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# LEM

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**Vanishing results for the cohomology of complex toric  
hyperplane complements**

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# Vanishing results for the cohomology of complex toric hyperplane complements

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## Abstract

Suppose  $\mathcal{R}$  is the complement of an essential arrangement of toric hyperplanes in the complex torus  $(\mathbb{C}^*)^n$  and  $\pi = \pi_1(\mathcal{R})$ . We show that  $H^*(\mathcal{R}; A)$  vanishes except in the top degree  $n$  when  $A$  is one of the following systems of local coefficients: (a) a system of nonresonant coefficients in a complex line bundle, (b) the von Neumann algebra  $\mathcal{N}\pi$ , or (c) the group ring  $\mathbb{Z}\pi$ . In case (a) the dimension of  $H^n$  is the Euler characteristic,  $e(\mathcal{R})$ , and in case (b) the  $n^{\text{th}}$   $\ell^2$  Betti number is also  $e(\mathcal{R})$ .

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## 1 Introduction

A *complex toric arrangement* is a family of complex subtori of a complex torus  $(\mathbb{C}^*)^n$ . The study of such objects is a relatively recent topic. Different versions of these arrangements, also known as toral arrangements, have been introduced and studied in works of Lehrer [15, 16], Dimca-Lehrer [10], Douglass [11], Looijenga [17] and Macmeikan [19, 20].

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The foundation of the topic can be traced to the paper [9] by De Concini and Procesi. There the main objects are defined and the cohomology of the complement of a toric arrangement is studied. An explicit goal of [9] is to generalize the theory of hyperplane arrangements. (For an extensive account of the work of De Concini and Procesi see [8].)

The next step is the work of Moci, in particular his papers [21, 22, 23], developing the theory with a special focus on combinatorics. In [24] Moci and the second author study the homotopy type of the complement of a special class of toric arrangements which they call *thick*. In [3] D’Antonio and Delucchi generalize results in [24] to a wider class of toric arrangements which they call *complexified* because of structural affinity with the case of hyperplane arrangements. They also provide a (finite) presentation for the fundamental group of the arrangement complement .

In this paper we generalize to toric arrangements a well known result for affine arrangements: vanishing conditions for the cohomology of the complement  $M(\mathcal{A})$  of an arrangement  $\mathcal{A}$  with coefficients in a complex local system  $A$ . Necessary conditions for  $H^k(M(\mathcal{A}); A) = 0$  if  $k \neq n$ , i.e., for the cohomology to be concentrated in top dimension, have been determined by a number of authors, including Kohno [14], Esnault, Schechtman and Viehweg [12], Davis, Januszkiewicz and Leary [4], Schechtman, Terao and Varchenko [27] and Cohen and Orlik [2]. In particular, in [27] (see also [2]) it is proved that the cohomology of the complement  $M(\mathcal{A})$  of an arrangement with coefficients in a complex local system is concentrated in top dimension provided certain *nonresonance* conditions for monodromies are fulfilled for a certain subset of edges (i.e., intersections of hyperplanes) that are called *denses*.

In order to generalize the above results we use techniques developed by the first author in a joint work with Januszkiewicz, Leary and Okun, [4, 5, 6, 7]. One considers an open cover of the complement  $M$  by “small” open sets each homeomorphic to the complement of a central arrangement. The  $E_1$  page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair  $(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ , which is homotopy equivalent to  $(\mathbb{C}^n, \Sigma)$  where  $\Sigma$  is the union of all hyperplanes in the arrangement. (The simplicial complex  $N(\mathcal{U})$  is the nerve of an open cover of  $\mathbb{C}^n$  and  $N(\mathcal{U}_{\text{sing}})$  is a subcomplex.)

It follows that the  $E_2$  page can be nonzero only in position  $(l, 0)$ . Using this, one can retrieve nonresonant conditions for affine arrangements. One also can prove that for an affine hyperplane arrangement of rank  $l$  only the

$l^{\text{th}}$   $\ell^2$ -Betti number of the complement  $M$  can be nonzero and that it is equal to the rank of the reduced  $(l - 1)$ -homology of  $\Sigma$  (cf. [4]). Similarly, with coefficients in the group ring,  $\mathbb{Z}\pi$ , for  $\pi = \pi_1(M)$ ,  $H^*(M; \mathbb{Z}\pi)$  is nonzero only in degree  $l$  (cf. [5]). We generalize all three of these vanishing results to the toric case in Theorems 5.1, 5.2 and 5.3.

In recent work [26], Papadima and Suciu generalize the result in [2] to arbitrary minimal CW-complex, i.e., a complex having as many  $k$ -cells as the  $k$ -th Betti number. It would be very interesting to decide if the complement of toric arrangement also could be minimal. In this case Theorem 5.1 would be a consequence of minimality.

Our paper begins with a review of some background about toric and affine arrangements. Then, in Section 3, we give a brief account of open covers by “small” convex sets. In Section 4 we recall basic definitions on systems of local coefficients. Finally in Section 5 we prove that the cohomology of the complement of a toric arrangement with coefficient in a local system  $A$  vanishes except in the top degree when  $A$  is a nonresonant local system, the von Neumann algebra  $\mathcal{N}\pi$  or the group ring  $\mathbb{Z}\pi$ .

## 2 Affine and toric hyperplane arrangements

**Affine hyperplanes arrangements** A *hyperplane arrangement*  $\mathcal{A}$  is a finite collection of affine hyperplanes in  $\mathbb{C}^n$ . A *subspace* of  $\mathcal{A}$  is a nonempty intersection of hyperplanes in  $\mathcal{A}$ . Denote by  $L(\mathcal{A})$  the poset of subspaces, partially ordered by inclusion, and let  $\bar{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$ . An arrangement is *central* if  $L(\mathcal{A})$  has a minimum element. Given  $G \in L(\mathcal{A})$ , its *rank*,  $\text{rk}(G)$ , is the codimension of  $G$  in  $\mathbb{C}^n$ . The minimal elements of  $L(\mathcal{A})$  form a family of parallel subspaces and they all have the same rank. The *rank* of an arrangement  $\mathcal{A}$  is the rank of a minimal element in  $L(\mathcal{A})$ .  $\mathcal{A}$  is *essential* if  $\text{rk}(\mathcal{A}) = n$ .

The *singular set*  $\Sigma(\mathcal{A})$  of  $\mathcal{A}$  is the union of hyperplanes in  $\mathcal{A}$ . The complement of  $\Sigma(\mathcal{A})$  in  $\mathbb{C}^n$  is denoted  $M(\mathcal{A})$ .

**Toric arrangements** Let  $T = (\mathbb{C}^*)^n$  be a complex torus and let  $\Lambda = \text{Hom}(T, \mathbb{C}^*)$  denote the group of characters of  $T$ . Then  $\Lambda \cong \mathbb{Z}^n$ . A character is *primitive* if it is a primitive vector in  $\Lambda$ . Given a primitive character  $\chi$  and an element  $a \in \mathbb{C}^*$  put

$$H_{\chi,a} = \{t \in T \mid \chi(t) = a\}.$$

The subtorus  $H_{\chi,a}$  is a *toric hyperplane*. A finite subset  $X \subset \Lambda \times \mathbb{C}^*$  defines a *toric arrangement*,

$$\mathcal{T}_X := \{H_{\chi,a}\}_{(\chi,a) \in X}$$

The projection of  $X$  onto the first factor is denoted  $p(X)$  and is called the *character set* of  $\mathcal{T}_X$ . (Thus,  $p(X) := \{\chi \mid (\chi, a) \in X\}$ .) The *singular set*,  $\Sigma_X$ , is the union of toric hyperplanes in the arrangement. Its complement,  $T - \Sigma_X$ , is denoted  $\mathcal{R}_X$ . The *intersection lattice*  $L_X$  is the set of nonempty intersections of toric hyperplanes and  $\bar{L}_X = L_X \cup \{T\}$ .  $\bar{L}_X$  is partially ordered by inclusion. The *rank* of the arrangement is the dimension of the linear subspace of  $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$  spanned by  $p(X)$ . The arrangement is *essential* if its rank is  $n$ .

Suppose  $G \in L_X$ . Choose a point  $x \in G$ . The *tangential arrangement along  $G$*  is the arrangement  $\mathcal{A}_G$  of linear hyperplanes which are tangent to the complex toric hyperplanes containing  $G$  (i.e., all hyperplanes of the form  $T_x(H_{\chi,a})$  where  $T_x(G) \subset T_x(H_{\chi,a})$ ). It is a central, affine arrangement of rank equal to  $n - \dim G$ .

Given a toric arrangement  $\mathcal{T}_X$  of rank  $l$ , put

$$K_X := \bigcap_{\chi \in p(X)} \text{Ker } \chi = \{t \in T \mid \chi(t) = 1, \forall \chi \in p(X)\}$$

and  $\bar{T}_X := T/K_X$ . Thus,  $K_X$  and  $\bar{T}_X$  are tori of dimensions  $n - l$  and  $l$ , respectively. ( $K_X \cong (\mathbb{C}^*)^{n-l}$  and  $\bar{T}_X \cong (\mathbb{C}^*)^l$ .) Suppose  $\bar{\Sigma}_X$  denotes the image of  $\Sigma_X$  in  $\bar{T}_X$ . We have a homeomorphism of pairs,

$$(T, \Sigma_X) \cong K_X \times (\bar{T}_X, \bar{\Sigma}_X). \quad (1)$$

In other words, the arrangement in  $T$  is just the product of the arrangement in  $\bar{T}_X$  with the torus  $K_X$ . ( $\bar{T}_X$  is called the *essentialisation* of  $\mathcal{T}_X$ .) So, it is not restrictive to consider essential toric arrangements.

**Lemma 2.1.** (cf. [4, Prop. 2.1]). *Suppose  $\mathcal{T}_X$  is an essential toric arrangement on  $T$  and  $\Sigma = \Sigma_X$ . Then  $H_*(T, \Sigma)$  is free abelian and concentrated in degree  $n$ .*

*Proof.* We follow the “deletion-restriction” argument in [4, Prop. 2.1] using induction on  $\text{Card}(\mathcal{T}_X)$ . Choose a toric hyperplane  $H \in \mathcal{T}_X$ . Let  $\mathcal{T}' = \mathcal{T}_X - \{H\}$  and let  $\mathcal{T}''$  be the restriction of  $\mathcal{T}_X$  to  $H$ , i.e.,  $\mathcal{T}'' = \{H \cap H' \mid H' \in \mathcal{T}_X\}$ .

Let  $\Sigma'$  and  $\Sigma''$  denote the singular sets of  $\mathcal{T}'$  and  $\mathcal{T}''$ , respectively. Consider the exact sequence of the triple  $(T, \Sigma, \Sigma')$ ,

$$\rightarrow H_*(T, \Sigma') \rightarrow H_*(T, \Sigma) \rightarrow H_{*-1}(\Sigma, \Sigma') \rightarrow \quad (2)$$

There is an excision,  $H_{*-1}(\Sigma, \Sigma') \cong H_{*-1}(H, \Sigma'')$ . The rank of  $\mathcal{T}'$  is either  $n$  or  $n - 1$ , while the rank of  $\mathcal{T}''$  is always  $n - 1$ . The argument breaks into two cases depending on the rank of  $\mathcal{T}'$ .

*Case 1:* the rank of  $\mathcal{T}'$  is  $n$ . By induction,  $H_*(T, \Sigma')$  and  $H_*(H, H \cap \Sigma)$  are free abelian and concentrated in degrees  $n$  and  $n - 1$ , respectively. So, (2) becomes,

$$0 \rightarrow H_n(T, \Sigma') \rightarrow H_n(T, \Sigma) \rightarrow H_{n-1}(H, H \cap \Sigma') \rightarrow 0$$

and all other terms are 0. Therefore,  $H_*(T, \Sigma)$  is concentrated in degree  $n$  and  $H_n(T, \Sigma)$  is free abelian.

*Case 2:* the rank of  $\mathcal{T}'$  is  $n - 1$ . Then the projection  $T \rightarrow \bar{T}$  takes  $H$  isomorphically onto  $\bar{T}$  and the arrangement  $\mathcal{T}'$  on  $H$  maps isomorphically to the arrangement  $\bar{\mathcal{T}}_X$  on  $\bar{T}$ . So,  $(H, H \cap \Sigma') \cong (\bar{T}, \bar{\Sigma})$ . By (1),  $(T, \Sigma') \cong K_X \times (H, H \cap \Sigma') \cong \mathbb{C}^* \times (H, H \cap \Sigma')$ . By the Künneth Formula,  $H_*(T, \Sigma') \cong H_*(\mathbb{C}^*) \otimes H_*(H, H \cap \Sigma')$ . So,

$$\begin{aligned} H_{n-1}(T, \Sigma') &\cong H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \quad \text{and} \\ H_n(T, \Sigma') &\cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma'); \end{aligned}$$

moreover, the first isomorphism is induced by the inclusion  $(H, \Sigma') \rightarrow (T, \Sigma')$ . So, (2) becomes,

$$\begin{aligned} 0 \rightarrow H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') &\rightarrow H_n(T, \Sigma) \rightarrow H_{n-1}(H, H \cap \Sigma') \\ &\rightarrow H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \end{aligned}$$

where the last map is an isomorphism. It follows that  $H_{n-1}(T, \Sigma) = 0$  and that  $H_n(T, \Sigma) \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$ , which, by inductive hypothesis, is free abelian. This proves the lemma.  $\square$

**Complexified toric arrangements** In [3] D'Antonio-Delucchi consider the case of “complexified toric arrangements.” This means that for each  $(\chi, a) \in X$ , the complex number  $a$  has modulus 1 (where  $X \subset \Lambda \times \mathbb{C}^*$  is a set defining a toric arrangement  $\mathcal{T}_X$ ). Let  $T^{\text{cpt}} = (S^1)^n \subset \mathbb{C}^n$  be the compact

torus. Then for each  $H \in \mathcal{T}_X$ ,  $H \cap T^{\text{cpt}}$  is a compact subtorus of  $T^{\text{cpt}}$ . The set of subtori,  $\mathcal{T}_X^{\text{cpt}} := \{H \cap \mathcal{S} \mid H \in \mathcal{T}\}$ , is called the associated *compact arrangement*.

Let  $\Sigma^{\text{cpt}} := \Sigma_X \cap T^{\text{cpt}}$ . We note that  $(T, \Sigma_X)$  deformation retracts onto  $(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ . Here are a few observations.

- (i) The universal cover of  $T^{\text{cpt}}$  is  $\mathbb{R}^n$  (actually the subspace  $i\mathbb{R}^n \subset \mathbb{C}^n$ ). Let  $\pi : \mathbb{R}^n \rightarrow T^{\text{cpt}}$  be the covering projection. Then for each  $H^{\text{cpt}} \in \mathcal{T}_X^{\text{cpt}}$ , each component of  $\pi^{-1}(H^{\text{cpt}})$  is an affine hyperplane and the collection of these hyperplanes is a periodic affine hyperplane arrangement in  $\mathbb{R}^n$ .
- (ii) If  $\mathcal{T}_X$  is essential, then  $\Sigma^{\text{cpt}}$  cuts  $T^{\text{cpt}}$  into a disjoint union of convex polytopes, called *chambers* (see [24]). The inverse images of these polytopes under  $\pi$  give a tiling of  $\mathbb{R}^n$ .
- (iii) When  $\mathcal{T}_X$  is essential, it follows from (ii) that for  $n \geq 2$ ,  $\Sigma^{\text{cpt}}$  is connected and that for  $n \geq 3$ ,  $\pi_1(\Sigma^{\text{cpt}}) = \pi_1(T^{\text{cpt}})$ .
- (iv) It is easy to prove Lemma 2.1 in the case of a compact arrangement. We have an excision  $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}}) \cong H_*(\coprod(P_i, \partial P_i))$  where each chamber  $P_i$  is an  $n$ -dimensional convex polytope. Hence,  $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$  is concentrated in degree  $n$  and is free abelian. Moreover, the rank of  $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$  is the number of chambers.
- (v) Let  $\tilde{\Sigma}^{\text{cpt}}$  denote the inverse image of  $\Sigma^{\text{cpt}}$  in  $\mathbb{R}^n$  and let  $\tilde{\Sigma}_X$  be the induced cover of  $\Sigma_X$ . Suppose  $\mathcal{T}_X$  is essential. Then  $\tilde{\Sigma}^{\text{cpt}}$  cuts  $\mathbb{R}^n$  into compact chambers. It follows that  $\tilde{\Sigma}^{\text{cpt}}$  (and hence,  $\tilde{\Sigma}$ ) is homotopy equivalent to a wedge of  $(n - 1)$ -spheres.

### 3 Certain covers and their nerves

Equip the torus  $T = (\mathbb{C}^*)^n$  with an invariant metric. This lifts to a Euclidean metric on  $\mathbb{C}^n$  induced from an inner product. Hence, geodesics in  $T$  lift to straight lines in  $\mathbb{C}^n$  and each component of the inverse image of a subtorus of  $T$  is an affine subspace of  $\mathbb{C}^n$ . A *convex subset* of  $T$  means a geodesically convex subset. Thus, each component of the inverse image of a convex subset of  $T$  is a convex subset of  $\mathbb{C}^n$ .

The intersection of an open convex subset of  $T$  with the toric hyperplanes in  $\mathcal{T}_X$  is equivalent to an affine arrangement. An open convex subset  $U \subset T$

is *small* (with respect to  $\mathcal{T}_X$ ) if this affine arrangement is central. In other words,  $U$  is *small* if the following two conditions hold (cf. [4, 5]):

- (i)  $\{G \in \overline{L}(\mathcal{T}_X) \mid G \cap U \neq \emptyset\}$  has a unique minimum element,  $\text{Min}(U)$ .
- (ii) A toric hyperplane  $H \in \mathcal{T}_X$  has nonempty intersection with  $U$  if and only if  $\text{Min}(U) \subset H$ .

If (i) and (ii) hold, then the arrangement in  $U$  is equivalent to the tangential arrangement along  $\text{Min}(U)$ , which we denote by  $\mathcal{A}_{\text{Min}(U)}$ . The intersection of two small convex open sets is also a small convex set; hence, the same is true for any finite intersection of such sets.

Let  $\mathcal{U} = \{U_i\}_{i \in I}$  be an open cover of  $T$  by small convex sets, put

$$\mathcal{U}_{\text{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma_X \neq \emptyset\}.$$

Given a nonempty subset  $\sigma \subset I$ , put  $U_\sigma := \bigcap_{i \in \sigma} U_i$ . The *nerve*  $N(\mathcal{U})$  of  $\mathcal{U}$  is the simplicial complex defined as follows. Its vertex set is  $I$  and a finite, nonempty subset  $\sigma \subset I$  spans a simplex of  $N(\mathcal{U})$  if and only if  $U_\sigma$  is nonempty. We have the following lemma.

**Lemma 3.1.** *Suppose  $\mathcal{T}_X$  is essential.  $N(\mathcal{U})$  is homotopy equivalent to  $T$  and  $N(\mathcal{U}_{\text{sing}})$  is a subcomplex homotopy equivalent to  $\Sigma_X$ . Moreover,  $H_*(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$  is concentrated in degree  $n$  and  $H_n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$  is free abelian.*

*Proof.*  $\mathcal{U}_{\text{sing}}$  is an open cover of a neighborhood of  $\Sigma_X$  which deformation retracts onto  $\Sigma_X$ . For each simplex  $\sigma$  of  $N(\mathcal{U})$ ,  $U_\sigma$  is contractible (in fact, it is a small convex open set). By a well-known result (see [13, Cor. 4G.3 and Ex. 4G(4)])  $N(\mathcal{U})$  is homotopy equivalent to  $T$  and  $N(\mathcal{U}_{\text{sing}})$  is homotopy equivalent to  $\Sigma_X$ . The last sentence of the lemma follows from Lemma 2.1.  $\square$

**Definition 3.2.**  $\beta(\mathcal{T}_X)$  is the rank of  $H_n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ .

Equivalently,  $\beta(\mathcal{T}_X)$  is the rank of  $H_n(T, \Sigma_X)$  (or of  $\overline{H}_{n-1}(\Sigma_X)$ ). Also, it is not difficult to see that  $(-1)^l \beta(\mathcal{T}_X) = e(T, \Sigma_X) = 1 - e(\Sigma_X) = e(\mathcal{R}_X)$ , where  $e(\cdot)$  denotes Euler characteristic.



## 4 Local coefficients

**Generic and nonresonant coefficients** Consider an affine arrangement  $\mathcal{A}$ . The fundamental group  $\pi$  of its complement,  $M(\mathcal{A})$ , is generated by loops  $a_H$  for  $H \in \mathcal{A}$ , where the loop  $a_H$  goes once around the hyperplane  $H$  in the “positive” direction. Let  $\alpha_H$  denote the image of  $a_H$  in  $H_1(M(\mathcal{A}))$ . Then  $H_1(M(\mathcal{A}))$  is free abelian with basis  $\{\alpha_H\}_{H \in \mathcal{A}}$ . So, a homomorphism  $H_1(M(\mathcal{A})) \rightarrow \mathbb{C}^*$  is determined by an  $\mathcal{A}$ -tuple  $\Lambda \in (\mathbb{C}^*)^{\mathcal{A}}$ , where  $\Lambda = (\lambda_H)_{H \in \mathcal{A}}$  corresponds to the homomorphism sending  $\alpha_H$  to  $\lambda_H$ . Let  $\psi_\Lambda : \pi \rightarrow \mathbb{C}^*$  be the composition of this homomorphism with the abelianization map  $\pi \rightarrow H_1(M(\mathcal{A}))$ . The resulting local coefficient system on  $M(\mathcal{A})$  is denoted  $A_\Lambda$ .

Returning to the case where  $\mathcal{T}_X$  is a toric arrangement, for each simplex  $\sigma$  in  $N(\widehat{\mathcal{U}})$ , let  $\mathcal{A}_\sigma := \mathcal{A}_{\text{Min}(U_\sigma)}$  be the corresponding central arrangement (so that  $\widehat{U}_\sigma \cong M(\mathcal{A}_\sigma)$ ). Given  $\Lambda_\sigma \in (\mathbb{C}^*)^{\mathcal{A}_\sigma}$ , put

$$\lambda_\sigma := \prod_{H \in \mathcal{A}_\sigma} \lambda_H.$$

Let  $A_{\Lambda_T} \in \text{Hom}(H_1(\mathcal{R}_X), \mathbb{C}^*)$  be a local coefficient system on  $\mathcal{R}_X$ . The localization of  $A_{\Lambda_T}$  on the open set  $\widehat{U}_\sigma$  has the form  $A_{\Lambda_\sigma}$ , where  $\Lambda_\sigma$  is a  $\mathcal{A}_\sigma$ -tuple in  $\mathbb{C}^*$ . We call  $\Lambda_T$  *generic* if  $\lambda_\sigma \neq 1$  for all  $\sigma \in N(\mathcal{U}_{\text{sing}})$ . We call  $\Lambda_T$  *nonresonant* if  $\Lambda_\sigma$  is nonresonant in the sense of [2] for all  $\sigma \in N(\mathcal{U}_{\text{sing}})$  i.e., if the Betti numbers of  $M(\mathcal{A}_\sigma)$  with coefficients in  $A_{\Lambda_\sigma}$  are minimal.

### $\ell^2$ -cohomology and coefficients in a group von Neumann algebra

For a discrete group  $\pi$ ,  $\ell^2\pi$  denotes the Hilbert space of complex-valued, square integrable functions on  $\pi$ . There are unitary  $\pi$ -actions on  $\ell^2\pi$  by either left or right multiplication; hence,  $\mathbb{C}\pi$  acts either from the left or right as an algebra of operators. The *associated von Neumann algebra*  $\mathcal{N}\pi$  is the commutant of  $\mathbb{C}\pi$  (acting from, say, the right on  $\ell^2\pi$ ).

Given a finite CW complex  $Y$  with fundamental group  $\pi$ , the space of  $\ell^2$ -cochains on the universal cover  $\widetilde{Y}$  is equal to  $C^*(Y; \ell^2\pi)$ , the cochains with local coefficients in  $\ell^2\pi$ . The image of the coboundary map need not be closed; hence,  $H^*(Y; \ell^2\pi)$  need not be a Hilbert space. To remedy this, one defines the *reduced*  $\ell^2$ -cohomology  $H_{\text{red}}^*(Y; \ell^2\pi)$  to be the quotient of the space of cocycles by the closure of the space of coboundaries. The von Neumann algebra admits a trace. Using this, one can attach a “dimension,”

$\dim_{\mathcal{N}\pi} V$ , to any closed,  $\pi$ -stable subspace  $V$  of a finite direct sum of copies of  $\ell^2\pi$  (it is the trace of orthogonal projection onto  $V$ ). The nonnegative real number  $\dim_{\mathcal{N}\pi}(H_{\text{red}}^p(Y; \ell^2\pi))$  is the  $p^{\text{th}}$   $\ell^2$ -Betti number of  $Y$ .

A technical advance of Lück [18, Ch. 6] is the use local coefficients in  $\mathcal{N}\pi$  in place of the previous version of  $\ell^2$ -cohomology. He shows there is a well-defined dimension function on  $\mathcal{N}\pi$ -modules,  $A \rightarrow \dim_{\mathcal{N}\pi} A$ , which gives the same answer for  $\ell^2$ -Betti numbers, i.e., for each  $p$  one has that  $\dim_{\mathcal{N}\pi} H^p(Y; \mathcal{N}\pi) = \dim_{\mathcal{N}\pi} H_{\text{red}}^p(Y; \ell^2\pi)$ .

**Group ring coefficients** Let  $Y$  be a connected CW complex,  $\pi = \pi_1(Y)$  and  $r : \tilde{Y} \rightarrow Y$  the universal cover. There is a well-defined action of  $\pi$  on  $\tilde{Y}$  and hence, on the cellular chain complex of  $\tilde{Y}$ . Given the left  $\pi$ -module  $\mathbb{Z}\pi$ , define the cochain complex with *group ring coefficients*

$$C^*(Y; \mathbb{Z}\pi) := \text{Hom}_{\pi}(C_*(\tilde{Y}), \mathbb{Z}\pi).$$

Taking cohomology gives  $H^*(Y; \mathbb{Z}\pi)$ .

## 5 The Mayer-Vietoris spectral sequence

**Statements of the main theorems** Suppose  $\mathcal{T}_X$  is an essential toric arrangement in  $T$  and  $\pi = \pi_1(\mathcal{R}_X)$ .

**Theorem 5.1.** *Let  $\Lambda_T$  a generic  $X$ -tuple with entries in  $k^*$ . Then  $H^*(\mathcal{R}_X; A_{\Lambda_T})$  is concentrated in degree  $n$  and*

$$\dim_k H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X).$$

**Theorem 5.2.** (cf. [6]). *The  $\ell^2$ -Betti numbers of  $\mathcal{R}_X$  are 0 except in degree  $n$  and  $\ell^2 b_n(\mathcal{R}_X) = \beta(\mathcal{T}_X)$ .*

**Theorem 5.3.** (cf. [5, 7]).  *$H^*(\mathcal{R}_X; \mathbb{Z}\pi)$  vanishes except in degree  $n$  and  $H^n(\mathcal{R}_X; \mathbb{Z}\pi)$  is free abelian.*

*Remark 5.4.* One of the most important instances of a toric arrangement is the following. Suppose  $W$  is a Euclidean reflection group on  $\mathbb{R}^n$  and that  $\mathbb{Z}^n \subset W$  is the subgroup of translations. The quotient  $W' := W/\mathbb{Z}^n$  is a finite Coxeter group. The reflection group  $W$  acts on the complexification  $\mathbb{C}^n$  and  $W'$  acts on the torus  $T = \mathbb{C}^n/\mathbb{Z}^n$ . The image of the affine reflection

arrangement in  $\mathbb{C}^n$  gives a toric arrangement  $\mathcal{T}_X$  in  $T$ . The fundamental group of  $\mathcal{R}_X$  is the Artin group  $A$  associated to  $W$  and  $\mathcal{R}_X$  is the Salvetti complex associated to  $A$ . The quotient of the compact torus by  $W'$  can be identified with the fundamental simplex  $\Delta$  of  $W$  on  $\mathbb{R}^n$ . (If  $W$  is irreducible, then  $\Delta$  is a simplex.) It follows that  $\beta(\mathcal{T}_x)$  is the order of  $W'$  (i.e., the index of  $\mathbb{Z}^n$  in  $W$ ). So, in this case Theorem 5.2 is a special case of the main result of [6] and Theorem 5.3 is a special case of a result of [7, Thm. 4.1].

**Lemma 5.5.** *Suppose  $\mathcal{A}$  is a finite, central arrangement of affine hyperplanes. Let  $\pi' = \pi_1(M(\mathcal{A}))$ . Then*

- (i) (cf. [27, 2, 4]). *For any generic system of local coefficients  $A$ ,  $H^*(M(\mathcal{A}); A)$  vanishes in all degrees.*
- (ii) (cf. [4]).  *$H^*(M(\mathcal{A}); \mathcal{N}\pi')$  vanishes in all degrees. Hence, all  $\ell^2$ -Betti numbers are 0.*
- (iii) (cf. [5]). *If the rank of  $\mathcal{A}$  is  $l$ , then  $H^*(M(\mathcal{A}); \mathbb{Z}\pi')$  vanishes except in the top degree,  $l$ .*

**Proofs using the Mayer-Vietoris spectral sequence** The proofs of these three theorems closely follow the argument in [6], [4] and particularly, in [5]. For  $\pi = \pi_1(\mathcal{R}_X)$ , let  $A$  denote one of the left  $\pi$ -modules in Section 4.

Let  $\mathcal{U} = \{U_i\}$  be an open cover of  $T$  by small convex sets. We may suppose that  $\mathcal{U}$  is finite and that it is closed under taking intersections. For each  $G \in \overline{L}_X$ , put

$$\begin{aligned} \mathcal{U}_G &:= \{U \in \mathcal{U} \mid \text{Min}(U) \leq G\}, \\ \mathcal{U}_G^{\text{sing}} &:= \{U \in \mathcal{U} \mid \text{Min}(U) < G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}. \end{aligned}$$

The open cover  $\mathcal{U}$  restricts to an open cover  $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$  of  $\mathcal{R}_X$ . Any element  $\widehat{U} = U - \Sigma_X$  of the cover is homotopy equivalent to the complement of a central arrangement  $M(\mathcal{A}_{\text{Min}(U)})$ .

Suppose  $N(\mathcal{U})$  is the nerve of  $\mathcal{U}$  and  $N(\mathcal{U}_G)$  is the subcomplex defined by  $\mathcal{U}_G$ . Since  $N(\mathcal{U}_G)$  and  $N(\mathcal{U}_G^{\text{sing}})$  are nerves of covers of  $G$  and  $\Sigma_X \cap G$ , respectively, by contractible open subsets, we have that for each  $G \in \overline{L}(\mathcal{A})$ ,

$$H^*(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}})) = H^*(G, \Sigma(\mathcal{T}_X \cap G)). \quad (3)$$

For each  $k$ -simplex  $\sigma = \{i_0, \dots, i_k\}$  in  $N(\mathcal{U})$ , let

$$U_\sigma := U_{i_0} \cap \dots \cap U_{i_k}$$

denote the corresponding intersection.

Let  $r : \widetilde{\mathcal{R}}_X \rightarrow \mathcal{R}_X$  be the universal cover. The induced open cover  $\{r^{-1}(\widehat{U})\}$  of  $\widetilde{\mathcal{R}}_X$  has the same nerve  $N(\widehat{\mathcal{U}})$  ( $= N(\mathcal{U})$ ). We have the Mayer–Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\widehat{U}_\sigma)),$$

where  $N^{(i)}$  denotes the set of  $i$ -simplices in  $N(\mathcal{U})$  (cf. [1, Ch. VII].) We get a corresponding double cochain complex,

$$E_0^{i,j} := \text{Hom}_\pi(C_{i,j}, A), \quad (4)$$

where  $\pi = \pi_1(\mathcal{R}_X)$ . The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\text{Gr } H^m(\mathcal{R}_X; A) = E_\infty := \bigoplus_{i+j=m} E_\infty^{i,j}.$$

By first using the horizontal differential, there is a spectral sequence with  $E_1$  page

$$E_1^{i,j} = C^i(N(\mathcal{U}); \mathcal{H}^j(A))$$

where  $\mathcal{H}^j(A)$  is the coefficient system on  $N(\mathcal{U})$  defined by

$$\sigma \mapsto H^j(\widehat{\mathcal{U}}_\sigma; A),$$

where  $\widehat{\mathcal{U}}_\sigma \cong M(\mathcal{A}_{\text{Min}(U_\sigma)})$ . For  $A = A_{\Lambda_T}$  or  $A = \mathcal{N}\pi$  these coefficients are 0 for  $G \neq T$ . For  $A = \mathbb{Z}\pi$ , they are 0 for  $j \neq \dim(G)$ . Hence, in all cases, for any coface  $\sigma'$  of  $\sigma$ , if  $G' := \text{Min}(U_{\sigma'}) < G$ , the coefficient homomorphism  $H^j(M(\mathcal{A}_G); A) \rightarrow H^j(M(\mathcal{A}_{G'}); A)$  is the zero map. Moreover, the  $E_1$  page of the spectral sequence decomposes as a direct sum (cf. [7, Lemma 2.2]). In fact, for a fixed  $j$ , by using Lemma 5.5, we see that the  $E_1^{i,j}$  term decomposes as

$$E_1^{i,j} = \bigoplus_{G \in \mathcal{L}_X^{n-j}} C^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}}); H^j(M(\mathcal{A}_G); A)),$$

where we have constant coefficients in each summand. Hence, at  $E_2$  we have

$$\begin{aligned} E_2^{i,j} &= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}}); H^j(M(\mathcal{A}_G); A)) \\ &= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(G, \Sigma_X \cap G; H^j(M(\mathcal{A}_G); A)), \end{aligned} \quad (5)$$

where the second equation follows from (3).

When  $A = A_{\Lambda_T}$  or  $A = \mathcal{N}\pi$ , all summands vanish for  $G \neq T$  and  $j \neq 0$ . So, we are left with  $E_2^{n,0} = H^n(T, \Sigma_X; A)$ , which is isomorphic to the tensor product free abelian group of rank  $\beta(\mathcal{T}_X)$  with  $A$ . It follows that  $H^*(\mathcal{R}_X; A)$  is concentrated in degree  $n$  and that  $\dim_{\mathbb{C}} H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X) = \dim_{\mathcal{N}\pi} H^n(\mathcal{R}_X; \mathcal{N}\pi)$ . This proves Theorems 5.1 and 5.2.

Consider formula (5) for  $A = \mathbb{Z}\pi$ . By Lemma 2.1,  $H^i(G, \Sigma_X \cap G)$  is concentrated in degree  $\dim G = n - j$ . Hence,  $E_2^{i,j}$  is nonzero (and free abelian) only for  $i + j = n$ . It follows that the spectral sequence degenerates at  $E_2$ , i.e.,  $E_2 = E_{\infty}$ . This proves Theorem 5.3.

*Remark 5.6.* Let us remark that the statement of Theorem 5.1 holds even if the local system  $\Lambda_T$  is nonresonant or if it verifies the Schechtman, Terao and Varchenko nonresonance conditions in all small open convex sets, i.e.  $\Lambda_{\sigma}$  verifies the nonresonance conditions in [27] for all  $\sigma \in N(\mathcal{U}_{\text{sing}})$ . Indeed under these conditions Lemma 5.5 holds.

## References

- [1] K. S. Brown, *Cohomology of Groups*, Springer-Verlag, Berlin and New York, 1982.
- [2] D. C. Cohen and P. Orlik, *Arrangements and local systems*, Math. Res. Lett. **7** (2000), 299-316.
- [3] G. D'Antonio and E. Delucchi, *A Salvetti complex for toric arrangements and its fundamental group*, International Mathematics Research Notices (2011), doi: 10.1093/imrn/rnr161.
- [4] M.W. Davis, T. Januszkiewicz and I.J. Leary, *The  $\ell^2$ -cohomology of hyperplanes complements*, Groups Geom. Dyn. **1** (2007), 301–309.

- [5] M.W. Davis, T. Januszkiewicz, I.J. Leary and B. Okun, *Cohomology of hyperplane complements with group ring coefficients*, IMRN (2011), 2110–2116.
- [6] M. W. Davis and I. J. Leary, *The  $\ell^2$ -cohomology of Artin groups*, J. London Math. Soc. (2) **68** (2003), 493–510.
- [7] M.W. Davis and B. Okun, *Cohomology computations for Artin groups, Bestvina–Brady groups, and graph products*, Groups Geom. Dyn. (to appear).
- [8] C. De Concini and C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*, Springer Verlag, 2010.
- [9] C. De Concini and C. Procesi *On the geometry of toric arrangements*, Transformation Groups **10** (2005), 387–422.
- [10] A. Dimca and G.I. Lehrer, *Purity and equivariant weight polynomials*, in: *Algebraic Groups and Lie Groups* Austral. Math. Soc. Lect. Ser., Vol. 9, Cambridge University Press, Cambridge, 1997, 161–181.
- [11] J.M. Douglass, *Toral arrangements and hyperplane arrangements*, Rocky Mountain J. Math. **28** (1998), 939–956.
- [12] H. Esnault, V. Schechtman and E. Viehweg, *Cohomology of local systems on the complement of hyperplanes*, Invent. math. **109** (1992), 557–561 (1992); Erratum. **112** (1993).
- [13] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge, 2001.
- [14] T. Kohno, *Homology of a local system on the complement of hyperplanes*, Proc. Japan. Acad. Ser. A **62** (1986), 144–147.
- [15] G. I. Lehrer, *A toral configuration space and regular semisimple conjugacy classes*, Math. Proc. Cambridge Philos. Soc. **118(1)** (1995), 105–113.
- [16] G. I. Lehrer, *The cohomology of the regular semisimple variety*, J. Algebra **199** (1998), no. 2, 666–689.

- [17] E. Looijenga, *Cohomology of  $M3$  and  $M1$  3. Mapping class groups and moduli spaces of Riemann surfaces*, (Gottingen, 1991/Seattle, WA, 1991), Contemp. Math., Vol. 150, Amer. Math. Soc., 1993, 205–228.
- [18] W. Lück,  *$L^2$ -invariants and  $K$ -theory*, Springer-Verlag, Berlin and New York, 2002.
- [19] C. Macmeikan, *Modules of derivations for toral arrangements*, Indag. Math. **15** (2004), 257–267.
- [20] C. Macmeikan, *The Poincaré polynomial of an  $mp$  arrangement*, Proc. Amer. Math. Soc. **132** (2004), 1575–1580.
- [21] L. Moci, *Combinatorics and topology of toric arrangements defined by root systems*, Rend. Lincei Mat. Appl. **19** (2008), 293–308.
- [22] L. Moci, *A Tutte polynomial for toric arrangements*, arXiv:0911.4823 [math.CO], to appear in Transaction of AMS.
- [23] L. Moci, *Wonderful models for toric arrangements*, ArXiv e-prints, December 2009.
- [24] L. Moci and S. Settepanella. *The homotopy type of toric arrangements*, Journal of Pure and Applied Algebra **215**, (2011), 1980–1989.
- [25] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer-Verlag, Berlin and New York, 1992.
- [26] S. Papadima and A. I. Suciu, *The spectral sequence of an equivariant chain complex and homology with local coefficients*, Trans. Amer. Math. Soc. **362** (2010), 2685–2721.
- [27] V. Schechtman, H. Terao and A. Varchenko, *Cohomology of local systems and the Kac-Kazhdan condition for singular vectors*, J. Pure Appl. Algebra **100** (1995), 93–102.

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