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# A BANZHAF SHARE FUNCTION FOR COOPERATIVE GAMES IN COALITION STRUCTURE

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# A Banzhaf share function for cooperative games in coalition structure *

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## Abstract

A *cooperative game with transferable utilities* –or simply a TU-game– describes a situation in which players can obtain certain payoffs by cooperation. A *solution concept* for these games is a function which assigns to every such a game a distribution of payoffs over the players in the game. Well-known solution concepts for TU-games are the *Shapley value* and the *Banzhaf value*. The Shapley value is efficient, i.e. the total payoff is equal to the worth of the 'grand coalition', but the Banzhaf value is not efficient. An alternative type of solution is the concept of *share functions*, being functions which assign to every player in a TU-game its share in the worth of the grand coalition. The Shapley (respectively Banzhaf) share function is the share function giving to each player his Shapley (Banzhaf) value divided by the sum of the Shapley (Banzhaf) values over all players.

In this paper we consider cooperative games in which the players are organized into a *coalition structure* being a finite partition of the set of players. A value function for games in coalition structure has been proposed by Owen. The *Owen value* can be considered as a direct generalization of the Shapley value to games in coalition structure. We define the Owen share function as the share function for games in coalition structure giving to each player his Owen value divided by the sum of the Owen values over all players. We then show that this Owen share function satisfies a *multiplicity property*, namely that the Owen share of a player in a coalition within the coalition structure is equal to the product of the Shapley share of the coalition in a first level game between the coalitions within the coalition structure and the Shapley share of the player in a second level game between the players within the coalition. We show that analogously a Banzhaf share function for games with coalition structure can be obtained by defining the share of a player in some coalition as the product of the Banzhaf share of the coalition in a first level game between the coalitions and the Banzhaf share of the player in a second level game between the players within the coalition. The application of the coalition structure share functions to simple majority games shows some appealing properties of these functions.

# 1 Introduction

A situation in which a finite set of $n$ agents can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utilities* –or simply a TU-game– being a pair $(N, v)$, where the finite set of players $N$ is defined by the set $N = \{1, \ldots, n\}$ representing the agents and where $v: 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$ is the *characteristic function* yielding for any subset $S$ of $N$ the payoff $v(S)$ that can be achieved by coalition $S$. Throughout the paper we use the notation $|S|$ for the number of players in coalition $S \subset N$.

In this paper we only consider *monotone* TU-games. A TU-game $(N, v)$ is *monotone* if $v(E) \leq v(F)$ for all $E \subset F \subset N$. The class of all monotone TU-games is denoted by $\mathcal{G}$. Note that *null* games $(N, v^0)$ defined by $v^0(E) = 0$ for all $E \subset N$ are monotone. In many applications we may restrict ourselves to the subclass $\mathcal{G}^0 = \{(N, v) \in \mathcal{G} \mid v \neq v^0\}$ of all monotone non-null games. Observe that $v(N) > 0$ for any $(N, v) \in \mathcal{G}^0$. The restriction of a game $(N, v) \in \mathcal{G}$ to a coalition $T \subset N$ is the $|T|$-player game $(T, v_T)$ with the characteristic function $v_T$ on $T$ defined by $v_T(E) = v(E)$ for all $E \subset T$.

A *value function* for monotone games is a function $f$ that assigns to every $n$-player game $(N, v) \in \mathcal{G}$ an $n$-dimensional real vector $f(N, v) \in \mathbb{R}^n$. This vector can be seen as a distribution of payoffs over the individual players in the game. Two well-known value functions are the *Shapley value* (Shapley (1953)) and the *Banzhaf value* (Banzhaf (1965)). A solution concept $f$ is called *efficient* if for every TU-game $(N, v)$ it exactly distributes the payoff that can be obtained by the 'grand coalition' $N$ consisting of all players, i.e. if for every TU-game $(N, v)$ it holds that the sum of the components of $f(N, v)$ is equal to $v(N)$. The Shapley value is indeed efficient, but the Banzhaf value is not. To apply the latter concept in distributing $v(N)$ we can use the *normalized Banzhaf value* which distributes the value $v(N)$ of the grand coalition proportional to the Banzhaf values of the players. For a characterization of the normalized Banzhaf value we refer to van den Brink and van der Laan (1995).

A new type of solution concepts has been introduced in van der Laan and van den Brink (1998), namely the concept of *share functions*. A share function assigns to every player in a TU-game $(N, v)$ his share in the worth of the grand coalition. In their paper van der Laan and van den Brink provide an axiomatic characterization of a class of share functions containing the Shapley share function and the Banzhaf share function as special cases. The Shapley (respectively Banzhaf) share function is the share function satisfying that for any player the share of this player multiplied with the payoff of the grand coalition is equal to the payoff of this player assigned to him by the Shapley value (respectively the normalized Banzhaf value).

Share functions yield a distribution of the worth of the grand coalition reflecting the individual bargaining position of the players. In this paper we consider situations in which the grand coalition forms in order to maximize the total payoff, but in which the players are also organized into smaller coalitions. These coalitions form a *coalition structure* being a finite partition into disjunct subsets of the player set $N$ and are assumed to be given exogenously. As motivated by Winter (1989), the coalitions can be seen as pressure groups for the division of $v(N)$. So, to divide the value of the grand coalition over all players, first this value is distributed over the coalitions in the a priori given coalition structure and then the payoff assigned to a coalition is distributed over all its players.

For games in a priori given coalition structure $P = \{P_1, \ldots, P_m\}$ of $m$ coalitions several value functions have been proposed in the literature. The Aumann-Drèze value assigns to any player in a coalition $P_k \in P$ the Shapley value of the restriction of the game $(N, v)$ to coalition $P_k$, see Aumann and Drèze (1974). Under this value concept, the total value of the players within a coalition $P_k$ is equal to the worth of this coalition and therefore the total payoff is equal to $\sum_{k=1}^{m} v(P_k)$, which need not to be equal to $v(N)$. According to the Aumann-Drèze value, the payoff to a player $i$ in some coalition $P_k$ does not depend upon his contribution to any coalition containing players outside $P_k$. In fact, it is supposed that the grand coalition is not formed but that the players agree to disagree and are satisfied with cooperation within the coalitions $P_k$, $k = 1, \ldots, m$. However, one can imagine many situations in which players form coalitions within the grand coalition. In such situations it is very reasonable that also the outside opportunities of the members of a coalition have to be taken into account. As argued by e.g. Hart and Kurz (1983) the existence of a coalition structure implies a two-level interaction between the players. Firstly, the worth of the grand coalition is distributed amongst the coalitions; secondly the payoff to each coalition is distributed amongst the players within this coalition. The outcome of such a two-level interaction is reflected by the so-called *coalition structure value* introduced by Owen (1977). The Owen coalition structure value has the property that the total payoff of the players in a coalition $P_k$ is equal to the Shapley value of the coalition $P_k$, when this coalition is considered to be a player in the first level game between the coalitions. As a consequence we have that the Owen value can be considered as a direct generalization of the Shapley value to games in coalition structure.

Analogously to the Owen generalization of the Shapley value, in this paper we want to generalize the Banzhaf value for games in coalition structure by applying the idea of share functions as introduced in van der Laan and van den Brink (1998). To do so we will redefine the Owen value as a *coalition structure share function.* We then show that the Owen share of a player $i \in P_k$ in the worth of the grand coalition is equal to the Shapley share of coalition $P_k$ in the first level game between the coalitions times the Shapley share of player $i$ in an appropriately defined second level game between the players in $P_k$. In the same way we will define the Banzhaf share of a player $i \in P_k$ in the worth of the grand coalition as the Banzhaf share of coalition $P_k$ in the first level game times the Banzhaf share of player $i$ in an appropriately defined second level game.

This paper is organized as follows. In the next section we recall the concept of share functions and state the main result given in van der Laan and van den Brink (1998). In section 3 we consider games in coalition structure and redefine the Owen values for games in coalition structure as a share function. We show that this Owen share function satisfies the above mentioned multiplicity property. Analogously we define a Banzhaf share function for games in coalition structure and show some of its properties. Finally in section 4 we apply the Owen and the Banzhaf coalition structure share function to simple majority games in coalition structure. This application of the coalition structure share functions shows some appealing properties of these functions.

## 2 Share functions

In this section we recall the concept of share functions as introduced in van der Laan and van den Brink (1998). First, given a game $(N, v) \in \mathcal{G}$, for all $E \subset N$ and all $i \in E$, let $m_E^i(N, v) = v(E) - v(E \setminus \{i\})$ be the marginal contribution of player $i$ to coalition $E$ in game $(N, v)$. Then the well-known *Shapley value* $\varphi^S$ and the *Banzhaf value* $\overline{\varphi}^B$ on the class $\mathcal{G}$ of monotone games are the functions defined by

$$\varphi_i^S(N, v) = \sum_{\substack{E \subset N \\ E \ni i}} \frac{(|E| - 1)!(n - |E|)!}{n!} m_E^i(N, v), \quad i \in N,$$

respectively,

$$\overline{\varphi}_i^B(N, v) = \sum_{\substack{E \subset N \\ E \ni i}} \frac{1}{2^{n-1}} m_E^i(N, v), \quad i \in N.$$

The Shapley value is characterized by the well-known axioms of efficiency, linearity, the dummy player property and anonymity. Axiomatizations of the Banzhaf value have been given by e.g. Lehrer (1988) and Haller (1994). In the latter contribution it has been shown that the Banzhaf value satisfies linearity, the dummy player property, anonymity and the proxy agreement property. The latter property says that the sum of the payoffs of two players does not change if one of them acts as a proxy for the other. Since the Banzhaf value is not efficient, this value is not adequate in allocating the value $v(N)$ of the 'grand coalition'. More precisely, summation over all components of the vector $\overline{\varphi}_i^B(N, v)$ gives

$$\sum_{i \in N} \overline{\varphi}_i^B(N, v) = \sum_{i \in N} \sum_{\substack{E \subset N \\ E \ni i}} \frac{1}{2^{n-1}} m_E^i(N, v) = \frac{1}{2^{n-1}} \sum_{E \subset N} (2|E| - n) v(E),$$

which is generically not equal to $v(N)$. Note that $\sum_{E \subset N}(2|E| - n)v(E) > 0$ and hence $\sum_{i \in N} \overline{\varphi}_i^B(N, v) > 0$ when $(N, v) \in \mathcal{G}^0$. Since $\overline{\varphi}_i^B(N, v^0) = 0$ for all $i \in N$, we have that $\overline{\varphi}^B$ satisfies efficiency on the subclass of null games. To divide the worth of the grand coalition according to the Banzhaf value, on the class of non-null games we have to replace the Banzhaf value by the *normalized Banzhaf value*, being the value function $\varphi^B$ given by

$$\varphi_i^B(N, v) = \frac{2^{n-1} v(N)}{\sum_{E \subset N}(2|E| - n)v(E)} \cdot \overline{\varphi}_i^B(N, v), \; i \in N, \; (N, v) \in \mathcal{G}^0.$$

The normalized Banzhaf value $\varphi^B(N,v)$ is an efficient value function that distributes the worth $v(N)$ of the grand coalition proportional to the Banzhaf values of the players. The normalized Banzhaf value satisfies anonymity. However, it does not satisfy linearity, the dummy player property and the proxy agreement property. An axiomatization of the normalized Banzhaf value has been given in van den Brink and van der Laan (1995).

An alternative approach to divide the worth of the grand coalition amongst its players is given by the concept of share function introduced by van der Laan and van den Brink (1998). A share function assigns to each player his share in the worth of the grand coalition, i.e. a share function on the class $\mathcal{G}$ of monotone games is a function $\rho$ on $\mathcal{G}$ giving player $i \in N$ the share $\rho_i(N,v)$ in the worth $v(N)$ of the grand coalition. We now state the following properties for share functions. First, a share function $\rho$ on $\mathcal{G}$ satisfies the *efficient shares* property if the shares assigned to the players sum up to one for all $(N,v) \in \mathcal{G}$, i.e. $\sum_{i\in N} \rho_i(N,v) = 1$. Second, $\rho$ satisfies the *null player property* [1] on $\mathcal{G}$ if for every $(N,v) \in \mathcal{G}^0$ and every *null player* $i$ in $(N,v)$ it holds that $\rho_i(N,v) = 0$. Third, $\rho$ satisfies *symmetry* if for every $(N,v) \in \mathcal{G}$ and every pair $i,j$ of *symmetric players* [2] in $(N,v)$ it holds that $\rho_i(N,v) = \rho_j(N,v)$. Finally, let be given some some real-valued function $\mu: \mathcal{G} \to \mathbb{R}$. Then $\rho$ satisfies $\mu$-linearity on $\mathcal{G}$ if for every pair of games $(N,v)$, $(N,w) \in \mathcal{G}$ and real numbers $a$, $b$ such that $(N, av+bw) \in \mathcal{G}$ it holds that [3] $\mu(N,av+bw)\rho(N,av+bw) = a\mu(N,v)\rho(N,v) + b\mu(N,w)\rho(N,w)$.

The last property is a generalization of the familiar linearity property which is obtained by taking $\mu(N,v) = 1$ for all $(N,v) \in \mathcal{G}$. A function $\mu: \mathcal{G} \to \mathbb{R}$ is called *positive* on $\mathcal{G}$ if $\mu(N,v^0) = 0$ and $\mu(N,v) > 0$ for every $(N,v) \in \mathcal{G}^0$. We call $\mu$ *linear* on $\mathcal{G}$ if for every pair of games $(N,v)$, $(N,w) \in \mathcal{G}$ and real numbers $a$, $b$ such that $(N,av+bw) \in \mathcal{G}$ it holds that $\mu(N,av+bw) = a\mu(N,v) + b\mu(N,w)$. Finally, we call $\mu$ *symmetric* on $\mathcal{G}$ if for every $(N,v) \in \mathcal{G}$, every pair of symmetric players $i$, $j$ in $(N,v)$,

[1] Player $i \in N$ is a *null player* in $(N,v)$ if $v(E) = v(E\setminus\{i\})$ for all $E \subset N$. The null player property is only assumed to hold on $\mathcal{G}^0$, because in a null game $(N,v^0)$ irrespective of the shares all players get a zero payoff when multiplying the shares with $v^0(N) = 0$.

[2] Players $i,j \in N$ are *symmetric* in $(N,v) \in \mathcal{G}$ if $v(E\cup\{i\}) = v(E\cup\{j\})$ for all $E \subset N$ with $E \cap \{i,j\} = \emptyset$.

[3] For a pair of games $(N,v)$, $(N,w) \in \mathcal{G}$ and real numbers $a$, $b$, the game $(N, av+bw)$ is given by $(av+bw)(E) = av(E) + bw(E)$ for all $E \subset N$.

and every $E \subset N$ such that $\{i,j\} \subset E$ and $(E \setminus \{i\}, v_{E\setminus\{i\}})$, $(E \setminus \{j\}, v_{E\setminus\{j\}}) \in \mathcal{G}$ it holds that $\mu(E \setminus \{i\}, v_{E\setminus\{i\}}) = \mu(E \setminus \{j\}, v_{E\setminus\{j\}})$. The following result follows from van der Laan and van den Brink (1998).

**Theorem 2.1**
*Let $\mu: \mathcal{G} \rightarrow \mathbb{R}$ be positive, symmetric and linear on $\mathcal{G}$. Then there exists a unique share function $\rho^\mu$ on $\mathcal{G}$ satisfying the property of efficient shares, the null player property, symmetry and $\mu$-linearity on $\mathcal{G}$.*

The proof of the theorem goes along the lines of the proof of a slightly different result in van der Laan and van den Brink (1998), see also van den Brink and van der Laan (1998).[4] Two specific examples of share functions are the *Shapley share* function and the *Banzhaf share* function.

**Definition 2.2 (Shapley and Banzhaf share function)**
(i) *The* **Shapley share function** $\rho^S$ *on $\mathcal{G}$ is defined by* $\rho_i^S(N,v) = \frac{\varphi_i^S(N,v)}{v(N)}$, $i \in N$, $(N,v) \in \mathcal{G}^0$ *and* $\rho_i^S(N,v) = \frac{1}{|N|}$, $i \in N$, *when* $v = v^0$.
(ii) *The* **Banzhaf share function** $\rho^B$ *on $\mathcal{G}$ is defined by* $\rho_i^B(N,v) = \frac{\varphi_i^B(N,v)}{v(N)} = \frac{2^{(n-1)}}{\sum_{E\subset N}(2|E|-n)v(E)}\overline{\varphi}_i^B(N,v)$, $i \in N$, $(N,v) \in \mathcal{G}^0$ *and* $\rho_i^B(N,v) = \frac{1}{|N|}$, $i \in N$, *when* $v = v^0$.

In van der Laan and van den Brink (1998) it is shown that the unique share function satisfying the properties stated in Theorem 2.1 is the Shapley share function when $\mu$ is defined as $\mu^S: \mathcal{G} \rightarrow \mathbb{R}$ given by $\mu^S(N,v) = v(N)$, respectively the Banzhaf share function when $\mu$ is defined as $\mu^B: \mathcal{G} \rightarrow \mathbb{R}$ given by $\mu^B(N,v) = \frac{1}{2^{n-1}}\sum_{E\subset N}(2|E| - n)v(E)$.

For null games the concept of share function is in itself not very interesting, because in such a game any player gets a payoff of zero irrespective of the shares. Moreover, in many applications we may restrict ourselves to monotone non-null games. However, in the next section we apply the concept of share functions to games in

[4]Observe that for a null game $(N, v^0)$ the axioms of symmetry and efficiency imply that for any $\mu$ we must have that $\rho_i^\mu(N, v^0) = \frac{1}{|N|}$, for all $i \in N$.

coalition structure. For such games the payoff of a player can be seen as the result of a first level game between coalitions and a second level gamel between the players within a coalition. In such a set-up we have to deal with null games, which may appear on the second level, even when the game itself is a non-null game. Therefore we extended the concept of share functions to null games by giving all players an equal share when $v = v^0$.

## 3 Coalition structure share functions

The share functions defined in the previous section yield a distribution of the worth of the grand coalition reflecting the individual bargaining position of the players. In many situations however, it is reasonable to suppose that players form coalitions which decide to act together against the other players in bargaining over $v(N)$. In this section we consider situations in which the players are organized in a priori given coalition structure.

A coalition structure is a finite partition $P = \{P_1, \ldots, P_m\}$ of $m$ non-empty, disjoint subsets of the player set $N$, i.e. $\cup_{k=1}^{m} P_k = N$ and $P_k \cap P_\ell = \emptyset$ for all $k$, $\ell \in \{1, \ldots, m\}$, $k \neq \ell$. In the following the set of coalitions in the coalition structure is denoted by $M = \{1, \ldots, m\}$. Furthermore, a game $(N, v) \in \mathcal{G}$ in coalition structure $P$ is denoted by $(N, v, P)$ and the collection of all coalition structures by $\mathcal{P}$. The collection of all monotone games in coalition structure is denoted by $\mathcal{G}(\mathcal{P})$. A coalition structure value (CS-value) function $\theta$ on the set $\mathcal{G}(\mathcal{P})$ of games assigns a payoff to any player for every game in coalition structure $(N, v, P) \in \mathcal{G}(\mathcal{P})$. The Aumann-Drèze value assigns to any player in a coalition $P_k \in P$ his Shapley value of the restriction of the game $(N, v)$ to $P_k$. Under this value concept, the total value of the players within a coalition $P_k$ is equal to the worth of this coalition. So, the players in $P_k$ ignore the foregone opportunities of forming coalitions with the players not in $P_k$. As already argued by Aumann and Drèze, it is very reasonable that in many situations also the outside opportunities of the members of a coalition are taken into account. Hart and Kurz (1983) argue that the existence of a coalition structure implies a two-level interaction between the players. The outcome of such a two-level interaction is reflected by the CS-value introduced by Owen (1977). The *Owen Coalition Structure value* (Owen CS-value) $\theta^O$ on $\mathcal{G}(\mathcal{P})$ is defined by

$$\theta_i^O(N,v,P) = \sum_{\substack{L\subset M \\ L\not\ni k}} \sum_{\substack{E\subset P_k \\ E\ni i}} \frac{|L|!(m-|L|-1)!}{m!} \frac{(|E|-1)!(|P_k|-|E|)!}{|P_k|!}$$

$$(v(E\cup P(L)) - v((E\setminus\{i\})\cup P(L))),\ i\in P_k,\ k\in M, \qquad (1)$$

where $P(L) = \cup_{j\in L}P_j$. We remark that the Owen CS-value reduces to the Shapley value when $P = \{\{N\}\}$ or when $P = \{\{i\}, i\in N\}$. In fact, the weight assigned to the marginal contribution of player $i \in E \subset P_k$ to the coalition $P(L)\cup E$, $k \notin L$, is the product of the Shapley weight of coalition $P_k$ when $k$ enters $L$ and the Shapley weight of player $i$ when $i$ enters coalition $E \subset P_k$. So, the weights reflect the fact that first coalitions enter subsequently in a random order and that within each coalition the players enter subsequently in a random order. From this it follows that the Owen CS-value has the property that the total payoff of the players in $P_k$ is equal to the Shapley value of coalition $P_k$ in the first level game between the coalitions. More precisely, for given game $(N,v,P)\in\mathcal{G}(\mathcal{P})$, with $P=\{P_1,\ldots,P_m\}$, $M=\{1,...,m\}$, the $m$-player game $(M,v^P)\in\mathcal{G}$ is defined by

$$v^P(L) = v(P(L)),\quad L\subset M,$$

i.e. for coalition structure $P=\{P_1,\ldots,P_m\}$, the game $(M,v^P)$ is the $m$-player game between the coalitions induced by the game $(N,v,P)$. Observe that $v^P(M)=v(N)$. Now, let $\varphi_k^S(M,v^P)$ be the Shapley value assigned to coalition $k$, $k\in M$, in the game $(M,v^P)$. Then, for all $k\in M$, it follows from equation (1) by summing up over all components $i\in P_k$ that $\sum_{i\in P_k}\theta_i^O(N,v,P) = \varphi_k^S(M,v^P)$. Since by the efficiency of the Shapley value $\sum_{k\in M}\varphi_k^S(M,v^P) = v^P(M) = v(N)$, it follows that also the Owen CS-value is efficient.

The discussion above shows that the Owen CS-value can be considered as a direct generalization of the Shapley value to games with coalition structure. The Owen CS-value satisfies the *consistency property* that the total payoff to the players in a coalition $P_k$ is equal to the payoff of player $k$ when applying the same value concept to the $m$-player game $(M, v^P, Q)$, being the induced game $(M, v^P)$ in coalition structure $Q = \{\{M\}\}$. For an axiomatization of the Owen CS-value we refer to Owen (1977), Hart and Kurz (1983) and Winter (1989). Winter shows that the Owen CS-value is the unique coalition structure value function satisfying the axioms of efficiency, null player property, additivity with respect to any two games $(N, v, P)$ and $(N, w, P)$ for given $P \in \mathcal{P}$, the symmetric players property and coalitional symmetry. The latter property means that when two coalitions $P_j$ and $P_k$ are symmetric players in the induced game $(M, v^P)$, then the total payoff of the players in coalition $P_j$ is equal to the total payoff to the players in coalition $P_k$. Clearly this property is implied by the consistency property and the symmetric players property.

Analogously to the generalization of the Shapley value, in the remaining of this section we want to generalize the Banzhaf value for games with coalition structure by applying the idea of share functions as discussed in section 2. Therefore we first define the *Owen CS-share* function as the share of player $i$ in the worth of $v(N)$ according to the Owen CS-value, i.e. this is the function $\psi^O$ given by

$$\psi_i^O(N, v, P) = \frac{\theta_i^O(N, v, P)}{v(N)}, \quad i \in P_k, \quad k \in M,$$

and reformulate the Owen CS-share of a player $i$ as the product of two Shapley shares. To do so, for $L \subset M$ and $k \notin L$ we define the $|P_k|$-player game $(P_k, v^{P_k, L})$ by

$$v^{P_k, L}(E) = v(E \cup P(L)) - v(P(L)), \quad E \subset P_k, \tag{2}$$

i.e. the game $(P_k, v^{P_k, L})$ assigns to each coalition $E$ of $P_k$ the marginal contribution of $E$ to the union $P(L)$ of the coalitions $P_j$, $j \in L$. Furthermore, for $k \in M$, we define the $|P_k|$-player game $(P_k, v^{P_k})$ by

$$v^{P_k}(E) = \sum_{\substack{L \subset M \\ L \not\ni k}} \frac{|L|!(m - |L| - 1)!}{m!} v^{P_k, L}(E), \quad E \subset P_k, \tag{3}$$

i.e. the game $(P_k, v^{P_k})$ is a weighted sum of the games $(P_k, v^{P_k, L})$, $L \subset M$, where the weight of the game $(P_k, v^{P_k, L})$ is equal to the Shapley weight assigned to coalition $k \in M$ if this coalition joins the collection $L \subset M$ of coalitions. We now have the following lemma, where $\rho_i^S(P_k, v^{P_k})$ is the Shapley share of player $i \in P_k$ in the game $(P_k, v^{P_k})$ and $\rho_k^S(M, v^P)$ the Shapley share of coalition $P_k$ in the game $(M, v^P)$.

**Lemma 3.1**

*Let be given a game $(N, v, P) \in \mathcal{G}(\mathcal{P})$ with $P = \{P_1, \ldots, P_m\}$ and $M = \{1, ..., m\}$. Then the Owen CS-share of player i in $P_k$ is equal to the product of the Shapley share of player i in the game $(P_k, v^{P_k})$ and the Shapley share of coalition k in the game $(M, v^P)$, i.e.*

$$\psi_i^O(N, v, P) = \rho_i^S(P_k, v^{P_k}) \cdot \rho_k^S(M, v^P), \;\; i \in P_k, \;\; k \in M.$$

**Proof.**

From equations (2) and (3) it follows that

$$\begin{aligned} v^{P_k}(P_k) &= \sum_{\substack{L \subset M \\ L \not\ni k}} \frac{|L|!(m - |L| - 1)!}{m!} \left(v(P_k \cup P(L)) - v(P(L))\right) \\ &= \sum_{\substack{L \subset M \\ L \not\ni k}} \frac{|L|!(m - |L| - 1)!}{m!} \left(v^P(\{k\} \cup L) - v^P(L)\right) \\ &= \sum_{\substack{L \subset M \\ L \not\ni k}} \frac{|L|!(m - |L| - 1)!}{m!} m^k_{L \cup \{k\}}(M, v^P) = \varphi_k^S(M, v^P). \end{aligned} \tag{4}$$

So, the value $v^{P_k}(P_k)$ is equal to the Shapley value of coalition $P_k$ in the game $(M, v^P)$. Using the definition of the games $(P_k, v^{P_k,L})$ as given in equation (2), we now rewrite equation (1) as

$$\begin{aligned} \theta_i^O(N, v, P) &= \sum_{\substack{L \subset M \\ L \not\ni k}} \sum_{\substack{E \subset P_k \\ E \ni i}} \frac{|L|!(m - |L| - 1)!}{m!} \frac{(|E| - 1)!(|P_k| - |E|)!}{|P_k|!} m_E^i(P_k, v^{P_k,L}) \\ &= \sum_{\substack{L \subset M \\ L \not\ni k}} \frac{|L|!(m - |L| - 1)!}{m!} \varphi_i^S(P_k, v^{P_k,L}), \;\; i \in P_k, \;\; k \in M. \end{aligned} \tag{5}$$

So, for given $k \in M$, the Owen CS-value to player $i \in P_k$ is a weighted sum of the Shapley values of player $i$ in the games $(P_k, v^{P_k,L})$. Because of the linearity property of the Shapley value it follows with (3) that

$$\theta_i^O(N, v, P) = \varphi_i^S(P_k, v^{P_k}), \quad i \in P_k, \ \ k \in M. \tag{6}$$

From the equations (4) and (6) and the fact van $v(N) = v^P(M)$ we obtain that

$$\psi_i^O(N, v, P) = \frac{\theta_i^O(N, v, P)}{v(N)} = \frac{\varphi_i^S(P_k, v^{P_k})}{v^{P_k}(P_k)} \cdot \frac{v^{P_k}(P_k)}{v(N)}$$

$$= \frac{\varphi_i^S(P_k, v^{P_k})}{v^{P_k}(P_k)} \cdot \frac{\varphi_k^S(M, v^P)}{v^P(M)} = \rho_i^S(P_k, v^{P_k}) \cdot \rho_k^S(M, v^P), \ i \in P_k, \ k \in M.$$

□

Considering the game $(M, v^P)$ as the first level game between the coalitions in $P$ and the game $(P_k, v^{P_k})$ as the game on the second level between the players in $P_k$, the lemma shows that the Owen CS-share of player $i$ in $P_k$ is equal to the Shapley share of player $i$ in the second level game times the Shapley share of coalition $P_k$ in the first level game. We use this multiplicity property of the Owen CS-share function to obtain a Banzhaf-type coalition structure share function. To do so as a first step we replace in equation (1) the two-level Shapley weights by the corresponding two-level Banzhaf weights. In the Banzhaf value each marginal contribution has an equal weight. Generalizing this to games in coalition structure we have to assign equal weights to each marginal contribution of a coalition within the coalition structure and within such a coalition equal weights to each marginal contribution of the players within that coalition. Doing so, we obtain the value function $\theta$ on $\mathcal{G}(\mathcal{P})$ defined by

$$\theta_i(N, v, P) = \sum_{\substack{L \subset M \\ L \not\ni k}} \sum_{\substack{E \subset P_k \\ E \ni i}} 2^{-(m-1)} \cdot 2^{-(|P_k|-1)} \left(v(E \cup P(L)) - v((E \setminus \{i\}) \cup P(L))\right)$$

$$= \sum_{\substack{L \subset M \\ L \not\ni k}} \sum_{\substack{E \subset P_k \\ E \ni i}} 2^{-(m-1)} \cdot 2^{-(|P_k|-1)} m_E^i(P_k, v^{P_k,L}), \ i \in P_k, \ k \in M. \tag{7}$$

Analogously to the second part of equation (5) this reduces to

$$\theta_i(N,v,P) = \sum_{\substack{L\subset M \\ L\not\ni k}} 2^{-(m-1)}\overline{\varphi}_i^B(P_k, v^{P_k,L}),\ i \in P_k,\ k\in M, \tag{8}$$

i.e. the value of a player $i \in P_k$ is a weighted sum of the Banzhaf values of player $i$ in the games $(P_k, v^{P_k,L})$, $k \in M$. Since the Banzhaf value is linear it follows that

$$\theta_i(N,v,P) = \overline{\varphi}_i^B(P_k, \overline{v}^{P_k}),\quad i\in P_k,\quad k\in M, \tag{9}$$

where the $|P_k|$-player game $(P_k, \overline{v}^{P_k})$, $k \in M$, is defined by

$$\overline{v}^{P_k}(E) = \sum_{\substack{L\subset M \\ L\not\ni k}} 2^{-(m-1)} v^{P_k,L}(E),\quad E\subset P_k. \tag{10}$$

Observe that analogously to the game $(P_k, v^{P_k})$, the game $(P_k, \overline{v}^{P_k})$ is a weighted sum of the games $(P_k, v^{P_k,L})$, $L \subset M$, where the weight of the game $(P_k, v^{P_k,L})$ is equal to the Banzhaf weight assigned to coalition $k \in M$ if this coalition joins the collection $L \subset M$ of coalitions.

The Banzhaf-type value $\theta_i(N, v, P)$ as defined in equation (9) is similar to the expression of the Owen CS-value as given in equation (6). However, the Banzhaf value is not efficient and so is not the value function $\theta$ on $\mathcal{G}(\mathcal{P})$ as defined above. Moreover, $\theta$ is not consistent in the sense that $\theta$ does not satisfy the property that the total value of the players in a coalition $P_k$ is equal to the Banzhaf value of coalition $P_k$ in the game $(M, v^P)$. However, by applying a similar multiplicity property as satisfied by the Owen CS-share function we can define a two-level normalization of $\theta$ as an efficient and consistent *Banzhaf CS-share* function. Therefore, let $\rho_i^B(P_k, \overline{v}^{P_k})$ be the Banzhaf share of player $i \in P_k$ in the game $(P_k, \overline{v}^{P_k})$ and let $\rho_k^B(M, v^P)$ be the Banzhaf share of coalition $P_k$ in the game $(M, v^P)$. Then analogously to the multiplicity property of the Owen CS-share function we define a Banzhaf-type CS-share function on $\mathcal{G}(\mathcal{P})$.

**Definition 3.2 (Banzhaf Coalition Structure share function)**
*The* **Banzhaf Coalition Structure share function** *is the function* $\psi^B$ *on* $\mathcal{G}(\mathcal{P})$ *given by*

$$\psi_i^B(N,v,P) = \rho_i^B(P_k,\overline{v}^{P_k})\cdot \rho_k^B(M,v^P),\ i\in P_k,\ k\in M.$$

Thus, the Banzhaf CS-share of player $i$ in $P_k$ is defined as the the Banzhaf share of player $i$ in the second level game $(P_k, \overline{v}^{P_k})$ between the players in $P_k$ times the Banzhaf share of coalition $P_k$ in the first level game $(M, v^P)$ between the coalitions in $P$. Clearly, multiplying this share with the worth $v(N)$ of the grand coalition we obtain a CS-value function satisfying both the efficiency and the consistency property. More precisely we have the following corollary.

**Corollary 3.3 (Efficiency and consistency properties)**
**Efficiency:** $\sum_{i \in N} \psi_i^B(N, v, P) = 1$.
**Consistency:**
(i) $\sum_{i \in P_k} \psi_i^B(N, v, P) = \rho_k^B(M, v^P)$.
(ii) $\psi_i^B(N, v, P) = \rho_i^B(N, v)$ *when* $P = \{\{N\}\}$.
(iii) $\psi_i^B(N, v, P) = \rho_i^B(N, v)$ *when* $P = \{\{i\},\ i \in N\}$.

The first consistency property says that the sum of the Banzhaf CS-shares of the players in a coalition $P_k$ is equal to the Banzhaf share of coalition $P_k$ in the game $(M, v^P)$ and follows immediately from the definition of $\psi^B$. The last two consistency properties say that the Banzhaf CS-share function is equal to the Banzhaf share function when $P = \{\{N\}\}$ or when $P = \{\{i\},\ i \in N\}$ and follow immediately from the fact that in these cases one of the two shares in the product is equal to one. So, $\psi^B$ can be seen as a generalization of the Banzhaf share function $\rho^B$ to games in coalition structure.

According to the properties of the Banzhaf share function the Banzhaf CS-share function satisfies also the null player property and the symmetry property. So a null player receives a share equal to zero, but also the sum of the shares of the players in a coalition being a null player in the game $(M, v^P)$ is equal to zero. Also the symmetry property holds for both two symmetric players within the same coalition and for two symmetric coalitions in the game $(M, v^P)$. The Banzhaf CS-share function is not additive. However, for given $(N, v, P)$ and $(N, w, P)$, let $(N, z, P)$ be given by $z = v + w$. Then we have that $z^P = v^P + w^P$ and that for $k \in M$ it holds that $\overline{z}^{P_k} = \overline{v}^{P_k} + \overline{w}^{P_k}$. Since according to Theorem 2.1 the Banzhaf share function is $\mu^B$-linear, it follows that the Banzhaf CS-share of player $i \in P_k$ in the game $(N, z, P)$ can be computed when we know his Banzhaf CS-shares in the games $(P_k, \overline{v}^{P_k})$ and $(P_k, \overline{w}^{P_k})$

and the Banzhaf shares of coalition $P_k$ in the games $(M, v^P)$ and $(M, w^P)$. So, the Banzhaf CS-share function satisfies the axioms of efficiency, null player property, the symmetric players property, coalitional symmetry property and can be decomposed into two $\mu^B$-additive share functions. From Theorem 2.1 it follows straightforwardly that the Banzhaf CS-share function is the unique CS-share function satisfying all these properties.

# 4 CS-share functions for simple games

In this section we consider the class of simple games. A game is called *simple* if $v(E) \in \{0, 1\}$ for any $E \subset N$. In a simple game a coalition is called *winning* when its value equals one and *losing* otherwise. An example of such a game is a weighted voting game, in which a coalition has the power to decide, and hence is winning, when it holds a certain number of votes. In such games it is common practice that the grand coalition will not form. For example, in parliament usually not the grand coalition $N = \{1, \ldots, n\}$ of all parties will form, but a coalition of parties just having enough seats to take the majority. However, in forming such a winning or *majority* coalition often the question arises about the relative power of each party within the coalition. For instance, in the Dutch parliamentary system the parties in a majority coalition $C \subset N$ forming the government not only have to agree on issues, but also on the division of the ministries between the governmental parties. This division is usually based on the number of seats within the coalition, i.e. the payoff to party $i \in C$ (in terms of ministries occupied by this party) is proportional to its number of seats within the coalition. Another rule to determine the number of ministries of each party in the majority coalition could be the Aumann-Drèze value of the original game restricted to the majority coalition $C$. Because in this case $v(C) = v(N) = 1$, we have indeed that the AD-values of the players in $C$ sum up to one. However, in both cases the division over the parties does not take into account the opportunities of the majority parties of breaking away and forming another majority coalition with parties not in $C$. To reflect these opportunities in determining the relative power of the governmental parties within the coalition we may consider the Owen and Banzhaf CS-share functions.

To do so we have to consider the question about the coalition structure with respect to the parties outside the majority coalition forming the government.

We consider a weighted voting game $(N, v)$ in which each player has a number of votes or seats, for instance each player represents a party in parliament. Let the integer number $s_i$, $i \in N$, be the number of votes of player $i$. A coalition has the majority (is winning) if it has at least some prespecified integer number $W > \frac{1}{2}\sum_{i\in N} s_i$ of votes. So, the game is *proper*, i.e. when $E \subset N$ is a majority coalition having value 1, then the complement $N \setminus E$ is losing. Moreover, for any subset $F$ of $N \setminus E$ we have that $v(F) = 0$ and $v(E \cup F) = 1$. Observe that in general there are many winning coalitions.

Let $C$ be a majority coalition and suppose that the members of $C$ indeed agree to cooperate, for instance in forming the government. To assign the CS-shares to each player (i.e. each party in the parliament) when $C$ is formed, we may consider two opposite possibilities with respect to the coalition structure of the parties outside $C$. The first possibility is that all players not in $C$ stay alone and do not cooperate together. In this case we have the coalition structure $P = \{C,\ \{h\}_{h\in N\setminus C}\}$ of $m = n - |C| + 1$ coalitions, namely the majority coalition $C$ and the $n - |C|$ parties not in $C$. The opposite situation is that the members not in $C$ join together in order to form an as powerful as possible opposition against $C$. This yields the coalition structure $Q = \{C,\ N \setminus C\}$ of $m = 2$ coalitions.

We first consider coalition structure $P$. Let $h$ be a player not in $C$. Since $C$ is winning we have that $\{h\}$ is a null player in the induced game $(M, v^P)$ and hence $\psi_h^O(N, v, P) = \psi_h^B(N, v, P) = 0$. So any player outside $C$ has zero power within the coalition structure $P$. Consequently it follows that for both the Banzhaf and Owen CS-share function the share of coalition $C$ in the game $(M, v^P)$ is equal to one, i.e. $\rho_C^S(M, v^P) = \rho_C^B(M, v^P) = 1$.

In case of the Banzhaf CS-share function it follows for $i \in C$ that $\psi_i^B(N, v, P) = \rho_i^B(C, \overline{v}^C)$ with the game $(C, \overline{v}^C)$ as the game $(P_k, \overline{v}^{P_k})$ as defined in equation (10) with $P_k = C$, i.e. $(C, \overline{v}^C)$ is a weighted sum of all $|C|$-player games $(C, v^{C,L})$, $L \subset N \setminus C$, given by

$$v^{C,L}(E) = v(L \cup E) - v(L) = v(L \cup E), \quad E \subset C.$$

Remark that $L \subset N \backslash C$ is losing and thus $v(L) = 0$. Now, due to the coalition structure $P = \{C, \{h\}_{h \in N \backslash C}\}$ equation (7) reduces for $i \in C$ to

$$\begin{aligned} \theta_i(N, v, P) &= \sum_{L \subset N \backslash C} \sum_{E \subset C} 2^{-(n-|C|)} \cdot 2^{-(|C|-1)} m^i_{L \cup E}(N, v) \\ &= \sum_{\substack{L \cup E \subset N \\ i \in L \cup E}} 2^{-(n-1)} m^i_{L \cup E}(N, v). \end{aligned}$$

So, the Banzhaf value $\overline{\varphi}^B_i(C, \overline{v}^C) = \theta_i(N, v, P)$, see equation (9), of a player $i \in C$ in the game $(C, \overline{v}^C)$ is the weigthed sum of all his marginal contributions with all weights equal to $2^{-(n-1)}$ and thus,

$$\overline{\varphi}^B_i(C, \overline{v}^C) = \overline{\varphi}^B_i(N, v), \quad i \in C,$$

i.e. the Banzhaf value of a player $i \in C$ in the game $(C, \overline{v}^C)$ is equal to the Banzhaf value in the game $(N, v)$ without coalition structure. The only difference is that in the coalition stucture game the value of the players outside $C$ is zero, while they may have positive value in the game without coalition structure. Normalizing the values of the players in $C$ to one we obtain that

$$\psi^B_i(N, v, P) = \rho^B_i(C, \overline{v}^C) = \frac{\rho^B_i(N, v)}{\sum_{j \in C} \rho^B_j(N, v)}, \text{ when } i \in C,$$

with $\rho^B(N, v)$ the Banzhaf share function of the game $(N, v)$. So, under the coalition structure $P$ the Banzhaf CS-share or power of the members within the majority coalition $C$ follows immediately from normalizing the Banzhaf shares of the members of the coalition in the game without coalition structure to one. When applying this result to the problem of determining the number of ministries to each party in a winning governmental coalition $C$ of the parliament we simply obtain that the number of ministries occupied by a party $i \in C$ should approximately be equal to $\frac{\rho^B_i(N,v)}{\sum_{j \in C} \rho^B_j(N,v)}$ times the number of ministries, which gives a simple rule for the distribution of the number of ministries over the parties in the coalition. Observe that this distribution may differ considerably from the usual distribution as mentioned in the beginning of this section. In general the distribution proportional to the number of seats in the parliament is in favor of the smaller parties within the government.

In case of the Owen CS-share we get for $i \in C$ that $\psi_i^O(N,v,P) = \rho_i^S(C,v^C)$ with the game $(C,v^C)$ as the game $(P_k, v^{P_k})$ defined in equation (3) with $P_k = C$, i.e. $(C,v^C)$ is a weighted sum of all $|C|$-player games $(C, v^{C,L})$, $L \subset N \setminus C$. So, the Owen CS-share of a player $i \in C$ is a weighted sum of the Shapley shares of the $|C|$-player games $(C, v^{C,L})$, $L \subset N \setminus C$. Looking to the weights of the marginal contributions of the players, remark that the Owen CS-value function is efficient and that $v(N) = 1$. Therefore we also have that $\psi^O(N,v,P) = \theta^O(N,v,P)$ and due to the coalition structure $P = \{C, \ \{h\}_{h \in N \setminus C}\}$, the formula for $\theta_i^O(N,v,P)$ as given in equation (1) reduces for $i \in C$ to

$$\theta_i^O(N,v,P) = \sum_{L \subset N \setminus C} \sum_{E \subset C} \frac{|L|!(n-|C|-|L|)!}{(n-|C|+1)!} \frac{(|E|-1)!(|C|-|E|)!}{|C|!} m_{L \cup E}^i(N,v)$$

Hence, for $i \in C$ the Owen CS-share is a weighted sum of all marginal contributions of player $i$. However, due to the coalition structure the weights differ from the standard Shapley weights of the marginal contributions. Therefore, for $i \in C$, generically $\psi_i^O(N,v,P) \neq \frac{\rho_i^S(N,v)}{\sum_{j \in C} \rho_j^S(N,v)}$. Consequently, the Owen CS-shares of the players in $C$ do not follow from normalizing the sum of the Shapley shares in the game $(N,v)$ of the players in $C$ to one, as is the case for the Banzhaf CS-shares. This maybe makes the Owen CS-share function less attractive compared to the Banzhaf CS-shares for simple games with a coalition structure containing one majority coalition and the other players as singletons. In fact, analogously to the formula for $\theta_i^O(N,v,P)$ as given in equation (5) it follows that

$$\theta_i^O(N,v,P) = \sum_{L \subset N \setminus C} \frac{|L|!(n-|C|-|L|)!}{(n-|C|+1)!} \varphi_i^S(C, v^{C,L}), \quad i \in C. \tag{11}$$

So, the Owen CS-shares are a weighted sum of the Shapley shares in the games $(C, v^{C,L})$, $L \subset N \setminus C$. We summarize the results stated above in the following theorem.

**Theorem 4.1** *Let $(N,v) \in \mathcal{G}$ be a simple game and let $P = \{C, \ \{h\}_{h \in N \setminus C}\}$ be a coalition structure with $C \subset N$ a majority coalition. Then the Banzhaf CS-shares are given by*

$$\psi_i^B(N,v,P) = \begin{cases} \frac{\rho_i^B(N,v)}{\sum_{j \in C} \rho_j^B(N,v)}, & \text{if } i \in C, \\ 0 & \text{if } i \notin C. \end{cases}$$

*The Owen CS-shares are given by*

$$\psi_i^O(N,v,P) = \begin{cases} \sum_{L\subset N\setminus C} \frac{|L|!(n-|C|-|L|)!}{(n-|C|+1)!}\varphi_i^S(C,v^{C,L}), & \textit{if } i \in C, \\ 0 & \textit{if } i \notin C. \end{cases}$$

**Example 4.2**

We consider a weigthed majority game $(N,v)$ with $N = \{1,2,3,4\}$. The number of votes is given by $s = (7,4,2,1)$ with $s_i$ the number of votes of player $i \in N$. Eight votes are needed to have the majority. Hence $v(S) = 1$ if $1 \in S$ and $|S| \geq 2$, $v(S) = 0$ otherwise. Without coalition structure the Shapley and Banzhaf shares of the players are given by $\rho^S(N,v) = (\frac{3}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})^\top$ and $\rho^B(N,v) = (\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})^\top$. For the game in coalition structure $P = \{\{1,2\}, \{3\}, \{4\}\}$, the Banzhaf CS-shares are zero for the players 3 and 4. For the players 1 and 2 these shares follow from normalizing the sum of the Banzhaf shares of the players 1 and 2 to one. Hence $\psi^B(N,v,P) = (\frac{7}{8}, \frac{1}{8}, 0, 0)^\top$. Also the Owen CS-shares are zero for the players 3 and 4. The Owen CS-shares of the players 1 and 2 are a weighted sum of their Shapley shares in the four 2-player games $(C, v^{C,L})$ with $C = \{1,2\}$ the majority coalition and $L \subset \{3,4\}$. The characteristic functions of these games and the Shapley shares of the two players in these games are given in Table 1. With formula (11) it follows that the Owen CS-shares of the players 1 and 2 are equal to $\frac{5}{6}$, respectively $\frac{1}{6}$, so that $\psi^O(N,v,P) = (\frac{5}{6}, \frac{1}{6}, 0, 0)^\top$. Observe that

$$\frac{\rho_2^S(N,v)}{\rho_1^S(N,v)} = \frac{1}{9} < \frac{1}{5} = \frac{\psi_2^O(N,v,P)}{\psi_1^O(N,v,P)}.$$

and that

$$\frac{\rho_2^B(N,v)}{\rho_1^B(N,v)} = \frac{1}{7} = \frac{\psi_2^B(N,v,P)}{\psi_1^B(N,v,P)}.$$

We now consider coalition structure $Q = \{C,\ N\setminus C\}$ of two coalitions. Let $C$ be the first player in the induced game $(M, v^Q)$ and let $N\setminus C$ be the second player in this game. Since $v(C) = v(N) = 1$ and $v(N\setminus C) = 0$ we have that the coalition $N\setminus C$ is a

| $S \subset \{1,2\}$ | $v^{C,\emptyset}(S)$ | $v^{C,\{3\}}(S)$ | $v^{C,\{4\}}(S)$ | $v^{C,\{3,4\}}(S)$ |
|---|---|---|---|---|
| $\emptyset$ | 0 | 0 | 0 | 0 |
| $\{1\}$ | 0 | 1 | 1 | 1 |
| $\{2\}$ | 0 | 0 | 0 | 0 |
| $\{1,2\}$ | 1 | 1 | 1 | 1 |
| $\varphi_1^S$ | $\frac{1}{2}$ | 1 | 1 | 1 |
| $\varphi_2^S$ | $\frac{1}{2}$ | 0 | 0 | 0 |

Table 1: Characteristic functions and Shapley shares

null player in the game $(M, v^Q)$ and hence $\rho_2^S(M, v^Q) = \rho_2^B(M, v^Q) = 0$, so that the sum of the shares of the players not in $C$ is equal to zero. Since all marginal contributions are nonnegative and in both the Banzhaf CS-share function and the Owen CS-share function the weights of these contributions are nonnegative, it follows immediately that each player gets a nonnegative share. Hence, $\psi_i^O(N, v, Q) = \psi_i^B(N, v, Q) = 0$ for all $i \notin C$. Consequently it also follows that $\rho_1^S(M, v^Q) = \rho_1^B(M, v^Q) = 1$, i.e. for both the Banzhaf and Owen CS-share function the sum of the shares of the players in the winning coalition $C$ is equal to one.

In case of the Owen CS-share function we get for $i \in C$ that $\psi_i^O(N, v, Q) = \rho_i^S(C, v^C)$ with the game $(C, v^C)$ as the game $(P_k, v^{P_k})$ defined in the equation (3) with $P_k = C$. Since the Owen CS-value function is efficient and because $v(N) = 1$, it follows that $\rho_i^S(C, v^C) = \varphi_i^S(C, v^C) = \theta_i^O(N, v, Q)$, $i \in C$. Due to the coalition structure $Q = \{C,\ N \setminus C\}$, the formula for $\theta_i^O(N, v, P)$ as given equation (1) reduces for $i \in C$ to

$$\begin{aligned}\varphi_i^S(C, v^C) &= \sum_{E \subset C} \frac{1}{2} \cdot \frac{(|E|-1)!(|C|-|E|)!}{|C|!} m_E^i(N, v) + \\ &\qquad \sum_{N \setminus C \subset E} \frac{1}{2} \cdot \frac{(|E|-1)!(|C|-|E|)!}{|C|!} m_E^i(N, v)\end{aligned}$$

$$= \frac{1}{2}\varphi_i^S(C, v^{C,\emptyset}) + \frac{1}{2}\varphi_i^S(C, v^{C,N\setminus C}), \quad i \in C,$$

where $v^{C,\emptyset}(E) = v(E)$, $E \subset C$ and $v^{C,N\setminus C}(E) = v(E \cup (N \setminus C))$, $E \subset C$. Hence the Owen CS-value of a player $i \in C$ under coalition structure $Q$ is the mean of the two Shapley values of $i$ in the games $(C, v^{C,\emptyset})$ and $(C, v^{C,N\setminus C})$. In the first game we consider all marginal contributions of a player $i \in C$ to coalitions not containing any of the players outside $C$, in the second game we consider all marginal contributions of a player $i \in C$ to coalitions containing all the players not in $C$. Observe that both Shapley values get an equal weight $\frac{1}{2}$. Since also $v^{C,\emptyset}(C) = v^{C,N\setminus C}(C) = 1$, in both games the Shapley values are equal to the Shapley shares and we finally obtain that

$$\psi_i^O(N, v, Q) = \frac{1}{2}\rho_i^S(C, v^{C,\emptyset}) + \frac{1}{2}\rho_i^S(C, v^{C,N\setminus C}), \quad i \in C,$$

so that the Owen CS-share is the mean of the two Shapley shares.

In case of the Banzhaf CS-share function we get for $i \in C$ that $\psi_i^B(N, v, Q) = \rho_i^B(C, \overline{v}^C)$ with the game $(C, \overline{v}^C)$ as the game $(P_k, \overline{v}^{P_k})$ as defined in equation (10) with $P_k = C$. Clearly, $\rho_i^B(C, \overline{v}^C) = \frac{\overline{\varphi}_i^B(C,\overline{v}^C)}{\sum_{h\in C} \overline{\varphi}_h^B(C,\overline{v}^C)}$, $i \in C$. Using the special form of the coalition structure and observing that $m = 2$ it follows that the expression of the unnormalized Banzhaf CS-value $\overline{\varphi}_i^B(C, \overline{v}^C) = \theta_i(N, v, Q)$ as given in equation (8) reduces to

$$\begin{aligned} \overline{\varphi}_i^B(C, \overline{v}^C) &= \sum_{E \subset C} \frac{1}{2} \cdot 2^{-(|C|-1)} m_E^i(N, v) + \sum_{N\setminus C \subset E} \frac{1}{2} \cdot 2^{-(|C|-1)} m_E^i(N, v) \\ &= \frac{1}{2}\overline{\varphi}_i^B(C, v^{C,\emptyset}) + \frac{1}{2}\overline{\varphi}_i^B(C, v^{C,N\setminus C}), \quad i \in C. \end{aligned}$$

So, the unnormalized Banzhaf CS-value of a player $i \in C$ under coalition structure $Q$ is the mean of the two Banzhaf values of $i$ in the games $(C, v^{C,\emptyset})$ and $(C, v^{C,N\setminus C})$. The Banzhaf CS-shares of the players in $C$ follow by normalizing the sum of the values $\overline{\varphi}_i^B(C, \overline{v}^C)$, $i \in C$, to one, which results in

$$\psi_i^B(N, v, Q) = \rho_i^B(C, \overline{v}^C) = \frac{1}{\mu^B(C, \overline{v}^C)}\overline{\varphi}_i^B(C, \overline{v}^C) =$$

$$= \frac{1}{\mu^B(C, \overline{v}^C)} \left( \frac{1}{2}\mu^B(C, v^{C,\emptyset})\rho_i^B(C, v^{C,\emptyset}) + \frac{1}{2}\mu^B(C, v^{C,N\setminus C})\rho_i^B(C, v^{C,N\setminus C}) \right)$$

$$= \frac{1}{2\mu^B(C,\overline{v}^C)} \left(\mu^B(C,v^{C,\emptyset})\rho_i^B(C,v^{C,\emptyset}) + \mu^B(C,v^{C,N\setminus C})\rho_i^B(C,v^{C,N\setminus C})\right).$$

So, the Banzhaf CS-share is a weighted sum of the two Banzhaf shares with the weights determined by the values of function $\mu^B$. Since the Owen CS-shares are simply the mean of the two Shapley shares this maybe makes the Banzhaf CS-share function less attractive compared to the Owen CS-shares for simple games with a coalition structure consisting of one majority coalition and one opposing coalition containing the other players. Again the results are summarized in the following theorem.

**Theorem 4.3** *Let $(N,v) \in \mathcal{G}$ be a simple game and let $Q = \{C,\ N \setminus C\}$ be a coalition structure with $C \subset N$ a majority coalition. Then the Owen CS-shares are given by*

$$\psi_i^O(N,v,Q) = \begin{cases} \frac{1}{2}\varphi_i^S(C,v^{C,\emptyset}) + \frac{1}{2}\varphi_i^S(C,v^{C,N\setminus C}), & \text{if } i \in C, \\ 0 & \text{if } i \notin C. \end{cases}$$

*The Banzhaf CS-shares are given by*

$$\psi_i^B(N,v,Q) = \begin{cases} \frac{\mu^B(C,v^{C,\emptyset})\rho_i^B(C,v^{C,\emptyset}) + \mu^B(C,v^{C,N\setminus C})\rho_i^B(C,v^{C,N\setminus C})}{2\mu^B(C,\overline{v}^C)}, & \text{if } i \in C, \\ 0 & \text{if } i \notin C. \end{cases}$$

Observe that analogously to the formula for the Banzhaf CS-shares the Owen CS-shares can also be written as

$$\psi_i^O(N,v,Q) = \begin{cases} \frac{\mu^S(C,v^{C,\emptyset})\rho_i^S(C,v^{C,\emptyset}) + \mu^S(C,v^{C,N\setminus C})\rho_i^S(C,v^{C,N\setminus C})}{2\mu^S(C,\overline{v}^C)}, & \text{if } i \in C, \\ 0 & \text{if } i \notin C. \end{cases}$$

Since $\mu^S(C,v^{C,\emptyset}) = \mu^S(C,v^{C,N\setminus C}) = \mu^S(C,\overline{v}^C) = 1$ and because for simple games the Shapley share function is equal to the Shapley value function this expression reduces to the expression as given in the theorem.

**Example 4.4**

We consider again the weigthed majority game $(N,v)$ of Example 4.2 and now consider the coalition structure $Q = \{\{1,2\},\ \{3,4\}\}$. Again the Owen and Banzhaf CS-shares are zero for the players 3 and 4. The Owen CS-shares of the players 1 and 2 are the mean of their Shapley shares of the two 2-player games $(C, v^{C,\emptyset})$ and $(C, v^{C,N\setminus C})$ with $C = \{1,2\}$. From Table 1 it follows immediately that the Owen CS-shares of the players 1 and 2 are equal to $\frac{3}{4}$, respectively $\frac{1}{4}$, so that $\psi^O(N, v, Q) = (\frac{3}{4}, \frac{1}{4}, 0, 0)^\top$. Analogously the unnormalized Banzhaf values of the players 1 and 2 are the mean of their unnormalized Banzhaf values in the two 2-player games $(C, v^{C,\emptyset})$ and $(C, v^{C,N\setminus C})$ with $C = \{1,2\}$. Since for two player games the unnormalized Banzhaf value is equal to the Shapley value, in this case the unnormalized Banzhaf value is efficient and hence it follows immediately that also $\psi^B(N, v, Q) = (\frac{3}{4}, \frac{1}{4}, 0, 0)^\top$.

# References


AUMANN, R.J., AND DRÈZE, J.H. (1974) "Cooperative Games with Coalition Structure", *International Journal of Game Theory*, 3, 217-237.

BANZHAF, J.F. (1965), "Weighted Voting Doesn't Work: A Mathematical Analysis", *Rutgers Law Review*, 19, 317-343.

BRINK, R. VAN DEN, AND LAAN, G. VAN DER (1995), "Axiomatizations of the Normalized Banzhaf Value and the Shapley Value", to appear in *Social Choice and Welfare*.

BRINK, R. VAN DEN, AND LAAN, G. VAN DER (1998), "The Normalized Banzhaf Value and the Banzhaf Share Function", to appear in: V. Mazalov and L. Petrosjan (eds.), *Game Theory and Applications*, Vol. 4.

HALLER, H. (1994), "Collusion Properties of Values", *International Journal of Game Theory*, 23, 261-281.

HART, S., AND KURZ, M. (1983), "Endogenous Formation of Coalitions", *Econometrica*, 51, 1047-1064.

LAAN, G. VAN DER, AND BRINK, R. VAN DEN (1998), "Axiomatization of a Class of Share Functions for N-Person Games", *Theory and Decision*, 44, 117-148.

LEHRER, E. (1988), "An Axiomatization of the Banzhaf Value", *International Journal of Game Theory*, 17, 89-99.

Shapley, L.S. (1953) "A Value for $n$-Person Games", in: H.W. Kuhn and A.W. Tucker (eds.), *Contributions to the Theory of Games*, Vol. II (*Annals of Mathematics Studies* 28), Princeton University Press, 307-317.

Owen, G. (1977) "Values of Games with a Priori Unions", in: *Essays in Mathematical Economics and Game Theory*, R. Hein and O. Moeschlin (eds.), Springer-Verlag, New York, 76-88.

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