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APPROXIMATE JUDGEMENT AGGREGATION

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Abstract. In this paper we analyze judgement aggregation problems in which a group of agents independently votes on a set of complex propositions that has some interdependency constraint between them (e.g., transitivity when describing preferences). We consider the issue of judgement aggregation from the perspective of approximation. That is, we generalize the previous results by studying approximate judgement aggregation. We relax the main two constraints assumed in the current literature, Consistency and Independence and consider mechanisms that only approximately satisfy these constraints, that is, satisfy them up to a small portion of the inputs. The main question we raise is whether the relaxation of these notions significantly alters the class of satisfying aggregation mechanisms. The recent works for preference aggregation of Kalai, Mossel, and Keller fit into this framework. The main result of this paper is that, as in the case of preference aggregation, in the case of a subclass of a natural class of aggregation problems termed 'truth-functional agendas', the set of satisfying aggregation mechanisms does not extend non-trivially when relaxing the constraints. Our proof techniques involve boolean Fourier transform and analysis of voter influences for voting protocols.
The question we raise for Approximate Aggregation can be stated in terms of Property Testing. For instance, as a corollary from our result we get a generalization of the classic result for property testing of linearity of boolean functions.

Keywords: judgement aggregation, truth-functional agendas, computational social choice, computational judgement aggregation, approximate aggregation, inconsistency index, dependency index

1 Introduction

A famous jury paradox shows that aggregating complex decisions might be non-trivial. Assume a jury is faced with a case in which a defendant is accused of murder. The legal doctrine (known by all of them) is that the defendant should be convicted if and only if they are convinced that **a)**The defendant indeed killed the victim and **b)**The defendant is sane. We assume that each of the jurors decides his opinion on the two issues independently and based on these decisions decides whether to convict. Then, the members cast their votes simultaneously and we assume no strategic behavior on their behalf. Kornhauser and Sager[25] noticed that it's possible to have an opinion profile in which, when applying issue-wise aggregation using majority, which seems natural[1], we get a discrepancy between the majority vote on the conviction question and the

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[1] although it is not the common aggregation method of jurors (or judges in a tribunal in most countries

conjunction of the majority vote on the two basic questions (whether the defendant killed and whether he is sane)[2]. This discrepancy is termed *The Doctrinal Paradox*. Lately, in [27], List showed that the probability to get such a discrepancy is non-negligible under the uniform distribution and also under other mild relaxations of it (still assuming the voters are i.i.d.).

This insight, that is common to many aggregation problems (e.g., Condorcet paradox for preference aggregation), started the field of 'Judgement Aggregation' and nowadays this field is the subject of a growing body of works in economics, computer science, political science, philosophy, law, and other related disciplines. We find this field highly applicable to agent systems, voting protocols in a network and other frameworks in which one needs to aggregate a lot of opinions in a systematic way without letting the voters deliberate. An aggregation problem in our context concerns a given **Agenda**, which is a set of $\{0,1\}$ vectors of length m (the number of issues), that defines the **consistent** (legal/rational/admissible) opinions that an individual might hold. Given an agenda, Aggregation Theory deals with exploring ways to aggregate the opinions of (often many) experts/judges while maintaining two main syntactical properties:

- **Consistency -** always returning an admissible opinion.
  In our example, the aggregated opinion should be to convict iff the aggregated opinion was that indeed the defendant killed and is sane.
- **Independence -** define the aggregated opinion on each issue independently of the votes on other issues. This criterion can be seen as respecting the structure of the agenda instead of handling it as a set of several different opinions (in the example above, four) disregarding the structure.

Most of these works find the set of 'acceptable' aggregation mechanisms (i.e., that satisfy the two criteria) to be very small and undesired (e.g., dictatorships) and hence are considered as impossibility results. A survey of this field can be found in [30,28]. Such impossibility results are quite strong, they show the impossibility of finding any reasonable aggregation mechanism that satisfies the two conditions and hence for (almost) every mechanism there will always be some judgement profile that leads to a breakdown of the mechanism.

In this work we extend the question to 'Approximate Judgement Aggregation'. We relax the above two properties and search for an aggregation mechanism that only *approximately* respects the structure of opinions and *up to a small fraction* of the inputs returns a consistent opinion. More specifically, we are interested in exploring the influence of relaxing the two properties on the set of 'acceptable' aggregation mechanisms.

We quantify being almost consistent by defining $\boldsymbol{\delta}$**-consistency** of an aggregation mechanism F as having a consistent aggregation mechanism G that disagrees with F on at most δ fraction of the inputs[3]. Similarly, we quantify being almost independent by defining $\boldsymbol{\delta}$**-independence** of an aggregation mechanism F as having an independent aggregation mechanism that disagrees with F on at most δ fraction of the inputs. Both terms can be equivalently defined as the failure probability of tests as we show in Section 2. Both definitions use the Hamming distance between mechanisms $d^{\mathbb{X}}(F,G) = \Pr\left[F(X) \neq G(X) \mid X \in \mathbb{X}^n\right]$. It includes two assumptions: uniform distribution over the opinions for each voter and assuming voters draw their opinions independently (**Impartial Culture Assumption**). These assumptions, while certainly unrealistic, are the natural choice in this kind of work and are discussed further in Section 2.

Lately there is a series of works coping with impossibility results in Social Choice Theory using approximations (e.g., [6,19]). In some cases allowing approximation enables significantly better results, while in other cases, hardly anything is gained by allowing it. For example, in [6] the authors deal with preference aggregation and show that when one approximates Dodgson's scoring rule one can achieve several desired properties (monotonicity, homogeneity, and low complexity) that cannot be achieved without this relaxation. On the other hand, in [19] the authors also deal with aggregation of preferences and show that relaxing the

[2] For instance, the following profile:

	Killed	Sane	Guilty
25% of the jurors:	✓	✓	✓
33% of the jurors:	✓	×	×
42% of the jurors:	×	✓	×

[3] Formally, $\Pr\left[F(X) \neq G(X) \mid X \in \mathbb{X}^n\right] \leqslant \delta$.

strategy-proofness property does not extend the set of satisfying aggregation mechanisms non-trivially and by that they strengthen the classic impossibility result of Gibbard & Satterthwaite. In this work we formalize (as far as we found for the first time) this question of quantifying the influence of relaxing the constraints and query whether one can use this in order to circumvent the impossibility results (as in [6]) or whether we strengthen the impossibility results (as in [19]).

In this paper we study a family of agendas: truth-functional agendas in which each conclusion is defined as conjunction or xor of several premises (up to input & output negation). In a truth-functional agenda the issues are divided into two types: premises and conclusions. Each conclusion j is characterized by a boolean function Φ^j over the premises and an opinion is consistent if the answers to the conclusion issues are attained by applying the function Φ^j on the answers to the premise issues.

$$\mathbb{X} = \left\{ x \in \{0,1\}^m \middle| x^j = \Phi^j(\text{premises}) \quad \text{for every conclusion issue } j. \right\}$$

For instance the (2-premises) conjunction agenda used in the example above is a truth-functional agenda with two premises and one conclusion and we notate the agenda by $\langle A, B, A \wedge B \rangle$.

For all the agendas we examined, we show that relaxing the two constraints, consistency and independence, does not extend the set of acceptable aggregation mechanisms in a non-trivial way.

We concentrated on two basic agendas: **Conjunction Agenda** $\left\langle A^1, \ldots, A^m, \wedge_{j=1}^m A^j \right\rangle$ (i.e., $m+1$ issues where the consistency means that the last one should be a conjunction of the first m) and **Xor Agenda** $\left\langle A^1, \ldots, A^m, \oplus_{j=1}^m A^j \right\rangle$ (i.e., $m+1$ issues where the consistency means that the last one should be a parity bit of the first m). For these agendas we prove

Theorem.

1. *For any $m \geqslant 2$, $\epsilon > 0$, and $n \geqslant 2$, there exists $\delta(\epsilon, n, m)$ polynomial in n and ϵ (but degrades exponentially in m) s.t. if an aggregation mechanism F over n voters for the m-premises conjunction agenda is δ-independent[4] and δ-consistent[5], then it is ϵ-close to a consistent independent aggregation mechanism G[6]. Moreover, $\delta = \frac{C}{n}\left(\frac{\epsilon}{8m}\right)^{2m-1}$ (for some constant $C>0$),*
2. *For any $m \geqslant 2$, $\epsilon > 0$, and $n \geqslant 2$, there exists $\delta(\epsilon, m)$ linear in ϵ (and degrades quadratically in m) s.t. if an aggregation mechanism F over n voters for the m-premises xor agenda is δ-independent[4] and δ-consistent[5], then it is ϵ-close to a consistent independent aggregation mechanism G[6]. Moreover, $\delta = \frac{\epsilon}{m(2m+3)}$*

We have a characterization for the sets of the independent and consistent aggregation mechanisms for the two agendas. For the conjunction agenda, an independent aggregation mechanism is consistent if either it returns constant `False` for one of the premises (and for the conjunction issue) or if it aggregates all the issues using the same oligarchy aggregation function (i.e., $\bigwedge_{i \in S} x_i$ for some coalition S - returns `True` if all the member of a coalition S voted `True`). This characterization is a direct corollary from a series of works in the more general framework of aggregation, e.g., [35,13] and for completeness we include a proof of it in the appendix. For the xor agenda, our proof implies a characterization of the independent and consistent aggregation mechanism which states that an independent aggregation mechanism is consistent if (essentially) all the issues are aggregated using the same linear aggregation function of the form $\chi(x) = \bigoplus_{i \in S} x_i$ (for some coalition $S \subseteq [n]$).

Hence, the above theorem can be seen as an impossibility result saying that it is impossible even to find a mechanism that is almost consistent and almost independent besides the trivial answers: independent consistent mechanism and perturbations of them which is (still) a relatively small and undesired collection of mechanisms.

[4] I.e., there exists an independent (not necessarily consistent) aggregation mechanism G that returns the same aggregated opinion as F for at least $(1-\delta)$ fraction of the profiles.

[5] I.e., F returns a consistent result for at least $(1-\delta)$ fraction of the profiles.

[6] I.e., F returns the same aggregated opinion as G for at least $(1-\epsilon)$ fraction of the profiles.

Our results are invariant to negation of issues (which is merely renaming), and hence we can easily generalize the results to other agendas such as $\left\langle A^1, A^2, A^3, A^1 \wedge A^2 \wedge \overline{A^3} \right\rangle$, $\langle A^1, A^2, A^1 \vee A^2 \rangle$, and $\left\langle A^1, A^2, A^3, \overline{A^1 \oplus A^2 \oplus A^3} \right\rangle$. Using induction we can generalize the result to more complex agendas that include several conclusion issues such as $\langle A^1, A^2, A^3,\ A^1 \vee A^2, A^2 \oplus A^3 \rangle$. We notice that this generalize our result to any agenda of the form $\langle A^1, A^2, \Phi\left(A^1, A^2\right) \rangle$ for **any function** Φ[7] and to **any affine agenda** (I.e., the set of admissible opinions form an affine space).

1.1 Previous works

There is a long line of works trying to circumvent impossibility results in Aggregation Theory (i.e., results which state that the set of consistent independent aggregation mechanisms is very small and undesired). Most of these works suggest consistent aggregating mechanisms while still trying to stay 'reasonably close' to independence (E.g., [25,24,37,29,10,5,26,11,38]). These classical works are heuristic, sometimes use the semantics of the agenda, and mainly do not prove bounds on the compliance to the independence property. In [27], List studies the asymptotic probability of getting an inconsistent result in the 2-premises conjunction agenda $\langle A, B, A \wedge B \rangle$ for voter-independent distributions and common (majority-based & supermajority-based) aggregation mechanisms. He mainly studies the conditions for the probability to converge to zero and to one. As far as we found, this is the only work that deals with quantifying, although only asymptotically, the property compliance of an aggregation mechanism for agendas other than the Arovian agenda (preference aggregation).

Another approach is Approximate Aggregation. This line of research started with [21] and was extended in [33,22]. In these works the authors deal with preference aggregation (although without stating the general framework of approximate aggregation) and show that relaxing the transitivity constraint (which is equivalent to consistency for this agenda) does not extend the set of satisfying aggregation mechanisms non-trivially.

Theorem ([22] Theorem 1.3). *There exists an absolute constant C such that the following holds: For any $\epsilon > 0$ and $k \geqslant 3$, if f is an aggregation mechanism for the preference agenda over k candidates that satisfies independence and $C \cdot \left(\epsilon/k^2\right)^3$-consistency, then there exists an aggregation mechanism G that satisfies independence and consistency such that $d(F, G) < \epsilon$.*

This result is neither derived by our results nor derives them because the agendas we deal with and the preference agenda are too different (For instance, the preference agenda cannot be represented as a truth-functional agenda and in some sense it is even far from it).

1.2 Connection to Property Testing

We think it might be useful to phrase the question of approximate aggregation using terminology of property testing. In this field we query a function at a small number of (random) points, testing for a global property (in our case, the property is being a consistent independent aggregation mechanism). For example, a corollary of the results we present in this paper (in property testing terms):

For any three binary functions $f, g, h : \{0,1\}^n \to \{0,1\}$, if the probability $\Pr\left[f(x) \oplus g(y) = h(x \oplus y)\right]$ is larger than $(1-\epsilon)$ (when the addition is in $\mathbb{Z}_2$ and $\mathbb{Z}_2^n$, respectively), then there exists three binary functions $f', g', h' : \{0,1\}^n \to \{0,1\}$ such that $\Pr[f(x) \neq f'(x)]$, $\Pr[g(x) \neq g'(x)]$, and $\Pr[h(x) \neq h'(x)]$ are smaller than $C\epsilon$ for some constant C independent of n and $\forall x, y\ :\ f'(x) \oplus g'(y) = h'(x \oplus y)$.

A special case of this result, $f = g = h$, is the classic result of Blum, Luby, and Rubinfeld ([3,1]) for linear testing of boolean functions. We discuss this connection further and its possible implications in in Section 5.

[7] The case of a function that ignores one of the two arguments (or both) is trivial.

1.3 Techniques

We prove the main theorem by proving the specific case of independent aggregation mechanism for two basic agenda families: the conjunction agendas (agendas in which there is exactly one conclusion that is constrained to be the conjunction of the premises. Theorem 3) and the xor agendas (agendas in which there is exactly one conclusion that is constrained to be the xor of the premises. Theorem 4). Later we extend these theorems to the general theorem of relaxing both constraints (Theorem 5) using an agenda-independent method.

We use two different techniques in the proofs. For the conjunction agendas we study influence measures of voters on the issue-aggregating functions[8]. and for the xor agendas we use Fourier analysis of the issue-aggregating functions.

An open question is whether one can find such bounds for any agenda or whether there exists an agenda for which the class of aggregation mechanisms that satisfy consistency and independence expands non trivially when we relax the consistency and independence constraints.

We proceed to describe the structure of the paper. In Section 2 we describe the formal model of aggregation mechanisms. In Section 3 we present the main agendas we deal with, truth-functional agendas, and specifically conjunction agendas and xor agendas. In Section 4 we state the motivation to deal with approximate aggregation. In Section 5 we describe the connection we find between Approximate Aggregation and the field of Property Testing. In Sections 6 and 7 we describe our main theorems and outline the proof. Section 8 concludes.

2 The Model

We define the model similarly to [13,14] (which is Rubinstein and Fishburn's model [41] for the boolean case).

We consider a **committee** of n individuals that needs to decide on m boolean issues[9]. An **opinion** is a vector $x = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$ denoting an answer to each of the issues. An opinion **profile** is a matrix $X \in (\{0,1\}^m)^n$ denoting the opinions of the committee members, so an entry X_i^j denotes the vote of the $i^{\underline{th}}$ voter for the $j^{\underline{th}}$ issue, the $i^{\underline{th}}$ row of it X_i states the votes of the $i^{\underline{th}}$ individual on all issues, and the $j^{\underline{th}}$ column of it X^j states the votes of each of the individuals on the $j^{\underline{th}}$ issue. In addition we assume that an **agenda** $\mathbb{X} \subseteq \{0,1\}^m$ of the **consistent** opinions is given.

The basic notion in this field is **Aggregation Mechanism** which is a function that returns an **aggregated opinion** (not necessarily consistent) for every profile[10] : $F : (\{0,1\}^m)^n \to \{0,1\}^m$.

An aggregation mechanism satisfies **Independence** (and we say that the mechanism is **independent**) if for any two consistent profiles X and Y and an issue j, if $X^j = Y^j$ (all individuals voted the same on the $j^{\underline{th}}$ issue in both profiles) then $(F(X))^j = (F(Y))^j$ (the aggregated opinion for the $j^{\underline{th}}$ issue is the same for both profiles). This means that F satisfies independence if one can find m boolean functions $f^1, f^2, \ldots, f^m : \{0,1\}^n \to \{0,1\}$ s.t. $F(X) \equiv \left(f^1\left(X^1\right), f^2\left(X^2\right), \ldots, f^m\left(X^m\right)\right)$[11]. An independent aggregation mechanism satisfies **systematicity** if all issues are aggregated using the same function, i.e., $F(X) = \left\langle f\left(X^1\right), \ldots, f\left(X^m\right)\right\rangle$ for some issue aggregating function f. We will use the notation $\left\langle f^1, f^2, \ldots, f^m\right\rangle$ for the independent aggregation mechanism that aggregates the $j^{\underline{th}}$ issue using f^j.

The main two measures we study in this paper are the **inconsistency index** $IC^{\mathbb{X}}(F)$ and the **dependency index** $DI^{\mathbb{X}}(F)$ of a given aggregation mechanism F and a given agenda $\mathbb{X}$. These measures are

[8] Both the known influence (Banzhaf power index) and a new measure we define: The ignorability of an individual and of a coalition of individuals.

[9] There is some literature on aggregating non-boolean issues, e.g., [41,15], but this is outside the scope of this paper.

[10] We define the function for all profiles for simplicity but we are not interested in the aggregated opinion in cases one of the voters voted an inconsistent opinion.

[11] Notice this property is a generalization of the IIA property for social welfare functions (aggregation mechanism for the preference agenda) so a social welfare function satisfies IIA iff it satisfies independence as defined here (when the issues are the pair-wise comparisons).

relaxations of the **consistency** and **independence** criterion that are usually assumed in current works[12]. We define the measures in the following way:

Definition 1 (Inconsistency Index).
For an agenda $\mathbb{X}$ *and an aggregation mechanism* F *for that agenda, the* **inconsistency index** *is defined to be the probability to get an inconsistent result.*[13]

$$IC^{\mathbb{X}}(F) = \Pr\left[F(X) \notin \mathbb{X} \mid X \in \mathbb{X}^n\right].$$

Definition 2 (Dependency Index[14]).
For an agenda $\mathbb{X}$ *and an aggregation mechanism* F *for that agenda, the* **dependency vector** $DI^{j,\mathbb{X}}(F)$ *is defined as*

$$DI^{j,\mathbb{X}}(F) = \mathop{\mathbb{E}}_{X \in \mathbb{X}^n}\left[\Pr_{Y \in \mathbb{X}^n}\left[(F(X))^j \neq (F(Y))^j | X^j = Y^j\right]\right].$$

The definition can be seen as a test for independence of the $j^{\underline{th}}$ *issue as discussed in Section 5*
The **dependency index** $DI^{\mathbb{X}}(F)$ *is defined by:* $DI^{\mathbb{X}}(F) = \max_{j=1,\ldots,m} DI^{j,\mathbb{X}}(F)$

In contexts where the agenda is clear we omit the agenda superscript and notate these as $IC(F)$, $DI^j(F)$, and $DI(F)$, respectively.

We define these two indices using local tests and prove that the more natural definition of distance to the class of aggregation mechanisms that satisfy consistency (or independence) is equivalent to the above (up to multiplication by a constant).

Proposition 1. *Let* F *be an aggregation mechanism for an agenda over* m *issues. Then* F *satisfies* $IC(F) \leqslant \delta$ *iff there exists a consistent aggregation mechanism* H *that satisfies* $d(F,H) \leqslant \delta$.

Proposition 2. *Let* F *be an aggregation mechanism and* j *an issue. If* $DI^j(F) \leqslant \delta$, *then there exists an aggregation mechanism* H *that satisfies* $DI^j(H) = 0$ *and* $d(F,H) \leqslant 2\delta$. *If* $DI^j(F) \geqslant \delta$, *then every aggregation mechanism* H *that satisfies* $DI^j(H) = 0$, *also satisfies* $d(F,H) \geqslant \frac{1}{2}\delta$

Proposition 3. *Let* F *be an aggregation mechanism for an agenda over* m *issues that satisfies* $DI(F) \leqslant \delta$. *Then there exists an independent aggregation mechanism* H *that satisfies* $d(F,H) \leqslant 2m\delta$. *If* $DI(F) \geqslant \delta$, *then every aggregation mechanism* H *that satisfies* $DI(H) = 0$, *also satisfies* $d(F,H) \geqslant \frac{1}{2}\delta$

These definitions include two major assumptions on the opinion profile distribution. First, we assume the voters pick their opinions independently and from the same distribution. Second, we assume a uniform distribution over the (consistent) opinions for each voter (**Impartial Culture Assumption**). The uniform distribution assumption, while certainly unrealistic, is the natural choice for proving 'lower bounds' on $IC(F)$. That is, proving results of the format "Every 'reasonable' aggregation mechanism of a given class has inconsistency index of at least $\gamma(n)$". In particular, the lower bound, up to a factor δ, applies also to any distribution that gives each preference profile at least a δ fraction of the probability given by the uniform distribution[15]. Note that we cannot hope to get a reasonable bound result for every distribution. For instance, since for every aggregation mechanism we can take a distribution on profiles for which it returns a consistent opinion.

[12] F satisfies consistency iff $IC(F) = 0$ and independence iff $DI(F) = 0$

[13] In [27] List presented this measure under the name 'Probability of a collective inconsistency' and studies its asymptotical behavior for the conjunction agenda and the issue-wise majority aggregation mechanism.

[14] In [33] Mossel defines similar measure for preference aggregation mechanism called η-IIA. Our definition coincides with his definition for this agenda.

[15] In successive works we relax this assumption and prove similar results for more general distributions.

2.1 The Independence Property

The independence criterion is sometimes criticized as being unjustified normatively in most real-life scenarios[16]. The impossibility results of judgement aggregation can also be seen as 'empirical' argument against independence since they show that it contradicts consistency which seems to be more desired. While we accept this argument, we think our work quantifies the tradeoff between the two criteria. Moreover, in this section we claim that this criterion can be justified on several different grounds.

First, in a lot of cases it is justified to expect, due to normative reasons or legal reasons, that changing an individual judgement on an issue should not change the collective judgement on another issue. The rational is usually that when the agenda is described as having a combinatorial structure (or perceived in such a way), the aggregation method should respect the structure and not treat the agenda as a simple set of alternatives.

Secondly, as in the case of multi-issue voting domains[43], when the number of voters is small compared to the agenda size (number of possible opinions) the natural ways of aggregating might be nonsignificant. For instance, using plurality when the number of voters is too small could well result in a situation where no outcome gets more than one vote, in which case plurality would give an extremely poor result.

In addition, there are works that defend this criterion by using manipulation-resistance arguments. In [12] Dietrich and List define the notion of manipulability of an aggregation mechanism[17] and prove that any aggregation mechanism that does not satisfy independence is manipulable. In this paper they further prove that this manipulability property is equivalent to a more game-theoretic property of strategy-proofness under some assumptions on players' preferences.

On the ground of simplicity of representation one can justify independence as a criterion that returns aggregation mechanisms that are easy to represent, calculate, or justify (for instance, justify an election result to the public).

Other grounds of justification for such aggregation mechanisms are from the voter point of view. There are situations in which the decisions are made over time and place (different meetings) or by different representatives of the same voting identity so it is fair to assume that when voting on an issue or aggregating the votes it is unreasonable to depend on votes on other issues. Another argument might be that there are scenarios in which you need to define the aggregation method and only at a later stage choose from the set of issues the relevant ones (For instance, the definition of Social Welfare Functions as returning a choice function so only at a later stage the society is faced with the menu of alternatives).

2.2 Binary Functions

Since this work deals with binary functions (for aggregating issues), we need to define several notions for this framework as well. To ease the presentation, throughout this paper we will identify `True` with 1 and `False` with 0 and use logical operators on bits and bit vectors (using entry-wise semantics).

Let $f : \{0,1\}^n \to \{0,1\}$ be a binary function. f is the **oligarchy** of a coalition S if it is of the form: $f(x) = \bigwedge_{i \in S} x_i$. This means that f returns 1 if all the members of S voted 1. We denote by **`Olig`** the class of all 2^n oligarchies. Two special cases of oligarchies are the constant 1 function which is the oligarchy of the empty coalition and the dictatorships which are oligarchies of a single voter.

f is a **linear** function if it is of the form $f(x) = \bigoplus_{i \in S} x_i$ for some coalition S[18]. This means that f returns 1 if an even number of the members of S voted 1. We denote by **`Lin`** the class of all 2^n linear functions.

[16] Chapman([7]) and Mongin([32]) attack this criterion and claim it removes the discipline of reason from social choice since it disregards the intra-issue dependencies which is the essence of the problem. According to this criterion the aggregation of 'complex' issues is done without regarding the reasons of the voters for their opinions and hence lacks the information for good aggregation.

[17] An aggregation mechanism F is manipulable at the profile X by individual i(the manipulator) on issue j if $X_j^i \neq (F(X))_j$, but $X_j^i = (F(X'))_j$ for some profile X' that differs form X in i's vote only. I.e., the manipulator disagrees with the aggregated opinion on issue j and can get his will on j by voting differently.

[18] An equivalent definition is: f is linear if $\forall x, y:\ f(x)+f(y) = f(x+y)$ when the addition is in $\mathbb{Z}_2$ and $\mathbb{Z}_2^n$, respectively.

Two special cases of linear functions are the constant 1 function which is the xor function over the empty coalition and the dictatorships which are xor of a single voter.

We say that f satisfies the **Pareto** criterion if $f(\bar{0}) = 0$ and $f(\bar{1}) = 1$[19]. I.e., when all the individuals voted unanimously 0 then f should return 0 and similarly for the case of 1.

We define the following measures for the influence of an individual or a coalition of individuals on a function $f : \{0,1\}^n \to \{0,1\}$. Both definitions use the uniform distribution over $\{0,1\}^n$ (which is consistent with the assumption we have on the profile distribution).

- The **Influence**[20] of a voter i on f is defined to be the probability that he can change the result by changing his vote.
$$I_i(f) = \Pr\left[f(x) \neq f(x \oplus e_i)\right]$$
($x \oplus e_i$ = adding to x, e_i(the $\text{i}^{\underline{\text{th}}}$ elementary vector)=flipping the $\text{i}^{\underline{\text{th}}}$ bit $0 \leftrightarrow 1$)
- The (zero-)**Ignorability** of a voter i on f is defined to be the probability that f returns 1 when i voted 0.
$$P_i(f) = \Pr\left[f(x) = 1 \mid x_i = 0\right]$$
(We did not find a similar index defined in the voting literature or in the cooperative games literature).
- A generalization of the above definition is the (zero-)**Ignorability** of a coalition $S \subseteq \{1, \ldots, n\}$. It is defined to be the probability that f returns 1 when one of the members of S voted 0. (So we get that $P_i(f) = P_{\{i\}}(f)$.)
$$P_S(f) = \Pr\left[f(x) = 1 \mid \exists i \in S \ x_i = 0\right]$$

In addition we define a distance function over the binary functions. The distance between two functions $f, g : \{0,1\}^n \to \{0,1\}$ is defined to be the probability of getting a different result (normalized Hamming distance). $d(f,g) = \Pr\left[f(x) \neq g(x)\right]$. From this measure we will derive a distance from a function to a set of functions by $d(f, \mathcal{G}) = \min_{g \in \mathcal{G}} d(f,g)$ One more notation we are using in this paper is x_J for a binary vector $x \in \{0,1\}^n$ and a coalition $J \subseteq \{1, 2, \ldots n\}$ for notating the entries of x that correspond to J.

3 Agenda Examples

A lot of natural problems can be formulated in the framework of aggregation mechanisms. It is natural to divide the agendas into two major classes **Truth-Functional Agendas** and **Non Truth-Functional Agendas**.

3.1 Truth-Functional Agendas

A (k-premise) truth-functional agenda is defined by a conclusions function ($\Phi : \{0,1\}^k \to \{0,1\}^{m-k}$) from the k premises to the $(m-k)$ conclusions. An opinion is consistent if the answers to the conclusion issues are attained by applying Φ on the answers to the premise issues.

$$\mathbb{X} = \left\{ x \in \{0,1\}^m \middle| x^j = \Phi^j(x_1, \ldots, x_k) \quad j = k+1, \ldots, m \right\}$$

There are cases in which there might be more than one way to classify the issues to premises and conclusions. For instance, the 2-premises xor agenda $\mathbb{X} = \{001, 010, 100, 111\}$ can be defined both as $\langle A, B, A \oplus B \rangle$ and as $\langle A, A \oplus C, C \rangle$. Since we choose to analyze the agenda as opinion sets (and not as a proposition set) we do not handle this point and notice that it is irrelevant for our results.

These agendas, due to their structure, seem to be a good point to start our work on approximate aggregation and in this paper we prove results for two families of truth-functional agendas. Later we derive results for a more general family of truth-functional agendas.

[19] In the literature this criterion is sometimes referred to as Unanimity, e.g., in [30]. We chose to follow [13,14] and refer to it as Pareto to distinguish between it and the unanimity function which is the oligarchy of $\{1, 2, \ldots, n\}$.

[20] In the simple cooperative games regime, this is also called the Banzhaf power index of player i in the game f.

Conjunction Agendas: In the m-premises conjunction agenda $\left\langle A^1, \ldots, A^m, \wedge_{j=1}^m A^j \right\rangle$ there are $m+1$ issues to decide on and the consistency criterion is defined to be that the last issue is a conjunction of the other issues. For instance the Doctrinal Paradox agenda is the 2-premises conjunction agenda.
Xor Agendas: Similarly, in the m-premises xor agenda $\left\langle A^1, \ldots, A^m, \oplus_{j=1}^m A^j \right\rangle$ there are $m+1$ issues to decide on and the consistency criterion is defined to be that the last issue is `True` if the number of true-valued opinions for the first m is even. An equivalent way to define this agenda is constraining the number of `True` answers to be odd.

3.2 Non Truth-Functional Agendas

One can think on a lot of agendas that cannot be represented as a truth-functional agenda. Among such interesting natural agendas that were studied one can find the equivalence agenda[18], the membership agenda [40][31], and the preference agenda described below.
Preference Aggregation: Aggregation of preferences is one of the oldest aggregation frameworks studied. In this framework there are s candidates and each individual holds a full strict order over them. We are interested in Social Welfare Functions which are functions that aggregate n such orders to an aggregated order. As seen in [34,9], this problem can be stated naturally in the aggregation framework we defined by defining $\binom{s}{2}$ issues[21].

4 Motivation

We find the motivation for dealing with the field of approximate judgement aggregation in three different disciplines.

- The consistent characterization are often regarded as 'impossibility results' in the sense that they 'permit' a very restrictive set of aggregation mechanisms. (e.g., Arrow's theorem tells us that there is no 'reasonable' way to aggregate preferences). Extending these theorems to approximate aggregation characterizations sheds light on these impossibility results by relaxing the constraints.
- The questions of Aggregation Theory have often roots in Philosophy, Law, and Political Science. There is a long line of works suggesting consistent aggregating mechanisms while still trying to stay 'reasonably close' to independence. The main general (not agenda-tailored) suggestions are premise-based mechanisms and conclusion-based aggregation for truth-functional agendas (see, among others, [25,24,37,29,10,5]), and a generalization of them to non-truth-functional agendas called sequential priority aggregation([26,11]). Another procedure in the literature is the distance-based aggregation([38]) which is well known for preference aggregation (E.g., Kemeny voting rule[23], Dodgson voting rule[2], and lately a more systematic analysis in [16]). Our work contribute to this discussion by pointing out where one should search for solutions while not leaving the consistency and independence constraints entirely.
- Connections to the Property Testing field as discussed in Section 5.

5 Connection to Property Testing

In the words of [39], the field of property testing deals with the following:

> Given the ability to perform (local) queries concerning a particular object (e.g., a function or a graph), the task is to determine whether the object has a predetermined (global) property (e.g., linearity or bipartiteness), or is far from having the property. The task should be performed by inspecting only a small (possibly randomly selected) part of the whole object, where a small probability of failure is allowed.

Property testing trades accuracy (the distance parameter) for efficiency (number of queries).

[21] The issue $\langle i,j \rangle$ (for $i<j$) represents whether an individual prefers c_i over c_j.

We think it might be useful to view the Approximate Aggregation problem in the framework of Property Testing. Below we highlight the connection between Approximate Aggregation and a special case of Property testing termed ‘one-sided non-adaptive program testing’. For a general survey of the field, one can read [17,39,20].

In our case the global property we are trying to test is ‘consistency and independence’ of an aggregation mechanism. The class of satisfying aggregation mechanism is characterized by the current state of research. It is clear that each of the components of this property separately, consistency and independence of an issue, can be tested trivially. The consistency test consists of picking a (consistent) profile uniformly at random and checking whether the aggregated opinion is consistent. The test for independence of issue j consists of picking a (consistent) profile uniformly at random, altering randomly the opinion for each voter without changing the $j^{\underline{th}}$ bit and check whether the aggregated opinion on the $j^{\underline{th}}$ issue is changed due to the altering. For each of the two tests the probability to accept a non-satisfying mechanism is linear in the distance to the satisfying set (and equals $IC(F)$ and $DI^j(F)$, respectively). The main question of this work can be stated using property testing terms as follows: What is the best test for being ‘consistent and independent’ one can assemble from running the $(m+1)$ tests as black boxes (and therefore get information only on $IC(F)$ and $DI^j(F)$).

Similar question was asked lately in [8]. In [8] the authors query (among other similar questions) the conditions needed in order to deduce from testability of two properties the testability of the intersection of the two properties. Our work can be seen as studying this question for a specific domain in which the question seems to be natural while adding the constraint that the test of the intersection property should be defined as a sequence of tests for the basic properties (in a non-adaptive way).

The main result of this paper is that for a class of mechanisms (corresponding to a natural class of agendas) one can assemble those tests to a test for the property ‘consistent and independent’.

Similarly one can state questions dealing with sub-families of aggregation mechanisms. For example, as we stated in the introduction, the classic result of Blum, Luby, and Rubinfeld for linearity testing of boolean functions is a direct corollary of our result for the 2-premises xor agenda when considering systematic aggregation mechanisms.

Still, the target of the two fields is different. While Property Testing deals with finding the most efficient (query-wise) algorithm for testing a property (functions family), Approximate Aggregation deals with analyzing a specific family of tests.

6 Main Results

The main result of this paper is

Theorem 1.
For any $\epsilon > 0$ and $m, n \geqslant 2$, there exists $\delta_{IC}, \delta_{DI} = n^{-1}\left(\frac{\epsilon}{m}\right)^{poly(m)}$, such that for every truth-functional agenda $\mathbb{X}$ over m issues, in which each conclusion issue is defined to be either conjunction of several premises or xor of several premises(up to negation of inputs or output)[22], if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying δ_{IC}-independence and δ_{DI}-consistency, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < \epsilon$
Moreover, one can take $\delta_{IC} = n^{-1}\left(\frac{(1-\beta_\epsilon)\epsilon}{8m}\right)^{2m-1} - \beta_\epsilon\epsilon$ and $\delta_{DI} = \frac{1}{2m}\beta_\epsilon\epsilon$ for any $\beta_\epsilon \in \left[0, n^{-1}\left(\frac{\epsilon}{8m}\right)^{2m-2}\right]$.

A direct corollary is the following impossibility result.

Corollary 1. *There exists a constant C such that for any $m, n \geqslant 2$ and $\epsilon, \delta \in [0,1]$ s.t. $\delta < C \cdot n^{-1}\left(\frac{\epsilon}{8m}\right)^{2m-1}$, and a truth-functional agenda $\mathbb{X}$ over m issues, in which each conclusion issue is defined to be either conjunction of several premises or xor of several premises(up to negation of inputs or output), no aggregation mechanism F for $\mathbb{X}$ over n voters satisfies the following three conditions:*

[22] For example, $\langle A, B, A \wedge B\rangle$, $\langle A, B, A \rightarrow B\rangle \equiv \left\langle A, B, \overline{A \wedge \overline{B}}\right\rangle$, $\langle A, B, C, A \wedge B, B \oplus C, A \vee C\rangle$.

– *δ-independence*
– *δ-consistency*
– *F is ϵ-far from any independent and consistent aggregation mechanism for $\mathbb{X}$.*

In the case of xor agenda (and its generalization, a truth-functional agenda in which all the conclusions are xor) we can get a better result (particulary, no dependency on the number of voters)

Theorem 2. *Let $m \geqslant 3$ and let the agenda be $\mathbb{X} = \left\langle A^1, \ldots, A^{m-1}, \bigoplus_{j=1}^{m-1} A^j \right\rangle$. For any $\delta < \frac{1}{6}$ and any aggregation mechanism F:*
If F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying δ-independence and δ-consistency, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < m(2m+3)\delta$

Noticing that any affine agenda (i.e., an agenda that is an affine subspace) can be represented as a truth-functional agenda that uses xor conclusions only (Lemma 10) we can get the following corollary

Corollary 2. *For any $\epsilon > 0$ and $m, n \geqslant 2$, there exists $\delta = \frac{\epsilon}{m(2m+3)}$, such that for every affine agenda $\mathbb{X}$ over m issues, if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying δ-independence and δ-consistency, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < \epsilon$*

7 Proof Sketch of the Main Theorem

In this section we sketch the techniques behind our proofs. The full proofs can be found in the appendices.

We prove the main theorem by proving three independent theorems. An approximation result for independent aggregation mechanisms for conjunction agendas (Theorem 3). An approximation result for independent aggregation mechanisms for xor agendas (Theorem 4). An agenda independent method of converting results for the independent case to the general case of relaxing both constraints (Theorem 5). Using induction on the number of conclusions and noticing that negating (of the inputs and of the output) is renaming of opinions in our framework (and hence does not change the approximation results) we get Theorem 1.

7.1 Conjunction Agenda

For the agenda $\left\langle A^1, \ldots, A^m, \bigwedge_{j=1}^{m} A^j \right\rangle$ we prove:

Theorem 3. *For the agenda $\mathbb{X} = \left\langle A^1, \ldots, A^m, \bigwedge_{j=1}^{m} A^j \right\rangle$ for $m \geqslant 2$:*

For any $\epsilon > 0$ and any independent aggregation mechanism F:
If $IC(F) \leqslant \epsilon$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < 8m\,(n\epsilon)^{\frac{1}{2m-1}}$.

There is a known characterization of the consistent independent aggregation mechanism for the conjunction agenda. (This characterization is a direct corollary from a series of works in the more general framework of aggregation, E.g., [35,13]. We include a proof of it in the appendix)

Lemma 1.
Let $f^1, \ldots, f^m, h : \{0,1\}^n \to \{0,1\}$ be $m+1$ voting functions satisfying $IC\left(\left\langle f^1, \ldots, f^m, h \right\rangle\right) = 0$. Then either there exists an issue j s.t. $f^j = h \equiv 0$ or $f^1 = f^2 = \ldots = f^m = h \in \mathtt{Olig}$.

A corollary from the above is a characterization of the approximate aggregation mechanisms for this agenda. Actually, in the proof of Theorem 3 we get a tighter characterization that distinguishes between the first case of having a constant zero issue-aggregation function and the second of aggregating using an oligarchy.

*Proof (***Proof sketch).**

Let $F = \langle f^1, \ldots, f^m, h\rangle$ be an aggregation mechanism that satisfies $IC(F) \leqslant \epsilon$. In case that f^j for some j is close to the constant zero function, F is close to the consistent aggregation mechanism that satisfies $f^j = h = 0$ (and all other issue-aggregating functions the same as in F).

Otherwise, all f^j are Δ-far from the constant zero function (for some Δ). The main insight in the proof is that for any two issue-aggregating functions f^j and f^k, we can bound the product of the ignorability of a coalition S for f^k and the minimal influence of its members on f^j using the inconsistency index of F by $P_S(f^k) \cdot \min_{i \in S} I_i(f^j) \leqslant C \cdot IC(F)$ for C a constant that depends solely on the size of S and on simple characteristics of the other issue-aggregating functions. Using this insight we can prove that any of the f^j functions is $\boldsymbol{4n\epsilon\Delta^{1-m}}$-close to a function g^j that is determined by $\boldsymbol{\log_2\left(\frac{2}{\Delta}\right)}$ voters. So if Δ is big enough (with respect to ϵ) we can deduce that there exists an issue aggregating function h' s.t. $G = \langle g^1, \ldots, g^m, h'\rangle$ is close to F and consistent. The latter is true since $IC(G)$ is close to $IC(F)$ (and hence small) but is determined by a small number of votes (and hence cannot be too small strictly positive number). □

7.2 Xor Agenda

For the agenda $\left\langle A^1, \ldots, A^{m-1}, \bigoplus_{j=1}^{m-1} A^j \right\rangle$ we prove:

Theorem 4. *Let $m \geqslant 3$ and let the agenda be $\mathbb{X} = \left\langle A^1, \ldots, A^{m-1}, \bigoplus_{j=1}^{m-1} A^j \right\rangle$.*

For any $\epsilon < \frac{1}{6}$ and any independent aggregation mechanism F: If $IC(F) \leqslant \epsilon$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) \leqslant m\epsilon$.

*Proof (***Proof sketch).** [23]

The proof uses the Fourier representation of the issue aggregating functions. That is, representing the functions as linear combinations of the linear boolean functions. We can represent any boolean function as $f = \sum_{\chi \in \mathtt{Lin}} \widehat{f}(\chi)\chi$ when $\widehat{f}(\chi) = \mathbb{E}\left[f(x)\chi(x)\right] = 1 - 2d\left(f, \chi\right) = 2d\left(f, -\chi\right) - 1$.

Given an independent aggregation mechanism $F = \langle f^1, \ldots, f^m, h\rangle$ we analyze the expression $\mathbb{E} = \mathbb{E}\left[\prod_{j=1}^{m} f^j(x_j) h\left(\prod_{j=1}^{m} f^j x_j\right)\right]$ (when x_j are sampled uniformly and independently). On one hand we show that $\mathbb{E} = 1 - 2IC(F)$. On the other hand we show that $\mathbb{E} = \sum_{\chi \in \mathtt{Lin}} \prod_{j=1}^{m} \widehat{f^j}(\chi)\widehat{h}(\chi)$. Hence, when $IC(F)$ is small, this expression is close to one and hence there exists a linear function such that all f^j, and h are close to it (up to negation). Noticing that for any linear function χ, $\langle \chi, \chi, \cdots, \chi\rangle$ (and the result of negation of any even number of functions) is a consistent independent aggregation mechanism for this agenda gives us the result. □

7.3 Extending to δ-independence Results

We prove

Theorem 5.
If

there exists a function $\delta(\epsilon, n)$ s.t. for any $\epsilon > 0$ and $n \geqslant 2$, if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying independence and $IC(F) \leqslant \delta(\epsilon)$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < \epsilon$.

Then,

[23] The proof is similar to the analysis of the BLR (Blum-Luby-Rubinfeld) linearity test done in [1].

for any $\epsilon > 0$ and $n \geqslant 2$, there exist $\delta_{IC}, \delta_{DI} > 0$, such that if F is an aggregation mechanism for $\mathbb{X}$ over n voters satisfying $IC(F) \leqslant \delta_{IC}$ and $DI(F) \leqslant \delta_{DI}$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) < \epsilon$.
Moreover, one can take $\delta_{IC} = \delta\left((1-\beta_\epsilon)\epsilon\right) - \beta_\epsilon\epsilon$ and $\delta_{DI} = \frac{1}{2m}\beta_\epsilon\epsilon$ for any $\beta_\epsilon \in [0,1]$ satisfying $\delta\left((1-\beta_\epsilon)\epsilon\right) \geqslant \beta_\epsilon\epsilon$

In order to extend the results for the δ-dependent case ($DI(F) \neq 0$) we prove the following agenda-independent proposition.

Proposition (Proposition 3). *Let F be an aggregation mechanism for an agenda over m issues that satisfies $DI(F) \leqslant \delta$. Then there exists an independent aggregation mechanism H that satisfies $d(F,H) \leqslant 2m\delta$. If $DI(F) \geqslant \delta$, then every aggregation mechanism H that satisfies $DI(H) = 0$, also satisfies $d(F,H) \geqslant \frac{1}{2}\delta$*

I.e., if F is δ-independent we can find a close consistent aggregation mechanism H and since it is close we can deduce bounds on the proximity of F to the consistent and independent aggregation mechanisms from bounds on this proximity of H. Similarly, since H is close to F, we can deduce that if F is δ-consistent then H is δ'-consistent for δ' close to δ. Combining these we get the theorem.

8 Summary and Future Work

In this paper we defined the question of approximate aggregation which is a generalization of the study of aggregation mechanisms that satisfy consistency and independence. We defined measures for the relaxation of the consistency constraint (inconsistency index IC) and for the relaxation of the independence constraint (dependency index DI). To our knowledge, this is the first time this question is stated in its general form.

We proved that relaxing these constraints does not extend the set of satisfying aggregation mechanisms in a non-trivial way for any truth-functional agenda in which every conclusion is either conjunction or xor up to negation of inputs or output. We notice that every conclusion of two premises can be stated as such as well as any affine agenda. Particulary we calculated the dependency between the extension of this class (ϵ) and the inconsistency index ($\delta(\epsilon)$) (although probably not strictly) for two families of truth-functional agendas with one conclusion. The relation we proved includes dependency on the number of voters (n). In similar works for preference agendas [21,33,22] the relation did not include such a dependency. An interesting question is whether such a dependency is inherent for conjunction agendas or whether it is possible to prove a relation that does not include it.

A major assumption in this paper is the uniform distribution over the inputs which is equivalent to assuming i.i.d uniform distribution over the premises. We think that our results can be extended for other distributions (still assuming voters' opinions are distributed i.i.d) over the space over premises' opinions which seem more realistic.

Immediate extensions for this work can be to extend our result to more complex truth-functional agendas and generalize our results to non-truth-functional agendas to get a result unifying our work and Kalai, Mossel, and Keller's works for the preference agenda.

A major open question is whether one can find an agenda for which relaxing the constraints of independence and consistency extends the class of satisfying aggregation mechanisms in a non-trivial way.

References

1. Bellare, M., Coppersmith, D., Hastad, J., Kiwi, M., Sudan, M.: Linearity testing in characteristic two. In: FOCS '95: Proceedings of the 36th Annual Symposium on Foundations of Computer Science. p. 432. IEEE Computer Society, Washington, DC, USA (1995) 4, 12
2. Black, D.: The theory of committees and elections. Kluwer Academic Publishers (1957 (reprint at 1986)) 9
3. Blum, M., Luby, M., Rubinfeld, R.: Self-testing/correcting with applications to numerical problems. Journal of Computer and System Sciences 47(3), 549 – 595 (1993) 4

4. Bollobás, B.: Combinatorics: set systems, hypergraphs, families of vectors, and combinatorial probability. Cambridge University Press, New York, NY, USA (1986) 24
5. Bovens, L., Rabinowicz, W.: Democratic answers to complex questions an epistemic perspective. Synthese 150, 131–153 (2006) 4, 9
6. Caragiannis, I., Kaklamanis, C., Karanikolas, N., Procaccia, A.D.: Socially desirable approximations for dodgson's voting rule. In: Proc. 11th ACM Conference on Electronic Commerce (2010) 2, 3
7. Chapman, B.: Rational aggregation. Politics, Philosophy & Economics 1(3), 337 (2002) 7
8. Chen, V., Sudan, M., Xie, N.: Property testing via set-theoretic operations. In: *Proc. 2nd Innovations in Computer Science*. pp. 211–222 (2011) 10
9. Dietrich, F., List, C.: Arrows theorem in judgment aggregation. Social Choice and Welfare 29(1), 19–33 (2007) 9
10. Dietrich, F.: Judgment aggregation: (im)possibility theorems. Journal of Economic Theory 126(1), 286–298 (January 2006) 4, 9
11. Dietrich, F., List, C.: Judgment aggregation by quota rules. Journal of Theoretical Politics 19(4), 391–424 (2007) 4, 9
12. Dietrich, F., List, C.: Strategy-proof judgment aggregation. Open Access publications from London School of Economics and Political Science http://eprints.lse.ac.uk/, London School of Economics and Political Science (2007) 7
13. Dokow, E., Holzman, R.: Aggregation of binary evaluations for truth-functional agendas. Social Choice and Welfare 32(2), 221–241 (February 2009) 3, 5, 8, 11
14. Dokow, E., Holzman, R.: Aggregation of binary evaluations. Journal of Economic Theory 145(2), 495 – 511 (2010) 5, 8
15. Dokow, E., Holzman, R.: Aggregation of non-binary evaluations. Advances in Applied Mathematics 45(4), 487 – 504 (2010) 5
16. Elkind, E., Faliszewski, P., Slinko, A.: Distance rationalization of voting rules. In: The Third International Workshop on Computational Social Choice (COMSOC-2010) (2010) 9
17. Fischer, E.: The art of uninformed decisions: A primer to property testing. Science 75, 97–126 (2001) 10
18. Fishburn, P., Rubinstein, A.: Aggregation of equivalence relations. Journal of Classification 3(1), 61–65 (March 1986) 9
19. Friedgut, E., Kalai, G., Nisan, N.: Elections can be manipulated often. In: FOCS '08: Proceedings of the 2008 49th Annual IEEE Symposium on Foundations of Computer Science. pp. 243–249. IEEE Computer Society, Washington, DC, USA (2008) 2, 3
20. Goldreich, O.: Combinatorial property testing (a survey). In: Randomization methods in algorithm design: DIMACS workshop, December 12-14, 1997. p. 45. Amer Mathematical Society (1999) 10
21. Kalai, G.: A fourier-theoretic perspective on the condorcet paradox and arrow's theorem. Adv. Appl. Math. 29(3), 412–426 (2002) 4, 13
22. Keller, N.: A tight quantitative version of Arrow's impossibility theorem. Arxiv preprint arXiv:1003.3956 (2010) 4, 13
23. Kemeny, J.G.: Mathematics without numbers. Daedalus 88(4), pp. 577–591 (1959) 9
24. Kornhauser, L.A.: Modeling collegial courts. ii. legal doctrine. Journal of Law, Economics and Organization 8(3), 441–70 (October 1992) 4, 9
25. Kornhauser, L.A., Sager, L.G.: Unpacking the court. The Yale Law Journal 96(1), 82–117 (1986) 1, 4, 9
26. List, C.: A model of path-dependence in decisions over multiple propositions. American Political Science Review 98(03), 495–513 (2004) 4, 9
27. List, C.: The probability of inconsistencies in complex collective decisions. Social Choice and Welfare 24(1), 3–32 (05 2005) 2, 4, 6
28. List, C.: Judgment aggregation: a short introduction (August 2008), `http://philsci-archive.pitt.edu/4319/` 2
29. List, C., Pettit, P.: Aggregating sets of judgments: An impossibility result. Economics and Philosophy 18, 89–110 (2002) 4, 9
30. List, C., Puppe, C.: Judgement aggregation: A survey. In: P. Anand, P.P., Puppe, C. (eds.) The Handbook of Rational and Social Choice. Oxford University Press, USA (2009) 2, 8
31. Miller, A.D.: Group identification. Games and Economic Behavior 63(1), 188 – 202 (2008) 9
32. Mongin, P.: Factoring out the impossibility of logical aggregation. Journal of Economic Theory 141(1), 100–113 (July 2008) 7
33. Mossel, E.: A quantitative arrow theorem. Probability Theory and Related Fields (Forthcoming) 4, 6, 13
34. Nehring, K.: Arrows theorem as a corollary. Economics Letters 80(3), 379–382 (2003) 9

35. Nehring, K., Puppe, C.: Consistent judgement aggregation: The truth-functional case. Social Choice and Welfare 31(1), 41–57 (2008) 3, 11
36. O'Donnell, R.: Some topics in analysis of boolean functions. In: STOC '08: Proceedings of the 40th annual ACM symposium on Theory of computing. pp. 569–578. ACM, New York, NY, USA (2008) 28
37. Pettit, P.: Deliberative democracy and the discursive dilemma. Philosophical Issues 11(1), 268–299 (2001) 4, 9
38. Pigozzi, G.: Belief merging and the discursive dilemma: An argument-based account to paradoxes of judgment aggregation. Synthese 152(2), pp. 285–298 (2006) 4, 9
39. Ron, D.: Property testing (a tutorial), vol. 2, chap. 15, pp. 597–649. Kluwer Academic Pub (2001) 9, 10
40. Rubinstein, A., Kasher, A.: On the question "Who is a J?": A social choice approach. Princeton Economic Theory Papers 00s5, Economics Department, Princeton University (Sep 1998) 9
41. Rubinstein, A., Fishburn, P.C.: Algebraic aggregation theory. Journal of Economic Theory 38(1), 63–77 (February 1986) 5
42. de Wolf, R.: A brief introduction to fourier analysis on the boolean cube. Theory of Computing, Graduate Surveys 1, 1–20 (2008) 28
43. Xia, L., Lang, J., Ying, M.: Sequential voting rules and multiple elections paradoxes. In: Proceedings of the 11th conference on Theoretical aspects of rationality and knowledge. pp. 279–288. TARK '07, ACM, New York, NY, USA (2007), http://doi.acm.org/10.1145/1324249.1324286 7
44. Nehama, I.: Approximate judgement aggregation. In: WINE. Lecture Notes in Computer Science, Springer (2011 (Forthcoming)) 1

A Lemmas Proof - General

A.1 Propositions 4,6

For a given pair of independent aggregation mechanisms, the following propositions connect between the pairwise distance between respective issue-aggregating functions (which we found easier to analyze in most cases) and both the distance between the mechanisms and the difference between the inconsistency indices of them.

Proposition 4.
For any agenda $\mathbb{X}$ *of* m *issues and any voting functions*
$f^1, \ldots, f^m, g^1, \ldots, g^m, : \{0,1\}^n \to \{0,1\}$,

$$d^{\mathbb{X}}\left(\left\langle f^1, \ldots, f^m \right\rangle \ , \ \left\langle g^1, \ldots, g^m \right\rangle\right) \leqslant \sum_{j=1}^{m} Pr\left[f^j\left(X^j\right) \neq g^j\left(X^j\right) \ \mid X \in \mathbb{X}^n\right].$$

Proof. Direct use of the union-bound inequality.

Proposition 5.
For any agenda $\mathbb{X}$ *of* m *issues and voting functions* $f^1, \ldots, f^m, g^1 : \{0,1\}^n \to \{0,1\}$,

$$\left|IC^{\mathbb{X}}\left(\left\langle f^1, f^2, \ldots, f^m \right\rangle\right) - IC^{\mathbb{X}}\left(\left\langle g^1, f^2, \ldots, f^m \right\rangle\right)\right| \leqslant Pr[f^1\left(X^1\right) \neq g^1\left(X^1\right) \ \mid X \in \mathbb{X}^n].$$

Proof.

$$\begin{aligned}
IC\left(\left\langle f^1, \ldots, f^m \right\rangle\right) &= \Pr\left[\left(f^1\left(X^1\right), f^2\left(X^2\right), \ldots, f^m\left(X^m\right)\right) \notin \mathbb{X} \mid X \in \mathbb{X}^n\right] \\
&\leqslant Pr[f^1\left(X^1\right) \neq g^1\left(X^1\right) \ \mid X \in \mathbb{X}^n] \\
&\quad + \Pr\left[\left(f^1\left(X^1\right), \ldots, f^m\left(X^m\right)\right) \notin \mathbb{X} \ \bigwedge f^1(x) = g^1(x) \mid X \in \mathbb{X}^n\right] \\
&\leqslant Pr[f^1\left(X^1\right) \neq g^1\left(X^1\right) \ \mid X \in \mathbb{X}^n] \\
&\quad + \Pr\left[\left(g^1\left(X^1\right), f^2\left(X^2\right), \ldots, f^m\left(X^m\right)\right) \notin \mathbb{X} \mid X \in \mathbb{X}^n\right] \\
&= IC^{\mathbb{X}}\left(\left\langle g^1, \ldots, f^m \right\rangle\right) + Pr[f^1\left(X^1\right) \neq g^1\left(X^1\right) \ \mid X \in \mathbb{X}^n]
\end{aligned}$$

Hence, $IC\left(\left\langle f^1, \ldots, f^m \right\rangle\right) - IC\left(\left\langle g^1, f^2, \ldots, f^m \right\rangle\right) \leqslant Pr[f^1\left(X^1\right) \neq g^1\left(X^1\right) \ \mid X \in \mathbb{X}^n]$.
Similarly we can prove that $IC\left(\left\langle g^1, \ldots, f^m \right\rangle\right) - IC\left(\left\langle f^1, f^2, \ldots, f^m \right\rangle\right) \leqslant Pr[f^1\left(X^1\right) \neq g^1\left(X^1\right) \ \mid X \in \mathbb{X}^n]$.

As a corollary of the above we derive

Proposition 6.
For any agenda $\mathbb{X}$ *of* m *issues and any voting functions*
$f^1, \ldots, f^m, g^1, \ldots, g^m, : \{0,1\}^n \to \{0,1\}$,

$$\left|IC^{\mathbb{X}}\left(\left\langle f^1, \ldots, f^m \right\rangle\right) - IC^{\mathbb{X}}\left(\left\langle g^1, \ldots, g^m \right\rangle\right)\right| \leqslant \sum_{j=1}^{m} Pr\left[f^j\left(X^j\right) \neq g^j\left(X^j\right) \ \mid X \in \mathbb{X}^n\right].$$

A.2 Proposition 2

Proposition.
Let F be an aggregation mechanism and j an issue. If $DI^j(F) \leqslant \delta$, then there exists an aggregation mechanism H that satisfies $DI^j(H) = 0$ and $d(F, H) \leqslant 2\delta$. If $DI^j(F) \geqslant \delta$, then every aggregation mechanism H that satisfies $DI^j(H) = 0$, also satisfies $d(F, H) \geqslant \frac{1}{2}\delta$

Proof. With no loss of generality assume that $j = 1$.

– Let F be an aggregation mechanism. We define the functions $G^1, \ldots, G^m : \mathbb{X}^n \to \{0, 1\}$ by:

$$G^1(X) = \begin{cases} 1 & \Pr_{Y \in \mathbb{X}^n}\left[(F(X))^1 = 1 \mid Y^1 = X^1\right] \geqslant \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$
$$j = 2, \ldots, m \; G^j(X) = (F(X))^j$$

and an aggregation mechanism $G(X) = \langle G^1(X), \ldots, G^m(X)\rangle$. Clearly $DI^1(G) = 0$.

$$\begin{aligned} d(F, G) &= \Pr_{X \in \mathbb{X}^n}\left[(F(X))^1 \neq G^1(X)\right] \\ &= \Pr_{X \in \mathbb{X}^n}\left[\Pr_{Y \in \mathbb{X}^n}\left[(F(X))^1 \neq (F(Y))^1 | X^1 = Y^1\right] \geqslant \frac{1}{2}\right] \\ &\leqslant 2 \mathop{\mathbb{E}}_{X \in \mathbb{X}^n}\left[\Pr_{Y \in \mathbb{X}^n}\left[(F(X))^1 \neq (F(Y))^1 | X^1 = Y^1\right]\right] \\ &= 2DI^1(F) \end{aligned}$$

– Let F be an aggregation mechanism that is ϵ-close to satisfy $DI^1(F) = 0$. That is, we can find an aggregation mechanism G such that $d(F, G) \leqslant \epsilon$ and $DI^1(G) = 0$.

$$\begin{aligned} DI^1(F) &= \mathop{\mathbb{E}}_{X \in \mathbb{X}^n}\left[\Pr_{Y \in \mathbb{X}^n}\left[(F(X))^1 \neq (F(Y))^1 | X^1 = Y^1\right]\right] \\ &\leqslant \Pr[F(X) \neq G(X)] + \sum_{X : F(X) = G(X)} \Pr_{Z \in \mathbb{X}^n}[Z = X] \Pr_{Y \in \mathbb{X}^n}\left[(G(X))^1 \neq (F(Y))^1 | X^1 = Y^1\right] \\ &\leqslant \epsilon + \sum_{X : F(X) = G(X)} \Pr_{Z \in \mathbb{X}^n}[Z = X] \Pr_{Y \in \mathbb{X}^n}\left[(G(Y))^1 \neq (F(Y))^1 | X^1 = Y^1\right] \\ &\leqslant 2\epsilon \end{aligned}$$

□

A.3 Proposition 3

Proposition.
Let F be an aggregation mechanism for an agenda over m issues that satisfies $DI(F) \leqslant \delta$. Then there exists an independent aggregation mechanism H that satisfies $d(F, H) \leqslant 2m\delta$. If $DI(F) \geqslant \delta$, then every aggregation mechanism H that satisfies $DI(H) = 0$, also satisfies $d(F, H) \geqslant \frac{1}{2}\delta$

Proof.

– We define issue aggregating functions $h^1, \ldots, h^m : \{0, 1\}^n \to \{0, 1\}$ by:

$$h^j(t) = \begin{cases} 1 & \Pr_{X \in \mathbb{X}^n}\left[F^j(X) = 1 \mid X^j = t\right] \geqslant \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and an (independent) aggregation mechanism $H = \left\langle h^1, \ldots, h^m \right\rangle$.

$$\begin{aligned}
d(F,H) &= \Pr_{X\in\mathbb{X}^n}\left[F(X) \neq H(X)\right] \\
&\leqslant \sum_{j=1}^{m} \Pr_{X\in\mathbb{X}^n}\left[F^j(X) \neq H^j(X)\right] \\
&= \sum_{j=1}^{m} \Pr_{X\in\mathbb{X}^n}\left[\Pr_{Y\in\mathbb{X}^n}\left[F^j(X) \neq H^j(Y) | X^j = Y^j\right] \geqslant \tfrac{1}{2}\right] \\
&\leqslant \sum_{j=1}^{m} 2 \mathop{\mathbb{E}}_{X\in\mathbb{X}^n}\left[\Pr_{Y\in\mathbb{X}^n}\left[F^j(X) \neq F^j(Y) | X^j = Y^j\right]\right] \\
&\leqslant 2mDI^j(F) \\
&\leqslant 2mDI(F)
\end{aligned}$$

– The other direction is a direct corollary of Proposition 2. □

A.4 Id Agenda

For completeness we add here an approximate aggregation theorem for the id agenda $\langle A, A\rangle$

Theorem 6.
For any $\epsilon > 0$ and any independent aggregation mechanism F:
If $IC^{\langle A,A\rangle} \leqslant \epsilon$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) \leqslant \epsilon$.

Proof. This theorem is trivial since

$$IC^{\langle A,A\rangle}(\langle f,g\rangle) \; = \Pr\left[f(x) \neq g(y) \mid x = y\right] \; = \Pr\left[f(x) \neq g(x)\right] \; = d(f,g)$$

Noticing that any aggregation mechanism of the form $\langle f, f\rangle$ is consistent for this agenda, we get the theorem.

B Lemmas Proof - Conjunction agenda

B.1 Theorem 3

Theorem.
For the agenda $\mathbb{X} = \left\langle A^1, \dots, A^m, \bigwedge_{j=1}^{m} A^j \right\rangle$ *for* $m \geqslant 2$*:*

For any $\epsilon > 0$ *and any independent aggregation mechanism* F*:*
If $IC(F) \leqslant \epsilon$*, then there exists an aggregation mechanism* G *that satisfies consistency and independence such that* $d(F,G) < 8m\,(n\epsilon)^{\frac{1}{2m-1}}$*.*

Proof.
Assume a mechanism $F = \langle f^1, \dots, f^m, h\rangle$ is given such that $IC(F) \leqslant \epsilon$ and define $\varDelta = 4\,(n\epsilon)^{\frac{1}{2m-1}}$.

If there exists $j \in \{1, \dots, m\}$ s.t. $d(f^j, 0) \leqslant \varDelta$ (with no loss of generality assume $j = 1$), then $\langle f^1, f^2, \dots, f^m, h\rangle$ is $(\epsilon + \varDelta)$-close to $\langle 0, f^2, \dots, f^m, 0\rangle$[24] which is a consistent mechanism.

If $\forall j \in \{1, \dots, m\} \ : \ d(f^j, 0) \geqslant \varDelta$, then, based on the following lemma each of the functions f^j is close to a function that depends on a small number of voters.

Lemma 2.
Let $f^1, \dots, f^m, h : \{0,1\}^n \to \{0,1\}$ *be* $(m+1)$ *voting functions and* $\epsilon, \delta > 0$ *constants. If*

$$IC^{\left\langle A^1,\dots,A^m, \bigwedge_{j=1}^{m} A^j \right\rangle}(f^1, \dots, f^m, h) \leqslant \epsilon.$$

Then there exists a function $g : \{0,1\}^n \to \{0,1\}$ *that depends on at most*

$$\log_2 \left(d\left(f^2, 0\right) - 2 \left(\prod_{j=3}^{m} d\left(f^j, 0\right) \right)^{-1} \frac{n\epsilon}{\delta} \right)^{-1}$$

voters and satisfies

$$d\left(f^1, g\right) \leqslant \delta.$$

(and similarly for $f^2, \dots, f^m$*)*

By choosing $\delta = 4\epsilon\varDelta^{1-m}$ we get that there exist functions $g^1, \dots, g^m : \{0,1\}^n \to \{0,1\}$ s.t.

- $\forall j \ : \ d\left(f^j, g^j\right) \leqslant 4n\epsilon\varDelta^{1-m}$
- g^j depends on a junta of voters, J^j, of size at most $\log_2\left(\frac{\varDelta}{2}\right)^{-1}$

Let $h' : \{0,1\}^n \to \{0,1\}$ be a issue-aggregating function satisfying
$\forall h \ : \ IC(f^1, \dots, f^m, h') \leqslant IC(f^1, \dots, f^m, h)$. Then based on Proposition 6 we get:

$$\begin{aligned} IC(g^1, \dots, g^m, h') &\leqslant IC(g^1, \dots, g^m, h) \\ &\leqslant IC(f^1, \dots, f^m, h) + \sum_{j=1}^{m} d\left(f^j, g^j\right) \\ &\leqslant \epsilon\left(1 + 4mn\varDelta^{1-m}\right) \\ &\leqslant 5mn\epsilon\varDelta^{1-m} \\ &= 2^{2-3m} 5m \cdot \tfrac{1}{2^m} \varDelta^m \\ &< \tfrac{1}{2^m} \varDelta^m \\ &\leqslant \prod_{j=1}^{m} \cdot 2^{-|J^j|} \end{aligned}$$

One the other hand, since the functions g^j depend on a small number of voters, the inconsistency index cannot be too small.

[24] $d\left(F\ ,\ \langle 0, f^2, \dots, f^m, 0\rangle\right) = \Pr[f^1\left(X^1\right) = 1 \vee h(X^{m+1}) = 1] = \Pr[f^1\left(X^1\right) = 1] + \Pr[f^1\left(X^1\right) = 0 \wedge h(X^{m+1}) = 1] \leqslant \varDelta + \epsilon$

Lemma 3. *Let $f^1, \dots, f^m : \{0,1\}^n \to \{0,1\}$ be m voting functions and $J^1, \dots, J^m \subseteq \{0,1,\dots,n\}$ coalitions such that each voting function f^j depends only on the votes of the members of the coalition J^j. Then*

$$\min_{h:\{0,1\}^n \to \{0,1\}} IC(f^1, \dots, f^m, h) = C \prod_{j=1}^{m} \cdot 2^{-|J^j|}$$

for some integer C.

So we get that actually $IC\left(\left\langle g^1, \dots, g^m, h' \right\rangle\right) = 0$ and $\left\langle g^1, \dots, g^m, h' \right\rangle$ is consistent.

Since $\max(\epsilon + \mathit{\Delta}, \epsilon + 4mn\epsilon\mathit{\Delta}^{1-m}) \leqslant 8m\,(n\epsilon)^{\frac{1}{2m-1}}$ (when $n^2\epsilon < 1$), we get the theorem.

B.2 Lemma 2

Lemma.
Let $f^1, \dots, f^m, h : \{0,1\}^n \to \{0,1\}$ be $(m+1)$ voting functions and $\epsilon, \delta > 0$ constants. If

$$IC^{\left\langle A^1,\dots,A^m,\bigwedge_{j=1}^{m} A^j\right\rangle}(f^1, \dots, f^m, h) \leqslant \epsilon.$$

Then there exists a function $g : \{0,1\}^n \to \{0,1\}$ that depends on at most

$$\log_2 \left(d\left(f^2, 0\right) - 2 \left(\prod_{j=3}^{m} d\left(f^j, 0\right) \right)^{-1} \frac{n\epsilon}{\delta} \right)^{-1}$$

voters and satisfies

$$d\left(f^1, g\right) \leqslant \delta.$$

(and similarly for $f^2, \dots, f^m$)

Proof. The proof of the lemma is constructive and defines a junta J and the function g that depends only on the votes of J. For proving the lemma, we define for a given function $f : \{0,1\}^n \to \{0,1\}$ and a coalition J (the junta), the junta function $f^J : \{0,1\}^n \to \{0,1\}$. It is derived from f in the following way:

$$f^J(x) = \text{majority}\left\{ f(y) \mid y_J = x_J \right\}.$$

I.e., for a given input, f^J reads only the votes of the junta members, iterates over all the possible votes for the members outside the junta, and returns the more frequent result (assuming uniform distribution over the votes of the voters outside J).
In our case, we define the junta to be all the voters with large influence

$$J = \{i \mid I_i(f^1) \geqslant \delta\}$$

and g to be $\left(f^1\right)^J$.

For these two definitions we prove the different claims of the lemma.

- $d\left(f^1, g\right) \leqslant n\delta$
  This is a direct corollary of the following lemma and the definition of J.
  Lemma 4. *Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function and $J \subseteq \{1, \dots, n\}$ a coalition. Then $d(f, f^J) \leqslant \sum_{i \notin J} I_i(f)$.*
- $|J| \leqslant \log_2 \left(d\left(f^2, 0\right) - 2\frac{\epsilon}{\delta} \left(\prod_{j=3}^{m} d\left(f^j, 0\right) \right)^{-1} \right)^{-1}$

  Lemma 5. *Let $f^1, \dots, f^m, h : \{0,1\}^n \to \{0,1\}$ be m voting functions, $S \subseteq \{1, \dots, n\}$ a coalition. Then $P_S(f^2) \cdot \min_{i \in S} I_i(f^1) \leqslant 2 \left(\prod_{j \geqslant 3} d(f^j, 0) \right)^{-1} \left(1 - 2^{-|S|}\right)^{-1} IC(f^1, \dots, f^m, h)$*

  Assigning in lemma 5 $S \leftarrow J$, we get:

$$\begin{aligned}\left(1 - 2^{-|S|}\right) P_J(f^2) \cdot \min_{i \in S} I_i(f^1) &\leqslant 2 \left(\prod_{j \geqslant 3} d(f^j, 0) \right)^{-1} \left(\min_{i \in J} I_i(f^j) \right)^{-1} IC(f^1, \dots, f^m, h) \\ &\leqslant 2 \left(\prod_{j=3}^{m} d(f^j, 0) \right)^{-1} \frac{\epsilon}{\delta}\end{aligned}$$

Lemma 6. *Let $f : \{0,1\}^n \to \{0,1\}$ be a voting function and $S \subseteq \{1,\dots,n\}$ a coalition. Then*

$$|S| \leqslant \log_2 \left(d(f,0) - \left(1 - 2^{-|S|} P_S(f)\right)\right)^{-1}$$

Assigning in lemma 6 $S \leftarrow J$, $f \leftarrow f^2$, we get:

$$|J| \leqslant \log_2 \left(d\left(f^2,0\right) - \left(1 - 2^{-|J|}\right) P_J(f^2)\right)^{-1} \leqslant \log_2 \left(d\left(f^2,0\right) - 2\frac{\epsilon}{\delta}\left(\prod_{j=3}^{m} d\left(f^j,0\right)\right)^{-1}\right)^{-1} \quad \square$$

B.3 Lemma 5

Lemma.
Let $f^1, \ldots, f^m, h : \{0,1\}^n \to \{0,1\}$ *be* m *voting functions,* $S \subseteq \{1, \ldots, n\}$ *a coalition.*
Then $P_S(f^2) \cdot \min_{i \in S} I_i(f^1) \leqslant 2 \left(\prod_{j \geqslant 3} d(f^j, 0) \right)^{-1} \left(1 - 2^{-|S|}\right)^{-1} IC(f^1, \ldots, f^m, h)$

Proof. With no loss of generality assume $S = \{1, 2, \ldots, t\}$. Denote by E the event $\left[\forall j \geqslant 3 \ : \ f^j(x^j) = 1\right]$ and by $\Pr_E[A]$ the probability $\Pr[A \mid E]$. Then (We use the notation $x \oplus e_i$ for adding e_i (the $\text{i}^{\underline{\text{th}}}$ elementary vector) which is equivalent to flipping the $\text{i}^{\underline{\text{th}}}$ bit $0 \leftrightarrow 1$)

$$\begin{aligned}
\Pr_E\left[\bigwedge_{j=1}^m f^j(x^j) \neq h\left(\bigwedge_{j=1}^m x^j\right)\right] &= \Pr_E\left[f^1(x^1) \wedge f^2(x^2) \neq h\left(\bigwedge_{j=1}^m x^j\right)\right] \\
&\geqslant \Pr_E\left[f^1(x^1) \wedge f^2(x^2) \neq h\left(\bigwedge_{j=1}^m x^j\right) \ ; \ x^2|_S \neq \bar{1} \wedge f^2(x^2) = 1\right] \\
&= \Pr_E\left[f^1(x^1) \neq h\left(\bigwedge_{j=1}^m x^j\right) \ ; \ x^2|_S \neq \bar{1} \wedge f^2(x^2) = 1\right] \\
&= \sum_{i=1}^t \Pr_E\left[f^1(x^1) \neq h\left(\bigwedge_{j=1}^m x^j\right) \ ; \ x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1\right] \\
&\geqslant \sum_{i=1}^t \tfrac{1}{2} \Pr_E\left[\begin{array}{c} f^1(x^1) \neq h\left(\bigwedge_{j=1}^m x^j\right) \ ; \ x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1 \\ \bigvee \\ f^1(x^1 + e_i) \neq h\left((x^1 + e_i) \wedge \bigwedge_{j=2}^m x^j\right) \ ; \ x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1 \end{array}\right] \\
&= \tfrac{1}{2} \sum_{i=1}^t \Pr_E\left[\begin{array}{c} f^1(x^1) \neq h\left(\bigwedge_{j=1}^m x^j\right) \ ; \ x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1 \\ \bigvee \\ f^1(x^1 + e_i) \neq h\left(x^1 \wedge \bigwedge_{j=2}^m x^j\right) \ ; \ x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1 \end{array}\right] \\
&\geqslant \tfrac{1}{2} \sum_{i=1}^t \Pr_E\left[f^1(x^1) \neq f^1(x^1 + e_i) \ ; \ x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1\right] \\
&= \tfrac{1}{2} \sum_{i=1}^k \Pr_E\left[f^1(x^1) \neq f^1(x^1 + e_i)\right] \cdot \Pr_E\left[x^2|_{[i-1]} = \bar{1} \wedge x^2_i = 0 \wedge f^2(x^2) = 1\right] \\
&\geqslant \tfrac{1}{2} \min_{i \in S} \Pr_E\left[f^1(x^1) \neq f^1(x^1 + e_i)\right] \cdot \Pr_E\left[x^2|_S \neq \bar{1} \wedge f^2(x^2) = 1\right]
\end{aligned}$$

$$\begin{aligned}
IC(f^1, \ldots, f^m, h) &= \Pr\left[\bigwedge_{j=1}^m f^j(x^j) \neq h\left(\bigwedge_{j=1}^m x^j\right)\right] \\
&= \Pr\left[\forall j \geqslant 3 \ : f^j(x^j) = 1\right] \cdot \Pr\left[\bigwedge_{j=1}^m f^j(x^j) \neq h\left(\bigwedge_{j=1}^m x^j\right) \middle| \forall j \geqslant 3 \ : f^j(x^j) = 1\right] \\
&\geqslant \prod_{j=3}^m \Pr[f^j(x^j) = 1] \cdot \tfrac{1}{2} \min_{i \in S} \Pr\left[f^1(x^1) \neq f^1(x^1 + e_i) \middle| \forall j \geqslant 3 \ : f^j(x^j) = 1\right] \\
&\quad \cdot \Pr\left[x^2|_S \neq \bar{1} \wedge f^2(x^2) = 1 \middle| \forall j \geqslant 3 \ : f^j(x^j) = 1\right] \\
&= \prod_{j=3}^m \Pr[f^j(x^j) = 1] \cdot \tfrac{1}{2} \min_{i \in S} I_i(f^1) \cdot P_S(f^2) \Pr\left[x^2|_S \neq \bar{1}\right] \\
&= \tfrac{1}{2}\left(1 - 2^{-|S|}\right) \prod_{j=3}^m \Pr[f^j(x^j) = 1] \cdot \min_{i \in S} I_i(f^1) \cdot P_S(f^2)
\end{aligned}$$

□

B.4 Lemma 4

Lemma.
Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function and $J \subseteq \{1,\dots,n\}$ a coalition. Then $d(f, f^J) \leqslant \sum_{i \notin J} I_i(f)$.

Proof. We define for a vector $c \in \{0,1\}^J$ the function $f_c^J : \{0,1\}^n \to \{0,1\}$ by $f_c^J(x) = f(y)$ where $y_J = c$ and $y_{-J} = x_{-J}$. Assume that c_i is sampled uniformly and independently at random. Then $f^J(x_J, x_{-J}) = \begin{cases} 0 & \mathbb{E}_c[f_c^J(x)] < \frac{1}{2} \\ 1 & \mathbb{E}_c[f_c^J(x)] \geqslant \frac{1}{2} \end{cases}$
We will use the following isoperimetric inequality on the boolean cube:

Proposition (The Isoperimetric Inequality for The Boolean Cube [4]).
Let $f : \{0,1\}^n \to \{0,1\}$ be a voting function. Then $\sum_i I_i(f) \geqslant \min(\mathbb{E}[f], 1 - \mathbb{E}[f])$.

$$\text{For any } c \in \{0,1\}^J : \sum_{i \notin J} I_i(f_c^J) = \sum_i I_i(f_c^J) \geqslant \min(\mathbb{E}[f_c^J], 1 - \mathbb{E}[f_c^J])$$

$$\begin{aligned} \text{For } i \notin J\text{: } I_i(f) &= \Pr[f(x) \neq f(x \oplus e_i)] \\ &= \mathbb{E}_c\left[\Pr[f_c^J(x) \neq f_c^J(x \oplus e_i)]\right] \\ &= \mathbb{E}_c\left[I_i(f_c^J)\right] \end{aligned}$$

$$\begin{aligned} \mathbb{E}_c\left[\sum_{i \notin J} I_i(f_c^J)\right] &= \sum_{i \notin J} \mathbb{E}_c\left[I_i(f_c^J)\right] \\ &= \sum_{i \notin J} I_i(f) \\ \mathbb{E}_c\left[\sum_{i \notin J} I_i(f_c^J)\right] &\geqslant \mathbb{E}_c\left[\min\left(\mathbb{E}\left[f_c^J\right], 1 - \mathbb{E}\left[f_c^J\right]\right)\right] \\ &= Pr\left[f^J(x) \neq f(x)\right] \\ d(f, f^J) &\leqslant \sum_{i \notin J} I_i(f) \end{aligned}$$

□

B.5 Lemma 6

Lemma.
Let $f:\{0,1\}^n \to \{0,1\}$ be a voting function and $S \subseteq \{1,\dots,n\}$ a coalition. Then

$$|S| \leqslant \log_2 \left(d(f,0) - \left(1 - 2^{-|S|} P_S(f)\right)\right)^{-1}$$

Proof.

$$\begin{aligned}\Pr\left[f(x)=1\right] &\leqslant \Pr\left[x_S = \bar{1}\right] + \Pr\left[x_S \neq \bar{1} \wedge f(x) = 1\right] \\ &= \Pr\left[x_S = \bar{1}\right] + P_S(f)\left(1 - \Pr\left[x_S = \bar{1}\right]\right) \\ &= 2^{-|S|} + P_S(f)\left(1 - 2^{-|S|}\right)\end{aligned}$$

□

B.6 Lemma 3

Lemma.
Let $f^1, \dots, f^m : \{0,1\}^n \to \{0,1\}$ be m voting functions and $J^1, \dots, J^m \subseteq \{0,1,\dots,n\}$ coalitions such that each voting function f^j depends only on the votes of the members of the coalition J^j. Then

$$\min_{h:\{0,1\}^n \to \{0,1\}} IC(f^1, \dots, f^m, h) = C \prod_{j=1}^{m} \cdot 2^{-|J^j|}$$

for some integer C.

Proof. For any function $g : \{0,1\}^n \to \{0,1\}$:[25]

$$IC(f^1, \dots, f^m, h) = \Pr\left[\bigwedge_{j=1}^{m} f^j(x^j) \neq g\left(\bigwedge_{j=1}^{m} x^j\right)\right]$$

$$= \sum_{\substack{c^1 \in \{0,1\}^{J^1} \\ \vdots \\ c^m \in \{0,1\}^{J^m}}} \prod_{j=1}^{m} \Pr\left[\left(x^j\right)_{J^j} = c^j\right] \cdot \Pr\left[\bigwedge_{j=1}^{m} f^j(x^j) \neq g\left(\bigwedge_{j=1}^{m} x^j\right) \middle| \forall j \ \left(x^j\right)_{J^j} = c^j\right]$$

Clearly there exists an issue-aggreagting function $h : \{0,1\}^n \to \{0,1\}$ that minimizes the expression $IC(f^1, \dots, f^m, h)$ and does not depend on voters outside of $\bigcup_{j=1}^{m} J^j$.

$$IC(f^1, \dots, f^m, h) = \prod_{j=1}^{m} \cdot 2^{-|J^j|} \# \left\{ c^1 \in \{0,1\}^{J^1}, \dots, c^m \in \{0,1\}^{J^m} \middle| \bigwedge_{j=1}^{m} f^j\left(c^j, \bar{0}\right) \neq h\left(\bigwedge_{j=1}^{m} \left(c^j, \bar{0}\right)\right) \right\} \quad \square$$

[25] We denote by $(c^j, \bar{0})$ the vector that equals to c^j on J^j and has zeroes elsewhere.

B.7 Lemma 1

Lemma.
Let $f^1, \ldots, f^m, h : \{0,1\}^n \to \{0,1\}$ be $m+1$ voting functions satisfying IC $\left(\langle f^1, \ldots, f^m, h\rangle\right) = 0$. Then either there exists an issue j s.t. $f^j = h \equiv 0$ or $f^1 = f^2 = \ldots = f^m = h \in$ `Olig`.

Proof. Assume that for issues j, f^j is not the constant zero function. We will prove that $f^1 = f^2 = \ldots = f^m = h \in$ `Olig` by proving the following series of claims.

- For all issues j, $f^j(\bar{1}) = 1$
  With no loss of generality, assume for contraction that $f^1(\bar{1}) = 0$. Let $x \in \{0,1\}^n$. Then

$$h(x) = h\left(\bar{1} \wedge \left(\bigwedge_{j=2}^{m} x\right)\right) = f^1\left(\bar{1}\right) \wedge \left(\bigwedge_{j=2}^{m} f^j\left(x\right)\right) = 0.$$

  I.e. $h \equiv 0$. From that we can conclude that there exists an issue j s.t. $f^j \equiv 0$ and get a contradiction.
- For all issues j $f^j = h$
  We will prove that $f^1 = h$. The proof is similar for all j.
  Let $x \in \{0,1\}^n$. Then $h(x) = h\left(x \wedge \left(\bigwedge_{j=2}^{m} \bar{1}\right)\right) = f^1\left(x\right) \wedge \left(\bigwedge_{j=2}^{m} f^j\left(\bar{1}\right)\right) = f^1(x)$
- $f^1 \in$ `Olig`
  Let $J = \{i \mid I_i(f^1) \neq 0\}$. Then f^1 is a function of $\{x_i\}_{i \in J}$. Based on lemma 5, for $i \in J$ $P_i(f^1) = 0$ and hence $\left[x_i = 0 \Rightarrow f^1(x) = 0\right]$. So we get that f^1 is the oligarchy of J. □

C Lemmas Proof - XOR agenda

C.1 Theorem 4

Theorem.
Let $m \geqslant 3$ and let the agenda be $\mathbb{X} = \left\langle A^1, \ldots, A^{m-1}, \overset{m-1}{\underset{j=1}{\oplus}} A^j \right\rangle$.

For any $\epsilon < \frac{1}{6}$ and any independent aggregation mechanism F: If $IC(F) \leqslant \epsilon$, then there exists an aggregation mechanism G that satisfies consistency and independence such that $d(F,G) \leqslant m\epsilon$.

Proof.
The theorem is a corollary of the following lemma:
(We rename the values from $\{0,1\}$ to $\{1,-1\}$ in order to ease the analysis (use multiplication instead of xor) and in particular use the Fourier transformation for the issue-aggregating functions f^j.[26])

Lemma 7. *Let $f^1, \ldots, f^m : \{-1,1\}^n \to \{-1,1\}$ be m voting functions and ϵ a constant such that*

$$\Pr\left[\prod_{j=1}^{m-1} f^j(x^j) \neq f^m\left(\prod_{j=1}^{m-1} x^j\right)\right] \leqslant \epsilon$$

Then,

- *There exists a linear function $\chi : \{-1,1\}^n \to \{-1,1\}$ defined as $\chi(x) = \prod_{i \in S} x_i$ for some coalition S and signs $\left(a^j\right)_{j=1,\ldots,m} \in \{-1,1\}$ such that*

$$\begin{aligned} \prod_{j=1}^{m} a^j &= 1 \\ d(f^1, a^1\chi) &\leqslant \epsilon \\ \forall j \ : \ d(f^j, a^j\chi) &\leqslant 2\epsilon \end{aligned}$$

- *If $\epsilon < \frac{1}{6}$, then there exists a linear function $\chi : \{-1,1\}^n \to \{-1,1\}$ defined as $\chi(x) = \prod_{i \in S} x_i$ for some coalition S and signs $\left(a^j\right)_{j=1,\ldots,m} \in \{-1,1\}$ such that $\prod_{j=1}^{m} a^j = 1$ and $d(f^j, a^j\chi) \leqslant \epsilon$ for all j*

Noticing that $\left\langle \left(a^j\chi\right) \right\rangle$ is a consistent mechanism for any linear function χ and signs a^j s.t. $\prod_{j=1}^{m} a^j = 1$ gives us the requested result by applying Proposition 4.

[26] Fourier transforms are widely used in mathematics, computer science, and engineering. The main idea is representing a function f over an orthonormal basis to the functions space χ_S when the inner product is defined to be $\langle f, g \rangle = \mathbb{E}[f(x)g(x)]$ and the basis vectors χ_S are defined to be $\chi_S(x) = \prod_{i \in S} x_i$ for $S \subseteq \{1, \ldots, n\}$. The coefficients of f according to the Fourier basis are notated $\widehat{f}(S)$. I.e., $f = \sum_S \widehat{f}(S)\chi_S$. For a good introduction to the subject see [36,42].
In this proof we are using the following:

- $\chi_S(xy) = \chi_S(x)\chi_S(y)$
- $\mathbb{E}[\chi_S(x)\chi_T(x)] = \begin{cases} 1 & S = T \\ 0 & otherwise \end{cases}$
- $\mathbb{E}[f^2(x)] = \sum_S \widehat{f}^2(S)$
- $\widehat{f}(S) = 1 - 2d(f, \chi_S) = 2d(f, -\chi_S) - 1$

Proof (**Proof of Lemma 7).**
The main ingredient in the proof is the following lemma that connects the inconsistency index with a simple expression over the Fourier coefficients of f^j.

Lemma 8. *Let* $f^1, \ldots, f^m : \{-1,1\}^n \to \Re$. *Then:* $\mathbb{E}\left[\prod_{j=1}^{m-1} f^j\left(x^j\right) f^m\left(\prod_{j=1}^{m-1} x^j\right)\right] = \sum_S \prod_{j=1}^{m} \widehat{f^j}(S)$.

Corollary 3. *For the aggregation mechanism* $F = \langle f^1, \ldots, f^m \rangle$*:*
$1 - 2IC(F) = \sum_S \prod_{j=1}^{m} \widehat{f^j}(S)$.

Now let $F = \langle f^1, \ldots, f^m \rangle$ be an independent aggregation mechanism that satisfies $IC(F) \leqslant \epsilon$.
First we claim there exists a coalition A and a sign $a^1 \in \{-1,1\}$ s.t. $d\left(f^1, a\chi_A\right) \leqslant \epsilon$

$$
\begin{aligned}
1 - 2IC(F) &= \sum_S \prod_{j=1}^{m} \widehat{f^j}(S) && \leqslant \sum_S \left|\widehat{f^1}(S)\right| \cdot \left|\prod_{j=2}^{m} \widehat{f^j}(S)\right| \\
&\leqslant \max_S \left|\widehat{f^1}(S)\right| \sum_S \prod_{j=2}^{m} \left|\widehat{f^j}(S)\right| && \leqslant^{\text{Lemma 9}} \max_S \left|\widehat{f^1}\right| \prod_{j=2}^{m} \sqrt[m]{\sum_S \left|\widehat{f^j}^{m-1}(S)\right|} \\
&\leqslant \max_S \left|\widehat{f^1}\right| \prod_{j=2}^{m} \sqrt{\sum_S \widehat{f^j}^2(S)} = \max_S \left|\widehat{f^1}\right| = 1 - 2 \min_{S, a \in \{-1,1\}} \left(d\left(f^1, a\chi_S\right)\right)
\end{aligned}
$$

and hence there exists a coalition A and a sign a^1 s.t. $\Pr[f^1(x) \neq a^1\chi_A(x)] \leqslant IC(f,g,h) = \epsilon$.

Based on Proposition 6, $IC(a^1\chi_A, f^2, \ldots, f^m) \leqslant IC(F) + d(f, a^1\chi_A) \leqslant 2\epsilon$. On the other hand based on corollary 3,

$$IC(a^1\chi_A, f^2, \ldots, f^m) = \frac{1}{2}\left(1 - a^1 \sum_S \widehat{\chi_A}(S) \prod_{j=2}^{m} \widehat{f^j}(S)\right) = \frac{1}{2}\left(1 - a^1 \prod_{j=2}^{m} \widehat{f^j}(A)\right).$$

So we get that $a^1 \prod_{j=2}^{m} \widehat{f^j}(A) \geqslant 1 - 4\epsilon$ and hence there exist signs $(a^j)_{j=1}^m$ such that $\prod_{j=2}^{m} a^j = 1$ and $a^j \widehat{f^j}(A) \geqslant 1 - 4\epsilon$ so $d(f^j, a^j\chi_A) \leqslant 2\epsilon$.

Due to symmetry there is also a coalition B and a sign b^2 such that $d(f^2, b^2\chi_B) \leqslant \epsilon$ and hence $d(b^2\chi_B, a^2\chi_A) \leqslant 3\epsilon$. On the other hand $d(b^2\chi_B, a^2\chi_A) = \begin{cases} 0 & a^2 = b^2 \wedge A = B \\ 1 & a^2 = b^2 \wedge A = B \\ \frac{1}{2} & A \neq B \end{cases}$.

Hence, if $\epsilon < \frac{1}{6}$, we get that $A = B$, $a^2 = b^2$.

Due to symmetry we can repeat this for all f^j. □

C.2 Lemma 8

Lemma.
Let $f^1, \ldots, f^m : \{-1,1\}^n \to \Re$. *Then:* $\mathbb{E}\left[\prod_{j=1}^{m-1} f^j\left(x^j\right) f^m\left(\prod_{j=1}^{m-1} x^j\right)\right] = \sum_S \prod_{j=1}^{m} \widehat{f^j}(S)$.

Proof.

$$\begin{aligned}
\mathbb{E}\left[\prod_{j=1}^{m-1} f^j\left(x^j\right) f^m\left(\prod_{j=1}^{m-1} x^j\right)\right] &= \mathbb{E}_{x^1,\ldots,x^{m-1}}\left[\sum_{S^1,\ldots,S^m} \prod_{j=1}^{m-1}\left(\widehat{f^j}\left(S^j\right)\chi_{S^j}\left(x^j\right)\right)\widehat{f^m}\left(S^m\right)\chi_{S^m}\left(\prod_{j=1}^{m-1} x^j\right)\right] \\
&= \sum_{S^1,\ldots,S^m} \prod_{j=1}^{m} \widehat{f^j}\left(S^j\right) \prod_{j=1}^{m} \mathbb{E}_{x^j}\left[\chi_{S^j}\left(x^j\right)\chi_{S^m}\left(x^j\right)\right] \\
&= \sum_S \prod_{j=1}^{m} \widehat{f^j}(S)
\end{aligned}$$

□

C.3 Lemma 9

Lemma 9.
Let $k \geqslant 2$ be an integer and $\{a_{i,j}\}_{i=1\ldots n, j=1\ldots k}$ positive reals. Then,

$$\left(\sum_{i=1}^{n}\prod_{j=1}^{k} a_{i,j}\right)^{k} \leqslant \prod_{j=1}^{k}\left(\sum_{i=1}^{n}(a_{i,j})^{k}\right)$$

Proof. We'll prove by induction over k.
If $k=2$, then by Cauchy-Swartz inequality $\left(\sum_{i=1}^{n} a_{i,1}a_{i,2}\right)^{2} \leqslant \left(\sum_{i=1}^{n}(a_{i,1})^{2}\right)\left(\sum_{i=1}^{n}(a_{i,2})^{2}\right)$
If $k>2$, then applying Hölder inequality

$$\begin{aligned}\left(\sum_{i=1}^{n}\prod_{j=1}^{k} a_{i,j}\right)^{k} &= \left(\sum_{i=1}^{n} a_{i,k}\prod_{j=1}^{k-1} a_{i,j}\right)^{k} \\ &\leqslant \left[\left(\sum_{i=1}^{n}(a_{i,k})^{k}\right)^{\frac{1}{k}}\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{k-1} a_{i,j}\right)^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}\right]^{k} \\ &= \left(\sum_{i=1}^{n}(a_{i,k})^{k}\right)\left(\sum_{i=1}^{n}\prod_{j=1}^{k-1}(a_{i,j})^{\frac{k}{k-1}}\right)^{k-1}\end{aligned}$$

By the induction hypothesis we get that

$$\begin{aligned}\left(\sum_{i=1}^{n}\prod_{j=1}^{k-1}(a_{i,j})^{\frac{k}{k-1}}\right)^{k-1} &\leqslant \prod_{j=1}^{k-1}\left(\sum_{i=1}^{n}\left((a_{i,j})^{\frac{k}{k-1}}\right)^{k-1}\right) \\ &= \prod_{j=1}^{k-1}\left(\sum_{i=1}^{n}(a_{i,j})^{k}\right)\end{aligned}$$

□

C.4 Affine Agenda - Lemma 10

Lemma 10. *Let $\mathbb{X}$ be an affine subspace of $\{0,1\}^m$ of degree k.*
Then $\mathbb{X}$ can be represented as a truth-functional agenda using xor conclusions only.

Proof.
$\mathbb{X}$ is an affine space and therefore can be represented as a linear subspace shifted by a constant vector. Shifting is merely renaming of the opinions so with no loss of generality, assume that $\mathbb{X}$ is a linear subspace defined by a matrix $A_{k\times m}$ of rank k in the following way $\mathbb{X} = \{x \in \{0,1\}^m \mid Ax = 0\}$. There exists an invertible matrix (representing the Gaussian elimination process) P s.t.

- $\{x \in \{0,1\}^m \mid Ax = 0\} = \{x \in \{0,1\}^m \mid PAx = 0\}$
- PA is in canonical form. I.e. for any row $t \in [k]$ there is a unique index $a_t \in [m]$ s.t. $(PA)_{t,j} = 1$ iff $j = a_t$.

Hence $\mathbb{X}$ is a truth-functional agenda for the premises $[m] \setminus \{a_t\}_{t\in[k]}$ and conclusions based on the row of PA.