



# A new approach to estimating equilibrium exchange rates for small open economies: The case of Canada

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# A new approach to estimating equilibrium exchange rates for small open economies: The case of Canada

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## Abstract

This paper proposes a new approach to estimating equilibrium exchange rates for small open economies. We set up a simple structural model of output, the rate of inflation and the real exchange rate. These observed variables are explained by unobserved equilibrium rates as well as unobserved transitory components in output and the exchange rate. Using Canadian data over 1974-2008 we jointly estimate the unobserved components and the structural parameters using the Kalman filter and Bayesian technique. We find that Canada's equilibrium exchange rate evolves smoothly and follows a trend depreciation. The transitory component is found to be very persistent but much more volatile than the equilibrium rate.

*Keywords:* equilibrium exchange rate, unobserved components, Kalman filter, Bayesian analysis, Importance sampling

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# 1 Introduction

The identification and estimation of equilibrium exchange rates is a controversial topic in international macroeconomics. The literature has come up with a number of different ways of determining equilibrium rates, and results strongly depend on which particular approach is used. Yet knowledge of equilibrium rates is indispensable for a variety of issues in exchange rate economics, including assessments of currency misalignments, the decision of opting for fixed or flexible exchange rates, or questions regarding the reform of the international monetary system. It is also of particular relevance when large movements in the exchange rate coincide with broad stability in economic fundamentals, as was recently experienced in Canada (OECD, 2004, p. 53).

In this paper we propose a new approach to estimating equilibrium exchange rates for small open economies. Our approach is based on a simple structural model of a small open economy introduced by Ball (1999). In this model observed macroeconomic aggregates are linked to unobserved equilibrium rates and unobserved transitory components. Specifically we disentangle output and the exchange rate into transitory and permanent components. The transitory components are then linked to each other and to the rate of inflation via a Phillips curve. The unobserved components and the structural parameters are jointly estimated in a Bayesian framework. The model is applied to Canadian data.

The remainder of the paper is structured as follows: Section 2 provides a brief overview of the various concepts of equilibrium real exchange rates, Section 3 introduces the stylized small open economy model, Section 4 elaborates on our estimation methodology, Section 5 presents the estimation results, and Section 6 concludes.

## 2 Equilibrium real exchange rates

Equilibrium real exchange rates can be identified in various different ways. The most commonly used are the (enhanced) purchasing power parity (PPP), the fundamental equilibrium exchange rate (FEER), the behavioral equilibrium exchange rate (BEER), and the permanent equilibrium exchange rate (PEER).<sup>1</sup>

The simplest approach to determining equilibrium exchange rates is based on PPP, according

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<sup>1</sup>For more complete taxonomies of equilibrium exchange rates, see MacDonald (2000) and Driver and Westaway (2004).

to which an exchange rate is in equilibrium if it equalizes the purchasing power of national currencies in terms of particular goods or output bundles. A variant of this paradigm is the so-called enhanced PPP approach, which incorporates the Balassa-Samuelson effect by linking nations' per-capita income levels with their effective real exchange rates. Consequently, equilibrium real exchange rates should be weaker for low-income and emerging economies in comparison to (technologically) more advanced countries. The empirical observation that exchange rates converge to their PPP equilibrium levels far too slowly to be compatible with any sensible notion of goods market arbitrage, the so-called PPP puzzle (Rogoff, 1996), implies that this equilibrium concept may determine the equilibrium exchange rate in the very long run only. In particular, this concept leaves out all factors which may account for deviations from PPP levels in terms of a time-varying equilibrium path of the exchange rate. These factors may include aggregate activity levels, net asset levels, or balance of payments positions, and are incorporated in various ways in the FEER and BEER equilibrium concepts.

FEERs have been popularized by Williamson (1983) as a concept of macroeconomic balance. This approach considers a country's internal and external balance, where the internal equilibrium corresponds to a zero output gap consistent with the NAIRU, and the external balance requires a sustainable current account position. This concept has been widely used by the IMF as the basis for the first and third approaches to estimating equilibrium exchange rates (IMF, 2006). However, the notion of a sustainable current account is not immediately operational. There is substantial uncertainty as to the exact magnitude of a "sustainable" external balance and whether divergences of the current account balance from target are transitory or permanent.

The natural real exchange rate (NATREX) has been introduced by Stein (1994) as an extension of the FEER based on dynamic stock-flow models, in which the external balance is explicitly modeled in terms of the key determinants of national savings and investment levels. These include the rate of time preference and the stock of foreign assets in the savings function, and the level and productivity of the capital stock in the investment function. Although appealing as a theoretical concept, empirical implementations of the NATREX have to rely on proxies for the most crucial variables in terms of the rate of time preference and the productivity of capital. For example, Stein uses the ratio of the sum of private and public consumption to GNP as the time preference measure, and a moving average of the growth of real GDP as the measure of productivity.

BEERs attempt to econometrically model the behavior of real exchange rates. Pioneered by [Clark and MacDonald \(1999\)](#), this approach tries to connect the observed real exchange rate with its long-run fundamental determinants, such as the terms of trade, the relative price of traded to nontraded goods, and net foreign assets. The relationship between the unobserved equilibrium exchange rate and the fundamentals is then assumed to be identical to the empirically estimated long-term relationship. BEERs are also used as the basis for the second of the IMF calculations of equilibrium exchange rates ([IMF, 2006](#)). A major drawback of BEERs lies in the assumption that the exchange rate is on average in equilibrium over the estimation period. Hence BEERs can only be used as an indicator of a country's under- or overvaluation relative to its own past averages and not as an absolute measure of the equilibrium exchange rate ([Cline and Williamson, 2007](#)).

A general problem of both the FEER and BEER approaches regards the selection and measurement of the appropriate fundamentals. There is a wide array of potentially important variables, and the outcomes depend critically on the set of variables included in the set of relevant fundamentals. Apart from selection issues, it is far from obvious how to measure the long-term values of the fundamentals themselves. Possibly most worrying is the observation that the influence of the variables most frequently included in the set of fundamentals of equilibrium exchange rates, such as the terms of trade or the stock of net foreign assets, is empirically not substantiated ([Égert et al., 2006](#)).

As an alternative approach which avoids problems associated with the selection of fundamental variables, PEERs use time-series estimators to decompose real exchange rates into their permanent and transitory components, with the permanent component defined as a measure of the equilibrium exchange rate. Such decompositions can be obtained by means of various statistical techniques, such as univariate or multivariate Beveridge-Nelson decompositions, structural vector-autoregressions, or cointegration-based estimation techniques.<sup>2</sup> A major disadvantage of PEERs lies in the fact that such decompositions are purely statistical and incorporate no economic determinants of exchange rate equilibrium.<sup>3</sup>

The approach suggested in this paper is also based on a decomposition into permanent and transitory components of real exchange rate movements as in the PEER approach. However,

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<sup>2</sup>See [MacDonald \(2000\)](#) for an overview and further references.

<sup>3</sup>However, a number of papers supplement the BEER approach with a PEER decomposition for assessment purposes, (see e.g. [Alberola et al., 1999](#); [Hoffmann and MacDonald, 2001](#); [Clark and MacDonald, 2004](#)).

instead of explicitly incorporating fundamental determinants of exchange rate determination, we base our decomposition on a structural model of a small open economy.

### 3 A stylized small open economy model

Recently Galí and Monacelli (2005) have extended the benchmark New Keynesian DSGE model to a small open economy. In their model households consume domestic and imported goods. Foreign shocks, such as the terms of trade, therefore affect consumption and thus the domestic business cycle. Based on a consumption Euler equation and optimal price setting behavior of domestic firms Galí and Monacelli derive an open economy forward-looking IS curve and a New Keynesian Phillips curve for an open economy. Given the importance of foreign variables, monetary policy is described by an interest rate rule that includes the exchange rate. Whether or not central banks react to exchange rate movements has been analyzed by Lubik and Schorfheide (2007) who estimate a structural forward-looking model in the spirit of Galí and Monacelli (2005). They find that the central banks of Canada and the UK target exchange rates while the central banks in New Zealand and Australia do not. A drawback of estimating structural DSGE models is that it requires stationarity. The common practice in this line of the literature is to assume some form of a deterministic trend for each variable and then to analyze the de-trended data. As a consequence linearized DSGE models can in general not be used to study the long-run trends in the variables under investigation.

The small open economy model outlined in this section is meant to capture the main effects of the exchange rate on business cycle fluctuations and the rate of inflation in the simplest possible way. It can be seen as a reduced form of a larger macroeconomic model. By explicitly specifying a stochastic law of motion for the equilibrium exchange rate the model shares some similarities with Laubach and Williams (2003) who estimate the natural rate of interest for the US economy.<sup>4</sup> In their model the natural rate of interest is linked to the growth rate of potential output. The interest rate gap, i.e. the deviation of the natural rate of interest from the actual data, is part of the output gap. In the context of a small open economy this suggests that the corresponding real exchange rate gap is linked to the output gap. Specifically we model the output gap,  $\tilde{y}_t$ , as an AR

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<sup>4</sup>The importance of allowing for a time-varying natural rate of interest for the analysis of monetary policy has been forcefully pointed out by Trehan and Wu (2007).

process plus the real exchange rate gap, i.e.

$$\tilde{y}_t = a_y(L)\tilde{y}_{t-1} + a_e(L)\tilde{e}_{t-1} + \eta_{1t}, \quad (1)$$

where  $\tilde{e}_t$  denotes deviations of the exchange rate from its equilibrium value and  $\eta_{1t}$  is a Gaussian mean zero white noise error term. The real exchange rate gap affects the output gap with a one-period delay and is defined such that an increase in  $\tilde{e}$  is a depreciation of the domestic currency. The equilibrium level of output, often referred to as potential output  $y_t^*$ , is modeled as a random walk with drift

$$y_t^* = y_{t-1}^* + \mu + \eta_{2t}, \quad (2)$$

where the drift term  $\mu$  measures potential output growth. [Basistha \(2007\)](#) analyzed alternative specifications for potential output growth in Canada, including a random walk drift or a constant drift without break, and found that a constant drift with one break in 1973:Q4 fits the data best.

The inflation rate is a function of expected inflation, the output gap and the exchange rate gap. The Phillips curve is specified in levels instead of first differences, which would result in an acceleration version of the Phillips curve. [Basistha \(2007\)](#) and [Kichian \(1999\)](#) examine both specifications of the Phillips curve for Canada and report a better fit when inflation is modeled in level-data. Another advantage of this method is that the mean of inflation can be modeled as a constant. As there is strong evidence of mean shifts in Canadian inflation we allow for breaks in this constant (see e.g. [Kichian, 1999](#); [Basistha, 2007](#)). The number and the timing of the breaks is to be determined by the ([Bai and Perron, 1998, 2001, 2003](#), hereafter BP) structural break test.<sup>5</sup> We further assume lagged inflation as a proxy for expected inflation.

$$\pi_t = \bar{\pi}_i + b_\pi(L)\pi_{t-1} + b_y\tilde{y}_{t-1} + b_e(L)\Delta\tilde{e}_{t-1} + \varepsilon_{1t}, \quad (3)$$

where  $\pi_t$  is the rate of inflation and  $\bar{\pi}_i$  is its long-run mean value. The subscript  $j$  refers to different values of  $\bar{\pi}_i$  with  $j = 1, \dots, m + 1$ ,  $m$  denoting the number of structural breaks. Note that the exchange rate gap affects inflation (i) directly through its first difference, and (ii) indirectly through the output gap. This way of modeling exchange rate effects in a small open economy was originally introduced by [Ball \(1999\)](#).

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<sup>5</sup>Briefly, BP suggest to first examine two tests (the so called  $UD_{max}$  and  $WD_{max}$  tests) to check if there are any structural breaks. If these tests reject the null of no breaks, a sequential procedure to determine the number of breaks is used. According to the BP notation this means computing a sequence of  $\text{Sup}F_T(l + 1|l)$  statistics to test the null of  $l$  breaks against the alternative of  $l + 1$  breaks. A detailed description of this test can be found in BP and [Rapach and Wohar \(2005\)](#).

Similar to the decomposition of output into trend and cycle, the real exchange rate is the sum of its long-run equilibrium level,  $e_t^*$ , and a transitory component,  $\tilde{e}_t$ . These two unobserved components are specified as follows:

$$e_t^* = e_{t-1}^* + \eta_{3t}, \quad (4)$$

$$\tilde{e}_t = C_e(L)\tilde{e}_{t-1} + \eta_{4t}. \quad (5)$$

The equilibrium exchange rate evolves according to a unit-root process. The unit-root specification of  $e_t^*$  implies that the actual exchange rate is non-stationary. The question whether exchange rates are stationary or not has received much attention in the literature. Although this still is an ongoing debate there is some consensus that the hypothesis of a unit-root in exchange rates over short or medium horizons cannot be rejected.<sup>6</sup> A non-stationary exchange rate can also be justified by the I(1) behavior of its fundamentals. The error term  $\eta_{3t}$  can thus be interpreted as changes in the exchange rate fundamentals. The transitory deviation from the equilibrium rate is modeled as an AR process, with the speed of adjustment captured by the sum of the AR parameters.

The decomposition of the equilibrium exchange rate into permanent and transitory components is not new. [Engel and Kim \(1999\)](#) model the long-run US/UK exchange rate as a unit-root process and the transitory deviations of the actual rate from the equilibrium as an AR(2) process. As [Engel and Kim](#) consider a time span in excess of 100 years, they allow for heteroscedasticity in both components by using a three-state Markov switching model for the variances. They find that a constant variance for the equilibrium rate and three distinct variance states for the transitory component provide the best fit. [Engel and Kim](#) show that the variance switches are related to different exchange rate regimes. Thus in the post 1973 era of floating nominal exchange rates the transitory components stays in a single variance state. The model presented here can be seen as an extension of the [Engel and Kim](#) approach. Similar to their analysis we decompose the exchange rate into transitory and permanent components and estimate the model using the Kalman filter and Bayesian technique. However, our multivariate model relates the transitory component to the output gap and the rate of inflation. Thus, it uses information contained in these variables to better identify the transitory component and consequently the equilibrium rate. Multivariate decompositions of macroeconomic variables have been shown to considerably

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<sup>6</sup>By applying standard unit-root tests we cannot reject the null hypothesis of a unit-root in the exchange rate analyzed here. Detailed results are available on request.



reduce the uncertainty regarding the estimation of the unobserved variables. [Basistha and Startz \(2008\)](#) show that a multivariate unobserved component model to estimate the natural rate of unemployment cuts in half uncertainty as measured by variance and leads to significantly tighter confidence bands as compared to a univariate decomposition.

## 4 Estimation methodology

### 4.1 State space representation of the model

The model given by equations (1)-(5) can be cast into a linear Gaussian state space model of the following general form<sup>7</sup>

$$y_t = Z\alpha_t + Ax_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, H), \quad (6)$$

$$\alpha_{t+1} = S + T\alpha_t + \eta_t, \quad \eta_t \sim N(0, Q), \quad t = 1, \dots, n, \quad (7)$$

where  $y_t$  is a  $p \times 1$  vector of  $p$  observed endogenous variables, modeled in the observation equation (6),  $x_t$  is a  $k \times 1$  vector of  $k$  observed exogenous or predetermined variables and  $\alpha_t$  is a  $m \times 1$  vector of  $m$  unobserved states, modeled in the state equation (7). The vectors  $\varepsilon_t$  and  $\eta_t$  are assumed to hold mutually independent Gaussian error terms with the former representing measurement errors and the latter structural shocks. The exact specification of the vectors  $y_t$ ,  $x_t$  and  $\alpha_t$  and the matrices  $Z$ ,  $S$ ,  $A$ ,  $T$ ,  $R$ ,  $H$  and  $Q$  is provided in [Appendix A.1](#).

### 4.2 Parameter estimation: a Bayesian framework

For given parameter matrices  $Z$ ,  $A$ ,  $T$ ,  $S$ ,  $H$ , and  $Q$ , the unobserved state vector  $\alpha_t$  can be identified from the observations  $y_1, \dots, y_n$  and  $x_1, \dots, x_n$  using the Kalman filter and smoother. In practice these matrices generally depend on elements of an unknown parameter vector  $\psi$ . One possible approach is to derive the log-likelihood function for the model under study from the Kalman filter (see e.g. [de Jong, 1991](#); [Koopman and Durbin, 2000](#); [Durbin and Koopman, 2001](#)) and replace the unknown parameter vector  $\psi$  by its maximum likelihood (ML) estimate. This is not the approach pursued in this paper. We analyze the state space model from a Bayesian point of view, i.e. we use prior information to down-weight the likelihood function in regions of the parameter space that are inconsistent with out-of-sample information and/or in which the structural model is not interpretable ([Schorfheide, 2006](#)). More formally, we treat  $\psi$  as a random

<sup>7</sup>See e.g. [Durbin and Koopman \(2001\)](#) for an extensive overview of state space models.

parameter vector with a known prior density  $p(\psi)$  and estimate the posterior densities  $p(\psi | y, x)$  for the parameter vector  $\psi$  and  $p(\hat{\alpha}_t | y, x)$  for the smoothed state vector  $\hat{\alpha}_t$ , where  $y$  and  $x$  denote the stacked vectors  $(y'_1, \dots, y'_n)'$  and  $(x'_1, \dots, x'_n)'$  respectively, by combining information contained in  $p(\psi)$  and the sample data. This boils down to calculating the posterior mean  $\bar{g}$

$$\bar{g} = E[g(\psi) | y, x] = \int g(\psi) p(\psi | y, x) d\psi, \quad (8)$$

where  $g$  is a function which expresses the moments of the posterior densities  $p(\psi | y, x)$  and  $p(\hat{\alpha}_t | y, x)$  in terms of the parameter vector  $\psi$ . In principle, the integral in equation (8) can be evaluated numerically by drawing a sample of  $n$  random draws of  $\psi$ , denoted  $\psi^{(i)}$  with  $i = 1, \dots, n$ , from  $p(\psi | y, x)$  and then estimating  $\bar{g}$  by the sample mean of  $g(\psi)$ . As  $p(\psi | y, x)$  is not a density with known analytical properties, such a direct sampling method is not feasible, though. Therefore, we use importance sampling (see Appendix A.2 for technical details).

As noted by Planas et al. (2008), Bayesian estimation of unobserved component models avoids the pile-up problem by specifying prior distributions that are strictly positive for the variance parameters. Another important advantage of the Bayesian framework over standard ML is that it is straightforward to calculate the posterior densities of both the parameter vector  $\psi$  and the smoothed state vector  $\hat{\alpha}_t$  where the latter takes both parameter and filter uncertainty into account (see Appendix A.3 for technical details).

## 5 Estimation Results<sup>8</sup>

### 5.1 Data

We use quarterly data for Canada from 1974Q1 to 2008Q4 taken from the OECD Economic Outlook and the International Monetary Fund (IMF) International Financial Statistics. For inflation we use the first difference of the log of the seasonally adjusted CPI. Output is the log of seasonally adjusted quarterly GDP multiplied by 100. For the exchange rate we use the log of Canada's real effective exchange rate.<sup>9</sup> Starting in 1974 implies (i) that we only focus on the post Bretton Woods era and (ii) that we do not need to address the productivity slowdown in Canada's real GDP in the early 70s.

<sup>8</sup>The GAUSS code to obtain the results presented in this section is available on request.

<sup>9</sup>Further information on the construction of the real effective exchange rate can be found at [www.oecd.org/std/finance](http://www.oecd.org/std/finance).

## 5.2 Structural breaks in the mean of inflation

Table 1 presents the results of the BP tests on structural breaks in inflation.<sup>10</sup>

Table 1 about here

Both, the  $WD_{max}$  and the  $UD_{max}$  test statistic clearly reject the null hypothesis of no structural breaks in inflation at conventional confidence levels. The sequential analysis also rejects the null hypothesis of no breaks against the alternative hypothesis of one break as well as the null of one break against the alternative hypothesis of two structural breaks. However, more than two breaks are not found. The detected break dates are 1982:Q3 and 1991:Q1.<sup>11</sup>

## 5.3 Prior distribution of the parameters

Prior information on the unknown parameter vector  $\psi$  is included in the analysis through the prior density  $p(\psi)$ . Detailed information on  $p(\psi)$  can be found in the first columns of Table 2. As stated above, the main motivation for setting these priors is to down-weight the likelihood function in regions of the parameter space that are inconsistent with out-of-sample information and/or in which the structural model is not interpretable. Previous estimates as well as economic theory give us an idea about the approximate value of the model's parameters. However, using previous studies to set priors should be done with caution particularly if these studies consider the same time period. We therefore use previous estimates only as a rough indication for the prior mean but choose the prior variance fairly loose. The bivariate unobserved component model for Canada of Basistha (2007) provides an indication for the parameter values in the output and inflation equations.

The output gap includes two lagged dependent variables. The prior distribution of the autoregressive parameters is chosen so that its 90% interval covers the range  $[0.10 - 0.90]$ . Thus the prior distribution does neither impose a volatile nor a very persistent output gap. For potential output growth we use a rather tight prior. Nevertheless the 90% interval for the annualized growth rate of potential output ranges from 2.44% to 3.48%. Regarding the impact of the lagged exchange rate gap on the output gap we cannot rely on previous estimates. Ball (1999) suggests a value of

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<sup>10</sup>The results of the BP tests have been obtained by using the original GAUSS program from P. Perron available on his webpage.

<sup>11</sup>The break dates are similar to the one in Basistha (2007).

0.2 for  $a_e$ . The same value has been used by [Zampolli \(2006\)](#), who calibrates the [Ball](#) model for the UK. Therefore we use a very uninformative prior for  $a_e$  with a mean equal to 0.2 and 10% and 90% percentiles of -0.60 and 1.00 respectively. The variance parameters for the shocks to the output gap and potential output have been set according to previous estimates. However we leave a considerable amount of uncertainty around their prior means.

Turning to inflation, the baseline model includes one lag of the dependent variable, the output gap and the change of the exchange rate gap. The prior distribution of inflation persistence, as measured by  $b_\pi$ , is very flat allowing for a unit-root process of the inflation rate. The prior mean of  $b_y$ , the slope of the Phillips curve, is set to 0.5. As in the output gap we do not have previous estimates for  $b_e$ . Thus we use a flat prior for this parameter. The exchange rate gap is modeled as an AR(2) process where the priors on the AR parameters are similar to the output gap AR parameters. The AR(2) specification for the transitory component in the exchange rate has also been used by [Engel and Kim \(1999\)](#). The mean of the variance parameters of the exchange rate components are set to very small values with a high degree of uncertainty.

## 5.4 Posterior distribution of the parameters

The last two columns of [Table 2](#) show the posterior mean and the 10% and 90% percentiles of the posterior distribution of all parameters. Similar to other studies for industrialized countries we find the output gap to be a relatively persistent process. The parameter of special interest in the output equation is  $a_e$  which is estimated to be 0.79 and statistically significantly different from zero. Thus the posterior mean of  $a_e$  is substantially higher as compared to its prior mean and moreover is estimated with lower uncertainty. This suggests that there is strong information in the data about  $a_e$ . Economically this result implies that transitory deviations of the effective exchange rate from its equilibrium value have a strong impact on Canada's output gap. However the impact of the change in the exchange rate gap on inflation is not very precisely estimated. Prior and posterior distribution of this parameter are very similar as can be seen in [Figure 4](#) which shows the prior together with the posterior distribution for all parameters. The persistence of inflation is somewhat lower than found in the literature. The long-run means of inflation as well as  $\sigma_{\varepsilon_1}^2$  are very similar to the estimates of [Basistha \(2007\)](#).<sup>12</sup>

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<sup>12</sup>Note that the long-run mean is given by  $\bar{\pi}_i/(1 - b_\pi)$ .

Table 2 about here

The sum of the AR parameters in the exchange rate gap is 0.95, implying a high degree of persistence. However, the result of very persistent transitory shocks is consistent with other estimates in the literature (see e.g Engel and Kim, 1999; Rogoff, 1996).

## 5.5 Posterior distribution of the states

Figure 1 shows the output gap and potential output together with actual output.<sup>13</sup> These two unobserved components are not the main focus of this paper. However, an exact estimation of the output gap is important since it is linked to the exchange rate gap.

Figure 1 about here

The shape and the magnitude of the output gap are very similar to the estimates of Basistha (2007). The shaded areas indicate Canadian recessions as defined by the Economic Cycle Research Institute. The estimated output gap picks up the business cycle turning points quite accurately.

Figure 2 about here

Figure 2 shows the real effective exchange rate together with the mean and the 10% and 90% percentiles of the posterior distribution of the equilibrium exchange rate. The equilibrium rate evolves very smoothly as compared to the transitory component. It turns out that the equilibrium exchange rate is far from being constant as it exhibits a trend depreciation over the sample period. Thus a first result is that simple demeaning of the Canadian real effective exchange rate leads to an incorrect measure of deviations from its equilibrium level. The transitory component is found to often deviate substantially from the equilibrium rate. Moreover these deviations are very persistent.

Figure 3 about here

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<sup>13</sup>The graph shows the smoothed estimates.

## 5.6 The Canadian dollar

Figure 2 shows that the equilibrium effective real exchange rate of the Canadian dollar depreciates continuously over most of the sample period. This finding is in line with previous studies and can be explained by a downward trend in the relative price of Canadian nontraded to traded goods (e.g. Clark and MacDonald, 2004). As in previous permanent-transitory decompositions, we find that the permanent component of the Canadian real exchange rate exhibits substantial time variability, but is more stable than the actual real exchange rate itself (Cumby and Huizinga, 1990; Clarida and Galí, 1994).

The deviations of the actual real exchange rate from its equilibrium are highly persistent, and our identification relates this low-frequency variability of the transitory component of the real exchange rate primarily to the cyclical nature of the output gap rather than to inflation dynamics. Due to the strong trade and financial linkages between the Canadian and US economies, the cyclical component of the Canadian effective real exchange rate is captured to a large extent by developments in the multilateral value of the US dollar. Figures 2 and 3 identify five periods of misalignment of the Canadian dollar, with significant undervaluations in the early 1980s and the late 1990s, and significant overvaluations in the mid-1970s, the early 1990s and the most recent period starting in the mid-2000s.

Both periods of undervaluation of the Canadian dollar follow in the wake of multilateral appreciations of the US dollar. In the early 1980s, the depreciation of the Canadian dollar can be associated with the Fed's monetary policy shift under Paul Volcker. The second instance of undervaluation at the end of the 1990s followed a series of major currency and banking crises in Southeast Asia, Brazil, and Russia, in which US-dollar denominated assets were considered to be safe investments. Similarly, the three periods of overvaluation of the Canadian real effective exchange rate are a direct consequence of the multilateral depreciations of the US-dollar. The weakness of the US-dollar in the mid-1970 arises in the wake of the breakdown of the Bretton-Woods system of fixed exchange rates. In the second half of the 1980s and starting in the late 1990s, the United States experienced two periods of substantial and persistent external imbalances. Both of these episodes are associated with appreciations of all major currencies relative to the US dollar, resulting in temporary overvaluations of the Canadian dollar (Bailliu et al., 2005).<sup>14</sup>

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<sup>14</sup>The value of the Canadian dollar against the US dollar rose further with the onset of the financial crises that

## 6 Conclusion

This paper proposes a new approach to estimating equilibrium exchange rate for small open economies. We identify the equilibrium real exchange rate on the basis of a permanent-transitory decomposition of output and the exchange rate using an unobserved components model. The unobserved components are themselves linked to the output gap and the rate of inflation by means of the simple structural model of a small open economy introduced by [Ball \(1999\)](#). We estimate the model using Canadian data. The long-run depreciation of the Canadian equilibrium rate found in previous studies is confirmed by our results. We also identify few but prolonged periods of currency misalignments. All these periods can be associated with external factors arising from shifts in the multilateral US dollar exchange rate.

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started in the subprime segment of the US real estate market in the summer of 2007. However, the resulting overvaluation of the Canadian dollar was short-lived as the US real effective exchange rate started to appreciate beginning in the second half of 2008.

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# Appendices

## Appendix A Technical details state space estimation

### A.1 State space representation of the model in (1)-(5)

$$y_t = [ y_t \quad \pi_t \quad e_t ]'; \quad x_t = [ D_1 \quad D_2 \quad D_3 \quad \pi_{t-1} ]'$$

$D_1, D_2$  and  $D_3$  are dummy variables to capture the mean shifts in inflation.

$$\alpha_t = [ \tilde{y}_t \quad \tilde{y}_{t-1} \quad y_t^* \quad \tilde{e}_t \quad \tilde{e}_{t-1} \quad \tilde{e}_{t-2} \quad e_t^* ]'; \quad S = [ 0 \quad 0 \quad \mu \quad 0 \quad 0 \quad 0 \quad 0 ]';$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bar{\pi}_1 & \bar{\pi}_2 & \bar{\pi}_3 & b_\pi \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad Z = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & b_y & 0 & 0 & b_e & -b_e & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix};$$

$$T = \begin{bmatrix} a_{y_1} & a_{y_2} & 0 & 0 & a_e & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{e_1} & c_{e_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad Q = \begin{bmatrix} \sigma_{\eta_{1t}}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{\eta_{2t}}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{\eta_{4t}}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{\eta_{3t}}^2 \end{bmatrix};$$

$$\varepsilon_t = [ 0 \quad \varepsilon_{1t} \quad 0 ]'; \quad \eta_t = [ \eta_{1t} \quad 0 \quad \eta_{2t} \quad \eta_{4t} \quad 0 \quad 0 \quad \eta_{3t} ]'; \quad H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\varepsilon_1}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

### A.2 Computational aspects of importance sampling

The idea is to use an importance density  $g(\psi | y, x)$  as a proxy for  $p(\psi | y, x)$ , where  $g(\psi | y, x)$  should be chosen as a distribution that can be simulated directly and is as close to  $p(\psi | y, x)$  as possible. By Bayes' theorem and after some manipulations, equation (8) can be rewritten as

$$\bar{g} = \frac{\int g(\psi) z^g(\psi, y, x) g(\psi | y, x) d\psi}{\int z^g(\psi, y, x) g(\psi | y, x) d\psi}, \quad (\text{A-1})$$

with

$$z^g(\psi, y, x) = \frac{p(\psi) p(y | \psi)}{g(\psi | y, x)}. \quad (\text{A-2})$$

Using a sample of  $n$  random draws  $\psi^{(i)}$  from  $g(\psi | y, x)$ , an estimate  $\bar{g}_n$  of  $\bar{g}$  can then be obtained as

$$\bar{g}_n = \frac{\sum_{i=1}^n g(\psi^{(i)}) z^g(\psi^{(i)}, y, x)}{\sum_{i=1}^n z^g(\psi^{(i)}, y, x)} = \sum_{i=1}^n w_i g(\psi^{(i)}), \quad (\text{A-3})$$

with  $w_i$

$$w_i = \frac{z^g(\psi^{(i)}, y, x)}{\sum_{i=1}^n z^g(\psi^{(i)}, y, x)}. \quad (\text{A-4})$$

The weighting function  $w_i$  reflects the importance of the sampled value  $\psi^{(i)}$  relative to other sampled values. Geweke (1989) shows that if  $g(\psi | y, x)$  is proportional to  $p(\psi | y, x)$ , and under a number of weak regularity conditions,  $\bar{g}_n$  will be a consistent estimate of  $\bar{g}$  for  $n \rightarrow \infty$ . As an importance density  $g(\psi | y, x)$ , we take a large sample normal approximation to  $p(\psi | y, x)$ , i.e.

$$g(\psi | y, x) = N(\hat{\psi}, \hat{\Omega}) \quad (\text{A-5})$$

where  $\hat{\psi}$  is the mode of  $p(\psi | y, x)$  obtained from maximizing

$$\log p(\psi | y, x) = \log p(y | \psi) + \log p(\psi) - \log p(y) \quad (\text{A-6})$$

with respect to  $\hat{\psi}$  and where  $\hat{\Omega}$  denotes the covariance matrix of  $\hat{\psi}$ . Note that  $p(y | \psi)$  is given by the likelihood function derived from the Kalman filter and we do not need to calculate  $p(y)$  as it does not depend on  $\psi$ .

As any numerical integration method delivers only an approximation to the integrals in equation (A-1), we monitor the quality of the approximation by estimating the probabilistic error bound for the importance sampling estimator  $\bar{g}_n$  ((Bauwens et al., 1999) chap. 3, eq. 3.34). This error bound represents a 95% confidence interval for the percentage deviation of  $\bar{g}_n$  from  $\bar{g}$ . It should not exceed 10%.

Note that the normal approximation in equation (A-5) selects  $g(\psi | y, x)$  in order to match the location and covariance structure of  $p(\psi | y, x)$  as good as possible. One problem is that the normality assumption might imply that  $g(\psi | y, x)$  does not match the tail behavior of  $p(\psi | y, x)$ . If  $p(\psi | y, x)$  has thicker tails than  $g(\psi | y, x)$ , a draw  $\psi^{(i)}$  from the tails of  $g(\psi | y, x)$  can imply an explosion of  $z^g(\psi^{(i)}, y, x)$ . This is due to a very small value for  $g(\psi | y, x)$  being associated with a relatively large value for  $p(\psi)p(y | \psi)$ , as the latter is proportional to  $p(\psi | y, x)$ . Importance sampling is inaccurate in this case as this would lead to a weight  $w_i$  close to one, i.e.  $\bar{g}_n$  is determined by a single draw  $\psi^{(i)}$ . This is signalled by instability of the weights and a probabilistic error bound that does not decrease in  $n$ . In order to help prevent explosion of the weights, we change the construction of the importance density in two respects (Bauwens et al., 1999, chap. 3). First, we inflate the approximate covariance matrix  $\hat{\Omega}$  by multiplying it by a factor of 1.1.

This reduces the probability that  $p(\psi | y, x)$  has thicker tails than  $g(\psi | y, x)$ . Second, we use a sequential updating algorithm for the importance density. This algorithm starts from the importance density defined by (A-5), with inflation of  $\widehat{\Omega}$ , estimates posterior moments for  $p(\psi | y, x)$  and then defines a new importance density from these estimated moments. This improves the estimates for  $\widehat{\psi}$  and  $\widehat{\Omega}$ . We continue updating the importance density until the weights stabilize. The number of importance samples  $n$  was chosen to make sure that the probabilistic error bound for the importance sampling estimator  $\bar{g}_n$  does not exceed 10%.

### A.3 Posterior distribution of parameter and states

An estimate  $\widetilde{\psi}$  for the posterior mean  $E[\psi | y, x]$  of the parameter vector  $\psi$  is obtained by setting  $g(\psi^{(i)}) = \psi^{(i)}$  in equation (A-3) and taking  $\widetilde{\psi} = \bar{g}_n$ . An estimate  $\widetilde{\alpha}_t$  for the posterior mean  $E[\widehat{\alpha}_t | y, x]$  of the smoothed state vector  $\widehat{\alpha}_t$  is obtained by setting  $g(\psi^{(i)}) = \widehat{\alpha}_t^{(i)}$  in equation (A-3) and taking  $\widetilde{\alpha}_t = \bar{g}_n$ , where  $\widehat{\alpha}_t^{(i)}$  is the smoothed state vector obtained from the Kalman smoother using the parameter vector  $\psi^{(i)}$ . In order to calculate the 10th and 90th percentiles of the posterior densities of both the parameter vector  $\psi$  and the smoothed state vector  $\widehat{\alpha}_t$ , let  $F(\psi_j | y, x) = \Pr(\psi_j^{(i)} \leq \psi_j)$  with  $\psi_j$  denoting the  $j$ -th element in  $\psi$ . An estimate  $\widetilde{F}(\psi_j | y, x)$  of  $F(\psi_j | y, x)$  is obtained by setting  $g(\psi^{(i)}) = I_j(\psi_j^{(i)})$  in equation (A-3) and taking  $\widetilde{F}(\psi_j | y, x) = \bar{g}_n$ , where  $I_j(\psi_j^{(i)})$  is an indicator function which equals one if  $\psi_j^{(i)} \leq \psi_j$  and zero otherwise. An estimate  $\widetilde{\psi}_j^{10\%}$  of the 10th percentile of the posterior density  $p(\psi | y, x)$  is chosen such that  $\widetilde{F}(\psi_j^{10\%} | y, x) = 0.10$ . An estimate  $\widetilde{\alpha}_{j,t}^{10\%}$  of the 10th percentile of the  $j$ th element of the posterior density  $p(\widehat{\alpha}_t | y, x)$  is obtained by setting  $g(\psi^{(i)}) = \widehat{\alpha}_{j,t}^{(i)} - 1.645\sqrt{\widehat{P}_{j,t}^{(i)}}$  in equation (A-3) and taking  $\widetilde{\alpha}_{j,t}^{5\%} = \bar{g}_n$ , where  $\widehat{\alpha}_{j,t}^{(i)}$  denotes the  $j$ -th element in  $\widehat{\alpha}_t^{(i)}$ , and  $\widehat{P}_{j,t}^{(i)}$  is the  $(j, j)$ th element of the smoothed state variance matrix  $\widehat{P}_t^{(i)}$  obtained using the parameter vector  $\psi^{(i)}$ . The 90th percentiles are constructed in a similar way. As such the posterior distribution of the smoothed state vector  $\widehat{\alpha}$  takes both parameter and filter uncertainty into account.

## Appendix B Prior and Posterior parameter distributions

Figure 4 about here

## Tables and Figures

**Table 1:** Test for structural breaks in inflation

$WD_{max}$	$UD_{max}$	$SupF_T(1 0)$	$SupF_T(2 1)$	$SupF_T(3 2)$	$SupF_T(4 3)$	$SupF_T(5 4)$
113.75*	97.38*	97.38*	44.46	3.42	2.68	1.57

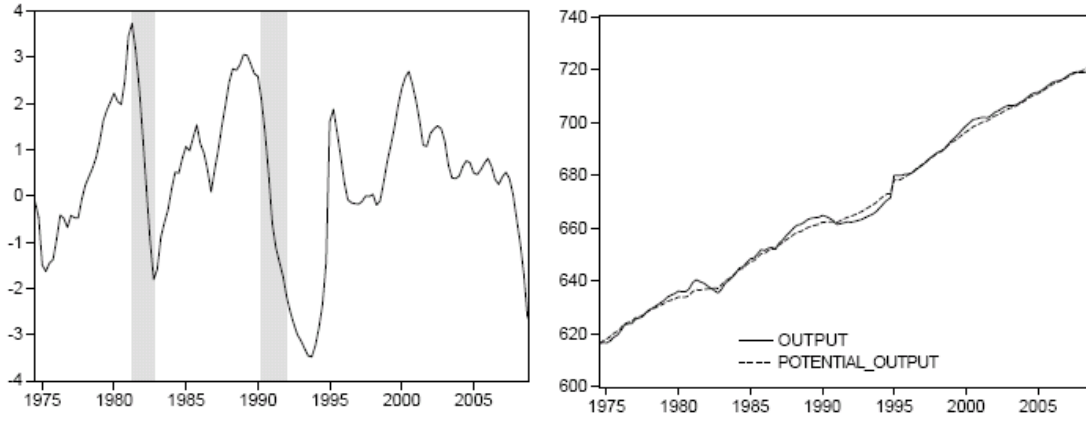
The maximum number of breaks is set to 5. The \* denotes significance at the 5% level. The 5% critical values are  $UD_{max} = 9.52$ ,  $WD_{max} = 10.39$ ,  $SupF_T(1|0) = 9.1$ ,  $SupF_T(2|1) = 10.55$ ,  $SupF_T(3|2) = 11.36$ ,  $SupF_T(4|3) = 12.35$ ,  $SupF_T(5|4) = 12.97$ .

**Table 2:** Prior and Posterior Parameter Distributions

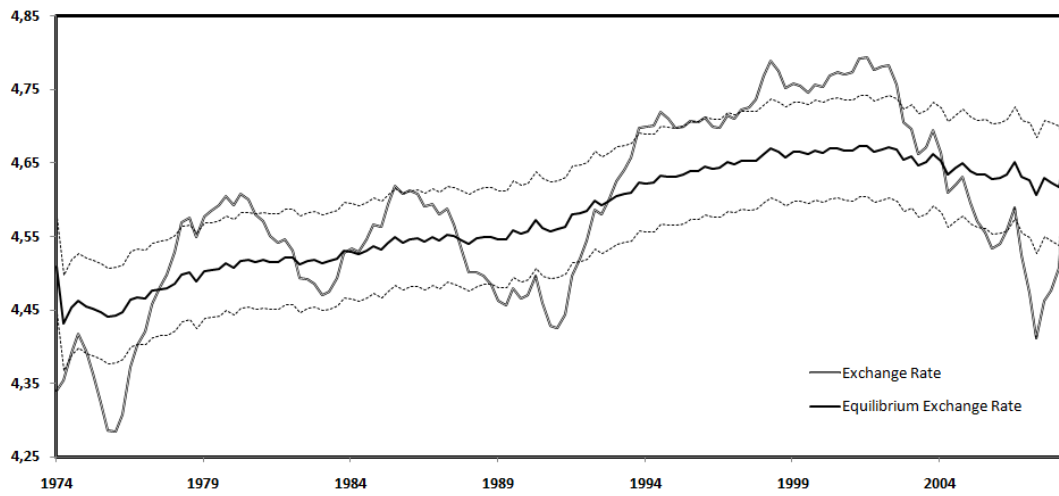
	Parameter	Prior Distribution		Posterior Distribution	
		Mean	90% Interval	Mean	90% Interval
<i>Output</i>	$a_{y_1}$	1.20	[1.00 – 1.40]	1.39	[1.27 – 1.51]
	$a_{y_2}$	-0.70	[-0.90 – -0.50]	-0.47	[-0.58 – -0.35]
	$a_e$	0.20	[-0.60 – 1.00]	0.79	[0.13 – 1.46]
	$\mu$	0.74	[0.61 – 0.87]	0.75	[0.68 – 0.82]
	$\sigma_{\eta_1}^2$	0.50	[0.21 – 0.89]	0.36	[0.28 – 0.47]
	$\sigma_{\eta_2}^2$	0.75	[0.21 – 0.89]	0.57	[0.46 – 0.71]
<i>Inflation</i>	$\bar{\pi}_1$	4.50	[2.69 – 6.31]	6.38	[5.53 – 7.23]
	$\bar{\pi}_2$	2.00	[0.19 – 3.81]	2.77	[2.14 – 3.39]
	$\bar{\pi}_3$	1.00	[-0.81 – 2.81]	1.36	[0.91 – 1.83]
	$b_\pi$	0.50	[0.00 – 1.00]	0.29	[0.21 – 0.38]
	$b_y$	0.50	[0.24 – 0.76]	0.23	[0.10 – 0.38]
	$b_e$	0.20	[-0.60 – 1.00]	0.30	[-0.50 – 1.11]
	$\sigma_{\varepsilon_1}^2$	1.25	[0.91 – 1.62]	3.02	[2.71 – 3.37]
<i>Exchange Rate</i>	$c_{e_1}$	1.20	[1.00 – 1.40]	1.49	[1.39 – 1.59]
	$c_{e_2}$	-0.70	[-0.90 – -0.50]	-0.54	[-0.64 – -0.44]
	* $\sigma_{\eta_3}^2$	0.08	[0.03 – 0.15]	0.14	[0.08 – 0.24]
	* $\sigma_{\eta_4}^2$	0.08	[0.03 – 0.15]	0.30	[0.21 – 0.40]

The prior distribution is assumed to be Gaussian for all elements in  $\psi$ , except the variance parameters which are assumed to be gamma distributed. With  $n=20,000$  for the initial importance function and all updates, the probabilistic error bound for the importance sampling estimator  $g_n$  is well below 10% for all coefficients. The number of subsequent updates of the importance density is 5 (see Appendix A for details). \* $\sigma_{\eta_i}^2 = \sigma_{\eta_i}^2 * 10^{-3}$ .

**Figure 1: Output Gap and Potential Output**



**Figure 2: Equilibrium exchange rate**



**Figure 3: Transitory component**

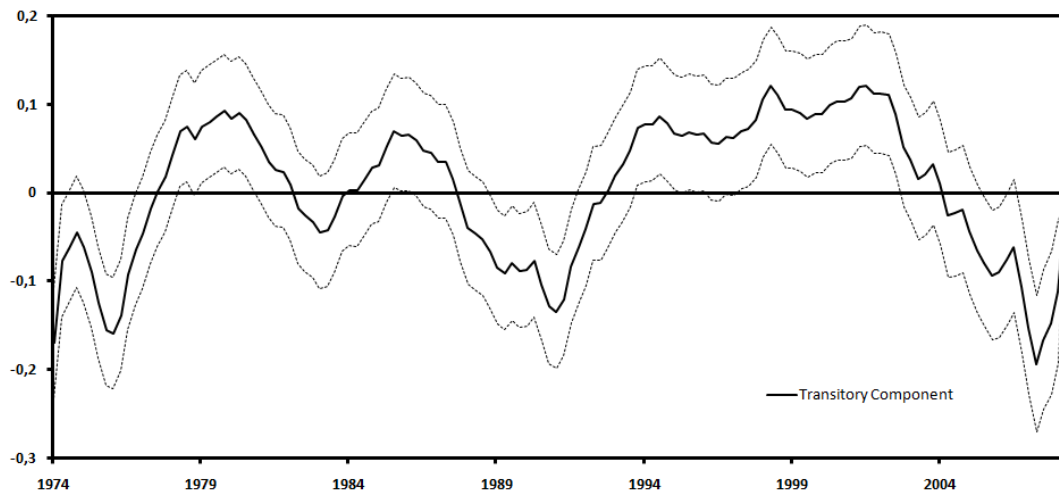


Figure 4: Prior and Posterior parameter distributions

