Dynamic Hotelling Duopoly with Linear Transportation Costs[1]

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Abstract

I investigate a spatial duopoly model with linear transportation costs as a di¤erential game where product di¤erentiation is the result of …rms' R&D investments. Two related results obtain, i.e., (i) the steady state R&D investment (product di¤erentiation) is negatively (positively) related to the cost of capital and time discounting; and (ii) if time discounting and the cost of capital are su¢ ciently high, the amount of di¤erentiation observed in steady state is su¢ ciently large to ensure the existence of a unique pure-strategy price equilibrium with prices above marginal cost.

J.E.L. Classi…cation: L13, O31

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1 Introduction

Ever since Hotelling's (1929) seminal contribution, the role of product differentiation as a remedy to the fragility of market equilibrium under price competition has represented a core issue in the field of industrial organization.

However, under horizontal product differentiation, an established result is that a pure-strategy equilibrium in prices may not always exist (see, inter alia, d'Aspremont et al., 1979; Gabszewicz and Thisse, 1986; Economides, 1986; Anderson, 1988). More precisely, a subgame perfect equilibrium with prices greater than marginal cost may fail to exist, because firms' location choices drive prices to marginal cost when transportation costs are linear (or not sufficiently convex) in the distance between the generic consumer and the firm he decides to patronise. This non-existence problem has generated a stream of literature proposing several remedies, either by adopting non-linear transportation cost functions (d'Aspremont et al., 1979; Stahl, 1982; Economides, 1986) or by adopting the Stackelberg equilibrium as the solution concept (Anderson, 1987), or by choosing appropriate distribution functions for the population of consumers (de Palma et al., 1985; Neven, 1986), or a mix thereof (Tabuchi and Thisse, 1995; Lambertini, 1997, 2000).[1]

In this paper, I illustrate an alternative route, which consists in nesting Hotelling's linear transportation cost problem into a differential game with R&D for product innovation. This entails that location is no longer a control variable, since product design is the outcome of firms' intertemporal R&D

[1] For exhaustive accounts of the debate, see Caplin and Nalebuff (1991); Anderson et al. (1992); Anderson et al. (1997).

efforts, i.e., investment is the control variable while location (and therefore product differentiation) becomes a state variable which varies over time. I show that there are conditions on time preferences (and the cost of capital) such that firms choose long run equilibrium locations where there exists no undercutting incentive, and therefore a price equilibrium does exist with prices above marginal production costs in correspondence to a steady state degree of differentiation which is not minimum. This result is derived under both open and closed loop solutions. There emerges that the range of time discounting (or the rental price of capital) wherein the game produces a price equilibrium in pure strategies is wider under the closed loop solution than under the open loop solution.

The reminder of the paper is structured as follows. The basic model and the non-existence problem are introduced in section 2. Section 3 describes the differential game. Section 4 contains concluding remarks.

2 The basic setup

Examine first the static problem, as originally formulated by Hotelling (1929). I consider a market for horizontally differentiated products where consumers are uniformly distributed with unit density along the unit interval $[0,1]$, the linear city. Two single-product profit-maximising firms, labelled as 1 and 2, sell a differentiated good along the segment. Product locations are x_1 and x_2. On the basis of the symmetry of the model, I assume that $x_1 \leq 1/2$ and $x_2 \geq 1/2$.[2] The generic consumer located at $a \in [0,1]$ buys one unit of the

[2] This assumption that firm 1 (respectively, 2) is located to the left (right) of 1/2 is meant to exclude the possibility of leapfrogging by either firm. As in Tabuchi and Thisse

good to maximise his utility:

$$U = s - p_i - c|x_i - a| \geq 0; \qquad i = 1; 2; \tag{1}$$

where x_i and p_i are firm's i location and mill price, respectively, and $c > 0$ is the transportation cost rate. In the remainder of the paper, I suppose that the reservation price s is never binding, so that full market coverage always obtains. One can easily derive from (1) the location $\hat{a} \in (x_1; x_2)$[3] of the consumer who is indifferent between the two goods at generic price and location pairs,

$$s - p_1 - c(\hat{a} - x_1) = s - p_2 - c(x_2 - \hat{a}); \tag{2}$$

as well as the demand functions:

$$y_1 = \frac{p_2 - p_1 + c(x_1 + x_2)}{2c}; \; y_2 = \frac{p_1 - p_2 + c(2 - x_1 - x_2)}{2c}: \tag{3}$$

Unit production cost is assumed to be constant and equal across varieties. Without further loss of generality, I normalise it to zero. Therefore, firm i's profit function is $\pi_i = p_i y_i$:

Firms play noncooperatively a two-stage game where they move simultaneously at both stages. In the first, firms choose locations, in the second they choose prices. The solution concept is the subgame perfect equilibrium by backward induction.

(1995) and Lambertini (1997), firms are allowed to locate also outside the city boundaries. This assumption is discussed in section 3.

[3] If this condition is not met, e.g., if the indifference condition is written under the assumption that $a \in (x_2; 1]$; then it can be immediately verified that the location of the indifferent consumer is undefined.

First order conditions (FOCs) at the market stage yield the following candidate equilibrium prices:

$$p_1^* = \frac{c\,(2 + x_1 + x_2)}{3}\,;\; p_2^* = \frac{c\,(4 - x_1 - x_2)}{3}\,. \tag{4}$$

The above prices are strictly positive for all $x_1 \in [0; 1/2]$ and $x_2 \in [1/2; 1]$. However, as proved by d'Aspremont et al. (1979), if firms locate in $[1/4; 3/4]$, there exists an incentive for each of them to undercut the rival's price by setting:

$$p_i^u = p_j - c\,|x_i - x_j|\,,\; \text{for all } p_j > 0. \tag{5}$$

That is, demand functions are discontinuous at p_i^u, since at that price, given any p_j, firm i becomes a monopolist and firm j is driven out of business. Therefore, the price pair (4) cannot be an equilibrium outcome for all locations. Moreover, at the first stage we have:

$$\frac{\partial \pi_1}{\partial x_1} > 0\,;\; \frac{\partial \pi_2}{\partial x_2} < 0 \tag{6}$$

for all admissible $\{x_1; x_2\}$, which entails that firms are lead towards the midpoint by profit incentives at the first stage of the game. That is, the choice of location drives firms precisely into the segment where the pure-strategy equilibrium with prices above marginal production cost (i.e., in this setting, with positive prices) fails to exist.[4] By altering the transportation cost function from linear to quadratic, d'Aspremont et al. (1979) obtain a tractable model where a unique pure-strategy price equilibrium exists for all location pairs. In particular, the adoption of quadratic disutility of transportation

[4] The price equilibrium always exists in mixed strategies (see Dasgupta and Maskin, 1986; Osborne and Pitchik, 1987).

eliminates the incentive to undercut, and equilibrium prices are zero if and only if differentiation is nil.

3 The differential game

The non-existence problem described above stems from the fact that, with linear transportation costs, the incentive to differentiate products is weaker than the incentive to move towards the middle of the market so as to increase the demand basin for a product. That is, firms are attracted by the median (and average) consumer. This creates a price war through undercutting, that drives equilibrium prices and profits to zero. A possible remedy to this problem consists in making it costly for firms to design their respective products according to the preferences of the median consumer. This is what I propose here.

The instantaneous profit is $\pi_i(t) = p_i(t)y_i(t) - \rho k_i(t)$, where $k_i(t)$ is the amount of resources invested in R&D by firm i at time t, and ρ is the rental price of capital, which in the remainder of the paper is assumed to be equal to the discount rate common to both firms. By symmetry, I focus upon the behaviour of firm 2. She can modify the location of her product through R&D investment according to:

$$\frac{\partial x_2(t)}{\partial t} = -\frac{k_2(t)}{1 + k_2(t)} \cdot x_2(t) \; ; \quad k_2(t) \geq 0 \; ; \; x_2(0) \geq 1 . \tag{7}$$

Notice that the condition $x_2(0) \geq 1$ potentially allows for any degree of differentiation to emerge at the long-run equilibrium. The R&D technology defined by (7) exhibits decreasing returns to scale.[5] The rationale behind (7)

[5] See Cellini and Lambertini (1999) for further discussion of (7).

is that the R&D technology is shaped so as to reproduce the tendency for firms to locate towards the middle of the linear city, which characterises the static model described in section 2.[6]

Firm 2 aims at maximizing the discounted value of her flow of profits $J_2 = \int_0^{\infty} e^{-\rho t}\pi_2(t)\,dt$ under the dynamic constraint (7) concerning the state variable $x_2(t)$. The control variables are $p_i(t)$ and $k_i(t)$. It is worth stressing that the undercutting incentive still exists within the second and third quartiles of the linear city. Yet, the fact that location $x_i(t)$ is no longer a control variable in the dynamic formulation of the duopoly game opens the possibility for the firms not to be affected by the undercutting problem in steady state.

The Hamiltonian function is:

$$H_2(t) = e^{-\rho t} \cdot \left\{ \frac{p_2\left[p_1 - p_2 + c(2 - x_1 - x_2)\right]}{2c} - \rho k_2(t) - \lambda_2(t)\frac{k_2(t)x_2(t)}{1 + k_2(t)} \right\}; \quad (8)$$

where $\lambda_2(t) = \mu_2(t)e^{\rho t}$; $\mu_2(t)$ being the co-state variable associated to $x_2(t)$:

[6] The relevant difference is that, in the present setting, relocation towards 1/2 is costly. Alternatively, one could examine a technology such that

$$\frac{\partial x_1(t)}{\partial t} < 0 \text{ and } \frac{\partial x_2(t)}{\partial t} > 0;$$

pulling firms outwards as time goes by. This, however, would appear as a rather ad hoc assumption to the aim of preventing firms from entering the product range where the undercutting incentive destroys the pure-strategy price equilibrium.

3.1 The open loop solution

In the open loop formulation of the game, the necessary and sufficient conditions for a path to be optimal are:[7]

$$\frac{\partial H_2(t)}{\partial p_2(t)} = \frac{p_1(t) - 2p_2(t) + c\,[2 - x_1(t) - x_2(t)]}{2c} = 0\,; \tag{9}$$

$$\frac{\partial H_2(t)}{\partial k_2(t)} = -\frac{\rho\,[1 + k_2(t)]^2 + \lambda_2(t)x_2(t)}{[1 + k_2(t)]^2} = 0\,; \tag{10}$$

$$-\frac{\partial H_2(t)}{\partial x_2(t)} = \frac{\partial \mu_2(t)}{\partial t} \Rightarrow \frac{\partial \lambda_2(t)}{\partial t} = \lambda_2(t)\left(\frac{k_2(t)}{1 + k_2(t)} + \rho\right) + \frac{p_2(t)}{2}\,; \tag{11}$$

$$\lim_{t \to \infty} \mu_2(t) \cdot x_2(t) = 0\,. \tag{12}$$

From the FOCs w.r.t. prices I obtain:

$$p_1^*(t) = \frac{c\,[2 + x_1(t) + x_2(t)]}{3}\,;\; p_2^*(t) = \frac{c\,[4 - x_1(t) - x_2(t)]}{3}\,; \tag{13}$$

which coincide with the optimal prices (4) characterising the static game.

From (10) I obtain:

$$\lambda_2(t) = -\frac{\rho\,[1 + k_2(t)]^2}{x_2(t)} \tag{14}$$

as well as

$$k_2(t) = \sqrt{\frac{-\lambda_2(t)x_2(t)}{\rho}} - 1\,; \tag{15}$$

which allows me to establish:

$$\frac{\partial k_2(t)}{\partial t} \propto -\frac{\partial \lambda_2(t)}{\partial t}x_2(t) - \frac{\partial x_2(t)}{\partial t}\lambda_2(t)\,. \tag{16}$$

[7] The FOC w.r.t. price for firm 1 is:

$$\frac{\partial H_1(t)}{\partial p_1(t)} = \frac{p_2(t) - 2p_1(t) + c\,[x_1(t) + x_2(t)]}{2c} = 0$$

Substituting (13) and (14) into (16), and using the symmetry condition $x_1(t) = 1 - x_2(t)$; I can rewrite (16) as follows:

$$\frac{\partial k_2(t)}{\partial t} = \rho^2 \left[1 + k_2(t)\right]^2 - \frac{c}{2} x_2(t) : \tag{17}$$

The expression on the r.h.s. of (17) is equal to zero at:[8]

$$k_2^*(t) = -1 + \frac{1}{\rho}\sqrt{\frac{c}{2} x_2(t)} ; \tag{18}$$

with

$$k_2^*(t) > 0 \text{ for all } \rho < \sqrt{\frac{c}{2} x_2(t)} \tag{19}$$

$$\frac{\partial k_2(t)}{\partial t} > 0 \text{ for all } k_2(t) > k_2^*(t) \tag{20}$$

$$\frac{\partial k_2(t)}{\partial t} < 0 \text{ for all } k_2(t) \in (0 ; k_2^*(t)) : \tag{21}$$

Obviously, $\frac{\partial x_2(t)}{\partial t} > 0$ always. Expression (18) immediately yields the following intuitive result:

Lemma 1 The open loop steady state R&D investment is decreasing in the cost of capital and in intertemporal discounting.

Now observe that

$$k_2^*(t) = 0 \text{ iff } x_2(t) = x_2^* = \frac{2}{c}\rho^2 \tag{22}$$

where

$$\frac{2}{c}\rho^2 \in \left(\frac{3}{4} ; 1\right) \text{ for all } \rho \in \left(\sqrt{\frac{3c}{8}} ; \sqrt{\frac{c}{2}}\right) : \tag{23}$$

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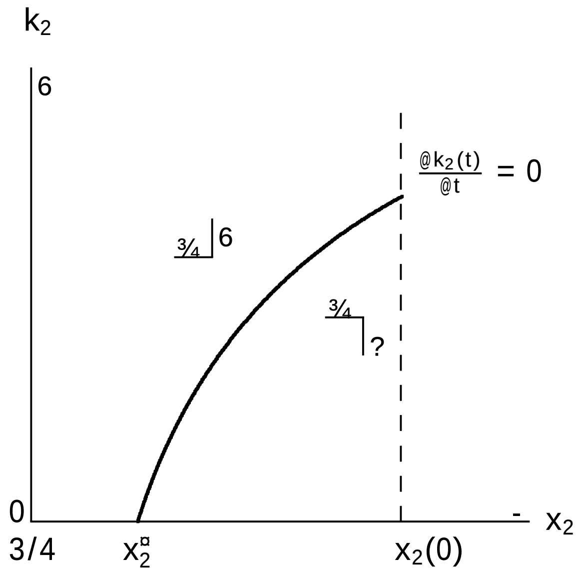
[8] The smaller root can be disregarded as it is always negative.

The phase diagram is illustrated in figure 1, where I describe a situation in which $x_2(0) > 3/4$ and $\rho \in \left(\sqrt{\frac{3c}{8}}; \sqrt{\frac{c}{2}}\right)$.[9]

Figure 1 : Dynamics in the space $(x_2; k_2)$



Considering the stability of the system, it remains to be stressed that, whenever $x_2^* > 3/4$; it is a saddle.[10] The above discussion can be summarised in the following:

Proposition 1 For all $\rho > \sqrt{\frac{3c}{8}}$; the open loop game reaches a steady state

[9] Whether $x_2(0)$ (respectively, $x_1(0)$) is larger or smaller than 1 (resp., 0) is irrelevant as to the graphical representation of the problem, as long as $x_2(0)$ (resp., $x_1(0)$) is larger (lower) than 3/4 (1/4).

[10] The complete proof is omitted, since this property immediately results from the dynamics of $x_2(t)$ and $k_2(t)$ as described by horizontal and vertical arrows in figure 1.

at

$$k_2^* = 0\ ;\ x_2^* = \frac{2}{c}\rho^2 > \frac{3}{4}$$

which is a saddle, where there exists no undercutting incentive.

The above Proposition produces the following Corollaries:

Corollary 1 The steady state degree of product differentiation is positively related with the cost of capital and time discounting.

Corollary 2 Given $x_2(0) \geq 1$; maximum differentiation obtains in steady state if $\rho = \sqrt{\frac{c}{2}}$.

The above property highlights that the dynamic model, where intertemporal investment is the relevant control variable, is intrinsically different from its static counterpart, where the first order conditions w.r.t. locations generate the well known minimum differentiation principle. When product design becomes costly, then the presence of linear disutility of transportation does not necessarily induce firms to seek for the product preferred by the median (and average) consumer, which in turn triggers the undercutting process. This, obviously, happens when the parameter measuring the rental prices of capital as well as time discounting is sufficiently low to drive firms within the second and third quartiles of the linear city. Then, as a final result, we have the following:

Corollary 3 For all $\rho > \sqrt{\frac{3c}{8}}$; the open loop solution of the game produces a unique price equilibrium in pure strategies, with strictly positive prices.

As a complement to Corollary 3, it is worth observing that the undercutting incentive still operates when $x_1^* \in [1/4\,; 1/2]$ and $x_2^* \in [1/2\,; 3/4]\,;$ which happens for all $\rho < \sqrt{\frac{3c}{8}}:$ That is, when the cost of capital and time discounting are sufficiently low, firms are driven into the region where the price equilibrium in pure strategies does not exist.

3.2 The closed loop solution

The characterisation of the Markov (subgame) perfect equilibrium (MPE) under the closed loop solution usually requires solving the relevant Bellman equation.[11] Given that the Hamiltonian problem defined in (8) is not written in a linear-quadratic form, the Bellman - Hamilton - Jacobi sufficient conditions for a MPE cannot be solved. However, I am going to show that, in the present case, the necessary conditions of the closed loop formulation suffice to characterise the MPE.

First order conditions are (9), (10), (12) and

$$-\frac{\partial H_2(t)}{\partial x_2(t)} - \frac{\partial H_2(t)}{\partial p_1(t)} \cdot \frac{\partial p_1^*(t)}{\partial x_1(t)} = \frac{\partial \mu_2(t)}{\partial t}\,; \tag{24}$$

where the term $\frac{\partial H_2(t)}{\partial p_1(t)} \cdot \frac{\partial p_1^{br}(t)}{\partial x_1(t)} = \frac{p_2(t)}{4}$ describes the feedback effect which does not appear in the open loop formulation.[12] The derivative $\frac{\partial p_1^{br}(t)}{\partial x_1(t)}$ is

[11] See Başar and Olsder (1982, 1995^2), Mehlmann (1988), Fudenberg and Tirole (1991, pp. 520-36), Vives (1999, pp. 336-47), inter alia.

[12] Notice that

$$-\frac{\partial H_2(t)}{\partial k_1(t)} \cdot \frac{\partial k_1(t)}{\partial x_1(t)}$$

does not appear in (24), in that $\frac{\partial H_2(t)}{\partial k_1(t)} = 0:$

calculated on the basis of firm 1's best reply function in the price space,

$$p_1^{br}(t) = \frac{p_2(t) + c\,(x_1 + x_2)}{2}; \tag{25}$$

which is the solution to:

$$\frac{\partial H_1(t)}{\partial p_1(t)} = \frac{\partial}{\partial p_1(t)}(p_1 y_1) = \frac{p_2(t) - 2p_1(t) + c\,(x_1 + x_2)}{2c} = 0: \tag{26}$$

Condition (24) yields:

$$\frac{\partial \lambda_2(t)}{\partial t} = \lambda_2(t)\left(\frac{k_2(t)}{1 + k_2(t)} + \rho\right) + \frac{p_2(t)}{3}: \tag{27}$$

The dynamics of $k_2(t)$ is defined as in (16), which now simplifies as follows:

$$\frac{\partial k_2(t)}{\partial t} \propto \rho^2\,[1 + k_2(t)]^2 - \frac{c}{4}x_2(t): \tag{28}$$

The only acceptable root of the r.h.s. of (28) is:

$$k_2^*(t) = -1 + \frac{1}{2\rho}\sqrt{c x_2(t)}; \tag{29}$$

$$k_2^*(t) > 0 \text{ for all } \rho < \frac{1}{2}\sqrt{c x_2(t)}; \tag{30}$$

with qualitatively the same properties as outlined in (20-21) as well as in figure 1. Obviously, the result stated in Lemma 1 applies in the closed loop formulation as well. Now observe that

$$k_2^*(t) = 0 \text{ iff } x_2(t) = x_2^* = \frac{4}{c}\rho^2 \tag{31}$$

where

$$\frac{3}{c}\rho^2 \in \left(\frac{3}{4}; 1\right] \text{ for all } \rho \in \left(\frac{\sqrt{3c}}{4}; \sqrt{\frac{c}{2}}\right]: \tag{32}$$

By comparing (31) with (22), the following Lemma obtains:

Lemma 2 The steady state degree of product differentiation is larger under the closed loop solution than under the open loop solution.

The above Lemma implies the final result:

Proposition 2 The critical threshold of the discount rate (or the cost of capital) above which there exists a price equilibrium in pure strategies is lower under the closed loop solution than under the open loop solution.

As a final remark, notice that, obviously, from (13) and given $x_1^* + x_2^* = 1;$ equilibrium prices (and consequently equilibrium outputs) are the same under both the open loop and the closed loop solution, i.e., $p_1^* = p_2 = c$ (and $y_1^* = y_2^* = 1/2$).

4 Concluding remarks

I have reformulated the spatial duopoly model with linear transportation costs as a differential game where location is costly and therefore product differentiation is the result of firms' R&D decisions over time. I have characterised both the open loop and the closed loop solution. This has generated two related results. The first is that the steady state R&D investment (product differentiation) is negatively (positively) related to the cost of capital and time discounting. The second is that, if time discounting and the cost of capital are sufficiently high, the amount of differentiation observed in steady state is sufficiently large to ensure the existence of a unique pure-strategy price equilibrium with prices above marginal cost. Product differentiation in steady state is larger under the closed loop solution than under the open loop solution. Consequently, the range of discount rates such that a pure strategy equilibrium exists is larger under the closed loop solution.

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