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Keywords

Stochastic programming, discrete time control problem, decomposition methods, iterative scheme

JEL Codes C61, C63, D81

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In this contribution we propose an approach to solve a multistage stochastic programming problem which allows us to obtain a time and nodal decomposition of the original problem. This double decomposition is achieved applying a discrete time optimal control formulation to the original stochastic programming problem in arborescent form. Combining the arborescent formulation of the problem with the point of view of the optimal control theory naturally gives as a first result the time decomposability of the optimality conditions, which can be organized according to the terminology and structure of a discrete time optimal control problem into the systems of equation for the state and adjoint variables dynamics and the optimality conditions for the generalized Hamiltonian. Moreover these conditions, due to the arborescent formulation of the stochastic programming problem, further decompose with respect to the nodes in the event tree. The optimal solution is obtained by solving small decomposed subproblems and using a mean valued fixed-point iterative scheme to combine them. To enhance the convergence we suggest an optimization step where the weights are chosen in an optimal way at each iteration.

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M.S.C. Classification: 49M27, 90C06, 90C15, 90C90. J.E.L. classification: C61, C63, D81.

1 Introduction

In this paper we propose a solution approach for a multistage stochastic programming problem. The main strength of our approach relies upon a time and nodal decomposition which naturally arises when we apply discrete time optimal control formulation to the multistage stochastic programming problem in its arborescent formulation.

Multistage stochastic optimization problems involve both dynamic and stochastic components. The stochastic component is introduced through a probabilistic specification (Ω, \mathcal{F}, P) of the process ω which affects the decisions strategy in subsequent stages. Hence they are complex to analyze and solve; nevertheless they are interesting since they allow for a great flexibility in modeling, see [35][37] for reviews of various applications.

The objective functions in stochastic programming problem are expected functionals. There are few manageable solution approaches which are mainly obtained by introducing restrictions on the description of the stochastic components.

One of the main approaches to obtain solvable problems is based on the assumption of a discrete and finite probability space $\Omega = \{\omega_1, \ldots, \omega_S\}$ with S, the number of distinct scenarios, of reasonable dimension. A useful representation of the probability space and of the process of arrival of information is given by a scenario tree, see [5]. For the discussion of some issues related to the generation of event trees for multistage stochastic programming see, for example, [10].

Once we have specified the scenario tree we can write the problem in the so-called arborescent formulation. One of the main advantages of this approach is that the original problem can be reformulated as a large-scale deterministic equivalent problem which can be solved relying on mathematical programming techniques and in particular on decomposition techniques suited for large-scale optimization problems. This facilitates the solution of problems with many different sources of uncertainty and dependence structures. There is a trade-off between an accurate specification of the underlying stochastic process, which might require very large value for S, and the need to control the dimension of the overall problem to retain solvability.

Optimization techniques for large-scale stochastic programming problems are typically based on special purpose algorithms which exploit the structure of the problem. There are a number of special decomposition methods which can be approximately classified into two main classes: primal decomposition methods and dual decomposition methods.

In the first case the attention is focused on subproblems related to time stages aiming at a nodal primal decomposition with respect to the scenario tree; see, for example, [4][28][30][33].

For dual approaches the decomposition is with respect to scenarios, that is the stochastic component of the problem, see, for example, [18][25][29]. For a decomposition approach which can be applied to obtain either a primal or dual decomposition see [27].

We propose a new decomposition approach which combine the stochastic programming arborescent formulation with the time decomposition properties of discrete time optimal control problems.

This approach results in a double decomposition with respect to time stages and to nodes in each time stage. The first step is to recast the problem as a discrete time optimal control problem. Through a discrete version of Pontryagin maximum principle we can obtain optimality conditions which lead to time separability. Besides exploiting the arborescent formulation of the stochastic programming problem we obtain a further nodal decomposition within each time stage.

The time decomposition feature brought in by an optimal control formulation and discrete maximum principle optimality conditions was only partially exploited by the authors in two recent contributions [1][3], with application to dynamic portfolio optimization problems. In those contributions the time decomposition was used as a solution method

for scenario subproblems and was not a stand-alone solution approach, but it contributed to enhance the convergence of the Progressive Hedging Algorithm [25] overwhelming some difficulties pointed out in the literature, see, for example, [34].

In this contribution we fully exploit the time decomposition feature combining it with an arborescent formulation of a stochastic programming problem. This method, has already been applied to efficiently solve a particular class of dynamic portfolio problems, see [2]. In this contribution we extend it to a broader class of multistage stochastic programming problem which can be formulated as discrete time stochastic optimal control problem.

2 Multistage stochastic programming problem

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We consider a multistage stochastic programming problem which can be written as (see, for example, [5])

$$\max_{z_1} \left\{ f_1(z_1) + \mathbb{E}_{\xi_1} \left[\max_{z_2} \left(f(z_2) + \ldots + \mathbb{E}_{\xi_T | \xi_1, \ldots, \xi_{T-1}} \left[\max_{z_T} f(z_T) \right] \right) \right] \right\}$$
(1)

$$A_1 z_1 - c_1 \tag{2}$$

$$B_2 z_1 + A_2 z_2 = c_2 \quad a.s. \tag{3}$$

$$B_t z_{t-1} + A_t z_t = c_t \quad a.s. \tag{4}$$

$$B_T z_{T-1} + A_T z_T = c_T \quad a.s. \tag{5}$$

$$z_t \ge 0 \quad t = 1, \dots, T \tag{6}$$

where $\xi_t = \{c_t, A_t, B_t\}$ is a random vector collecting the random elements of time t. The objective function is additive in time with f_i , $i = 1, \ldots, T \in C^2$ and concave; the constraints involve decisions from different stages, as quite usual in stochastic programming formulation. The lagged dynamic structure involves only decisions from two adjacent periods.

If we introduce a probability description for these random quantities and specify a discrete and finite probability distribution we can rewrite the problem in its deterministic equivalent formulation. In more detail we assume a general structure of a scenario tree, see, for example, [10][23]. We denote with k_1 the root node, from which the tree originates, and with $k_t = K_{t-1} + 1, \ldots, K_t$ a node in the tree at time t. For each node k_t there is an unique ancestor $b(k_t)$ at time t-1 and a set of descendants $\{d_i(k_t), i = 1, \ldots, D(k_t)\}$ at time t+1. At the final stage, T, there are $S = K_{T-1}+1, \ldots, K_T$ terminal nodes (leaves). A sequence of nodes connecting the root of the tree with a leaf node is called scenario. There are S scenarios in the tree and the probability of each scenario, π_s , $s = 1, \ldots, S$, is the product of the transition probabilities from a node to its successor in the sequence of nodes in the scenario. Let $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_T$ be such a sequence, then $\pi_s = \prod_{t=1}^T \pi_{\bar{k}_t}$ is the probability of scenario s, where the generic π_{k_t} denotes the probability of moving from node $b(k_t)$ to node k_t .

The multistage stochastic programming problem in its arborescent formulation, i.e. with implicit non-anticipativity constraints (see, for example, [5]) can thus be written as

max

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$$\left\{ f_1(z_{k_1}) + \sum_{k_2=2}^{K_2} \pi_{k_2} f_2(z_{k_2}) + \dots + \sum_{k_{T-1}=K_{T-1}+1}^{K_T} \pi_{k_T} f_T(z_{k_T}) \right\}$$
(7)
$$A_{k_1} z_{k_1} = c_{k_1}$$
(8)

 K_{π}

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$$B_{k_1} z_{b(k_2)} + A_{k_2} z_{k_2} = c_{k_2} \quad k_2 = 2, \dots, K_2$$
(9)

$$B_{k_t} z_{b(k_t)} + A_{k_t} z_{k_t} = c_{k_t} \quad k_t = K_{t-1} + 1, \dots, K_t$$
(10)

$$B_{k_T} z_{b(k_T)} + A_{k_T} z_{k_T} = c_{k_T} \quad k_T = K_{T-1} + 1, \dots, K_T$$
(11)

$$z_{k_t} \ge 0 \quad k_t = K_{t-1} + 1, \dots, K_t, \ t = 1, \dots, T.$$
(12)

The size of the resulting problem can be very large and solution approaches based on traditional optimization algorithms may not be useful; for a survey of applications see, for example, [35]. This has motivated the introduction of a number of solution approaches for stochastic programming problems mainly based on decomposition methods which exploit both the structure of the problems and parallelize the solution algorithm.

These approaches together with a continuous enhancement of computer capabilities facilitate the solution of problems with thousands of scenarios.

3 Formulation as a discrete time optimal control problem

Our aim is to analyze the arborescent formulation of the problem from the point of view of a discrete time optimal control problem. An interesting feature in optimal control theory is the presence of state and control variables, this separation is useful in devising the solution approach based on Pontryagin's Maximum Principle, see [31]. In the following we discuss how we can recast the original multistage stochastic programming problem in such a way to apply discrete time optimal control results.

To move from the arborescent stochastic programming formulation to the discrete time optimal control formulation we need to distinguish between control variables and state variables and to reformulate the constraints to show the dynamics of state variables.

We consider only problems in which this distinction is naturally present in the model. An example is a dynamic portfolio management problem where the amounts of each asset in an investor's portfolio represent the state variables while the purchases and sales, which determine the variations in the state amounts, are the controls. Many problems from financial optimization theory and other fields can be put into this control and state formulation.

The first step for our reformulation is, thus, to "separate" the optimization variables into two subset and let x_{k_t} be the vector of state variables and u_{k_t} the vector of control variables, in node k_t . As a consequence we can partition the vectors of decision variables and the corresponding matrices as follows

$$z_{k_t} = \left(\begin{array}{c} x_{k_t} \\ u_{k_t} \end{array}\right) \qquad z_{k_T} = \left(\begin{array}{c} x_{k_T} \end{array}\right)$$

$$A_{kt} = \begin{pmatrix} A_{kt}^s & A_{kt}^c \end{pmatrix} \qquad B_{kt} = \begin{pmatrix} B_{kt}^s & B_{kt}^c \end{pmatrix}$$

where $A_{k_t}^s B_{k_t}^s$ and $A_{k_t}^c B_{k_t}^c$ denote the sub-matrices related to the state and control variables, respectively.

The general constraints for time t of the stochastic programming problem

$$B_{k_t} z_{b(k_t)} + A_{k_t} z_{k_t} = c_{k_t} \tag{13}$$

$$z_{k_t} \ge 0$$
 $k_t = K_{t-1} + 1, \dots, K_t$ (14)

can now be rewritten as

$$B_{k_t}^s x_{b(k_t)} + B_{k_t}^c u_{b(k_t)} + A_{k_t}^s x_{k_t} + A_{k_t}^c u_{k_t} = c_{k_t}$$
(15)

$$x_{k_t} \ge 0 \quad u_{k_t} \ge 0 \qquad k_t = K_{t-1} + 1, \dots, K_t. \tag{16}$$

Since we want to keep the reformulation from a stochastic programming problem to a discrete time optimal control problem as simple as possible we do not discuss how to treat constraints in which there is a dependence between states and controls from arbitrary different periods. However, many if not most actual applications are covered by our theory since a way to treat these cases is to introduce new state variables and rewrite the problem in order to obtain dependence only among subsequent periods.

The second step for our reformulation is to put on evidence the dynamics of the state variables. In the arborescent formulation of the problem there is an implicit dynamics already determined by the relation of each node with its (unique) ancestor and its descendants. Regarding constraints (13), let $x_{b(k_t)}$ be the state variables, inherited from the ancestor node $b(k_t)$, which enter node k for the period [t - 1, t], and let u_{k_t} be the control variables at time t in node k, then in equation (15) we can see how this implicit dynamics gives rise to a dynamics for the state variables connecting states in two subsequent periods.

Our objective is to rewrite problem (7)-(12) as a discrete time optimal control problem in a form where we distinguish between equality and inequality constraints.

The general form for a discrete time optimal control problem with mixed inequality constraints is (see [31])

$$\max \{\sum_{t=0}^{T-1} F_t(x(t), u(t)) + F_T(x(T))\}$$
(17)

$$x(t+1) = A(t)x(t) + B(t)u(t) + q(t)$$
(18)

$$x(0) = x_0 \tag{19}$$

$$C(t)x(t) + D(t)u(t) + r(t) \ge 0$$
(20)

$$u(t) \ge 0 \tag{21}$$

$$t = 0, \dots, T - 1.$$

From our reformulation of the original stochastic programming problem we have that the vectors of state and control variables at time t are obtained from the collection of state and control vectors of each node at time t and are thus defined as $x(t) = (x_{K_{t-1}+1}, \ldots, x_{K_t})$ and $u(t) = (u_{K_{t-1}+1}, \ldots, u_{K_t})$. Moreover

$$F_t(x(t), u(t)) = \sum_{k_t = K_{t-1} + 1}^{K_t} \pi_{k_t} f_t(z_{k_t})$$
(22)

$$F_T(x(T)) = \sum_{k_T = K_{T-1}+1}^{K_T} \pi_{k_T} f_T(z_{k_T}).$$
(23)

The matrices A(t), B(t), C(t), D(t) and the vectors q(t) and r(t) are obtained rearranging terms from constraints (8)-(11).

Any equality constraints which does not represent a dynamic can be included into (20) with two opposite inequalities. Non negativity conditions (12) include both non negativity constraints on state and control variables. Non negativity constraints on controls are easily tractable, while non negativity conditions on state variables, which are usually more difficult to treat, are transformed by means of the dynamics into mixed state-control inequality constraints and incorporated in (20).

Given this reformulation as a discrete time optimal control problem we can now apply a solution approach, commonly used for this class of problems, which leads to the definition of the Hamiltonian and of the generalized Hamiltonian (see, for example, [31]).

We denote with $\psi(t+1), t = 0, \dots, T-1$ and $\lambda(t), t = 0, \dots, T-1$ the multipliers associated with the dynamics of the state variables and the multipliers associated to the mixed constraints, respectively. The Hamiltonian for each t, with $t = 0, \dots, T-1$, is

$$H(x(t), u(t), \psi(t+1)) = F_t(x(t), u(t)) + \psi(t+1)'[A(t)x(t) + B(t)u(t) + q(t) - x(t+1)]$$
(24)

and the generalized Hamiltonian for each t, with $t = 0, \ldots, T - 1$, is

$$\tilde{H}(x(t), u(t), \psi(t+1), \lambda(t)) = H(x(t), u(t), \psi(t+1)) + \lambda(t)'[C(t)x(t) + D(t)u(t) + r(t)].$$
(25)

Moreover the Lagrangian for the whole problem denoted with $\mathcal{L}(x, u, \psi, \lambda)$ is given by

$$\mathcal{L}(x, u, \psi, \lambda) = \sum_{t=0}^{T-1} F_t(x(t), u(t)) + F_T(x(T)) + \sum_{t=0}^{T-1} \psi(t+1)' [A(t)x(t) + B(t)u(t) + q(t) - x(t+1)] + \psi(0)'(x_0 - x(0)) + \sum_{t=0}^{T-1} \lambda(t)' [C(t)x(t) + D(t)u(t) + r(t)].$$
(26)

In the following we explicit the relation between these quantities

$$\mathcal{L}(x, u, \psi, \lambda) = F_T(x(T)) + \sum_{t=0}^{T-1} H(x(t), u(t), \psi(t+1)) + \sum_{t=0}^{T-1} \lambda(t)' [C(t)x(t) + D(t)u(t) + r(t)] + \psi(0)'(x_0 - x(0)) = F_T(x(T)) + \sum_{t=0}^{T-1} \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t)) + \psi(0)'(x_0 - x(0)).$$
(27)

Using the discrete time optimal control formulation we can obtain a natural decomposition of the problem with respect to time. In particular in the next section we will derive the optimality conditions for this problem and exploit their decomposability features.

4 Time and nodal decomposition

There are several efficient algorithms available to solve special types of the problem (17)-(21). In particular for the unconstrained case and for the control constrained one, see for example [11] and [21]. In the linear quadratic case Rockafellar and Wets [24], and Rockafellar and Zhu [26], reformulated the discrete time optimal control problem as an extended linear quadratic programming problem in order to exploit duality properties and propose solution algorithms based on a Lagrangian approach.

Following [6] and [31] problem (17)-(21) is approached as a mathematical programming problem with equality and inequality constraints (see also [12][19][20]). The optimality conditions for this problem are

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial x(t)} = 0 \tag{28}$$

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial x(T)} = 0 \tag{29}$$

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial \psi(t+1)} = 0 \tag{30}$$

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial \psi(0)} = 0 \tag{31}$$

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial u(t)} \le 0 \tag{32}$$

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial \lambda(t)} \ge 0 \tag{33}$$

$$\lambda(t) \ge 0 \tag{34}$$

$$\begin{aligned} u(t) &\ge 0 \\ \chi(t)' \left[\partial \mathcal{L}(x, u, \psi, \lambda) \right] = 0 \end{aligned} \tag{35}$$

$$\lambda(t)' \left[\frac{\partial \lambda(t)}{\partial \lambda(t)} \right] = 0$$
(36)
$$u(t)' \left[\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial u(t)} \right] = 0$$
(37)

$$u(t)' \left[\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial u(t)} \right] = 0$$

$$t = 0, \dots, T - 1.$$
(3)

Provided that constraints qualification conditions and generalized concavity conditions for the objective function hold, conditions (28)-(37) are then the necessary and sufficient optimality conditions that can be obtained applying a discrete version of Pontryagin Maximum Principle for a discrete time optimal control problem with mixed constraints. To solve conditions (28)-(37) Wright [36] proposed an interior point method, we refer also to [13] for a review of solution approaches for a discrete time optimal control problem with general constraints.

In the following we analyze in more detail these optimality conditions, we present the natural time and nodal decomposition which can be obtained and discuss how these conditions can be reorganized in order to devise an efficient solution scheme.

4.1 Time decomposition

As for the time decomposition feature this is a direct consequence of the reformulation of the original problem as a discrete optimal control problem since all variables related to the same time stage t are collected in vectors x(t) and u(t) and the optimality conditions (28)-(37) must be satisfied for each t = 0, ..., T - 1.

Analyzing conditions in more detail we can see that conditions (28)-(29) become

$$\frac{\partial F_t(x(t), u(t))}{\partial x(t)} + A(t)'\psi(t+1) - \psi(t) + C(t)'\lambda(t) = 0$$
(38)
$$t = 0 \qquad T - 1$$

$$\frac{\partial F_T(x(T))}{\partial x(T)} - \psi(T) = 0.$$
(39)

which can be equivalently written as

$$\psi(t) = \frac{\partial F_t(x(t), u(t))}{\partial x(t)} + A(t)'\psi(t+1) + C(t)'\lambda(t)$$
(40)

$$t = 0, \dots, T - 1$$

$$\psi(T) = \frac{\partial F_T(x(T))}{\partial x(T)}.$$
(41)

and give the dynamics of the adjoint state variables $\psi(t+1)$, $t = 0, \ldots, T-1$, in the terminology used for optimal control problems.

In the same way conditions (30)-(31) give the dynamics of the state variables x(t)

$$x(t+1) = A(t)x(t) + B(t)u(t) + q(t)$$

$$t = 0, \dots, T-1$$
(42)

$$x(0) = x_0.$$
 (43)

Given that

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial u(t)} = \frac{\partial H(x(t), u(t), \psi(t+1))}{\partial u(t)} + D(t)'\lambda(t)$$
$$= \frac{\partial \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))}{\partial u(t)}$$
(44)

$$\frac{\partial \mathcal{L}(x, u, \psi, \lambda)}{\partial \lambda(t)} = \frac{\partial \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))}{\partial \lambda(t)}$$

$$t = 0, \dots, T-1$$
(45)

conditions (32)-(37), for each t, t = 0, ..., T - 1, are the necessary and sufficient conditions for the following problem

$$\max_{u(t)} H(x(t), u(t), \psi(t+1))$$
(46)

$$C(t)x(t) + D(t)u(t) + r(t) \ge 0$$
(47)

$$u(t) \ge 0. \tag{48}$$

If we consider the problem in terms of the generalized Hamiltonian we have that (32)-(37) are the necessary and sufficient conditions for a saddle point of the generalized Hamiltonian $\tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))$ in the controls $u(t), t = 0, \ldots, T-1$ and in the multipliers associated with the mixed constraints $\lambda(t), t = 0, \ldots, T-1$.

For sake of clarity in the following we recall the optimality conditions written in terms of generalized Hamiltonian and present a possible way of dealing with them.

The optimality conditions for problem (46)-(48) are

$$\frac{\partial H(x(t), u(t), \psi(t+1), \lambda(t))}{\partial u(t)} \le 0$$
(49)

$$u(t)' \left[\frac{\partial \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))}{\partial u(t)} \right] = 0$$
(50)

$$u(t) \ge 0 \tag{51}$$

$$\frac{\partial H(x(t), u(t), \psi(t+1), \lambda(t))}{\partial \lambda(t)} \ge 0$$
(52)

$$\lambda(t)' \left[\frac{\partial \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))}{\partial \lambda(t)} \right] = 0$$
(53)

$$\lambda(t) \ge 0 \tag{54}$$

$$t = 0, \ldots, T - 1$$

where $\lambda(t)$ are the multipliers associated with the inequality constraints.

For each t, the set of conditions (49)-(54) is equivalent to the optimality conditions of the following pair of optimization problems

$$\max_{u(t)} \quad \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))$$
(55)

$$\frac{\partial H(x(t), u(t), \psi(t+1), \lambda(t))}{\partial \lambda(t)} \ge 0$$
(56)

$$u(t) \ge 0 \tag{57}$$

$$\min_{\lambda(t)} \quad \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t)) \tag{58}$$

$$\frac{\partial H(x(t), u(t), \psi(t+1), \lambda(t))}{\partial u(t)} \le 0$$
(59)

$$\lambda(t) \ge 0 \tag{60}$$

Following Intriligator [15] for the solution of the system of conditions (49)-(54) we suggest to solve, for each t, problems (55)-(60).

Exploiting this reorganization of the optimality conditions for the discrete time optimal control problem for each $t = 0, \ldots, T-1$ we obtain four subproblems. Two of them represent

a two-point boundary value problem while the other two are optimization problems. In more detail the optimality conditions (28)-(37) can be rewritten as follow

$$x(t+1) = A(t)x(t) + B(t)u(t) + q(t)$$
(61)

$$x(0) = x_0 \tag{62}$$

$$\psi(t) = A(t)'\psi(t+1) - \frac{\partial L_t(x(t), u(t))}{\partial x(t)} - C(t)'\lambda(t)$$
(63)

$$\psi(T) = \frac{\partial L_T(x(T))}{\partial x(T)} \tag{64}$$

$$\max_{u(t)} \qquad \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t)) \tag{65}$$

$$\frac{\partial \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))}{\partial \lambda(t)} \ge 0$$
(66)

$$u(t) \ge 0 \tag{67}$$

$$\min_{\lambda(t)} \qquad \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t)) \tag{68}$$

$$\frac{\partial \tilde{H}(x(t), u(t), \psi(t+1), \lambda(t))}{\partial u(t)} \le 0$$
(69)

$$\lambda(t) \ge 0 \tag{70}$$
$$t = 0 \qquad T - 1$$

Using these reformulation we have obtained a time decomposition of the optimality conditions and a reorganization of them into subproblems that are easier and less memory demanding to solve.

4.2 Nodal decomposition

A further interesting result can be obtained if we reintroduce the arborescent notation from the stochastic programming problem formulation. To this aim, we recall that for each tthe vectors of state and control variables are obtained from the collection of corresponding variables of nodes at time t and, in particular, that we have $x(t) = (x_{K_{t-1}+1}, \ldots, x_{K_t})$ and $u(t) = (u_{K_{t-1}+1}, \ldots, u_{K_t})$. This allow us to obtain the decomposability of previous conditions which result to be separable with respect to the nodes in the scenario tree. In more detail, we obtain that conditions (61)-(70), for all t, can be written in a nodal form as follows

$$x_{k_t}(t+1) = \tilde{A}_{k_t}(t)x_{b(k_t)}(t) + \tilde{B}_{k_t}(t)u_{k_t}(t) + \tilde{q}_{k_t}$$
(71)

$$x_{b(k_1)}(0) = x_0 \tag{72}$$

$$\psi_{k_t}(t) = \tilde{A}_{k_t}(t)'\psi_j(t+1) - \pi_{k_t}\frac{\partial f_t(x_{k_t}(t), u_{k_t}(t))}{\partial x_{k_t}(t)} - \tilde{C}_{k_t}(t)'\lambda_{k_t}(t)$$
(73)

$$\psi_{k_T}(T) = \pi_{k_T} \frac{\partial f_T(x_{k_T}(T))}{\partial x_{k_T}(T)}$$
(74)

$$\max_{u_{k_t}(t)} \{\tilde{H}_{k_t}(x_{k_t}(t), u_{k_t}(t), \psi_{d(k_t)}(t+1), \lambda_{k_t}(t))\}$$
(75)

$$\frac{\partial \tilde{H}_{k_t}(x_{k_t}(t), u_{k_t}(t), \psi_{d(k_t)}(t+1), \lambda_{k_t}(t))}{\partial \lambda_{k_t}(t)} \ge 0$$
(76)

$$u_{k_t}(t) \ge 0 \tag{77}$$

$$\min_{\lambda_{k_t}(t)} \quad \{\tilde{H}_{k_t}(x_{k_t}(t), u_{k_t}(t), \psi_{d(k_t)}(t+1), \lambda_{k_t}(t))\}$$
(78)

$$\frac{\partial H_{k_t}(x_{k_t}(t), u_{k_t}(t), \psi_{d(k_t)}(t+1), \lambda_{k_t}(t))}{\partial u_{k_t}(t)} \le 0$$
(79)

$$A_{k_t}(t) \ge 0$$

$$k_t = 1, \dots, K_T \quad t = 0, \dots, T - 1.$$
(80)

The matrices
$$\tilde{A}$$
, \tilde{B} , \tilde{C} are obtained by rearranging from the correspondent matrices A , B and C .

This further decomposability feature is really interesting and allows us to obtain smaller subproblems easier to be solved.

Thus, our approach which combines discrete time optimal control and stochastic programming is able to exploit features from both approaches obtaining a time decomposability from the optimality conditions of the optimal control formulation and further nodal decomposition from arborescent formulation.

This approach has already been applied in the case of dynamic portfolio management problems with implicit non-anticipativity constraints (see [2]), which represent special cases of the general problem treated in this contribution. In this case, conditions (75) and (78) simplify considerably.

For a different approach which uses the time decomposition, in the framework of Progressive Hedging Algorithm [25], without nodal decomposition see [1][3].

5 Iterative solution scheme

Optimality conditions given in (61)-(70) can be expressed as a system of linear and nonlinear equations with nonnegativity constraints. This problem can be tackled using different solutions approaches. The most widely used solution approaches proposed in the literature

rely on Newton's method or successive modifications of it. One of the issue which arise in the solution of this kind of system is related to the presence of nonnegativity constraints. Interior point methods introduce barriers to avoid the non negativity constraints in the original problem and then deal with the solution of a saddle point systems at each Newton iteration. Primal-dual methods cope with the nonnegativity constraints modifying the search direction and the step length as that the non negativity constraints are satisfied strictly at every iteration. The iterative solution approach, we propose, can easily handle the nonnegativity constraints at each iteration as a part of an optimization problem, this guarantees us that each iteration is feasible, since it satisfies all the constraints.

In this contribution we want to take advantage of the decomposability features induced by the optimal control formulation and thus, to solve problem (17)-(21) we propose to solve the smaller subproblems obtained in conditions (71)-(80) for t = 0, ..., T - 1 and $k_t = 1, ..., K_T$ and we suggest an iterative method based on a fixed-point scheme to aggregate the solutions.

In more detail, the solution of the four subproblems, for each $k_t = 1, \ldots, K_T$ and each $t = 0, \ldots, T - 1$, represents an iteration of the fixed point problem that we can summarize as follows

$$y_{new} = MP4(y)$$

where $y = (x(1), \ldots, x(T))$ is the vector collecting the state variables and MP4 represents the transformation given by the four subproblems.

We suggest a solution approach based on a mean value iterative fixed-point scheme. This approach, originally proposed by Mann [16][17], is obtained applying a sequence transformation to a standard fixed point iterative scheme. We refer to [8] for a discussion on sequence transformations built specifically for accelerating fixed point iterations in the linear and nonlinear cases.

We cast the problem in such a way that we can solve iteratively the four subproblems and then aggregate the solutions using the iterative point scheme to obtain convergence to the solution of the original whole problem. In more detail, solutions obtained from these subproblems are aggregated according to the mean value iteration method of Mann for which an extensive literature exists (see [16] [17] [7] [8] [9] [14] [22]).

Different methods can be designed considering as starting point different group of variables involved in the optimization process. In the following we propose an iterative procedure which starts from a feasible solution for the controls u(t), t = 0, ..., T - 1 and the states x(t), t = 1, ..., T and has two main steps.

The first is a descending step of backward type from t = T - 1 to t = 0 and allows us to calculate the values of the adjoint variables $\psi(t+1)$ and of the multipliers $\lambda(t)$ solving, for each $k_t = 1, \ldots, K_T$, problems (73)-(74) and (78)-(80).

The second is an *ascending step* that allows us to calculate the values of the state and control variables in a *forward direction* from t = 0 to t = T, solving, for each $k_t = 1, \ldots, K_T$, problems (71)-(72) and (75)-(77).

Accordingly to this approach at each step of the algorithm we consider a weighted average of the solutions found in previous steps. Denote by w^{ν} the weighted average of

optimal solutions up to iteration ν and $w^0 = y^0$. The mean value iteration scheme proposed by Mann is defined as

$$y^{\nu+1} = F(w^{\nu})$$
 (81)

$$w^{\nu+1} = \sum_{i=1}^{\nu+1} \delta_i y^i$$
 (82)

where δ_i , $i = 1, \ldots, \nu + 1$ satisfy

$$\delta_i \ge 0 \quad \forall i \tag{83}$$

$$\sum_{i=1}^{\nu+1} \delta_i = 1. \tag{84}$$

The weights δ_i can be chosen in different ways. A first method, proposed in the work of Mann, applies as weighting matrix the Cesáro matrix (see [16]) which yields the arithmetic mean. This iterative scheme in our experiments converges but is rather slow.

We now analyze an approach to improve the convergence speed. We introduce an optimization step which allows us to choose the weights in an optimal way with respect to the objective function of the original problem. We tested two different methods.

In the first method we choose the best new point $w^{\nu+1}$ as follows

$$w^{\nu+1} = \delta^* w^{\nu} + (1 - \delta^*) y^{\nu+1}$$

where $y^{\nu+1}$ is given by (81) and the weight δ^* is determined as the solution of the optimization problem $\max_{\delta} f(\delta)$ with $0 \le \delta \le 1$ and f denotes the objective function of the original problem in (17) expressed as a function of δ . This approach, which is faster than the Cesáro method, still has rather slow convergence.

In the second method we take into account the whole region obtained as a convex combination of all y^{ν} obtained in the previous iterations and the optimal weights $\delta = (\delta_1, \ldots, \delta_{\nu})$ are obtained as solution of the optimization problem

$$\max_{\delta} \quad f(\delta) \tag{85}$$

$$\sum_{i=1}^{\nu} \delta_i = 1 \tag{86}$$

$$\delta_i \ge 0 \quad \forall i = 1, \dots, \nu. \tag{87}$$

where f is the objective function (17) written as a function of δ .

At each step of the iterative scheme we do not fix a priori the weights, δ_i , as in the Cesàro matrix, but we look for the best choice of the coefficients solving an optimization problem. This considerably improves the convergence rate. Nevertheless in this method the dimension of the vector of weights increases with the number of iterations.

We now briefly present the structure of the solution algorithm. Other approaches can be applied considering different starting points or solving subproblems according to a different sequence.

1. Set $\nu = 0$, $u(t) = \bar{u}$, $\forall t = 0, ..., T - 1$, with \bar{u} globally admissible. Given the initial value $x(0) = x_0$, obtain the starting values $\bar{x}(t)$, t = 1, ..., T from conditions (71)-(72). Set $y^0 = (\bar{x}(1), ..., \bar{x}(T))$ and $w^0 = y^0$.

2. Using the sequence for x(t), u(t) obtained from previous step we recursively compute a sequence for $\psi(t)$ and $\lambda(t)$, from conditions (78)-(80) and (73)-(74) starting from t = T - 1 backward to t = 0. That is we first solve the optimization problem (78)-(80) for t = T - 1 and thus we obtain the optimal value for $\lambda(T-1)$ together with the value $\psi(T)$ from (74). We use these values in (73) to compute optima $\psi(T-1)$. We solve conditions (78)-(80) and (73) backward up to t = 0.

3. Set $\nu = \nu + 1$. Using values from previous iteration obtain new values for $u(t), t = 0, \ldots, T - 1$ and $x(t), t = 1, \ldots, T$ solving problem (75)-(77) and conditions (71)-(72) recursively in a forward direction. Set $y^{\nu} = (x(1), \ldots, x(T))$.

4. Compute the weighted average $w^{\nu} = \sum_{i=1}^{\nu} \delta_i y^i$ choosing the weights $\delta = (\delta_1, \ldots, \delta_{\nu})$ in an optimal way as solutions of (85)-(87). Update $x(t), t = 1, \ldots, T$ setting $(x(1), \ldots, x(T)) = w^{\nu}$.

5. If $||w^{\nu} - w^{\nu-1}|| \leq \epsilon_1$ and/or $||f^{\nu} - f^{\nu-1}|| \leq \epsilon_2$ stop, otherwise go to step 2.

If the sequence w^{ν} is convergent to w^* and the stopping criteria in step 5 are satisfied, then we can consider the values of x(t) and u(t) computed in the last iteration as solution for the necessary and sufficient optimality conditions for the four subproblems, i.d. for the optimality conditions of the original problem, and thus they are a solution for our problem.

A sufficient condition to guarantee the convergence of the sequence w^{ν} is that the transformation MP4 is Lipschtzian, see [32]. This condition depends on the particular problem at hand. Many different conditions, to be verified in each case, can be set to guarantee the convergence, see, for example, the papers which extend the mean value iterative method of Mann ([7] [8] [9] [14] [22]). In our experiments on a dynamic portfolio management problem, see [1], the iterative scheme converges.

In the following we present an example of the computational advantage we can obtain applying our double decomposition approach to a dynamic portfolio management problem.

The calculation are drawn from an example presented in detail in [2]. We have considered a set of test problems with increasing number of scenarios and risky assets using the data from the Italian stock market. In figures 1 and 2 we present the convergence behavior of a pair of components of the vector w, the same for all cases. We note that there is a significant improvement in the convergence when we introduce the optimization of the weights with respect to the standard Cesàro weights.

For more detailed results and computation time and for a comparisons with other solution approaches for the considered portfolio problem we refer to [2]



Figure 1: Example of convergence behavior for (w_i, w_j) , two components of the vector w, along the iterations with Cesàro matrix.



Figure 2: Example of convergence behavior for (w_i, w_j) , two components of the vector w, along the iterations with optimized weights.

6 Conclusion

We proposed a time and nodal decomposition approach to solve a wide class of multistage stochastic programming problems.

The method, starting from a multistage stochastic programming problem in arborescent formulation, reformulates the problem as a discrete time optimal control problem and applies a discrete version of Pontryagin's maximum principle to obtain the necessary conditions for optimality.

These steps decompose the problem with respect to time and nodes obtaining small subproblems to be solved. To aggregate the solutions of the subproblems, which are optimal only with respect to each subproblem in each iteration, and to converge to the optimal solution of the original problem an iterative scheme is introduced. In this step it is important to improve the speed of convergence. To this aim we propose to introduce a further optimization step choosing in each iteration the optimal weights to combine solutions from previous iterations. This can be done in different ways, in this contribution we suggest two methods.

The results obtained for some test problems show that the proposed decomposition method with the further optimization step over the convex region obtained from previous iterations, can solve very large optimization problems reaching the optimal solutions in a very efficient way requiring a relative low computational time and a small number of iterations.

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