb{R}^J$, and broker's profit $\pi \in \mathbb{R}$ is given by :*

$$B^i(p, q, \pi) = \left\{ (x, z) \in X^i \times Z^i \;\middle|\; \begin{array}{lcc} p_0 x_0 & \leq & p_0 w_0^i - qz - c|q||z| + \nu_i \pi \\ (px)_1 & = & (pw^i)_1 + Az \end{array} \right\}.$$

It might seem artificial to introduce the budget sets of the agents with an inequality in the date zero constraint, and an equality in the date one constraints. We put an inequality at date zero in order to get a convex budget set (the absolute value of z causes non convexity in the case of an equality). For technical reasons (see for instance lemma 4.1), we introduce equalities for the date one constraints. Of course it is clear that, under strict monotonicity of the preferences, both constraints will be equalities at equilibrium. Therefore, we can define budget sets as in definition 3.1 without loss of generality.

Each agent will try to maximize his utility over his budget set. Therefore, a financial market equilibrium is defined as follows :

Definition 3.2 : *A financial market equilibrium is a vector of consumption bundles $\{\overline{x}^i, i \in \{1, 2, \ldots, I\}\}$, asset portfolios $\{\overline{z}^i, i \in \{1, 2, \ldots, I\}\}$, spot prices $p \in \mathbb{R}^{(S+1)\times L}$, asset prices $q \in \mathbb{R}^J$, and broker's profit $\pi \in \mathbb{R}$ satisfying :*

for all $i \in \{1, 2, \ldots, I\}$:
i) there does not exist $(y, v) \in B^i(p, q, \pi)$ such that $y \succ^i \overline{x}^i$,
ii) $(\overline{x}^i, \overline{z}^i) \in B^i(p, q, \pi)$
and :
iii) $\sum_{i=1}^I \overline{x}^i = \sum_{i=1}^I w^i$,
iv) $\sum_{i=1}^I \overline{z}^i = 0$,
v) $\pi = c \sum_{i=1}^I |q||\overline{z}^i|$.

Conditions $i)$ to $iv)$ are the usual conditions for utility maximization and market clearing on financial markets. Condition $v)$ expresses that the broker's profit equals the total amount of transaction costs paid by all the agents. It is clear that, if preference relations are strictly monotone, we can apply Walras' law. Therefore, if iv) is satisfied, $S+1$ of the $(S+1)L+1$ market clearing conditions in iii) and v) are redundant, one for each state $s \in \{0,1,\ldots,S\}$. For instance, since the sum of all shares equals 1, i.e. $\sum_{i=1}^{I} \nu_i = 1$, it is clear that $i) \rightarrow iv)$ imply v). Therefore, condition v) is in fact redundant.

As stated before, the aim in this paper is to prove existence of equilibria on a (potentially) incomplete financial market with portfolio constraints as well as transaction costs. Therefore, we need to study the properties of the demand and excess demand correspondences. Before we do so, we introduce some assumptions and notations.

Notations : For a subset $Z \subset \mathbb{R}^n$ for some $n \geq 1$, we will denote $\overline{Z}$ for the closure, ∂Z for the boundary, $\text{int}(Z)$ for the interior of the set Z, and Z^c for its complement, i.e. $Z^c = \mathbb{R}^n \setminus Z$.

For a matrix $A \in \mathbb{R}^{n \times m}$, we will denote $\text{Ker}(A)$ for the null space of the matrix, i.e. $\text{Ker}(A) = \{z \in \mathbb{R}^m \mid Az = 0\}$.

For a vector $q \in \mathbb{R}^J$, and an $\epsilon > 0$, we will denote $B(q,\epsilon)$ for the open ball with center q and radius ϵ.

Furthermore, for a spot price vector $p \in \mathbb{R}^{(S+1)\times L}$, an asset price vector $q \in \mathbb{R}^J$, a consumption bundle $x \in \mathbb{R}^{L\times(S+1)}$, and a trade vector $z \in \mathbb{R}^J$, we will denote

$$(p,q)(x,z) = \sum_{s=0}^{S} p_s x_s + \sum_{j=1}^{J} q_j z_j.$$

We will denote $AS(i)$ for the asymptotic cone of the trade set of agent i, i.e.

$$AS(i) = \{z \in Z^i \mid tz \in Z^i \text{ for all } t \in \mathbb{R}_+\}.$$

Finally, we denote $\mathbf{1}_j^n$ for the j^{th} unit column vector in $\mathbb{R}^n$. If n is clear from the context, we will omit it in the notation.

Assumptions A :

A_1) preference relations $\succeq^i, i \in \{1,2,\ldots,I\}$ are continuous, strictly monotone and convex,
A_2) $w^i >> 0$ for each agent $i \in \{1,2,\ldots,I\}$,
A_3) the trade sets $Z^i, i \in \{1,2,\ldots,I\}$ satisfy
  A_{31}) Z^i is closed and convex,
  A_{32}) $0 \in Z^i$,

A_{33}) for each asset $j \in \{1, 2, \dots, J\}$, there exists an $\epsilon > 0$ and agents $i_1, i_2 \in \{1, 2, \dots, I\}$ such that $\epsilon \mathbf{1}_j \in Z^{i_1}$ and $-\epsilon \mathbf{1}_j \in Z^{i_2}$,

A_4) $\mathrm{Ker}(A) \cap AS(i) = \{0\}$ for all $i \in \{1, 2, \dots, I\}$.

Remarks :

1) It is clear (see for instance [Debreu (1959)]) that under assumption A_1, there exist utility functions $u^i : X^i \to \mathbb{R}_+, i \in \{1, 2, \dots, I\}$, such that $x \succeq^i y \Leftrightarrow u^i(x) \geq u^i(y)$. Now for a set $B \subset \mathbb{R}_+^{(S+1)L} \times \mathbb{R}^J$, we will denote

$$\underset{(x,z)\in B}{\operatorname{argmax}}\, u^i(x) := \{(x, z) \in B \mid \text{ there does not exist } (y, v) \in B : y \succ^i x\}.$$

2) Under assumptions A_{31} and A_{32}, the definition of the asymptotic cone $AS(i)$ is equivalent to the definition of the asymptotic cone of a set as it is given in [Debreu (1959)].
3) Assumption A_4 weakens the assumption of no redundancy ($\mathrm{Ker}(A) = \{0\}$).

4. Demand and excess demand

As usual, the demand and excess demand correspondences are defined as follows :

Definition 4.1 : *For each agent $i \in \{1, 2, \dots, I\}$, we define*

$$\begin{aligned} f^i(p, q, \pi) &= \underset{(x,z)\in B^i(p,q,\pi)}{\operatorname{argmax}} u^i(x) \\ &= \{(x, z) \in B^i(p, q, \pi) \mid \textit{ there is no } (y, v) \in B^i(p, q, \pi) : y \succ^i x\}. \end{aligned}$$

The excess demand relation is defined by :

$$F(p, q, \pi) = \sum_{i=1}^{I} (f^i(p, q, \pi) - (w^i, 0)).$$

In this section, we study the properties of these demand and excess demand relations. As in the case of incomplete financial markets without portfolio constraints or transaction costs, we start by defining the set of asset prices and spot prices for which budget sets are compact. It is clear that this will still be closely related to the non existence of arbitrage opportunities. It is important however to remark that, under the presence of trade sets $Z^i, i \in \{1, 2, \dots, I\}$, one has to make a difference between two kinds of arbitrage possibilities, those that can yield an infinite profit for at least one agent (type one), and those that can only yield a finite profit for each agent (because of their portfolio constraints) (type two). Consider for instance the case where a certain agent has a compact trade set Z^i. Then it is clear that his budget set will be compact whatever the prices of the assets may be. Indeed, since his trade set is bounded, the profit of arbitrage possibilities within his trade set will also be bounded. So it becomes clear that, in order to have compact budget sets, one only has to exclude those arbitrage possibilities that can yield an infinite profit, i.e. those price vectors for which there exists a portfolio in the asymptotic cone of the trade set of at least one agent, yielding a non-negative return in each state and a strictly positive return in at least one state. Therefore, we have the following definition :

Definition 4.2 : *We define the set of no arbitrage asset prices as follows :*

$$Q := \left\{ q \in I\!R^J \;\middle|\; \forall z \in \cup_{i=1}^I AS(i) : \begin{pmatrix} -qz - c|q||z| \\ Az \end{pmatrix} \notin I\!R_+^{S+1} \setminus \{0\} \right\}.$$

Since spot prices have to be positive, we define the price set P as follows

$$P := I\!R_{++}^{(S+1)\times L} \times Q.$$

In order to see the equivalence between the existence of a utility maximizing consumption bundle for each agent and the fact that prices (p, q) are in the set P, we first need some lemmas. It is clear that utility can be maximized as soon as the set

$$\left\{ x \in I\!R_+^{(S+1)L} \mid \exists z \in Z^i : (x, z) \in B^i(p, q, \pi) \right\},$$

is compact. To see that this is equivalent to the compactness of $B^i(p, q, \pi)$ for prices (p, q) in P, we need to study the relationship between x and z for $(x, z) \in B^i(p, q, \pi)$. This is what we do in lemma 4.1 and lemma 4.2.

Lemma 4.1 : *Let $\{(p^n, q^n) \in P : n \in I\!N\}$ be a sequence converging to $(p, q) \in \overline{P}$, and $\{\pi^n : n \in I\!N\}$ a sequence in $I\!R$, and let for each $n \in I\!N$: $(x^n, z^n) \in B^i(p^n, q^n, \pi^n)$. Suppose that assumptions A_{31}, A_{32} and A_4 are satisfied. Then*

$$||z^n|| \to \infty \Rightarrow \begin{cases} \{\dfrac{z^n}{||x^n||} : n \in I\!N\} \text{ is bounded,} \\ \{||x^n|| : n \in I\!N\} \to \infty. \end{cases}$$

If $p >> 0$ and $\{\pi^n : n \in I\!N\}$ is a bounded sequence, then

$$||x^n|| \to \infty \Rightarrow ||z^n|| \to \infty.$$

Proof : Suppose that $\{||z^n|| : n \in \mathbb{N}\} \to \infty$ and suppose that the sequence $\{\frac{x^n}{||z^n||} : n \in \mathbb{N}\}$ is not bounded. Then there is a subsequence $\{n_k : k \in \mathbb{N}\}$ such that

$$\lim_{k\to\infty} \frac{||x^{n_k}||}{||z^{n_k}||} = 0.$$

From the budget constraints, it follows that for all $s \in \{1, 2, \ldots, S\}$ we have :

$$p_s^n \frac{(x_s^n - w_s^i)}{||z^n||} = (A \frac{z^n}{||z^n||})_s.$$

Since $\lim_{n\to\infty} ||z^n|| = +\infty$, we know that $\lim_{n\to\infty} \frac{z^n}{||z^n||} = y \in AS(i) \setminus \{0\}$. This implies that there exists $y \in AS(i), y \neq 0$ satisfying $Ay = 0$, which is in contradiction to assumption A_4. Therefore, we know that the sequence $\{\frac{z^n}{||x^n||} : n \in \mathbb{N}\}$ is bounded. Now it is clear that this also implies that $\{||x^n|| : n \in \mathbb{N}\} \to \infty$.

If $p >> 0$ and $\{\pi^n : n \in \mathbb{N}\}$ is a bounded sequence, then it follows from the budget constraints that

$$||x^n|| \to \infty \Rightarrow ||z^n|| \to \infty.$$

Lemma 4.2 : *Let $\{(p^n, q^n) \in P : n \in \mathbb{N}\}$ be a sequence converging to $(p, q) \in P$, and $\{\pi^n : n \in \mathbb{N}\}$ a bounded sequence in $\mathbb{R}$. Suppose that the sequence $\{(x^n, z^n) \in B^i(p^n, q^n, \pi^n) : n \in \mathbb{N}\}$ satisfies*

$$\lim_{n\to\infty} ||(x^n, z^n)|| = +\infty.$$

Suppose furthermore that assumptions A_{31}, A_{32} and A_4 are satisfied. Then the sequence $\{\frac{||z^n||}{||x^n||} : n \in \mathbb{N}\}$ is bounded, $||z^n|| \to \infty$ and $||x^n|| \to \infty$.

Proof : Since $p >> 0$ and $\{\pi^n : n \in \mathbb{N}\}$ is bounded, we know by lemma 4.1 that $||z^n|| \to \infty$. Then lemma 4.1 gives the desired result.

Lemma 4.3 : *Let $\{(p^n, q^n) \in P : n \in \mathbb{N}\}$ be a sequence converging to $(p, q) \in P$. Let $\{\pi^n : n \in \mathbb{N}\}$ be a bounded sequence in $\mathbb{R}$. Suppose that assumptions A_{31}, A_{32} and A_4 are satisfied. Then for each $i \in \{1, 2, \ldots, I\}$, the set $\bigcup_{n\in\mathbb{N}} B^i(p^n, q^n, \pi^n)$ is bounded.*

Proof: Suppose that there exists an $i \in \{1, 2, \ldots, I\}$ such that the set $\bigcup_{n\in\mathbb{N}} B^i(p^n, q^n, \pi^n)$ is not bounded. Then there is a sequence $\{m_n \in \mathbb{N} : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$,

there exists (x^n, z^n) satisfying $(x^n, z^n) \in B^i(p^{m_n}, q^{m_n}, \pi^{m_n})$ and $\lim_{n\to\infty} ||(x^n, z^n)|| = +\infty$.
Then, since $\lim_{n\to\infty} p^n = p >> 0$, it follows from lemma 4.2 that :

$$\begin{aligned} & ||x^n|| \to \infty, \\ \text{and} \quad & ||z^n|| \to \infty, \\ \text{and} \quad & \{\frac{z^n}{||x^n||} : n \in I\!N\} \text{ is bounded.} \end{aligned}$$

Therefore, there exists a subsequence such that $\lim_{n\to\infty} \frac{z^n}{||x^n||} = y \in AS(i)$. Then it follows from the budget constraints that

$$\lim_{n\to\infty} p_0^n \frac{x_0^n}{||x^n||} \leq \lim_{n\to\infty} \left(p_0^n \frac{w_0^i}{||x^n||} - q^n \frac{z^n}{||x^n||} - c|q^n| \frac{|z^n|}{||x^n||} + \nu_i \frac{\pi^n}{||x^n||} \right),$$
$$\lim_{n\to\infty} p_s^n \frac{x_s^n}{||x^n||} = \lim_{n\to\infty} \left(p_s^n \frac{w_s^i}{||x^n||} + A \frac{z^n}{||x^n||} \right).$$

Therefore, by taking limits for $n \to \infty$, and since $p >> 0$, it follows that there exists $y \in AS(i)$ satisfying :

$$\begin{pmatrix} -qy - c|q||y| \\ Ay \end{pmatrix} \in I\!R_+^{S+1} \setminus \{0\}.$$

This is a contradiction to the fact that $q \in Q$.

We immediately have the following corollary :

Corollary 4.1 : *Let (p, q) be a price vector in P, and $\pi \in I\!R$. Suppose that assumptions A_{31}, A_{32} and A_4 are satisfied. Then for each $i \in \{1, 2, \ldots, I\}$, the budget set $B^i(p, q, \pi)$ is bounded.*

Proof : Trivial consequence of lemma 4.3.

In the following proposition we show that each agent $i \in \{1, 2, \ldots, I\}$ will be able to maximize his utility over his budget set $B^i(p, q, \pi)$ if and only if the spot prices and the asset prices (p, q) are in the set P.

Proposition 4.1 : *Suppose that assumptions A_1, A_{31}, A_{32} and A_4 are satisfied . Then for every $\pi \geq 0$, the following statements are equivalent :*

$$\begin{aligned} & i)\ \forall i \in \{1, 2, \ldots, I\} : B^i(p, q, \pi) \text{ is compact,} \\ & ii)\ (p, q) \in P, \\ & iii)\ \forall i \in \{1, 2, \ldots, I\} : \operatorname*{argmax}_{(x,z)\in B^i(p,q,\pi)} u^i(x) \neq \emptyset. \end{aligned}$$

Proof : It is clear that budget sets are closed. Therefore, *ii*) $\Rightarrow$ *i*) follows directly from corollary 4.1.

i) $\Rightarrow$ *iii*) is trivial since it is clear that for $(p,q) \in P$, and $\pi \geq 0$, the budget set $B^i(p,q,\pi)$ is non-empty, since $(w^i, 0) \in B^i(p,q,\pi)$.

iii) $\Rightarrow$ *ii*) : suppose that $(p,q) \notin P$. Then either there exists $(s,l) \in \{0,1,\ldots,S\} \times \{1,2,\ldots,L\}$ such that $p_{s,l} \leq 0$, and then clearly agents will not be able to maximize their utility, or there exists an agent $i \in \{1,2,\ldots,I\}$ and a portfolio $z \in AS(i)$ such that

$$\begin{pmatrix} -qz - c|q||z| \\ Az \end{pmatrix} \in \mathbb{R}_+^{S+1} \setminus \{0\}.$$

Now let $(\overline{x},\overline{z}) \in B^i(p,q,\pi)$, and $\overline{x} \in \mathrm{argmax}_{(x,z)\in B^i(p,q,\pi)}\, u^i(x)$. Since $z \in AS(i), \overline{z} \in Z^i$, it is well known that $\overline{z} + z \in Z^i$. Then consider the set of those $x \in X^i$, satisfying :

$$\begin{array}{lcr} p_0 x_0 & \leq & p_0 w_0^i - q(\overline{z}+z) - c|q||\overline{z}+z| + \nu_i \pi \\ (px^i)_1 & = & (pw^i)_1 + A(\overline{z}+z), \end{array}$$

i.e., $(x, \overline{z}+z) \in B^i(p,q,\pi)$. Now since $\binom{-qz-c|q||z|}{Az} \in \mathbb{R}_+^{S+1} \setminus \{0\}$, it follows that

$$\begin{aligned} \begin{pmatrix} -q(\overline{z}+z) - c|q||\overline{z}+z| \\ A(\overline{z}+z) \end{pmatrix} &\geq \begin{pmatrix} -qz - c|q||z| \\ Az \end{pmatrix} + \begin{pmatrix} -q\overline{z} - c|q||\overline{z}| \\ A\overline{z} \end{pmatrix} \\ &> \begin{pmatrix} -q\overline{z} - c|q||\overline{z}| \\ A\overline{z} \end{pmatrix}. \end{aligned}$$

This implies that there exists an $(x, z+\overline{z})$ satisfying :

$$\begin{cases} (x, \overline{z}+z) \in B^i(p,q,\pi), \\ x \succ^i \overline{x}, \end{cases}$$

which is clearly a contradiction to the fact that $(\overline{x},\overline{z}) \in f^i(p,q,\pi)$.

Proposition 4.1 implies that demand and excess demand relations are correspondences, i.e. their values are non-empty, on the set $P \times \mathbb{R}_+$. Since we can use Walras' law it is clear that, if we want to prove existence of equilibria (and are not necessarily concerned with the multiplicity of the equilibria), we can introduce $S+1$ price normalizations, i.e. one for each state $s \in \{0,1,\ldots,S\}$. Therefore, we define the "normalized" price set as being the set of those prices in P for which the price of good one equals 1 in each state $s \in \{0,1,\ldots,S\}$. This yields the following definition :

Definition 4.3 : *The normalized price set is defined by :*

$$\mathcal{P} := \{(p,q) \in P \mid \forall s \in \{0,1,\ldots,S\} : p_{s1} = 1\}.$$

In the next proposition, we will study the properties of the demand and excess demand correspondences on the normalized price set $\mathcal{P}$.

Proposition 4.2 : *Under assumptions A, we have :*

1) $B^i(p,q,\pi)$ is a non-empty, closed and convex subset of $X^i \times Z^i$ for every vector $(p,q,\pi) \in \overline{\mathcal{P}} \times I\!R_+$.

2) $f^i(p,q,\pi)$ is a non-empty, convex, compact subset of $X^i \times Z^i$ for every vector $(p,q,\pi) \in \mathcal{P} \times I\!R_+$.

3) $f^i(p,q,\pi)$ has a closed graph on $\overline{\mathcal{P}} \times I\!R_+$, and is upper hemi continuous on $\mathcal{P} \times I\!R_+$.

4) Let $\{(p^n,q^n) : n \in I\!N\} \subset \mathcal{P}$, and $\{\pi^n : n \in I\!N\} \subset I\!R_+$. If

i) $\lim_{n\to\infty} ||p^n|| = +\infty$

or ii) $\lim_{n\to\infty}(p^n,q^n) = (p,q) \in \overline{\mathcal{P}} \setminus \mathcal{P}$ *and* $\{\pi^n : n \in I\!N\} \subset I\!R_+$ *is bounded*

or iii) $\lim_{n\to\infty} ||q^n|| = +\infty$

or iv) $\lim_{n\to\infty} \pi^n = +\infty$ *and* $\{(p^n,q^n) : n \in I\!N\}$ *bounded,*

then all sequences $\{y^n : n \in I\!N\}$ for which for each $n \in I\!N$ there exists a portfolio $z^n \in I\!R^J$ such that $(y^n,z^n) \in F(p^n,q^n,\pi^n)$ are such that the set $\{\sum_{s=0}^{S}\sum_{l=1}^{L} y^n_{s,l} : n \in I\!N\}$ is unbounded above.

Proof :

1) It is clear that for each vector $(p,q,\pi) \in \overline{\mathcal{P}} \times I\!R_+$, the budget set $B^i(p,q,\pi)$ is non-empty and closed. Now suppose that $(x,z) \in B^i(p,q,\pi)$ and $(y,v) \in B^i(p,q,\pi)$. Then

$$\begin{aligned} p_0 x_0 &\le p_0 w_0^i - qz - c|q||z| + \nu^i\pi, \\ p_0 y_0 &\le p_0 w_0^i - qv - c|q||v| + \nu^i\pi, \\ (px)_1 &= (pw^i)_1 + Az, \\ (py)_1 &= (pw^i)_1 + Av. \end{aligned}$$

Therefore, it follows that for all $t \in [0,1]$, we have

$$\begin{aligned} p_0(tx_0 + (1-t)y_0) &\le t\left(p_0 w_0^i - qz - c|q||z| + \nu^i\pi\right) + (1-t)\left(p_0 w_0^i - qv - c|q||v| + \nu^i\pi\right) \\ &\le p_0 w_0^i - q(tz + (1-t)v) - c|q||tz + (1-t)v| + \nu^i\pi \\ (p(tx + (1-t)y))_1 &= (pw^i)_1 + A(tz + (1-t)v). \end{aligned}$$

So clearly

$$(tx + (1-t)y, tz + (1-t)v) \in B^i(p,q,\pi).$$

Therefore, budget sets are convex.

2),3) The proof of 2) and 3) goes along the usual lines if one keeps in mind the price normalization, the assumptions A, and the results of lemma 4.1, lemma 4.2, and lemma 4.3, and proposition 4.1.

4) It is clear that, since consumption sets are bounded from below, it is sufficient to prove that there exists an agent $i \in \{1,2,\dots,I\}$ such that all sequences $\{x^n : n \in I\!N\}$ for which for each $n \in I\!N$ there exists a portfolio $z^n \in I\!R^J$ such that $(x^n, z^n) \in f^i(p^n, q^n, \pi^n)$ are such that the sequence $\{||x^n|| : n \in I\!N\}$ is unbounded above.

4i) We consider the case where there exists a state $s \in \{0,1,\dots,S\}$ and a good $l \in \{2,3,\dots,L\}$ such that $\lim_{n\to\infty} p^n_{s,l} = +\infty$.

Let $i \in \{1,2,\dots,I\}$ be arbitrary but fixed. Suppose that there exists a sequence $\{(x^n,z^n) \in f^i(p^n,q^n,\pi^n) : n \in I\!N\}$ such that the sequence $\{||x^n|| : n \in I\!N\}$ is bounded. Then there is a convergent subsequence $\{x^{n_k} : k \in I\!N\} \to x$.

- Suppose that $x_{s,l} > 0$. Then we define a new sequence as follows :

$$\begin{cases} y^k_{s,l} := x^{n_k}_{s,l} - \dfrac{1}{p^{n_k}_{s,l}} & \\ y^k_{s,1} := x^{n_k}_{s,1} + \dfrac{1}{p^{n_k}_{s,1}} & \\ y^k_{t,v} := x^{n_k}_{t,v} & \text{for all } t \neq s, v \in \{1,2,\dots,L\} \\ y^k_{s,v} := x^{n_k}_{s,v} & \text{for all } v \in \{2,3,\dots,L\} \setminus \{l\}. \end{cases}$$

Clearly $p^{n_k}_s y^k_s = p^{n_k}_s x^{n_k}_s$ for all $k \in I\!N$ and for all $s \in \{0,1,\dots,S\}$. Therefore, since $x_{s,l} > 0$, there exists $k_0 \in I\!N$ such that $y_k \geq 0$ and therefore $(y^k, z^{n_k}) \in B^i(p^{n_k}, q^{n_k}, \pi^{n_k})$ for all $k \geq k_0$. This implies that for all $k \geq k_0$ we have $y^k \preceq^i x^{n_k}$. Therefore also $\lim_{k\to\infty} y^k \preceq^i x$. This is a contradiction since by monotonicity of the preferences, by definition of $\{y^k : k \in I\!N\}$, and since $p^{n_k}_{s,1} = 1$ we know that $\lim_{k\to\infty} y^k \succ^i x$.

- Suppose that $x_{s,l} = 0$. Then we define for each $t \in]0,1]$ the sequence

$$y^{t,k} := (1-t)y^k + tw^i.$$

Then for each $t \in]0,1]$, there exists $k_t \in I\!N$ such that for all $k \geq k_t$ we have $y^{t,k} \geq 0$. Since (y^k, z^{n_k}) and $(w^i, 0)$ are in $B^i(p^{n_k}, q^{n_k}, \pi^{n_k})$ for all $k \geq k_t$, it follows that $(y^{t,k}, (1-t)z^{n_k}) \in B^i(p^{n_k}, q^{n_k}, \pi^{n_k})$ for all $k \geq k_t$. So for each $t \in]0,1]$ we must have $\lim_{k\to\infty} y^{t,k} = y^t \preceq^i \lim_{k\to\infty} x^{n_k} = x$, and therefore $\lim_{t\to 0} y^t \preceq^i x$. This is a contradiction, since clearly $\lim_{t\to 0} y^t > x$.

4ii) Suppose that $\lim_{n\to\infty}(p^n, q^n) = (p,q) \in \overline{\mathcal{P}} \setminus \mathcal{P}$, and there is a subsequence such that $\lim_{n\to\infty} \pi^n = \pi$. By proposition 4.1, we know that there is an agent $i \in \{1, 2, \ldots, I\}$ satisfying $f^i(p,q,\pi) = \emptyset$. Suppose now that there exists a sequence $\{(x^n, z^n) \in f^i(p^n, q^n, \pi^n) : n \in \mathbb{N}\}$ such that $\{x^n : n \in \mathbb{N}\}$ is a bounded sequence. By lemma 4.1, we know that $\{z^n : n \in \mathbb{N}\}$ is also a bounded sequence. Therefore, there exists a convergent subsequence $\lim_{k\to\infty}(x^{n_k}, z^{n_k}) = (x,z)$. Then the contradiction follows from the the fact that f^i has a closed graph on $\overline{\mathcal{P}} \times \mathbb{R}_+$ and the fact that $f^i(p,q,\pi) = \emptyset$.

4iii) Suppose that $\lim_{n\to\infty} ||q^n|| = \infty$. Suppose for instance that there is an asset j with $\lim_{n\to\infty} q_j^n = +\infty$.

Then, by assumption A_2 and assumptions A_3 we know that there exists an $\epsilon > 0$, and an agent $i \in \{1, 2, \ldots, I\}$ such that $\overline{z} = -\epsilon \mathbf{1}_j \in Z^i$. Since Z^i is convex and $0 \in Z^i$, we can always choose ϵ small enough such that also $w^i_{1,s} + (A\overline{z})_s \geq 0$ for all $s \in \{1, 2, \ldots, S\}$. Then we define the following consumption bundles

$$\begin{cases} y^n_{0,1} = p^n_0 w^i_0 - q^n \overline{z} - c|q^n||\overline{z}| + \nu^i \pi^n & \\ y^n_{0,l} = 0 & \forall l \geq 2 \\ y^n_{s,1} = p^n_s w^i_s + (A\overline{z})_s & \\ y^n_{s,l} = 0 & \forall l \geq 2. \end{cases}$$

Clearly, there is a subsequence such that $q^n \overline{z} + c|q^n||\overline{z}| = -\epsilon(1-c)q^n_j$ for all n. Therefore, since $c \leq 1$, $(y^n, \overline{z}) \in B^i(p^n, q^n, \pi^n)$, and $\lim_{n\to\infty} y^n_{0,1} = +\infty$. Suppose that there exists a sequence $\{(x^n, z^n) \in f^i(p^n, q^n, \pi^n) : n \in \mathbb{N}\}$ such that the sequence $\{x^n : n \in \mathbb{N}\}$ is bounded. Then consider the consumption bundles

$$v^n = \frac{1}{y^n_{0,1}} y^n + (1 - \frac{1}{y^n_{0,1}}) x^n.$$

Clearly $(v^n, \frac{1}{y^n_{0,1}}\overline{z} + (1 - \frac{1}{y^n_{0,1}})z^n) \in B^i(p^n, q^n, \pi^n)$. Therefore for each $n \in \mathbb{N}$ we know that $v^n \preceq^i x^n$. But taking limits for $n \to \infty$, this yields a contradiction.

The proof in the case where there exists $j \in \{1, 2, \ldots, J\}$ such that $\lim_{n\to\infty} q^n_j = -\infty$ is analogous.

4iv) Finally, suppose that $\{(p^n, q^n) : n \in \mathbb{N}\}$ is bounded and $\lim_{n\to\infty} \pi^n = +\infty$. Since $\sum_{i=1}^I \nu_i = 1$, we know that there is at least one agent $i \in \{1, 2, \ldots, I\}$

who has a strictly positive share in the broker, i.e. $\nu_i > 0$. Then it follows from the budget constraints that for a sequence $\{(x^n, z^n) \in f^i(p^n, q^n, \pi^n) : n \in \mathbb{N}\}$ it must be the case that $\lim_{n\to\infty} ||(x^n, z^n)|| = +\infty$. Now we know that there exists a convergent subsequence $(p^{n_k}, q^{n_k}) \to (p, q) \in \overline{\mathcal{P}}$. Therefore, applying lemma 4.1 yields that the sequence $\{x^n : n \in \mathbb{N}\}$ is unbounded since it has a subsequence converging to $+\infty$.

5. Existence of equilibria

We will use Kakutani's fixed point theorem to prove existence of equilibria. Therefore, we first construct a compact subset of the price set $\mathcal{P}$ such that equilibrium prices cannot be on the boundary of this set.

Lemma 5.1 : *If assumptions A_{31}, A_{32} and A_4 are satisfied, then Q is an open subset of $\mathbb{R}^J$. Furthermore $Q \supset \{q \in \mathbb{R}^J \mid \exists \pi \in \mathbb{R}^S_{++} : q = \pi A\}$.*

Proof : Suppose that $\{q^n : n \in \mathbb{N}\}$ is a sequence in $\mathbb{R}^J \backslash Q$. Then it is clear that for each $n \in \mathbb{N}$, there exists a portfolio $z^n \in \cup_{i=1}^I AS(i) \setminus \{0\}$, such that $-qz^n - c|q||z^n| \geq 0$, and $Az^n \geq 0$, with at least one strict inequality (by assumption A_4, $Ay = 0$ and $y \in \cup_{i=1}^I AS(i)$ implies that $y = 0$. Therefore, all arbitrage possibilities $y \in \cup_{i=1}^I AS(i)$ are such that $Ay > 0$). Since $z^n \neq 0$, and by definition of $AS(i)$, this implies that

$$\begin{aligned}
&-q\frac{z^n}{||z^n||} - c|q|\frac{|z^n|}{||z^n||} \geq 0,\\
&A\frac{z^n}{||z^n||} \geq 0,\\
&\frac{z^n}{||z^n||} \in \cup_{i=1}^I AS(i) \setminus \{0\}.
\end{aligned}$$

Now there exists a subsequence such that $\lim_{k\to\infty} \frac{z^{n_k}}{||z^{n_k}||} = y, \lim_{k\to\infty} \frac{|z^{n_k}|}{||z^{n_k}||} = |y|$, and

$$\begin{aligned}
&-qy - c|q||y| \geq 0,\\
&Ay \geq 0.
\end{aligned}$$

By assumption A_{31} and by definition of y, it follows that $y \in \cup_{i=1}^I AS(i) \setminus \{0\}$. Then it follows from assumption A_4 that $Ay \neq 0$. So, we can conclude that $q \notin Q$. Therefore, it follows that Q is open.

It is well known that

$$\{q \in \mathbb{R}^J \mid \exists \pi \in \mathbb{R}^S_{++} : q = \pi A\} = \{q \in \mathbb{R}^J \mid \forall z \in \mathbb{R}^J : \begin{pmatrix} -qz \\ Az \end{pmatrix} \notin \mathbb{R}^{S+1}_+ \setminus \{0\}\}.$$

Therefore, it is clear that

$$Q \supset \{q \in \mathbb{R}^J \mid \exists \pi \in \mathbb{R}^S_{++} : q = \pi A\}.$$

It is clear that the normalized price set

$$\mathcal{P} = \left\{(p,q) \in \mathbb{R}^{(S+1)L}_{++} \times Q \mid p_{s,1} = 1 \quad \forall s \in \{0,1,\ldots,S\}\right\},$$

is not an open set because of the normalization. Therefore, we introduce the following notation :

Notation : For each matrix $p \in \mathbb{R}^{(S+1)\times L}$ we denote $p_{|1}$ for the matrix in $\mathbb{R}^{(S+1)\times(L-1)}$ containing $p_{s,l}, s \in \{0,1,\ldots,S\}, l \in \{2,3,\ldots,L\}$.

For each matrix $p = (p_{s,l}, (s,l) \in \{0,1,\ldots,S\} \times \{2,3,\ldots,L\}) \in \mathbb{R}^{(S+1)\times(L-1)}$ we denote p_{+1} for the matrix in $\mathbb{R}^{(S+1)\times L}$ with $p_{s,1} = 1$ for all states $s \in \{0,1,\ldots,S\}$.

For a subset C of $\mathbb{R}^{(S+1)(L-1)}$, we denote

$$C_{+1} := \{p_{+1} \mid p \in C\}.$$

In order to be able to prove existence of equilibria by means of Kakutani's fixed point theorem, we need to compactify the price set in such a way that we know for sure that equilibrium prices have to be in the interior of this compact subset of prices. To construct this compact set, we need a lemma.

Lemma 5.2 : *Let K be a compact subset of the set Q. Let $\tilde{q}$ be a vector in Q. Then there exists a compact subset C of Q such that :*

i) $\tilde{q} \in \operatorname{int}(C)$,
ii) $K \subset \operatorname{int}(C)$.

Proof : Trivial consequence of lemma 5.1.

For technical reasons, we define a special price vector as follows :

Definition 5.1 : *The price vector $(\tilde{p},\tilde{q}) \in \mathcal{P}$ is defined as follows :*

$$\begin{aligned} \tilde{p}_{s,l} &= 1 \qquad && \forall s \in \{0,1,\ldots,S\}, l \in \{1,2,\ldots,L\}, \\ \tilde{q}_j &= \sum_{s=1}^{S} A_{sj} \qquad && \forall j \in \{1,2,\ldots,J\}. \end{aligned}$$

Clearly, $\tilde{p} >> 0$. Furthermore, $\tilde{q} = \pi A$ with $\pi = (1,1,\ldots,1) >> 0$. Therefore, it follows from lemma 5.1 that $\tilde{q} \in Q$. So clearly $(\tilde{p},\tilde{q}) \in \mathcal{P}$.

In the following lemma we construct compact price sets C_p for spot prices and C_q for asset prices, and an interval for the broker's profit π such that we know that equilibrium prices and profits cannot be on the boundary of these sets.

Lemma 5.3 : *There is a non-empty subset C_p of $\mathbb{R}^{(S+1)(L-1)}_{++}$, a non-empty subset C_q of Q, and a real number $K > 0$, such that $C = C_p \times C_q$ is compact, and*

1) for all $q \in \partial C_q \cup (Q \setminus C_q), p \in \mathbb{R}^{(S+1)L}_{++}, \pi \geq 0$, we have

$$(y,z) \in F(p,q,\pi) \Rightarrow \sum_{s=0}^{S}\sum_{l=1}^{L} y_{s,l} > 0,$$

2) for all $p \in \partial C_p \cup (\mathbb{R}^{(S+1)L}_{++} \setminus C_p), \pi \geq 0, q \in Q$, we have

$$(y,z) \in F(p_{+1},q,\pi) \Rightarrow \sum_{s=0}^{S}\sum_{l=1}^{L} y_{s,l} > 0,$$

3) for all $\pi \geq K, p \in \mathbb{R}^{(S+1)L}_{++}, q \in Q$, we have

$$(y,z) \in F(p,q,\pi) \Rightarrow \sum_{s=0}^{S}\sum_{l=1}^{L} y_{s,l} > 0,$$

4) $(\tilde{p}_{|1},\tilde{q}) \in \operatorname{int}(C_p) \times \operatorname{int}(C_q)$.

Proof :

1) We define the following set

$$K_a = \left\{ q \in Q \;\middle|\; \exists \pi \in \mathbb{R}_+, p \in \mathbb{R}^{(S+1)L}_{++} : \exists (y,z) \in F(p,q,\pi) : \sum_{s=0}^{S}\sum_{l=1}^{L} y_{s,l} \leq 0 \right\}.$$

Now suppose that $\{q^n : n \in \mathbb{N}\}$ is a sequence in K_a. Then for all $n \in \mathbb{N}$, there exists a spot price matrix $p^n \in \mathbb{R}^{(S+1)L}_{++}$ and a profit $\pi^n \geq 0$ such that there exists a vector $(y^n,z^n) \in F(p^n,q^n,\pi^n)$ satisfying

$$\sum_{s=0}^{S}\sum_{l=1}^{L} y^n_{s,l} \leq 0.$$

Then it follows from proposition 4.2 that :

$$\begin{cases} \{||(p^n, q^n)|| : n \in I\!N\} \text{ is a bounded sequence (4.2.i),iii)),} \\ \{\pi^n : n \in I\!N\} \text{ is a bounded sequence (4.2.iv)),} \\ \text{if } \lim_{n\to\infty} (p^n, q^n) = (p, q) \text{ then } (p, q) \notin \mathcal{P}(4.2.ii)), \end{cases}$$

Therefore, it follows that there exists a price vector $(p, q) \in I\!R_{++}^{(S+1)L} \times Q$ and a profit $\pi \geq 0$ such that for a subsequence we have $\lim_{n\to\infty}(p^n, q^n, \pi^n) = (p, q, \pi) \in \mathcal{P} \times I\!R_+$. By upper hemi continuity of the excess demand correspondence F on $\mathcal{P}$, it follows that there exists a vector (y, z) such that for a subsequence :

$$\begin{cases} (y, z) = \lim_{k\to\infty} (y^{n_k}, z^{n_k}), \\ (y, z) \in F(p, q, \pi), \\ \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} \leq 0. \end{cases}$$

Therefore, the price vector q is in K_a. This implies that K_a is a compact subset of Q. Then by lemma 5.2, we know that there exists a compact set C_q such that $\tilde{q} \in \text{int}(C_q)$ and $K_a \subset \text{int}(C_q)$.

Now $q \in \partial C_q \cup (Q \setminus C_q)$ implies $q \notin K_a$ and this implies, by definition of K_a, that for all $\pi \geq 0$, $p \in I\!R_{++}^{(S+1)L}$, and for all $(y, z) \in F(p, q, \pi)$ we have :

$$\sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} > 0.$$

Therefore, the set C_q satisfies 1).

2) We now define the set :

$$K_s = \left\{ p \in I\!R_{++}^{(S+1)(L-1)} \;\middle|\; \exists \pi \geq 0, q \in Q : \exists\, (y, z) \in F(p_{+1}, q, \pi) : \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} \leq 0 \right\}.$$

As in 1), it can be shown that K_s is compact. Then we construct a compact set $C_p \subset I\!R_{++}^{(S+1)(L-1)}$ such that $\tilde{p}_{|1} \in C_p$ and $K_s \subset \text{int}(C_p)$.

3) We take the set :

$$K_b = \left\{ \pi \in I\!R_+ \;\middle|\; \exists (p, q) \in I\!R_{++}^{(S+1)L} \times Q : \exists\, (y, z) \in F(p, q, \pi) : \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} \leq 0 \right\}.$$

As in 1), we prove that K_b is a compact set. Now it is clear that

$$K = \max\{\pi \mid \pi \in K_b\} + 1,$$

gives the desired result.

4) is satisfied by construction.

To prove existence we proceed in the traditional way, i.e. we use Kakutani's fixed point theorem and show that this fixed point is an equilibrium. To be able to use Kakutani's fixed point theorem however, we need a convex price set. Since Q will in general (for $c > 0$) not be convex, we cannot simply take the convex hull of C_q to get a convex subset of Q. If there would be an argument that yields that (for assets with positive payoff) excess demand is strictly positive as soon as an asset price q_j is non-positive, we could restrict to the set

$$Q_{++} := \left\{ q \in I\!R^J_{++} \;\middle|\; \forall z \in \cup_{i=1}^{I} AS(i) : \begin{pmatrix} -qz - cq|z| \\ Az \end{pmatrix} \notin I\!R^{S+1}_{+} \setminus \{0\} \right\},$$

which is an open and convex subset of $I\!R^J$.

But, even in the case where asset returns are non negative in each state, i.e. $A_j \in I\!R^S_+ \setminus \{0\}$ for each $j \in \{1, 2, \ldots, J\}$, we cannot in general exclude the possibility that there is no excess demand of the asset if the price is zero. To prove this statement, we give an example of a market with portfolio constraints and no transaction costs ($c = 0$) where assets pay off in positive amounts, and yet, the equilibrium price of one of the assets is strictly negative.

Example : Negative equilibrium prices

We consider an economy with two agents, two assets, one good, two time periods, and two possible states at date one. Spot prices are considered to be equal to one. The matrix of asset returns A is given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The trade set for agent one is given by $Z^1 = \{(z_1, z_2) \in I\!R^2 \mid z_1 + z_2 \le 1\}$. The trade set for agent two is given by $Z^2 = \{(z_1, z_2) \in I\!R^2 \mid z_1 \le 1\}$.

There are no transaction costs ($c = 0$). The initial endowment for both agents is equal to $(w_0^i, w_1^i, w_2^i) = (4, 3, 3), i = 1, 2$. The utility function of agent one equals $u^1(x) =$

$\sqrt{x_0}+\sqrt{x_1}+a\sqrt{x_2}$. The utility function of agent two equals $u^2(x) = b(\sqrt{x_0}+\sqrt{x_1})+\sqrt{x_2}$. So budget sets are given by :

$$B^i(q) = \left\{ (x_0, x_1, x_2) \in \mathbb{R}^3 \;\middle|\; \exists z \in Z^i : \begin{array}{r} x_0 = 4 - qz \\ x_1 = 3 + z_1 \\ x_2 = 3 + z_2 \end{array} \right\}.$$

We see quite easily that for $a = \sqrt{5}(2+\frac{\sqrt{2}}{2})$, and $b = \sqrt{7}$, $q = (-1, 1)$ is an equilibrium price vector with equilibrium allocations $\overline{x}^1 = (1, 2, 5)$, and $\overline{x}^2 = (7, 4, 1)$.

So asset 1 has a negative equilibrium price although its payoff is a positive vector. The explanation of the possibility of a negative equilibrium price in this case is simple. The price of contingent commodity one is strictly negative. Yet, agent one will not buy as much as he is allowed to buy of this contingent commodity because every unit of contingent commodity one that he buys implies that he is allowed to buy one unit of contingent commodity two less, since $z_1 + z_2$ is bounded from above. Therefore, we choose the utility function of agent one such that he prefers contingent commodity two more than contingent commodity one and date zero consumption. To compensate this, we choose the utility function of agent two such that he prefers contingent commodity two less than contingent commodity one and date zero consumption. This turns out to lead to an equilibrium with a negative price for contingent commodity one.

This example clearly shows that, even in the case where asset returns are positive, we cannot a priori say that equilibrium prices will be strictly positive. This will depend on the structure of the trade sets $Z^i, i \in \{1, 2, \ldots, I\}$. This implies that we cannot solve the problem of the non-convexity of the set Q by restricting to positive prices $q \in Q_{++}$, even if we restrict to asset structures $A \in \mathbb{R}_+^{S\times J}$.

Therefore, we compactify the economy such that in the new economy, demand and excess demand are well defined and u.h.c. on the set

$$\tilde{\mathcal{P}} := \{(p, q, \pi) \in \mathbb{R}^{(S+1)L} \times \mathbb{R}^J \times \mathbb{R}_+ \mid p_{s,1} = 1, s \in \{0, 1, \ldots, S\}\},$$

and yield excess demand for each asset price on the boundary of the convex hull of the set C_q. Then we can use Kakutani's fixed point theorem to prove that this compactified economy has an equilibrium which is also an equilibrium for the original economy.

Definition 5.2 :

i) For each agent we define the compact consumption set :

$$\tilde{X}^i = \left\{ x \in X^i \;\middle|\; x^i_{s,l} \leq \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{s=0}^{S} w^i_{s,l} + 1 \right\}.$$

ii) For each agent we define the budget set :

$$\tilde{B}^i(p,q,\pi) = \left\{ (x,z) \in \tilde{X}^i \times Z^i \;\middle|\; \begin{array}{lcc} p_0 x_0 & \leq & p_0 w_0^i - qz - c|q||z| + \nu_i \pi \\ (px)_1 & = & (pw^i)_1 + Az \end{array} \right\}.$$

iii) For each agent we define the correspondences :

$$\tilde{f}^i(p,q,\pi) = \underset{(x,z)\in\tilde{B}^i(p,q,\pi)}{\operatorname{argmax}} \quad u^i(x)$$

and

$$\tilde{F}(p,q,\pi) = \sum_{i=1}^{I} (\tilde{f}^i(p,q,\pi) - (w^i, 0)).$$

Lemma 5.4 : *Suppose that assumptions A are satisfied. Then*

i) the budget sets $\tilde{B}^i(p,q,\pi), i \in \{1,2,\ldots,I\}$ are non-empty, compact and convex for every $(p,q,\pi) \in \tilde{\mathcal{P}}$.

ii) the correspondences $\tilde{f}^i, i \in \{1,2,\ldots,I\}$ and $\tilde{F}$ are upper hemi continuous on $\tilde{\mathcal{P}}$.

Proof : i) It is clear that $(w^i, 0) \in \tilde{B}^i(p,q,\pi)$ for all $i \in \{1,2,\ldots,I\}$. So budget sets are non-empty. Now suppose that $\{(x^n, z^n) : n \in \mathbb{N}\} \subset \tilde{B}^i(p,q,\pi)$. Then $\{x^n : n \in \mathbb{N}\}$ is a bounded sequence and has a converging subsequence. Then it follows from lemma 4.1 that $\{z^n : n \in \mathbb{N}\}$ is also bounded. So there exists a converging subsequence $\{(x^n, z^n) : n \in \mathbb{N}\} \to (x,z)$, and clearly $(x,z) \in \tilde{B}^i(p,q,\pi)$. Therefore $\tilde{B}^i(p,q,\pi)$ is compact. The proof of the convexity goes as in proposition 4.2 1).

The proof of *ii)* goes along the usual lines.

Lemma 5.5 : *There is a non-empty subset $\tilde{C}_p$ of $\mathbb{R}_{++}^{(S+1)(L-1)}$, a non-empty subset $\tilde{C}_q$ of $\mathbb{R}^J$, and a real number $K > 0$, such that $C = \tilde{C}_p \times \tilde{C}_q$ is compact and convex, and satisfies :*

1) for all $q \in \partial\tilde{C}_q \cup Q^c, p \in \mathbb{R}_{++}^{(S+1)L}, \pi \geq 0$, we have

$$(y,z) \in \tilde{F}(p,q,\pi) \Rightarrow \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} > 0,$$

2) for all $p \in \partial\tilde{C}_p, \pi \geq 0, q \in \mathbb{R}^J$, we have

$$(y,z) \in \tilde{F}(p_{+1},q,\pi) \Rightarrow \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} > 0,$$

3) for all $\pi \geq K, p \in \mathbb{R}^{(S+1)L}_{++}, q \in \mathbb{R}^J$*, we have*

$$(y, z) \in \tilde{F}(p, q, \pi) \Rightarrow \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} > 0,$$

4) $(\tilde{p}_{|1}, \tilde{q}) \in \mathrm{int}(\tilde{C}_p) \times \mathrm{int}(\tilde{C}_q)$.

Proof : Let C_p, C_q and $K \in \mathbb{R}$ be as constructed in lemma 5.3. Then we denote $\tilde{C}_q$ for the convex hull of C_q, and $\tilde{C}_p$ for the convex hull of C_p.

1) Let $q \in \partial\tilde{C}_q \cup Q^c, p \in \mathbb{R}^{(S+1)L}_{++}, \pi \geq 0$ and suppose that there exists $(x^i, z^i) \in \tilde{f}^i(p, q, \pi)$ for each agent $i \in \{1, 2, \ldots, I\}$, such that

$$\sum_{s=0}^{S} \sum_{l=1}^{L} \sum_{i=1}^{I} (x^i_{s,l} - w^i_{s,l}) \leq 0.$$

Then by definition of $\tilde{X}^i$, it must be the case that for all agents $i \in \{1, 2, \ldots, I\}$, we have $x^i_{s,l} < \sum_{i=1}^{I} \sum_{s=0}^{S} \sum_{l=1}^{L} w^i_{s,l} + 1$ for $s \in \{0, 1, \ldots, S\}, l \in \{1, 2, \ldots, L\}$. Therefore, the strong convexity of the utility functions yields $(x^i, z^i) \in f^i(p, q, \pi)$ for all agents $i \in \{1, 2, \ldots, I\}$.

But this is a contradiction because by lemma 5.3, if $q \in \partial\tilde{C}_q \cup Q^c$, then either $q \in \partial C_q \cup (Q \setminus C_q)$ and this would imply that $\sum_{s=0}^{S} \sum_{l=1}^{L} \sum_{i=1}^{I} (x^i_{s,l} - w^i_{s,l}) > 0$, or $q \notin Q$ and then there exists an agent such that $f^i(p, q, \pi) = \emptyset$.

2),3) If $q \notin Q$, then 1) yields the desired result. Otherwise, we can again apply lemma 5.3.

4) Trivial.

Theorem 5.2 : *Under assumptions A, there exists a financial market equilibrium with portfolio constraints and transaction costs, i.e. there exist prices and profit* $(\overline{p}, \overline{q}, \overline{\pi}) \in \mathbb{R}^{(S+1)L}_{++} \times \mathbb{R}^J \times \mathbb{R}_+$ *such that*

i) $\overline{x}^i \in \mathrm{argmax}_{(x,z) \in B^i(\overline{p}, \overline{q}, \overline{\pi})} u^i(x)$ *for all* $i \in \{1, 2, \ldots, I\}$,
ii) $(\overline{x}^i, \overline{z}^i) \in B^i(\overline{p}, \overline{q}, \overline{\pi})$ *for all* $i \in \{1, 2, \ldots, I\}$,
ii) $\sum_{i=1}^{I} \overline{x}^i = \sum_{i=1}^{I} w^i$,
iii) $\sum_{i=1}^{I} \overline{z}^i = 0$,
iv) $\overline{\pi} = c \sum_{i=1}^{I} |\overline{q}||\overline{z}^i|$.

Proof : We define the following correspondence :

$$h(p, q, \pi) = \prod_{i=1}^{I} \left(\tilde{f}^i(p, q, \pi) - (w^i, 0) \right).$$

Let $\tilde{C}_p, \tilde{C}_q$ and K be as constructed in lemma 5.5. We denote $C = \tilde{C}_p \times \tilde{C}_q$. Since the demand correspondences $\tilde{f}^i$ are u.h.c. on C_{+1}, we know that there exists a compact, convex set B such that $h(C_{+1} \times [0, K]) \subset B$. We define the correspondence

$$\phi : C_{+1} \times B \times [0, K] \to C_{+1} \times B \times [0, K],$$

with

$$\phi(p, q, ((y, z)), \pi) = \operatorname*{argmax}_{(p',q') \in C_{+1}} ((p', q')(\sum_{i=1}^{I} y^i, \sum_{i=1}^{I} z^i)) \times h(p, q, \pi) \times \{\min\{c|q| \sum_{i=1}^{I} |z^i|, K\}\},$$

where $((y, z))$ denotes the vector $((y^1, z^1), (y^2, z^2), \ldots, (y^I, z^I))$.

This is clearly a non-empty, convex valued correspondence which has a closed graph. Therefore we know by Kakutani's fixed point theorem that there is a fixed point $(\overline{p}, \overline{q}, ((\overline{y}, \overline{z})), \overline{\pi})$. Clearly, this fixed point satisfies :

$$(\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) \in \tilde{F}(\overline{p}, \overline{q}, \overline{\pi}),$$

$$\text{and} \qquad ((p, q) - (\overline{p}, \overline{q}))(\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i)) \leq 0 \qquad \forall (p, q) \in C_{+1},$$

$$\text{and} \qquad \overline{\pi} = \min\{c|\overline{q}| \sum_{i=1}^{I} |\overline{z}^i|, K\}.$$

Furthermore, it is straightforward to see that for $(y, z) = (\sum_{i=1}^{I} y^i, \sum_{i=1}^{I} z^i) \in \tilde{F}(p, q, \pi)$, one has

$$((\tilde{p}, \tilde{q}) - (p, q))(y, z) \geq \sum_{s=0}^{S} \sum_{l=1}^{L} y_{s,l} + c|q| \sum_{i=1}^{I} |z^i| - \pi,$$

where $(\tilde{p}, \tilde{q})$ denotes the special vector defined in definition 5.1. Now suppose that $\overline{\pi} = K < c|\overline{q}| \sum_{i=1}^{I} |\overline{z}^i|$. Then by lemma 5.5 3), it follows that :

$$\begin{aligned} ((\tilde{p}, \tilde{q}) - (\overline{p}, \overline{q}))(\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) &> \sum_{i=1}^{I} \sum_{s=0}^{S} \sum_{l=1}^{L} \overline{y}^i_{s,l} \\ &> 0. \end{aligned}$$

Since $(\tilde{p}, \tilde{q}) \in C_{+1}$, this is contradictory to the fixpoint conditions. Therefore, we know that $\overline{\pi} = c\overline{q} \sum_{i=1}^{I} |\overline{z}^i|$. So it follows that

$$\sum_{i=1}^{I} \sum_{s=0}^{S} \sum_{l=1}^{L} \overline{y}^i_{s,l} \leq ((\tilde{p}, \tilde{q}) - (\overline{p}, \overline{q}))(\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) \leq 0.$$

Therefore, by lemma 5.5 2) we know that $\overline{p}_{|1} \notin \partial \tilde{C}_p$.

Now let $(s,l) \in \{0,1,\ldots,S\} \times \{2,3,\ldots,L\}$ be arbitrary but fixed. Since $\overline{p}_{|1}$ is in the interior of $\tilde{C}_p$, we can choose $\epsilon > 0$ small enough such that the vector $(p,q) = (\overline{p} + \epsilon \mathbf{1}_{(s,l)}, \overline{q})$ is again in the set C_{+1}. This leads to the conclusion that for each state $s \in \{0,1,\ldots,S\}$ and for each $l \in \{2,3,\ldots,L\}$, we have $\sum_{i=1}^{I} \overline{y}^i_{s,l} \leq 0$. Repeating this same argument with $-\epsilon$ instead of ϵ gives us that for all states $s \in \{0,1,\ldots,S\}$, and for all $l \in \{2,3,\ldots,L\}$, we have $\sum_{i=1}^{I} \overline{y}^i_{s,l} = 0$.

Analogously, since

$$\sum_{i=1}^{I} \sum_{s=0}^{S} \sum_{l=1}^{L} \overline{y}^i_{s,l} \leq ((\tilde{p},\tilde{q}) - (\overline{p},\overline{q}))(\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) \leq 0,$$

it follows from lemma 5.5 1) that $\overline{q} \notin \partial \tilde{C}_q$. Therefore we can find $\epsilon > 0$ such that for each asset $j \in \{1,2,\ldots,J\}$, the vectors $(p,q) = (\overline{p}, \overline{q} \pm \epsilon \mathbf{1}_j), j \in \{1,2,\ldots,J\}$ are again in C_{+1}. Therefore, we can conclude that $\sum_{i=1}^{I} \overline{z}^i_j = 0$ for each asset $j \in \{1,2,\ldots,J\}$.

Introducing these conclusions into Walras' Law, we get

$$\begin{cases} \overline{\pi} = c|\overline{q}| \sum_{i=1}^{I} |\overline{z}^i|, & \\ \sum_{i=1}^{I} \overline{z}^i = 0, & \\ \sum_{i=1}^{I} \overline{y}^i_{s,l} = 0 & s \in \{0,1,\ldots,S\}, l \in \{2,3,\ldots,L\}, \\ \overline{p}_{0,1} \sum_{i=1}^{I} \overline{y}^i_{0,1} \leq 0, & \\ \overline{p}_{s,1} \sum_{i=1}^{I} \overline{y}^i_{s,1} = 0 & s \in \{1,2,\ldots,S\}. \end{cases}$$

Now since we know that $\overline{p}_{s,1} = 1$ for all $s \in \{0,1,\ldots,S\}$, this implies that $\sum_{i=1}^{I} \overline{y}^i_{0,1} \leq 0$ and $\sum_{i=1}^{I} \overline{y}^i_{s,1} = 0$ for all $s \in \{1,2,\ldots,S\}$. Therefore, it follows that $\overline{x}^i_{s,l} := \overline{y}^i_{s,l} + w^i_{s,l} < \sum_{s=0}^{S} \sum_{l=1}^{L} \sum_{i=1}^{I} w^i_{s,l} + 1$, for all $s \in \{0,1,\ldots,S\}, l \in \{1,2,\ldots,L\}$, and therefore :

$$(\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) \in F(\overline{p}, \overline{q}, \overline{\pi}).$$

This implies that the date zero budget constraint is satisfied in equality, so we can conclude that :

$$\begin{cases} (\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) \in F(\overline{p}, \overline{q}, \overline{\pi}), \\ (\sum_{i=1}^{I} \overline{y}^i, \sum_{i=1}^{I} \overline{z}^i) = (0, 0). \end{cases}$$

6. Concluding remarks

In this paper, we study a general equilibrium model for potentially incomplete financial markets with two types of constraints on the trade : portfolio constraints and transaction costs. We prove that equilibria exist under some rather general assumptions on the mathematical structure of the trade sets of the agents. Furthermore, we show that there are essentially two types of arbitrage possibilities, from which only one of these types is a priori incompatible with equilibrium. By giving an example, we show that equilibrium prices exist which allow for the second type of arbitrage possibilities.

As mentioned before, the results in this paper can easily be generalized to the case where :

i) there is more than one broker,
ii) there are different classes of assets, and each class has it's own commission c.

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