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THE SPLIT CORE FOR SEQUENCING GAMES

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Abstract

The split core is a refinement of the core for sequencing games. The split core arises from a generalization of the Equal Gain Splitting (EGS) rule that is introduced by *Curiel, Pederzoli* and *Tijs (1989)*. It is pointed out that the split core is the convex hull of permutation based gain splitting allocations and the EGS allocation is in the barycenter of the split core. Finally, an axiomatic characterization of the split core is provided.

KEYWORDS: Sequencing games, EGS-rule, Scheduling, Split core.

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1 Introduction

In one-machine sequencing situations each agent (player) has one job that has to be processed on a single machine. Each job is specified by its processing time, the time the machine takes to handle the job. We assume that the cost of a player depends linearly on the completion time of his job. Furthermore, there is an initial order on the jobs of the agents before the processing of the machine starts.

Each group of agents (coalition) is allowed to obtain cost savings by rearranging their jobs in a way that is admissible with respect to the initial order. An optimal order of a coalition is an admissible rearrangement that maximizes the cost savings of this coalition. By defining the worth of a coalition as the (maximum) cost savings a coalition can make by an optimal rearrangement, we obtain a cooperative sequencing game, related to the one machine sequencing situation. This game theoretic approach has been taken in *Curiel, Pederzoli* and *Tijs (1989)*. They introduced the equal gain splitting (EGS) rule on the class of sequencing situations. The EGS rule is based on the fact that the optimal order of the grand coalition can be obtained from the initial order by consecutive switching of neighbours. According to the EGS rule each agent obtains half of the gains of all neighbour switches he is actually involved in to reach an optimal order. Note that the EGS rule is independent of the chosen optimal order and that the gain of a neighbour switch is independent of the position of the neighbours in the queue. It was shown that each EGS allocation is in the core of the corresponding sequencing game. Further, an axiomatic characterization of the EGS rule was provided. *Curiel, Potters, Rajendra Prassad, Tijs* and *Veltman (1993)* showed that the EGS allocation is the average of two marginal vectors of the corresponding sequencing game. *Curiel, Hamers, Potters* and *Tijs(1993)* presented an alternative characterization of the EGS rule. Moreover, they introduced the head-tail core for sequencing games and showed that the corresponding EGS allocation is in the barycenter of this core. They also showed that the EGS rule can be regarded as a general nucleolus (*Maschler, Potters* and *Tijs (1992)*).

This paper considers a generalization of the EGS rule. We study division rules for sequencing situations where each player obtains an arbitrary non-negative part of the gains of all neighbour switches he is actually involved in to reach the optimal order. The union of all corresponding allocations is called the split core. Obviously, the EGS allocation is an element of the split core. It is shown that the split core of a sequencing situation is a subset of the core of the corresponding sequencing game. Further, it is shown that the split core is the convex hull of so-called permutation based gain splitting allocations and that the corresponding EGS allocation is the average of these vectors. Finally, it is shown that the split core is the largest set-valued solution concept satisfying efficiency, the dummy property and a monotonicity condition.

2 Sequencing games

This section recalls the definitions of a sequencing game and the EGS rule.

In a one machine sequencing situation there is a queue of agents, each with one job, to be processed by one machine. The finite set of agents is denoted by $N = \{1, ..., n\}$. The position of the agents in the queue is described by a bijection $\sigma : N \to \{1, ..., n\}$. Specifically, $\sigma(i) = j$ means that player i is in position j. We assume that there is an initial order $\sigma_0 : N \to \{1, ..., n\}$ on the jobs of the players before the processing of the machine starts. The processing time p_i of the job of player i is the time the machine takes to handle this job. Further, it is assumed that every agent has an affine cost function $c_i : [0, \infty) \to \mathbf{R}$ defined by $c_i(t) = \alpha_i t + \beta_i$ with $\alpha_i > 0, \beta_i \in \mathbf{R}$. So $c_i(t)$ is the cost for agent i if he has t units of time in the system.

A sequencing situation as described above is denoted by (N, σ_0, p, α), where $N = \{1, ..., n\}, \sigma_0 : N \to \{1, ..., n\}, p = (p_i)_{i \in N} \in (0, \infty)^n$ and $\alpha = (\alpha_i)_{i \in N} \in (0, \infty)^n$. The vector $\beta = (\beta_i)_{i \in N} \in \mathbf{R}^n$ is omitted in the description of the sequencing situation since the fixed costs it represents are independent of the positions of the players in the queue.

The set of predecessors (followers) of a player $i \in N$ w.r.t. a rearrangement σ is defined by $P(\sigma, i) = \{j \mid \sigma(j) < \sigma(i)\}$ $(F(\sigma, i) = \{j \mid \sigma(j) > \sigma(i)\})$.

If the processing order is given by $\sigma : N \to \{1, ..., n\}$ then the completion time of player i is equal to $C(\sigma, i) = \sum_{j \in P(\sigma, i)} p_j + p_i$. The total costs $c_S(\sigma)$ of a coalition $S \subset N$, is given by $c_S(\sigma) = \sum_{i \in S} \alpha_i(C(\sigma, i)) + \beta_i$.

The (maximal) cost savings of a coalition S depend on the set of admissible rearrangements of this coalition. A bijection $\sigma : N \to \{1, ..., n\}$ is called admissible for S if $P(\sigma_0, i) = P(\sigma, i)$ for all $i \in N \backslash S$. This implies that the completion time in σ of each player outside the coalition S is equal to his completion time in the initial order. Moreover, players of S are not allowed to jump over players outside S. The set of admissible rearrangements for a coalition S is denoted by Σ_S.

A *cooperative game* is a pair (N, v) where N is a finite set of players and v is a mapping $v : 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$ and 2^N denotes the collection of all subsets of N.
A game (N, v) is called *convex* if for all coalitions $S, T \in 2^N$ and all $i \in N$ with $S \subset T \subset N \backslash \{i\}$ it holds that

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Cooperative game theory focuses on 'fair' and/or 'stable' division rules for the worth $v(N)$ of the grand coalition. A core element $x = (x_i)_{i \in N} \in \mathbf{R^N}$ is such that no coalition has an incentive to split off, i.e.

$$\sum_{i \in N} x_i = v(N) \quad \text{and} \quad x(S) \geq v(S) \text{ for all } S \in 2^N.$$

where $x(S) = \sum_{i \in S} x_i$. The core $C(v)$ consists of all core elements. A game is called balanced if its core is non-empty.
Let (N, v) be a game and let Π_N be the set of all permutations of N. Then the $k - th$ coordinate of the marginal vector $m^\pi(v), \pi \in \Pi_N$, is defined by

$$m_k^\pi(v) = v(\{j \mid \pi(j) \leq \pi(k)\}) - v(\{j \mid \pi(j) < \pi(k)\}).$$

Shapley (1971) and *Ichiishi (1980)* showed that the marginal vectors are the extreme points of the core if and only if the game is convex. Since the core is a convex set we have that the core of a convex game is the convex hull of its marginals. Obviously, a convex game is balanced.

Given a sequencing situation (N, σ_0, p, α) the worth of a coalition S of the corresponding sequencing game *(Curiel et al.(1989))* is defined as the maximal cost savings the coalition can achieve by means of an admissible rearrangement. Formally,

$$v(S) = \max_{\sigma \in \Sigma_S} \{\sum_{i \in S} (\alpha_i C(\sigma_0, i) + \beta_i) - \sum_{i \in S} (\alpha_i C(\sigma, i) + \beta_i)\} \tag{1}$$

A set S is called connected if for all $i, j \in S$ and $k \in N$ such that $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ it holds that $k \in S$. *Curiel et al. (1989)* showed that expression (1) for any connected coalition S is equivalent to

$$v(S) = \sum_{i,j \in S: \sigma_0(i) < \sigma_0(j)} g_{ij},$$

where $g_{ij} := \max\{\alpha_j p_i - \alpha_i p_j, 0\}$ represents the gain attainable for player i and j in case player i is directly in front of player j. For a coalition T that is not connected it follows that

$$v(T) = \sum_{S \in T \backslash \sigma_0} v(S) \tag{2}$$

where $T\backslash\sigma_0$ is the set of maximally connected components of T.

The Equal Gain Splitting (EGS) rule of a sequencing situation (N, σ_0, p, α) is defined by

$$EGS_i(N, \sigma_0, p, \alpha) = \frac{1}{2} \sum_{j: \sigma_0(j) > \sigma_0(i)} g_{ij} + \frac{1}{2} \sum_{k: \sigma_0(k) < \sigma_0(i)} g_{ki}$$

for all $i \in N$. We note that the optimal order of a queue can be obtained from the initial order by consecutive switches of neighbours i, j with $g_{ij} > 0$ (*cf. Smith (1956)*). In the EGS rule a player obtains half of the gains of all neighbour switches he is actually involved in. *Curiel et al. (1989)* showed that sequencing games are convex games and that the EGS rule assigns to each sequencing situation an allocation that is in the core of the corresponding sequencing game.

Example 1 Let $N = \{1, 2, 3\}, \sigma_0(i) = i$ for all $i \in N, p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. It follows that $g_{12} = g_{23} = 4$ and $g_{13} = 6$. Then $EGS_1(N, \sigma_0, p, \alpha) = \frac{1}{2}(4 + 6) = 5, EGS_2(N, \sigma_0, p, \alpha) = \frac{1}{2}(4 + 4) = 4$ and $EGS_3(N, \sigma_0, p, \alpha) = \frac{1}{2}(6 + 4) = 5$.

3 The split core

This section introduces the split core corresponding to a sequencing situation. It is shown that the core of a sequencing game contains the split core of the corresponding sequencing situation as a subset. Further, we describe the extreme points of the split core by introducing permutation based gain splitting allocations. It is shown that the EGS allocation is in the barycenter of the corresponding split core. Finally, the split core is axiomatically characterized by efficiency, the dummy property and a form of monotonicity.

Generalizing the EGS rule we consider gain splitting (GS) rules in which each player obtains a non-negative part of the gain of all neighbour switches he is actually involved in to reach the optimal order. The total gain of a neighbour switch is divided among both players that are involved. Formally, we define for all $i \in N$ and all $\lambda \in \Lambda$

$$GS_i^\lambda(N,\sigma_0,p,\alpha) = \sum_{j:\sigma_0(i)<\sigma_0(j)} \lambda_{ij} g_{ij} + \sum_{k:\sigma_0(k)<\sigma_0(i)} (1-\lambda_{ki}) g_{ki}.$$

where $\Lambda = \{\{\lambda_{ij}\}_{i,j\in N, i\neq j} \mid 0 \leq \lambda_{ij} \leq 1\}$. Note that for each $\lambda \in \Lambda$ we possibly obtain another allocation. Moreover, $GS^\lambda(N,\sigma_0,p,\alpha) = EGS(N,\sigma_0,p,\alpha)$ in case $\lambda_{ij} = \frac{1}{2}$ for all $i,j \in \{1,...,n\}, i \neq j$.

Example 2 If we take $\lambda_{12} = \frac{3}{4}, \lambda_{13} = \frac{1}{3}$ and $\lambda_{23} = 1$ in the game of example 1, then $GS_1^\lambda(N,\sigma_0,p,\alpha) = 5, GS_2^\lambda(N,\sigma_0,p,\alpha) = 5$ and $GS_3^\lambda(N,\sigma_0,p,\alpha) = 4$.

The split core of a sequencing situation (N,σ_0,p,α) is defined by

$$SPC(N,\sigma_0,p,\alpha) = \{GS^\lambda(N,\sigma_0,p,\alpha) \mid \lambda \in \Lambda\}.$$

First it is shown that the split core is a subset of the core.

Theorem 1 *Let (N,σ_0,p,α) be a sequencing situation and let (N,v) be the corresponding sequencing game. Then $SPC(N,\sigma_0,p,\alpha) \subset C(v)$.*

PROOF: Let $\lambda \in \Lambda$ and let S be a connected set. Then

$$\begin{aligned}\sum_{i\in S} GS_i^\lambda(N,\sigma_0,p,\alpha) &= \sum_{i\in S}\Big[\sum_{j:\sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij} + \sum_{k:\sigma_0(k)<\sigma_0(i)} g_{ki}(1-\lambda_{ki})\Big] \\ &\geq \sum_{i\in S}\Big[\sum_{j\in S:\sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij} + \sum_{k\in S:\sigma_0(k)<\sigma_0(i)} g_{ki}(1-\lambda_{ki})\Big] \\ &= \sum_{i,j\in S:\sigma_0(i)<\sigma_0(j)} g_{ij} = v(S).\end{aligned}$$

In case $S = N$ the inequality becomes an equality. Hence, $GS^\lambda(N,\sigma_0,p,\alpha) \in C(v)$. □

For describing the extreme points of the split core we assign to each permutation $\tau \in \Pi_N$ a vector $\lambda(\tau) \in \Lambda$ in the following way. For all $i,j \in \{1,...,n\}, i \neq j$

$$\lambda_{ij}(\tau) = \begin{cases} 0 & \text{if } \tau(i) < \tau(j) \\ 1 & \text{if } \tau(i) > \tau(j) \end{cases} \tag{3}$$

Then for each sequencing situation (N,σ_0,p,α) the collection of *permutation based gain splitting* allocations is defined by

$$PBGS(N,\sigma_0,p,\alpha) = \{GS^{\lambda(\tau)}(N,\sigma_0,p,\alpha) \mid \tau \in \Pi_N\}.$$

Let (N,σ_0,p,α) be a sequencing situation. Then the corresponding *switching game* (N,w) is defined by

$$w(S) = \sum_{i,j\in S:\sigma_0(i)<\sigma_0(j)} g_{ij} \text{ for all } S \subset N.$$

With (N,v) the corresponding sequencing game we have that $w(S) \geq v(S)$ if S is disconnected and that $w(S) = v(S)$ if S is a connected. Hence, $C(w) \subset C(v)$. Further, it is easy to verify that each switching game is a convex game.

Lemma 1 *Let (N,σ_0,p,α) be a sequencing situation and (N,w) be the corresponding switching game. Then $m^\tau(w) = GS^{\lambda(\tau)}(N,\sigma_0,p,\alpha)$ for each $\tau \in \Pi_N$.*

PROOF: Let $i \in N$. Then

$$GS_i^{\lambda(\tau)}(N,\sigma_0,p,\alpha) = \sum_{j:\sigma_0(i)<\sigma_0(j)} g_{ij}\lambda_{ij}(\tau) + \sum_{k:\sigma_0(k)<\sigma_0(i)} g_{ki}(1-\lambda_{ki}(\tau))$$

$$= \sum_{j\in P(\tau,i)\cap F(\sigma_0,i)} g_{ij}\lambda_{ij}(\tau) + \sum_{j\in F(\tau,i)\cap F(\sigma_0,i)} g_{ij}\lambda_{ij}(\tau)$$

$$+ \sum_{k\in P(\tau,i)\cap P(\sigma_0,i)} g_{ki}(1-\lambda_{ki}(\sigma)) + \sum_{k\in F(\tau,i)\cap P(\sigma_0,i)} g_{ki}(1-\lambda_{ki}(\sigma))$$

$$= \sum_{j\in P(\tau,i)\cap F(\sigma_0,i)} g_{ij} + \sum_{k\in P(\tau,i)\cap P(\sigma_0,i)} g_{ki}$$

$$= \sum_{k,l\in P(\tau,i)\cup\{i\}:\sigma_0(k)<\sigma_0(l)} g_{kl} - \sum_{k,l\in P(\tau,i):\sigma_0(k)<\sigma_0(l)} g_{kl}$$

$$= v(P(\tau,i)\cup\{i\}) - v(P(\tau,i)) = m_i^\tau(w)$$

where the third equality follows from (3). □

The following theorem shows that the the split core is the convex hull of all corresponding permutation based gain allocations.

Theorem 2 *Let (N,σ_0,p,α) be a sequencing situation.*
Then $SPC(N,\sigma_0,p,\alpha) = conv\{GS^{\lambda(\tau)}(N,\sigma_0,p,\alpha) \mid \tau \in \Pi_N\}$.

PROOF: Let (N,w) be the switching game corresponding to (N,σ_0,p,α). Since (N,w) is a convex game we have that $C(w) = conv\{m^\tau(w) \mid \tau \in \Pi_N\}$. Lemma 1 implies that $conv\{m^\tau(w) \mid \tau \in \Pi_N\} = conv\{GS^{\lambda(\tau)}(N,\sigma_0,p,\alpha) \mid \tau \in \Pi_N\}$. Since $SPC(N,\sigma_0,p,\alpha)$ is a convex set we have $C(w) \subset SPC(N,\sigma_0,p,\alpha)$.

On the other hand, let $\lambda \in \Lambda$ and let $S \subset N$. Then

$$\sum_{i \in S} GS_i^{\lambda}(N, \sigma_0, p, \alpha) \geq \sum_{i \in S}[\sum_{j \in S: \sigma_0(i) < \sigma_0(j)} g_{ij}\lambda_{ij} + \sum_{k \in S: \sigma_0(k) < \sigma_0(i)} g_{ki}(1 - \lambda_{ki})]$$

$$= \sum_{i,j \in S: \sigma_0(i) < \sigma_0(j)} g_{ij} = w(S).$$

In case $S = N$ the inequality in the above calculation becomes an equality. Hence, $SPC(N, \sigma_0, p, \alpha) \subset C(w)$. □

The next corollary follows from the proof of theorem 1.

Corollary 1 *Let (N, σ_0, p, α) be a sequencing situation. Then $PBGS(N, \sigma_0, p, \alpha)$ is the set of all extreme points of $SPC(N, \sigma_0, p, \alpha)$.*

The following theorem shows that the EGS allocation of a sequencing situation is the average of all corresponding permutation based gain splitting allocations.

Theorem 3 *Let(N, σ_0, p, α) be a sequencing situation. Then*

$$EGS(N, \sigma_0, p, \alpha) = \frac{1}{n!} \sum_{\tau \in \Pi_N} GS^{\lambda(\tau)}(N, \sigma_0, p, \alpha).$$

PROOF: For each $\tau \in \Pi_N$ there exists a unique $\tau^c \in \Pi_N$ such that $\lambda(\tau) + \lambda(\tau^c) = \lambda(e)$ where $\lambda(e) \in \Lambda$ with $\lambda(e)_{ij} = 1$ for all $i, j \in N, i \neq j$. Note that the definition of $\lambda(\tau)$ implies that $\{\tau \mid \tau \in \Pi_N\} = \{\tau^c \mid \tau \in \Pi_N\}$. Since $GS^{\lambda(\tau)}(N, \sigma_0, p, \alpha) + GS^{\lambda(\tau^c)}(N, \sigma_0, p, \alpha) = GS^{\lambda(e)}(N, \sigma_0, p, \alpha) = 2EGS(N, \sigma_0, p, \alpha)$ we have that

$$\frac{1}{n!} \sum_{\tau \in \Pi_N} GS^{\lambda(\tau)}(N, \sigma_0, p, \alpha)$$

$$= \frac{1}{n!} \sum_{\tau \in \Pi_N} (\frac{1}{2} GS^{\lambda(\tau)}(N, \sigma_0, p, \alpha) + \frac{1}{2} GS^{\lambda(\tau^c)}(N, \sigma_0, p, \alpha))$$

$$= \frac{1}{n!} \sum_{\tau \in \Pi_N} EGS(N, \sigma_0, p, \alpha) = EGS(N, \sigma_0, p, \alpha).$$

□

Example 3 Let $N = \{1, 2, 3\}, \sigma_0(i) = i$ for all $i \in N, p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. Then $g_{12} = g_{23} = 4$ and $g_{13} = 6$ and the corresponding sequencing game is given by $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1, 2\}) = v(\{2, 3\}) = 4, v(\{1, 3\}) = 0$ and $v(\{1, 2, 3\}) = 14$. The extreme points of $SPC(N, \sigma_0, p, \alpha)$ are

$$GS^{\lambda(\tau_0)}(N, \sigma_0, p, \alpha) = (0, 4, 10) \quad \text{where } \tau_0 = (1, 2, 3)$$

$$GS^{\lambda(\tau_1)}(N, \sigma_0, p, \alpha) = (0, 8, 6) \quad \text{where } \tau_1 = (1, 3, 2)$$

$$GS^{\lambda(\tau_2)}(N,\sigma_0,p,\alpha) = \quad (4,0,10) \quad \text{where } \tau_2 = (2,1,3)$$
$$GS^{\lambda(\tau_3)}(N,\sigma_0,p,\alpha) = \quad (10,0,4) \quad \text{where } \tau_3 = (2,3,1)$$
$$GS^{\lambda(\tau_4)}(N,\sigma_0,p,\alpha) = \quad (6,8,0) \quad \text{where } \tau_4 = (3,1,2)$$
$$GS^{\lambda(\tau_5)}(N,\sigma_0,p,\alpha) = \quad (10,4,0) \quad \text{where } \tau_5 = (3,2,1)$$

and $EGS(N,\sigma_0,p,\alpha) = \frac{1}{6}\sum_{\tau\in\Pi_N} GS^{\lambda(\tau)}(N,\sigma_0,p,\alpha) = (5,4,5)$.
Note that $m^{\tau_i}(v) = GS^{\lambda(\tau_i)}(N,\sigma_0,p,\alpha)$ for $i \in \{0,2,3,5\}$ and that $m^{\tau_i}(v) \neq GS^{\lambda(\tau_i)}(N,\sigma_0,p,\alpha)$ for $i \in \{1,4\}$. (see figure 1)

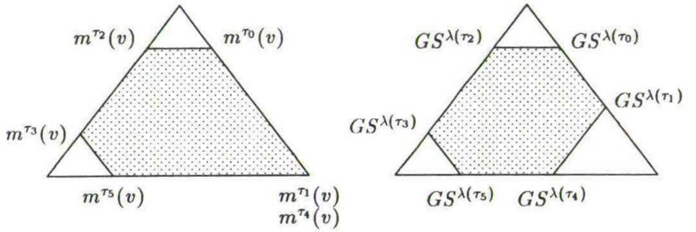

figure 1

In example 3 an extreme point of the core of a sequencing game coincides with an extreme point of the corresponding split core if the corresponding permutation is connected. Here, a permutation $\tau \in \Pi_N$ is called connected if for any $i \in N$ the set $P(\tau,i)$ is a connected set. The next proposition shows that this property holds for any sequencing situation.

Proposition 1 *Let (N,σ_0,p,α) be a sequencing situation and (N,v) the corresponding sequencing game. Then $m^\tau(v) = GS^{\lambda(\tau)}(N,\sigma_0,p,\alpha)$ if τ is connected.*

PROOF: For any connected $\tau \in \Pi_N$ we have $m^\tau(v) = m^\tau(w)$. Lemma 1 completes the proof. □

Example 3 shows that in case a permutation is not connected the corresponding permutation based gain splitting allocation does not necessarily coincide with a marginal vector of the sequencing game. The following example illustrates that a $\lambda \in \Lambda$ with $\lambda_{ij} \in \{0,1\}$ for all $i,j \in N, i \neq j$ that does not arise by a permutation $\tau \in \Pi_N$ as defined in (3) is not necessarily an extreme point of $SPC(N,\sigma_0,p,\alpha)$.

Example 4 Consider the game of example 3. Consider λ defined by $\lambda_{12} = 1, \lambda_{13} = 0$ and $\lambda_{23} = 1$. Obviously λ can not be constructed by means of a permutation τ as in (3). Note that $GS^{\lambda}(N, \sigma_0, p, \alpha) = (4, 6, 4)$ is not an extreme point of $SPC(N, \sigma_0, p, \alpha)$.

In the following we will give an axiomatic characterization of the split core. Let $SEQ(N)$ represent the class of all sequencing situations with player set N. A set-valued solution concept γ assigns to each sequencing situation $SEQ(N)$ a non-empty subset of $\mathbf{R}^{\mathbf{N}}$. We consider the following three properties of a solution concept γ.

(i) *Efficiency*: Let $(N, \sigma_0, p, \alpha) \in SEQ(N)$ and let $\hat{\sigma}$ be an optimal rearrangement of N. Then for any $x \in \gamma((N, \sigma_0, p, \alpha))$ we have that $\sum_{k \in N} x_k = C_N(\sigma_0) - C_N(\hat{\sigma})$.
(ii) *Dummy property:* Let $(N, \sigma_0, p, \alpha) \in SEQ(N)$ and let $\hat{\sigma}$ be an optimal rearrangement of N. If $P(\sigma_0, k) = P(\hat{\sigma}, k)$ for some $k \in N$, then for all $x \in \gamma((N, \sigma_0, p, \alpha))$ it holds that $x_k = 0$.
(iii) *Monotonicity:* Let $(N, \sigma_0, p, \alpha), (N, \sigma_1, p, \alpha) \in SEQ(N)$ and $i, j \in N$ be such that $\sigma_0(i) = \sigma_0(j) - 1, \sigma_1(i) = \sigma_0(j), \sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N \backslash \{i, j\}$. Then for all $x \in \gamma((N, \sigma_0, p, \alpha))$ there exists a $y \in \gamma((N, \sigma_1, p, \alpha))$ such that
(a) $x_k = y_k$ for all $k \in N \backslash \{i, j\}$ and $x_i \geq y_i,\ x_j \geq y_j$
or
(b) $x_k = y_k$ for all $k \in N \backslash \{i, j\}$ and $x_i \leq y_i,\ x_j \leq y_j$.

Efficiency states that the maximum cost savings of the grand coalition is divided among the players. The dummy property states that if a player does not contribute to the cost savings of the grand coalition, then he will obtain no share of these profits. Two sequencing situations are called ***neighbour related*** if the initial order of the one can be obtained from the other by only one neighbour switch. Monotonicity states that for each solution (element) of a sequencing situation there exists a solution (element) for a neighbour related sequencing situation such that all players that are in the same position in both sequencing situations receive the same, and that the two players who switched both will either gain or lose.

The following theorem shows that the split core is the maximal solution concept that satisfies efficiency, the dummy property and monotonicity. Here, maximality means that any solution concept that satisfies these three properties assigns to each sequencing situation a subset of the split core.

Theorem 4 *The split core is a solution concept on $SEQ(N)$ that satisfies efficiency, the dummy property and monotonicity. Let γ be a solution concept on $SEQ(N)$ that satisfies efficiency, the dummy property and monotonicity.*
Then $\gamma((N, \sigma_0, p, \alpha)) \subset SPC(N, \sigma_0, p, \alpha)$ for all $(N, \sigma_0, p, \alpha) \in SEQ(N)$.

PROOF: Obviously, the split core assigns to each sequencing situation in $SEQ(N)$ a non-empty subset of $\mathbf{R}^{\mathbf{N}}$. First we show that the split core satisfies the three properties. Let $(N, \sigma_0, p, \alpha) \in SEQ(N)$. Efficiency follows immediately from the fact that $SPC(N, \sigma_0, p, \alpha)$ is a subset of the core of the corresponding sequencing game. If player k is a dummy player we have that $g_{ik} = 0$ for all $i \in N$ with $\sigma_0(i) < \sigma_0(k)$ and $g_{kj} = 0$ for all $j \in N$ with $\sigma_0(k) < \sigma_0(j)$. This implies that $GS_k^{\lambda}(N, \sigma_0, p, \alpha) = 0$ for any $\lambda \in \Lambda$ and consequently the split core satisfies the dummy property. For monotonocity, let $\lambda \in \Lambda$ and let $i, j \in N$ be such that $\sigma_0(i) = \sigma_0(j) - 1$ and take σ_1 such that $\sigma_1(i) = \sigma_0(j), \sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N\backslash\{i, j\}$. From the definition of a gain splitting allocation it readily follows that

$$GS_k^{\lambda}(N, \sigma_0, p, \alpha) = GS_k^{\lambda}(N, \sigma_1, p, \alpha) \text{ for all } k \in N\backslash\{i, j\}. \tag{4}$$

Further, we have that

$$GS_i^{\lambda}(N, \sigma_0, p, \alpha) - GS_i^{\lambda}(N, \sigma_1, p, \alpha) = g_{ij}\lambda_{ij} - g_{ji}(1 - \lambda_{ji}) \tag{5}$$

and

$$GS_j^{\lambda}(N, \sigma_0, p, \alpha) - GS_j^{\lambda}(N, \sigma_1, p, \alpha) = g_{ij}(1 - \lambda_{ij}) - g_{ji}\lambda_{ji}. \tag{6}$$

If $\alpha_j p_i - \alpha_i p_j \geq 0$ then $g_{ij} \geq 0$ and $g_{ji} = 0$ which implies that $GS_m^{\lambda}(N, \sigma_0, p, \alpha) \geq GS_m^{\lambda}(N, \sigma_1, p, \alpha)$ for $m \in \{i, j\}$. On the other hand, if $\alpha_j p_i - \alpha_i p_j < 0$ then $g_{ij} = 0$ and $g_{ji} > 0$ which implies $GS_m^{\lambda}(N, \sigma_0, p, \alpha) \leq GS_m^{\lambda}(N, \sigma_1, p, \alpha)$ for $m \in \{i, j\}$. Hence, the split core satisfies monotonicity.

Let γ be a set-valued solution concept on $SEQ(N)$ that satisfies efficiency, the dummy property and monotonicity, and let $(N, \sigma_0, p, \alpha) \in SEQ(N)$. To show that

$\gamma((N,\sigma_0,p,\alpha)) \subset SPC(N,\sigma_0,p,\alpha)$ we proceed by induction to the number of misplacements $M_\sigma = \{(i,j) \mid \sigma_0(i) < \sigma_0(j), g_{ij} > 0\}$. If $\mid M_{\sigma_0} \mid = 0$ then σ_0 is an optimal order and the dummy property implies that $\gamma((N,\sigma_0,p,\alpha)) = \{(0,...,0)\} = SPC(N,\sigma_0,p,\alpha)$. Assume that $\gamma(N,\sigma,p,\alpha) \subset SPC(N,\sigma,p,\alpha)$ for all $\sigma \in \Sigma_N$ with $\mid M_\sigma \mid \leq m$. Let σ_0 be such that $\mid M_{\sigma_0} \mid = m+1$. We show that $\gamma(N,\sigma_0,p,\alpha) \subset SPC(N,\sigma_0,p,\alpha)$. Take $x \in \gamma((N,\sigma_0,p,\alpha))$ and let $i,j \in N$ be such that $\sigma_0(i) = \sigma_0(j) - 1$ and (i,j) is a misplacement of σ_0. Take σ_1 such that $\sigma_1(i) = \sigma_0(j), \sigma_1(j) = \sigma_0(i)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in N\backslash\{i,j\}$. Note that $g_{ij} > 0$ since (i,j) is a misplacement. For any $z \in \gamma((N,\sigma_1,p,\alpha))$ we have by efficiency that

$$\sum_{k\in N} x_k - \sum_{k\in N} z_k = C_N(\sigma_0) - C_N(\sigma_1) = g_{ij} \tag{7}$$

where the last equality follows from the definition of σ_1 and the fact that (i,j) is a misplacement. From (7) and monotonicity follows that there exists a $y \in \gamma((N,\sigma_1,p,\alpha))$ such that

$$(x_i + x_j) - (y_i + y_j) = g_{ij} \text{ and } x_i \geq y_i,\ x_j \geq y_j,\ x_k = y_k \text{ for all } k \in N\backslash\{i,j\}. \tag{8}$$

Since $\mid M_{\sigma_1} \mid = m$ the induction hypothesis yields that there exist a $\lambda \in \Lambda$ such that $y = GS^\lambda((N,\sigma_1,p,\alpha))$. Substitution in (8) gives

$$\begin{cases} x_i + x_j &= GS_i^\lambda(N,\sigma_1,p,\alpha) + GS_j^\lambda((N,\sigma_1,p,\alpha)) + g_{ij} \\ x_i &\geq GS_i^\lambda(N,\sigma_1,p,\alpha) \\ x_j &\geq GS_j^\lambda(N,\sigma_1,p,\alpha) \\ x_k &= GS_k^\lambda(N,\sigma_1,p,\alpha) \qquad \text{for all } k \in N\backslash\{i,j\}. \end{cases} \tag{9}$$

Since x is a solution of the system (9), there exists an $s^* \in [0,1]$ such that

$$\begin{aligned} x_i &= GS_i^\lambda(N,\sigma_1,p,\alpha) + s^* g_{ij} \\ x_j &= GS_j^\lambda(N,\sigma_1,p,\alpha) + (1-s^*) g_{ij} \\ x_k &= GS_k^\lambda(N,\sigma_1,p,\alpha) \qquad \text{for all } k \in N\backslash\{i,j\} \end{aligned} \tag{10}$$

Take $\lambda^* \in \Lambda$ such that $\lambda^*_{ij} = s^*$ and $\lambda^*_{kl} = \lambda_{kl}$ for all $k,l \in \{1,...,n\}, k \neq l$ and $(k,l) \neq (i,j)$. Then from (4), (5),(6), (10) and the fact that $g_{ji} = 0$ we have $x = GS^{\lambda^*}(N,\sigma_0,p,\alpha)$. Consequently, $x \in SPC(N,\sigma_0,p,\alpha)$. □

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