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A SIMPLICIAL ALGORITHM FOR COMPUTING ROBUST STATIONARY POINTS OF A CONTINUOUS FUNCTION ON THE UNIT SIMPLEX

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A SIMPLICIAL ALGORITHM FOR COMPUTING ROBUST STATIONARY POINTS OF A CONTINUOUS FUNCTION ON THE UNIT SIMPLEX

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Abstract A simplicial algorithm is proposed to compute a robust stationary point of a continuous function f from the $(n-1)$-dimensional unit simplex S^{n-1} into R^n. The concept of robust stationary point is a refinement of the concept of stationary point of f on S^{n-1}. Starting from an arbitrarily chosen interior point v in S^{n-1}, the algorithm generates a piecewise linear path of points in S^{n-1}. This path is followed by alternating linear programming pivot steps and replacement steps in a specific simplicial subdivision of the relative interior of S^{n-1}. In this way an approximate robust stationary point of any given a prior chosen accuracy is reached within a finite number of steps. The algorithm leaves the starting point along one out of $n!$ rays. When the path approaches the boundary of S^{n-1}, the mesh size of the triangulation goes to zero. This makes the algorithm different from all simplicial restart algorithms and homotopy algorithms known so far. Roughly speaking, the algorithm is a combination of a restart and a homotopy algorithm. However the algorithm does not need an extra dimension as homotopy algorithms do. Some examples are discussed.

Keywords: Robust stationary point, variational inequality, simplicial algorithm, subdivision, piecewise linear approximation.

1 Introduction

Let the $(n-1)$-dimensional unit simplex S^{n-1} be defined by

$$S^{n-1} = \{ x \in R^n_+ \mid \sum_{i=1}^{n} x_i = 1 \}.$$

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We assume that $f : S^{n-1} \longrightarrow R^n$ is a continuous function. Then the stationary point problem or variational inequality problem for f on S^{n-1} is to find a point $x^* \in S^{n-1}$ such that

$$(x^* - x)^T f(x^*) \geq 0$$

for any point x in S^{n-1}. We call x^* a stationary point of f on S^{n-1}. It is well known that this problem is equivalent to the Brouwer fixed point problem on S^{n-1} (see e.g Eaves [7]).

To compute a fixed point or a stationary point of a continuous function on S^{n-1}, several simplicial algorithms have been developed (Scarf [17, 18], Kuhn [11], Eaves [8], Kuhn and MacKinnon [12], van der Laan and Talman [13, 14], Doup and Talman [5], and Doup, van der Laan and Talman [6]). Todd [21] and Doup [4] presented an excellent survey on the development of simplicial algorithms. In a simplicial subdivision of S^{n-1} such algorithms search for a simplex which provides an approximate solution, by generating a sequence of adjacent simplices. The simplex with which the algorithm terminates is found within a finite number of steps. The so-called variable dimension restart algorithm, originated in van der Laan and Talman [13], can be started in an arbitrarily chosen grid point of the subdivision and generates a sequence of adjacent simplices of varying dimension. When the end simplex does not yield an approximate solution with a satisfactory accuracy, the algorithm can be restarted at the approximate solution with a finer triangulation in the hope of finding a better approximate solution within a small number of iterations.

The concept of robust stationary point is a refinement of the concept of stationary point on the unit simplex and essentially motivated from economic equilibrium problems, noncooperative games, biology and engineering applications (see e.g. Myerson [16], Yamamoto [22], and also van Damme [2]). Because a continuous function from S^{n-1} into R^n may have multiple stationary points and some of them are undesirable, we need to refine the concept of stationary point.

In this paper we propose a simplicial algorithm to compute a robust stationary point. Starting from an arbitrarily chosen interior point v in S^{n-1}, the algorithm generates a piecewise linear path of points in S^{n-1}. This path is traced by alternating linear programming pivot steps to follow a linear piece of the path and replacement steps in a simplicial subdivision of the relative interior of S^{n-1}. Within a finite number of function evaluations and linear programming pivot steps the algorithm finds an approximate robust stationary point of any a prior chosen accuracy. The path generated by the algorithm corresponds to a sequence of θ-robust stationary points of the piecewise linear approximation $\bar{f}$ of f with respect to the underlying simplicial subdivision, where $0 < \theta < 1$. This simplicial subdivision differs from other triangulations of S^{n-1}. We call it the P-triangulation. When the variable θ goes to zero, the mesh size of the triangulation converges to zero. This makes the algorithm different from all other simplicial algorithms. Roughly speaking, the algorithm is a combination of a simplicial restart algorithm and a homotopy algorithm. However, it should be mentioned that the algorithm does not need an extra dimension as homotopy algorithms do.

Although it may not be apparent from the arguments of this paper, the algorithm is implicitly related to the procedure proposed by Yamamoto [22] for the determination of a proper Nash equilibrium of finite-person games. However, the reader can easily see the difference between the procedure and the algorithm.

The remainder of this paper is summarized next. In Section 2 we introduce the definition of a robust stationary point and prove the existence of a robust stationary point for a continuous function on the unit simplex. In Section 3 we specify the P-triangulation of the unit simplex. In Section 4 we give a detailed description of the algorithm. Section 5 is devoted to some numerical examples.

2 The concept of robust stationary point

In this section we first give the definition of a robust stationary point and then show the nonemptiness of the set of robust stationary points of a continuous function on the unit simplex. Let a function $f : S^{n-1} \longrightarrow R^n$ be given and N the set of the integers $\{1, \dots, n\}$.

Definition 2.1 *For given $\theta > 0$ the point $x \in S^{n-1}$ is a θ-robust stationary point of f if*

(1) x is a relative interior point of S^{n-1};

(2) $x_k \leq \theta x_l$ if $f_k(x) < f_l(x)$, for k, l, $1 \leq k, l \leq n$.

Definition 2.2 *A point $x^* \in S^{n-1}$ is a robust stationary point of f on S^{n-1} if there exist sequences $\{\theta_t\}_1^\infty$ of positive numbers and $\{x(\theta_t)\}_1^\infty$ of θ_t-robust stationary points $x(\theta_t)$ of f such that*

$$\lim_{t\to\infty} \theta_t = 0 \ \ and \ \ \lim_{t\to\infty} x(\theta_t) = x^*.$$

We remark that if a stationary point x^* of f lies in the relative interior of S^{n-1}, then x^* must be a robust stationary point of f. Some examples given in Section 5 will demonstrate that the concept of robust stationary point is a refinement of the concept of stationary point.

Lemma 2.3 *Let $f : S^{n-1} \longrightarrow R^n$ be a continuous function. If $x^* \in S^{n-1}$ is a robust stationary point of f, then x^* is also a stationary point of f.*

Proof: We only need to consider two cases. If x^* lies in the relative interior of S^{n-1}, it implies that $f_i(x^*) = f_j(x^*)$ for $i, j \in N$. Hence we have

$$(x^* - x)^T f(x^*) = \sum_{i=1}^{n} (x_i^* - x_i) f_i(x^*) = 0$$

for any $x \in S^{n-1}$. It means that x^* is a stationary point of f. On the other hand, if x^* is on the boundary of S^{n-1}, there exists a proper subset J of N such that $x_j^* = 0$ for $j \in J$. It follows from Definitions 2.1 and 2.2 that $f_i(x^*) = f_j(x^*)$ for $i, j \in N \backslash J$ and $f_i(x^*) \geq f_j(x^*)$ for $i \in N \backslash J$ and $j \in J$. Now for given $l \in N \backslash J$, we have

$$(x^* - x)^T f(x^*) = \sum_{i \in N \backslash J} (x_i^* - x_i) f_i(x^*) - \sum_{j \in J} x_j f_j(x^*) \geq \sum_{i=1}^{n} (x_i^* - x_i) f_l(x^*) = 0$$

for any $x \in S^{n-1}$. It also implies that x^* is a stationary point of f. □

Theorem 2.4 *Let $f : S^{n-1} \longrightarrow R^n$ be a continuous function. Then f has at least one robust stationary point in S^{n-1}.*

Proof: We first show that there exists at least one θ-robust stationary point, for any θ, $0 < \theta < 1$. Given such a θ, let $\delta = \frac{1}{n}\theta^n$ and define

$$S(\theta) = \{ x \in S^{n-1} \mid x_i \geq \delta, \ i = 1, ..., n \}.$$

It is clear that $S(\theta)$ is a nonempty, convex, compact subset of S^{n-1}. We further define a set-valued correspondence F on $S(\theta)$ by

$$F(x) = \{ y \in S(\theta) \mid \text{ if } f_i(x) < f_j(x) \text{ then } y_i \leq \theta y_j \text{ for any } i, j \}, \ x \in S(\theta).$$

$F(x)$ is obviously a closed convex set for every $x \in S(\theta)$. Given $x \in S(\theta)$ and $i \in \{ 1, ..., n \}$, let $\Delta(i)$ be the number of j's such that $f_i(x) < f_j(x)$ and let

$$y_i^* = \theta^{\Delta(i)} / \sum_{l=1}^{n} \theta^{\Delta(l)}.$$

Then $y_i^* \geq \delta$ for $i = 1, ..., n$. Hence $y^* \in F(x)$ and therefore $F(x)$ is nonempty. Moreover the continuity of f guarantees that F is upper semicontinuous. Thus F satisfies all conditions of the Kakutani fixed point theorem and so there exists a point $x(\theta) \in S(\theta)$ such that $x(\theta) \in F(x(\theta))$. It is easily seen that $x(\theta)$ is a θ-robust stationary point of f.

So for every $0 < \theta < 1$, f has a θ-robust stationary point $x(\theta)$. Now let us take a sequence $\{ \theta_t \}_1^\infty$ of numbers between 0 and 1 converging to zero and a sequence of θ_t-robust stationary points of f. Since S^{n-1} is a compact set, there exists a subsequence converging to a cluster point $x^* \in S^{n-1}$. Clearly, x^* is a robust stationary point of f. □

In the subsequent sections we will design an algorithm to compute a robust stationary point.

3 The P-triangulation of the unit simplex

We first introduce some notations to be used below. Z and Z_0 represent the set of positive integers and the set of nonnegative integers, respectively. The i-th unit vector in R^n is denoted by $e(i)$, $i \in N$. Moreover, $J \subset N$ denotes a proper subset J of N. Let v be a point in the relative interior of S^{n-1}. The point v will be the starting point of the algorithm. We define a vector $p \in S^{n-1}$ by

$$\begin{aligned} p_i = &\ v_{j_i}, \text{ for } i \in N \\ p_l \geq &\ p_m, \text{ for } l \leq m \end{aligned}$$

where $(j_1, j_2, \ldots, j_n)$ is a permutation of $(1, 2, ..., n)$. For $t \in [0, 1]$, let

$$p_i(t) = p_i t^{i-1} / \sum_{j \in N} p_j t^{j-1}, \text{ for } i \in N.$$

It is readily seen that $p_1(t) \geq p_2(t) \geq ... \geq p_n(t)$ for $t \in [0, 1]$.

Definition 3.1

For $t \in [0, 1]$, the set $A(t)$ is defined by

$$\begin{aligned} A(t) = \{ x \in R^n \quad | & \sum_{i \in N} x_i = 1 \\ & \sum_{j \in J} x_j \leq \sum_{j=1}^{k} p_j(t) \text{ for any } J \subset N \text{ with } k = |J| \}. \end{aligned}$$

It is easily seen that $A(0) = S^{n-1}$, and that if v is the barycenter of S^{n-1}, then $A(1) = \{ v \}$. More generally for every $t \in [0, 1]$ we have that $v \in A(t)$ and v is a vertex of $A(1)$. Moreover $A(t)$ is a polytope for every $t \in [0, 1]$.

For $J \subset N$ and $t \in [0, 1]$, we define $a(J)$ and $b_J(t)$ by

$$\begin{aligned} a(J) &= \textstyle\sum_{j \in J} e(j), \\ b_J(t) &= \textstyle\sum_{j=1}^{l} p_j(t) \quad \text{with } l = |J|. \end{aligned}$$

Let $\mathcal{I} = \{ I = (I_1, I_2, \ldots, I_m) \mid \emptyset \neq I_1 \subset \ldots \subset I_m \subset N \}$. We say that $I \in \mathcal{I}$ conforms to $J \in \mathcal{I}$, if it holds that every component of I is also a component of J. For $I \in \mathcal{I}$ and a positive integer k, let

$$F(k, I) = \{ x \in A(2^{-k}) \mid a^T(I_i) x = b_{I_i}(2^{-k}) \text{ for every } i \in \{1, 2, ..., m\} \}.$$

Then $F(k, I)$ is a face of $A(2^{-k})$ with dimension equal to $n - 1 - m$. For $I \in \mathcal{I}$, let

$$F(0, 1; I) = \{ x \mid x = av + (1 - a)z \text{ for some } z \in F(1, I) \text{ and some } a \in [0, 1] \}$$

and for $k \in Z$

$$F(k, k+1; I) = \{\, x \mid \quad x = ay + (1-a)z \ \text{ for some } y \in F(k, I), \ \text{some } z \in F(k+1, I), \ \text{ and some } a \in [0,1] \,\}.$$

Figure 1 shows the subdivision of S^{n-1} for $n = 3$ and $v = (1/2, 1/3, 1/6)^T$.

Figure 1. The subdivision of S^{n-1} for $n = 3$ and $v = (1/2, 1/3, 1/6)^T$.

For $I \in \mathcal{I}$, we denote the union of $F(k, k+1; I)$ over all $k = 0, 1, \ldots$ by $F(I)$. Notice that the dimension of $F(I)$ is equal to $t = n - m$. A simplicial subdivision underlying the algorithm must be such that every set $F(k, k+1; I)$ is subdivided into t-dimensional simplices. Such a triangulation can be described as follows. For $I \in \mathcal{I}$, we denote $v(0, I) = v$ and for $k \in Z$, let $v(k, I)$ be a relative interior point (e.g. the barycenter) of $F(k, I)$. For $I \in \mathcal{I}$, if I consists of $n-1$ components, then $F(k, I)$ is a vertex of $A(2^{-k})$. For general $I \in \mathcal{I}$, let $F(k, I(n-1))$ be a vertex of $F(k, I)$, i.e. $I(n-1)$ has $n-1$ components and I conforms to $I(n-1)$. Moreover let $(J_1, J_2, \ldots, J_t) = \gamma(I, I(n-1))$ be a conformation of I and $I(n-1)$, where $t = n-m$, i.e., $J_1 = I(n-1)$, $J_k \in \mathcal{I}$ for $k = 2, \ldots, t-1$, $J_t = I$, J_k conforms to J_{k-1} and has one component less than J_{k-1} for $k = 2, \ldots, t$. For given $k \in Z_0$, $I \in \mathcal{I}$ and $\gamma(I, I(n-1))$, the subset $F(k, k+1; I, \gamma(I, I(n-1)))$ of $F(k, k+1; I)$ is defined to be the convex hull of $v(k, J_1)$, $v(k, J_2), \ldots, v(k, J_t)$, $v(k+1, J_1)$, $v(k+1, J_2), \ldots,$ and $v(k+1, J_t)$, so

$$\begin{aligned} F(k, k+1; I, \gamma(I, I(n-1))) \;=\; & \{\, x \in S^{n-1} \mid x = v(k, I(n-1)) + \alpha q_0 \\ & + \sum_{j=1}^{t-1} \alpha_j q_j(\alpha), \\ 0 \;& \le \alpha \le 1, \text{ and } 0 \le \alpha_{t-1} \le \ldots \le \alpha_1 \le 1 \,\} \end{aligned}$$

where $q_0 = (v(k+1, J_1) - v(k, J_1))$, and for $j = 1, \ldots, t-1$, $0 \le \alpha \le 1$,

$$q_j(\alpha) = \alpha(v(k+1, J_{j+1}) - v(k+1, J_j)) + (1-\alpha)(v(k, J_{j+1}) - v(k, J_j)).$$

The dimension of $F(k, k+1; I, \gamma(I, I(n-1)))$ is equal to t and $F(k, k+1; I)$ is the union of $F(k, k+1; I, \gamma(I, I(n-1)))$ over all conformations $\gamma(I, I(n-1))$ and over all index sets $I(n-1)$ conformed by I.

Let d be an arbitrary positive integer.

Definition 3.2 *For given $k \in Z_0$, $I \in \mathcal{I}$ and $\gamma(I, I(n-1))$, the set $G^d(k, k+1; I, \gamma(I, I(n-1)))$ is the collection of $t-$simplices $\sigma(a, \pi)$ with vertices $y^1, \ldots, y^{t+1}$ in $F(k, k+1; I, \gamma(I, I(n-1)))$ such that* . .

(1) $y^1 = v(k, I(n-1)) + a(0)d^{-1}q_0 + \sum_{j=1}^{t-1} a(j)q_j(a(0)/d)/(a(0)+kd)$ *where* $a = (a(0), a(1), \ldots, a(n-2))^T$ *is a vector of integers such that* $0 \leq a(0) \leq d-1$ *and* $a(n-2) = \ldots = a(t) = 0 \leq a(t-1) \leq \ldots \leq a(2) \leq a(1) \leq a(0) + kd$;

(2) $\pi = (\pi_1, \ldots, \pi_t)$ *is a permutation of* $(0, 1, \ldots, t-1)$ *such that* $s < s'$ *if for some* $q \in \{0, 1, \ldots, t-2\}$ *it holds that* $\pi_s = q$, $\pi_{s'} = q+1$, $a(q) = a(q+1)$ *in case* $q \geq 1$, *and* $a(0) + kd = a(1)$ *in case* $q = 0$;

(3) *Let* i *be such that* $\pi_i = 0$. *Then*

$$\begin{aligned} y^{j+1} &= y^j + q_{\pi_j}(a(0)/d)/(a(0)+kd), \; j = 1, \ldots, i-1, \\ y^{i+1} &= v(k, I(n-1)) + (a(0)+1)d^{-1}q_0 \\ &+ \sum_{j=1}^{t-1} a(j)q_j((a(0)+1)/d)/(a(0)+1+kd) \\ &+ \sum_{j=1}^{i-1} q_{\pi_j}((a(0)+1)/d)/(a(0)+1+kd), \\ y^{j+1} &= y^j + q_{\pi_j}((a(0)+1)/d)/(a(0)+1+kd), \; i < j \leq t. \end{aligned}$$

The set $G^d(k, k+1 : I, \gamma(I, I(n-1)))$ is a simplicial subdivision of $F(k, k+1 : I, \gamma(I, I(n-1)))$ with grid size d^{-1}. Moreover, the union $G^d(k, k+1 : I)$ of $G^d(k, k+1 : I, \gamma(I, I(n-1)))$ over all conformations $\gamma(I, I(n-1))$ and $I(n-1)$ conformed by I is a simplicial subdivision of $F(k, k+1 : I)$, and the union $G^d(k, k+1)$ of $G^d(k, k+1 : I)$ over all sets $I \in \mathcal{I}$ induces a triangulation of $A(2^{-k-1}) \backslash A(2^{-k})$. Taking the union $G^d(k)$ of $G^d(j, j+1)$ over $j = 0, 1, \ldots, k-1$, we obtain a simplicial subdivision of $A(2^{-k})$ with grid size d^{-1}. The union of $G^d(k)$ over all $k \in Z_0$ is a simplicial subdivision of the relative interior of S^{n-1} and is called the P-triangulation of S^{n-1} with grid size d^{-1}. Observe that for $I \in \mathcal{I}$ the union $G^d(I)$ of $G^d(k, k+1 : I)$ over $k = 0, 1, \ldots$ is a simplicial subdivision of the set $F(I)$. The P-triangulation of S^{n-1} for $n = 3$, $d = 2$ and $v = (1/3, 1/3, 1/3)$ is illustrated in Figure 2.

Figure 2. The P-triangulation of S^{n-1} for $n = 3$, $d = 2$ and $v = (1/3, 1/3, 1/3)$.

As norm we use the Euclidean norm $\|\cdot\|$ in R^n. For a set B in R^n, we define the diameter of B by

$$diam(B) = \sup\{\, \|y^1 - y^2\| \mid y^1, y^2 \in B \,\}.$$

Then for given $k \in Z_0$ the mesh size of $G^d(k, k+1)$ is equal to

$$\delta_{k,d} = \sup\{\, diam(\sigma) \mid \sigma \in G^d(k, k+1) \,\}.$$

Now we have the following property.

Lemma 3.3 *For the* P-*triangulation of* S^{n-1} *with grid size* d^{-1}, *it holds that*

$$\lim_{k \to \infty} \delta_{k,d} = 0.$$

4 The algorithm

In this section we discuss how to operate the algorithm in the P-triangulation of S^{n-1} to approximate a robust stationary point of a continuous function on S^{n-1}. Starting at the point v, the algorithm will generate a sequence of adjacent simplices of the P-triangulation in the set $F(I)$ having I-complete common facets, for varying $I \in \mathcal{I}$.

Definition 4.1

Let be given the function $f : S^{n-1} \longrightarrow R^n$. For given $I = (I_1, ..., I_m) \in \mathcal{I}$ and $s = t-1$ or t, where $t = n-m$, an s-simplex σ with vertices $y^1, ..., y^{s+1}$ is I-complete if the system of linear equations

$$\sum_{i=1}^{s+1} \lambda_i \begin{pmatrix} f(y^i) \\ 1 \end{pmatrix} - \sum_{j=1}^{m} \mu_j \begin{pmatrix} a(I_j) \\ 0 \end{pmatrix} - \beta \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.1}$$

where e is an n-vector of 1's, has a solution λ_i^*, $i = 1, ..., s+1$, μ_j^*, $j = 1, ..., m$, and β^* with $\lambda_i^* \geq 0$, $i = 1, ..., s+1$, and $\mu_j^* \geq 0$, $j = 1, ..., m$.

A solution λ_i^*, $i = 1, ..., s+1$, μ_j^*, $j = 1, ..., m$, and β^* will be denoted by $(\lambda^*, \mu^*, \beta^*)$. For $s = t-1$ we assume that the system (4.1) has a unique solution with $\lambda_i^* > 0$, $i = 1, ..., t$, and $\mu_j^* > 0$, $j = 1, ..., m$, and that for $s = t$ at most one variable of (λ^*, μ^*) is equal to zero (nondegeneracy assumption).

The algorithm starts to leave the point v in one out of $n!$ directions. This direction is uniquely determined by $f(v)$. Because of the nondegeneracy assumption, all components of the vector $f(v)$ are different. Let $(i_1, ..., i_n)$ be a permutation of the set $(1, ..., n)$ such that $f_{i_1}(v) > ... > f_{i_n}(v)$. Then the 0-dimensional simplex $\{v\}$ is I^0-complete with $I^0 = (I_1^0, ..., I_{n-1}^0)$ where $I_j^0 = \{i_1, ..., i_j\}$ for $j = 1, ..., n-1$. Moreover, $\{v\}$ is a facet of a unique 1-simplex σ^0 in $F(I^0)$, where $\sigma^0 = \sigma(a, \pi)$ with $a = 0$ and $\pi = (0)$. Since for given $I \in \mathcal{I}$ an I-complete t-simplex has at most two I-complete facets and a facet of a t-simplex in $F(I)$ is a facet of at most one other t-simplex in $F(I)$, we obtain that the I-complete t-simplices $\sigma(a, \pi)$ in $F(I)$, determine sequences of adjacent t-simplices in $F(I)$ with I-complete common facets. As described below, the sequences of the I-complete t-simplices in $F(I)$ can be uniquely linked together for varying $I \in \mathcal{I}$ to obtain sequences of adjacent simplices of varying dimension. Under the nondegeneracy assumption, one of these sequences starts with σ^0 in $F(I^0)$ and is followed by the algorithm, so starting at the point v, the algorithm generates a unique sequence of I-complete adjacent t-simplices in $F(I)$ of varying dimension. In this way within a finite number of steps either the algorithm reaches a ponit $\bar{x}$ in an $(n-1)$-dimensional simplex for which $\bar{f}_i(\bar{x}) = \bar{f}_j(\bar{x})$ for every i and $j \in N$, where $\bar{f}$ is the piecewise linear approximation of f with respect to the P-triangulation, or for $k = 1, 2 ...$ the algorithm finds an

I-complete $(t-1)$-simplex in $F(k, I)$ for some $I \in \mathcal{I}$. Suppose the latter case holds, then we have the following result.

Lemma 4.2 *For $k \in Z$ and $I \in \mathcal{I}$, let σ with vertices $y^1, \ldots, y^t$ be an I-complete $(t-1)$-simplex lying in $F(k, I)$. Let $(\lambda^*, \mu^*, \beta^*)$ be the corresponding unique solution of system (4.1). Then $x = \sum_{i=1}^{t} \lambda_i^* y^i$ is a 2^{-k}-robust stationary point of the piecewise linear approximation $\bar{f}$ of f with respect to the P-triangulation. Moreover, x is a stationary point of $\bar{f}$ on $A(2^{-k})$.*

Proof: Since $I = (I_1, I_2, \ldots, I_m) \in \mathcal{I}$, there exist $l_1 < l_2 < \ldots < l_m$ such that

$$\begin{aligned} I_1 &= \{ i_1, \ldots, i_{l_1} \} \\ I_2 &= \{ i_1, \ldots, i_{l_1}, i_{l_1+1}, \ldots, i_{l_2} \} \\ &\ldots\ldots \\ I_m &= \{ i_1, \ldots, i_{l_m} \} \\ N \setminus I_m &= \{ i_{l_m+1}, \ldots, i_n \}. \end{aligned}$$

Then it follows from equation (4.1) that

$$\begin{aligned} & \bar{f}_{i_1}(x) = \ldots = \bar{f}_{i_{l_1}}(x) = \mu_1^* + \ldots + \mu_m^* + \beta^* \\ > & \bar{f}_{i_{l_1+1}}(x) = \ldots = \bar{f}_{i_{l_2}}(x) = \mu_2^* + \ldots + \mu_m^* + \beta^* > \\ & \ldots\ldots \\ > & \bar{f}_{i_{l_{m-1}+1}}(x) = \ldots = \bar{f}_{i_{l_m}}(x) = \mu_m^* + \beta^* \\ > & \bar{f}_{i_{l_m+1}}(x) = \ldots = \bar{f}_{i_n}(x) = \beta^*, \end{aligned}$$

where $\mu_i^* > 0$ for $i = 1, \ldots, m$. Now it is not difficult to check that

$$x_i \leq 2^{-k} x_j \text{ whenever } \bar{f}_i(x) < \bar{f}_j(x).$$

It means that x is a 2^{-k}-robust stationary point of the piecewise linear approximation $\bar{f}$ of f with respect to the P-triangulation.

Moreover, for each face $F(k, I)$, $I \in \mathcal{I}$, let $F^*(I)$ be the set of all n-dimensional vectors y such that every point of $F(k, I)$ is a solution of the linear programming problem

$$\max\ y^T \hat{x} \quad \text{subject to } \hat{x} \in A(2^{-k}).$$

Then the stationary point problem for $\bar{f}$ on $A(2^{-k})$ is the problem of finding a point x in $A(2^{-k})$ such that $\bar{f}(x) \in F^*(I)$ for a minimum face $F(k, I)$ of $A(2^{-k})$ containing x. Duality theory implies that $F^*(I) = \{ y \mid y = \sum_{i=1}^{m} \mu_i a(I_i) + \beta e,\ \mu_i \geq 0 \text{ for } i = 1, \ldots, m, \text{ and } \beta \in R \}$. It follows from equation (4.1) that $\bar{f}(x) \in F^*(I)$. Hence x is a stationary point of $\bar{f}$ on $A(2^{-k})$. □

The next lemma shows that a 2^{-k}-robust stationary point of $\bar{f}$ is an approximate 2^{-k}-robust stationary point of f.

Lemma 4.3 *Let* $\eta_{k,d} = \sup\{ diam(f(\sigma)) \mid \sigma \in G^d(k-1,k) \}$. *Let* x *be a* 2^{-k}-*robust stationary point of the piecewise linear approximation* $\bar{f}$ *of* f *with respect to the* P-*triangulation obtained by the algorithm, so that* $x \in F(I,k)$ *for some* $I \in \mathcal{I}$. *Then* $f(x)$ *lies in the* $\eta_{k,d}$-*neighborhood of* $F^*(I)$, *i.e. there is a* $y \in F^*(I)$ *such that* $||y - f(x)|| \leq \eta_{k,d}$.

Proof: Let $y^1, \ldots, y^t$ be the vertices of a $(t-1)$-simplex of $G^d(k-1,k)$ in $F(k,I)$ containing x. Then $\bar{f}(x) = \sum_{j=1}^t \lambda_j^* f(y^j)$ lies in $F^*(I)$, where $\lambda_1^*, \ldots, \lambda_t^*$ are convex combination coefficients such that $x = \sum_{j=1}^t \lambda_j^* y^j$. Therefore

$$\begin{aligned} ||\bar{f}(x) - f(x)|| &= ||\sum_{j=1}^{t} \lambda_j^* f(y^j) - f(x)|| \\ &= ||\sum_{j=1}^{t} \lambda_j^* (f(y^j) - f(x))|| \\ &= \sum_{j=1}^{t} \lambda_j^* ||f(y^j) - f(x)|| \\ &\leq \eta_{k,d}. \end{aligned}$$

□

Since S^{n-1} is compact and f is continuous on S^{n-1}, the error $\eta_{k,d}$ tends to zero as the mesh size $\delta_{k,d}$ goes to zero when k goes to infinity. Let x^k be a 2^{-k}-robust stationary point of $\bar{f}$ and $\eta_{k,d}$ the error in Lemma 4.3. Then the algorithm generates a sequence $\{ x^h \mid h = 1, 2, \ldots \}$ of approximate 2^{-k}-robust stationary points of f which therefore has a cluster point x^*. For simplicity of notation we can assume that this sequence itself converges to x^*. We are now ready to state the following corollary.

Corollary 4.4 *Suppose* x^k *be an approximate* 2^{-k}-*robust stationary point generated by the algorithm, for* $k = 1, 2, \ldots$. *Then the sequence* $\{ x^k \mid k = 1, 2, \ldots \}$ *has a cluster point and any cluster point is a robust stationary point of* f *on* S^{n-1}.

Proof: The continuity of f, the property of the P-triangulation and the compactness of S^{n-1} imply that for any given $\epsilon > 0$, there exists a positive integer M, such that for $k \in Z$ with $k \geq M$, there is a 2^{-k}-robust stationary point $\bar{x}^k \in A(2^{-k})$ of f which is in the ϵ-neighborhood of x^k. On the other hand, since $\lim_{k\to\infty} x^k = x^*$, it immediately follows that

$$\lim_{k\to\infty} \bar{x}^k = x^*.$$

Hence x^* is a robust stationary point of f on S^{n-1}.

□

In case the algorithm terminates with an $(n-1)$-dimensional simplex σ with vertices $y^1, \dots, y^n$, then $\bar{x} = \sum_{i=1}^{n} \lambda_i^* y^i$ is a robust stationary point of $\bar{f}$. If the accuracy of approximation is not satisfactory, the algorithm can be restarted at the point $\bar{x}$ with a smaller grid size d^{-1} to find a better approximate robust stationary point hopefully within a small number of steps. Without loss of generality we assume that the algorithm generates a sequence $\{ \bar{x}^h \mid h = 1, 2, \dots \}$, where $\bar{x}^h$ is the robust stationary point of $\bar{f}$ corresponding to the grid size d_h^{-1} for an increasing sequence of positive integers $\{ d_h \mid h = 1, 2 \dots \}$. It is readily seen that for every $k \in Z_0$, the mesh size δ_{k,d_h} tends to zero as h goes to infinity. Therefore the sequence $\{ \bar{x}^h \mid h = 1, 2, \dots \}$ has a subsequence converging to a cluster point x^*. Clearly, x^* is a robust stationary point of f on S^{n-1}.

As described above, starting at the point v, the algorithm generates a unique sequence of adjacent t-simplices $\sigma(a, \pi)$ in $F(I)$ for varying $I \in \mathcal{I}$ of varying dimension $t = n - m$. When, with respect to some $\sigma(a, \pi)$ with vertices $y^1, \dots, y^{t+1}$ in some $G^d(k, k+1; I, \gamma(I, I(n-1)))$ for some $k \in Z_0$ and $\gamma(I, I(n-1))$, the variable λ_q, for some q, $1 \leq q \leq t+1$, becomes zero through a linear programming pivot step in (4.1), then the replacement step determines the unique t-simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in $F(k, k+1; I, \gamma(I, I(n-1)))$ sharing with σ the common facet τ opposite vertex y^q unless this facet lies in the boundary of $F(k, k+1; I, \gamma(I, I(n-1)))$. If τ does not lie in the boundary of the set $F(k, k+1; I, \gamma(I, I(n-1)))$, then $\bar{\sigma}(\bar{a}, \bar{\pi})$ can be obtained from a and π as given in Table 1, where $E(j-1)$ is the j-th unit vector in R^{n-1}, $j = 1, \dots, n-1$.

Table 1. Parameters of $\bar{\sigma}$ if the vertex y^q of $\sigma(a, \pi)$ is replaced.

	$\bar{\pi}$	$\bar{a}$
$q = 1$	$(\pi_2, \dots, \pi_t, \pi_1)$	$a + E(\pi_1)$
$1 < q < t+1$	$(\pi_1, \dots, \pi_{q-2}, \pi_q, \pi_{q-1}, \pi_{q+1}, \dots, \pi_t)$	a
$q = t+1$	$(\pi_t, \pi_1, \dots, \pi_{t-1})$	$a - E(\pi_t)$

The algorithm continues with $\bar{\sigma}$ by making a linear programming (lp) pivot step in (4.1) with $(f(\bar{y})^T, 1)^T$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet τ. In case a facet τ of a simplex in $G^d(k, k+1; I, \gamma(I, I(n-1)))$ is not a facet of another simplex in $G^d(k, k+1; I, \gamma(I, I(n-1)))$, then τ lies in the boundary of $F(k, k+1; I, \gamma(I, I(n-1)))$. According to Definition 3.2 we have the following lemma.

Lemma 4.5 *Let $\sigma(a, \pi)$ be a t-simplex in $F(k, k+1; I, \gamma(I, I(n-1)))$. The facet τ of σ opposite the vertex y^q, $1 \leq q \leq t+1$, lies in the boundary of this set if and only if one of the following cases occurs:*

(i) $1 < q < t+1$, $\pi_q = h+1$, $\pi_{q-1} = h$ *for some* $h \in \{0, 1, ..., t-2\}$, *and* $a(h) = a(h+1)$ *in case* $h \geq 1$, *and* $a(0) + kd = a(1)$ *in case* $h = 0$;

(ii) $q = t+1$, $\pi_t = t-1$, *and* $a(t-1) = 0$;

(iii) $q = 1$, $\pi_1 = 0$, *and* $a(0) = d-1$;

(iv) $q = t+1$, $\pi_t = 0$, *and* $a(0) = 0$.

Suppose the algorithm generates the simplex $\sigma(a, \pi)$ as given in Lemma 4.5 and λ_q becomes zero after making an lp pivot step in (4.1). Then the facet τ of σ opposite to the vertex y^q is I-complete. In case *(iii)* the facet τ lies in the face $F(k+1, I)$ of $A(2^{-k-1})$ and the algorithm reaches a 2^{-k-1}-robust stationary point $\bar{x} = \sum_{i=2}^{t+1} \lambda_i^* y^i$ of $\bar{f}$ lying in $F(k+1, I)$. If k is large enough, then $\bar{x}$ is an approximate robust stationary point of f. Otherwise, the algorithm proceeds with $\bar{\sigma}$ by making an lp pivot step in (4.1) with $(f^T(\bar{y}), 1)^T$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet τ and $\bar{\sigma}$ in $F(k+1, k+2; I, \gamma(I, I(n-1)))$ is obtained according to Table 1.

In case *(iv)* the facet τ lies in the face $F(k, I)$ of $A(2^{-k})$ and the algorithm continues with $\bar{\sigma}$ by making an lp pivot step in (4.1) with $(f^T(\bar{y}), 1)^T$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet τ and $\bar{\sigma}$ in $F(k-1, k; I, \gamma(I, I(n-1)))$ is obtained also from Table 1.

In case *(i)* and if $h \geq 1$, the facet τ is a facet of the t-simplex $\bar{\sigma} = \sigma(a, \pi)$ in $F(k, k+1; I)$ lying in the subset $F(k, k+1; I, \bar{\gamma}(I, I(n-1)))$ with

$$\bar{\gamma}(I, I(n-1)) = (J_1, ..., J_h, \bar{J}_{h+1}, J_{h+2}, ..., J_t),$$

where $\bar{J}_{h+1} \in \mathcal{I}$, $\bar{J}_{h+1} \neq J_{h+1}$, is uniquely determined by the properties that $\bar{J}_{h+1}$ conforms to J_h, has one component less than J_h, and is conformed by J_{h+2}. In case *(i)* and if $h = 0$, then τ is a facet of the t-simplex $\bar{\sigma} = \sigma(a, \pi)$ in $F(k, k+1; I, \bar{\gamma}(I, \bar{I}(n-1)))$ with $\bar{I}(n-1)$ and $\bar{\gamma}$ defined as follows. Let $J_1 = I(n-1) = (I_1, ..., I_{n-1})$. In case $J_2 = (I_1, ..., I_{n-2})$, we have $\bar{I}(n-1) = (I_1, ..., I_{n-2}, \bar{I}_{n-1})$ with $\bar{I}_{n-1} = I_{n-2} \bigcup N \backslash I_{n-1}$. In case $J_2 = (I_2, ..., I_{n-1})$, let $\bar{I}(n-1) = (\bar{I}_1, I_2, ..., I_{n-1})$ with $\bar{I}_1 = I_2 \backslash I_1$. Finally if $J_2 = (I_1, ..., I_k, I_{k+2}, ..., I_{n-1})$ for some $k \in \{1, ..., n-3\}$, we have $\bar{I}(n-1) = (I_1, ..., I_k, \bar{I}_{k+1}, I_{k+2}, ..., I_{n-1})$ with $\bar{I}_{k+1} = I_k \bigcup I_{k+2} \backslash I_{k+1}$. Then $\bar{\gamma}(I, \bar{I}(n-1)) = (\bar{I}(n-1), J_2, ..., J_t)$. In both subcases of case *(i)* the algorithm continues with making a pivot step in (4.1) with $(f^T(\bar{y}), 1)^T$, where $\bar{y}$ is the vertex of the new t-simplex $\bar{\sigma}$ opposite the facet τ.

In case *(ii)* the facet lies in the set $F(k, k+1; J_{t-1})$ of $F(I)$. More precisely, τ is the $(t-1)$-simplex $\sigma(a, \bar{\pi})$ in $F(k, k+1; \bar{I}, \bar{\gamma}(\bar{I}, I(n-1)))$, where $\bar{I} = J_{t-1}$, $\bar{\gamma}(\bar{I}, I(n-1)) = (J_1, ..., J_{t-1})$, and $\bar{\pi} = (\pi_1, ..., \pi_{t-1})$. The algorithm now proceeds with making a pivot step in (4.1) with $(-a^T(I_h), 0)^T$, where I_h is the unique component of J_{t-1} but not of J_t.

Finally, if through a linear programming pivot step in (4.1), the variable μ_h becomes 0 for some $h \in \{ 1, \dots, m \}$, then the algorithm terminates with the approximate robust stationary point $\bar{x} = \sum_i \lambda_i^* y^i$ of f if $m = 1$ and restarts at the point $\bar{x}$ with a smaller grid size in case the accuracy is not satisfactory. Otherwise, the simplex $\sigma(a, \pi)$ is a facet of a unique $(t+1)$-simplex $\bar{\sigma}$ in $F(\bar{I})$ with $\bar{I} = (I_1, \dots, I_{h-1}, I_{h+1}, \dots, I_m)$. More precisely, $\bar{\sigma} = \sigma(a, \bar{\pi})$ lies in $F(k, k+1; \bar{I}, \bar{\gamma}(\bar{I}, I(n-1)))$, where $\bar{\gamma}(\bar{I}, I(n-1)) = (\gamma, \bar{I})$, and $\bar{\pi} = (\pi_1, \dots, \pi_t, t)$. The algorithm continues by making a pivot step in (4.1) with $(f^T(\bar{y}), 1)^T$, where $\bar{y}$ is the vertex of $\bar{\sigma}$ opposite the facet σ. This concludes the description of how the algorithm works in the P-triangulation of S^{n-1}.

5 Examples

In the current section we give some examples to show the power of robust stationary point concept. Let us briefly review the standard model of a pure exchange economy. For detail, we refer to Debreu [3]. In such an economy there are, say, n commodities and a finite number of consumers, each having a vector of initial endowments. Exchange of commodities are based on relative prices. All consumers exchange goods in order to maximize their utility under their initial wealth constraints. This economy can be characterized by an excess demand function $z : R^n_+ \backslash \{0\} \longrightarrow R^n$ which satifies the following standard conditions:

(i) z is a continuous function;

(ii) $z(\lambda p) = z(p)$ for any $\lambda > 0$ and $p \in R^n_+ \backslash \{0\}$ (homogeneity);

(iii) $p^T z(p) = 0$ for $p \in R^n_+ \backslash \{0\}$ (Walras' law).

The element $p^* \in R^n_+ \backslash \{0\}$ is an equilibrium price vector if $z(p^*) \leq 0$. Note that homogeneity permits us to normalize the price vectors to the $(n-1)$-dimensional unit simplex S^{n-1}. Now it is not hard to show that this problem is equivalent to the stationary point problem on S^{n-1}. We present two examples. Example 1: there are two goods. The excess demand function is given by $z(p) = (p_1 p_2^2 (1 - p_1^2), -p_1^2 p_2 (1 - p_1^2))^T$ for $p \in S^1$. There are two equilibria (i.e. stationary points) $x = (1, 0)^T$, $y = (0, 1)^T$. However only x is a robust stationary point. Example 2: there are three goods. The excess demand function is given by $z(p) = (p_2 p_3, p_1 p_3^2, -p_1 p_2 (1 + p_3))^T$ for $p \in S^2$. The set of stationary points is $\{ p \in S^2 | p_3 = 0 \}$. But z only has one robust stationary point: $p^* = (1, 0, 0)^T$.

Finally, we conclude with one more example: the function is defined by $f(x) = (x_1 + x_2, x_2 + x_3, x_3 + x_1)^T$ for $x \in S^2$. The set of stationary points is

$$\{ (1/3, 1/3, 1/3)^T, (1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T \}.$$

However, f just has one robust stationary point: $(1/3, 1/3, 1/3)^T$.

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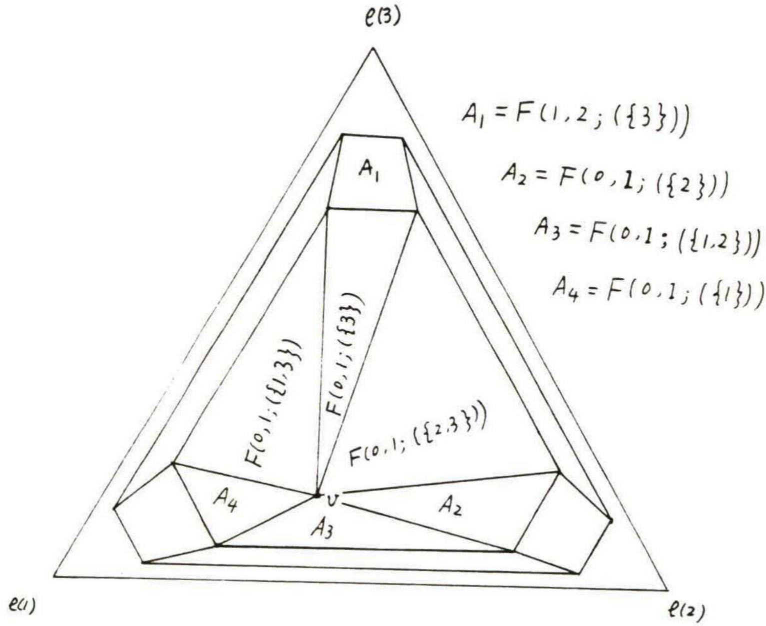

Figure 1. The subdivision of S^{n-1} for $n = 3$ and $v = (1/2, 1/3, 1/6)^T$.

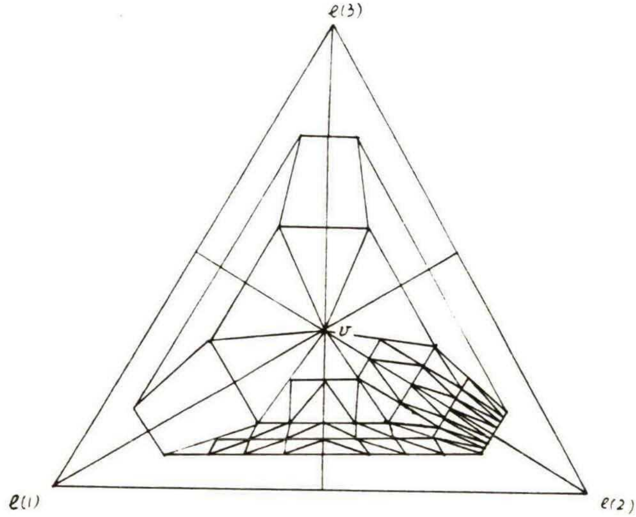

Figure 2. The P-triangulation of S^{n-1} for $n=3$, $d=2$ and $v=(1/3, 1/3, 1/3)$.

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