Sequential Entry in a Vertically Differentiated Duopoly

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Abstract

We analyse a model of vertical differentiation focusing on the trade-off between entering early and exploiting monopoly power with a low quality, versus waiting and enjoying a dominant market position with a superior product. We show that there exists a unique equilibrium where the leader enters with a lower quality than the follower, for low discount factors, for high costs of quality and for low consumers' willingness to pay for quality.

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1 Introduction

An apparently well established result in the theory of vertically differentiated oligopoly states that earlier entrants supply goods of higher quality than later entrants, in that the high-quality products earn higher profits than low-quality alternatives (see, *inter alia*, Gabszewicz and Thisse, 1979, 1980; Shaked and Sutton, 1982, 1983; Donnenfeld and Weber, 1992, 1995). A general proof of this result for every convex fixed-cost function of quality improvement is provided by Lehmann-Grube (1997).[1]

Two basic assumptions are at the basis of this result. The first is that consumers' marginal willingness to pay for quality is uniformly distributed over a given support. Since the density of consumers (i.e., demand) is the same at any income level, the top-quality market niche is the most profitable. Therefore, in a static game, firms obviously prefer to enter with a product characterised by the highest possible quality.

The second assumption concerns the time horizon considered in the above mentioned literature. Entry in a vertically differentiated market is usually analyzed within a single-period extensive form game. However, if one models the entry problem in an explicit dynamic setup, an obvious trade-off immediately appears, even maintaining the previous assumption. In order to enter with an high quality product, the firm has to wait for the R&D activity to take place and consequently it looses monopoly profits. However, postponing entry, the firm is able to produce a higher quality good, obtaining thus higher profits. A static model does not allow to assess the possibility that there exists such a trade-off between early innovation and the attainment of a dominant position in the market.

Although it is generally asserted that quality may result from firms' R&D

[1] Aoki and Prusa (1997) adopt a specific case of the cost function analysed by Lehmann-Grube (1997), to investigate the consequences on profits, consumer surplus and social welfare of the timing of investment in product quality in a vertically differentiated duopoly where the market stage is played in the price space. To this regard, see also Lambertini (1999).

efforts, this aspect of vertical product differentiation has received a relatively scanty attention, the development phase being summarised by a cost function which does not account for the time elapsed before the good is produced and then marketed. To our knowledge, relevant contributions dealing explicitly with the R&D activity are Beath *et al.* (1987); Motta (1992); Rosenkranz (1995, 1997) and Dutta *et al.* (1995). These papers investigate the incentive towards R&D cooperation (Motta, 1992; Rosenkranz, 1995) and the relationship between R&D and the persistence of quality leadership (Beath *et al.*, 1987; Rosenkranz, 1997). Dutta *et al.* (1995) analyse strategic timing in the adoption of a new technology leading to product differentiation and quality improvements. All of these papers maintain that being the quality leader (i.e., supplying the highest quality in the market) entails higher profits than the rivals.

We present a simple model of vertical differentiation focusing upon the trade-off between entering early and exploiting monopoly power with a low quality, versus waiting and enjoying a dominant market position with a superior product. We retain the assumption of a uniform income distribution, that would make it profitable to produce a high quality good in a static game, but relax the assumption of a static extensive form game. Namely, in our model there exists a unique equilibrium where the leader enters with a lower quality than the follower, for a large set of parameter values.[2]

This highlights that an unfavourable position in duopoly (or oligopoly), due to a lower quality than the rivals', may well be more than balanced by the monopoly rent enjoyed *ad interim* with lower development costs. Therefore, it appears that the established wisdom stating that early entry goes along with high quality (and profits) is not robust to a fully fledged investigation of the role of calendar time in shaping endogenously firms' incentives.

The remainder of the paper is structured as follows. The basic model

[2]From a different setting, Dutta *et al.* (1995) also derive an equilibrium where the first entrant produces a lower quality than the second entrant. However, in their model the later entrant makes more profits. As it will become clear in the remainder, this conclusion rests upon the shape of the cost function.

of vertical differentiation is laid out in section 2. Section 3 describes the solution of all admissible subgames. The subgame perfect equilibrium of the whole game is derived in section 4. Finally, section 6 provides concluding remarks.

2 The Model

Consider a market for vertically differentiated products. Let this market exist over time t, with $t \in [0, \infty)$. Two single-product firms, labelled 1 and 2, produce goods of different qualities, q_1 and $q_2 \in [0, \infty)$, through the same technology. Without loss of generality we can assume that firms production costs are nought, while development costs are

$$C_i(q_i) = c \int_{\underline{q}}^{\underline{q}+q_i} e^{-rt} dt \tag{1}$$

with $i = 1, 2$ and $\underline{q} \geq 0$. Development costs $C_i(q_i)$ are evaluated at the beginning of the period of investment, therefore in 0 for firm 1 and in t_1 for firm 2. As usual, these costs can be interpreted as fixed cost due to the R&D effort needed to produce a certain quality. We characterize the technology represented by the above cost function as follows:

Assumption 1 *The R&D costs are constant over time and equal to c. If firm i searches for a period of length t_i, then it can produce a good at most of quality t_i and any other lower quality. Once entered into the market the firm cannot invest anymore in R&D.*

The above amounts to assuming that any change in the quality level implies adjustment costs if and only if the change takes the form of a quality increase. Conversely, once firm i has borne the cost of developing a given quality, she may decide to decrease the quality of her product costlessly. For the sake of simplicity we assume that quality is strictly correlated with the time of entry. More precisely, if firm 1 enters at time t_1, its maximum feasible quality is $t_1 = q_1$. Firm 2's cost of imitation, however, are exactly equal to the costs of innovation.[3] Therefore, firm 2's time of entry satisfies the equality $t_2 = q_1 + q_2$. In the remainder, we shall label the first entrant as the *leader*. Firm 2 enters at date $t_2 \in [t_1, \infty)$, and we shall refer to her as the *follower*.

Assumption 2 *Products are offered on a market where consumers have unit demands, and buy if and only if the net surplus derived from consumption* $v_\theta(q_k, p_i(q_k)) = \theta q_k - p_i(q_k) \geq 0$, *where* $p_i(q_k)$ *is the unit price charged by firm* i *on a good of quality* q_k, *purchased by a generic consumer whose marginal willingness to pay is* $\theta \in [\underline{\theta}, \bar{\theta}]$, *with* $\underline{\theta} = \bar{\theta} - 1$. *We assume that* θ *is uniformly distributed with density one over such interval, so that the total mass of consumer is one. Throughout the following analysis, we assume partial market coverage.*

The above assumption is rather common in vertically differentiated product models. More relevant are the assumptions relative to the timing of the game.

Assumption 3 *Firm 1 chooses when to enter the market with the new product and simultaneously chooses the quality and the price to be offered. Then firm 2 decides whether to imitate firm 1 and when to enter the market. Once firm 2 has entered, the two firms choose simultaneously the quality levels, which become common knowledge. Finally both firms choose simultaneously the price levels.*

This timing can be justified as follows. Suppose that firm 1 has invented a new product, but it has to decide the quality level of that product before entry. Since nobody knows the existence of this new product, only firm 1 can enter first. Thereafter, other firms can imitate firm 1. Suppose only firm 2 has the necessary technology. However, firm 2 has to sustain the R&D

[3]The case for very high imitation costs is supported by empirical findings (see Mansfield *et al.*, 1981; and Levin *et al.*, 1987).

costs before being able to enter and this takes time and precisely the period between t_1 and t_2.[4]

3 Solution of the Game

As usual we will solve the game backwards. However, it is useful before solving the model to introduce two definitions, concerning firms' behavior. In the remainder, we shall refer to the first entrant (firm 1) as the *leader*, and to the second entrant (firm 2) as the *follower*. We are going to examine two alternative perspectives:

A. The follower enters at t_2 with a product whose quality is lower than the leader's. We label this case as *high-quality leadership*.

B. The follower enters at t_2 with a product whose quality is higher than the leader's. We label this case as *low-quality leadership*.

3.1 The Price Game

In both cases, over $t \in [t_2, \infty)$, firms compete in prices. We borrow from Aoki and Prusa (1997) and Lehmann-Grube (1997) the assumption that downstream Bertrand competition is simultaneous. Market demands for the high- and low-quality good are, respectively:

$$x_H = \bar{\theta} - \frac{p_H - p_L}{q_H - q_L} \text{ and } x_L = \frac{p_H - p_L}{q_H - q_L} - \frac{p_L}{q_L} \tag{2}$$

Duopoly revenue functions are $R_H = p_H x_H$ and $R_L = p_L x_L$. Solving for the equilibrium prices, we obtain:

$$p_H = 2\bar{\theta} q_H \frac{q_H - q_L}{4q_H - q_L} \;;\; p_L = \bar{\theta} q_L \frac{q_H - q_L}{4q_H - q_L} \tag{3}$$

[4] To solve the game we adopt subgame perfection, and we look for simultaneous Nash equilibria in each stage. Considering the Stackelberg solution would make calculations more cumbersome without affecting significantly the main results.

which allow to rewrite the revenue function of firms in terms of qualities only, as follows:[5]

$$R_H = \frac{4\bar{\theta}^2 q_H^2 (q_H - q_L)}{(4q_H - q_L)^2} \tag{4}$$

$$R_L = \frac{\bar{\theta}^2 q_H q_L (q_H - q_L)}{(4q_H - q_L)^2} \tag{5}$$

On the basis of expressions (4-5), previous literature, dealing with single-period models, establishes that the first entrant would choose to supply the high-quality good, given that $R_H > R_L$. In the remainder, we label the leader's quality as q_1 and the follower's quality as either q_H or q_L, with the understanding that $q_H \geq q_1$ and $q_1 \geq q_L$.

3.2 The Follower's Quality Choice

We determine the conditions which induce the follower to enter either with a lower or with a higher quality than the leader. We will define the two situations entry from below and entry form above, and will be analyzed in a sequel.

3.2.1 Entry from below

The follower's profits when entering from below are:

$$\begin{aligned} \Pi_{2L} &= \int_{q_1+q_L}^{\infty} R_L e^{-rt} dt - c \int_{q_1}^{q_1+q_L} e^{-rt} dt = \\ & R_L \frac{e^{-(q_1+q_L)r}}{r} - \frac{c}{r}\left(e^{-q_1 r} - e^{-(q_1+q_L)r}\right) \end{aligned}$$

which using (5) can be rewritten as:

$$\begin{aligned} R_{2L}(q_L, q_1) = \frac{\bar{\theta}^2 q_1 q_L (q_1 - q_L)}{(4q_1 - q_L)^2} \frac{e^{-(q_1+q_L)r}}{r} - \frac{c}{r}\left(e^{-q_1 r} - e^{-(q_1+q_L)r}\right) = \\ \frac{\bar{\theta}^2}{r} e^{-rq_1} \left(\frac{q_1 q_L (q_1 - q_L)}{(4q_1 - q_L)^2} e^{-rq_L} + \frac{c}{\bar{\theta}^2} e^{-rq_L} - \frac{c}{\bar{\theta}^2} \right) \end{aligned}$$

[5] The proof is omitted here, as it is provided by several authors (Gabszewicz and Thisse, 1979; Choi and Shin, 1992; Motta, 1993; Aoki and Prusa, 1997; Lehmann-Grube, 1997).

and setting $\gamma \equiv \frac{c}{\bar{\theta}^2}$, $\gamma\tilde{q}_L \equiv q_L$ and $\gamma\tilde{q}_1 \equiv q_1$, and substituting, we obtain:

$$\frac{r}{c}R_{2L}\left(\gamma\tilde{q}_L, \gamma\tilde{q}_1\right) = \left(\left(\frac{\tilde{q}_1\tilde{q}_L(\tilde{q}_1 - \tilde{q}_L)}{(4\tilde{q}_1 - \tilde{q}_L)^2} + 1\right)e^{-\delta\tilde{q}_L} - 1\right)e^{-\delta\tilde{q}_1} \tag{6}$$

where $\delta \equiv \gamma r \equiv rc/\bar{\theta}^2$. Differentiating (6) for $\tilde{q}_L$ we obtain the first order condition:

$$-e^{-\delta(\tilde{q}_1+\tilde{q}_L)}\cdot$$

$$\frac{(7\tilde{q}_L - 4\tilde{q}_1)(\tilde{q}_1)^2 + \delta\tilde{q}_1\tilde{q}_L(\tilde{q}_1 - \tilde{q}_L)(4\tilde{q}_1 - \tilde{q}_L) + \delta(4\tilde{q}_1 - \tilde{q}_L)^3}{(4\tilde{q}_1 - \tilde{q}_L)^3} = 0$$

If we set $\tilde{q}_L = x\tilde{q}_1$, the numerator becomes:

$$\tilde{q}_1^3\left(7x - 4 + \delta\tilde{q}_1 x(1-x)(4-x) + \delta(4-x)^3\right) = 0$$

hence:

$$\tilde{q}_1 = \frac{4 - 7x - \delta(4-x)^3}{\delta x(1-x)(4-x)} \tag{7}$$

and:

$$\tilde{q}_L = x\tilde{q}_1 = \frac{4 - 7x - \delta(4-x)^3}{\delta(1-x)(4-x)}$$

Remark 1 *The follower entering with the low quality chooses the following quality level:*

$$\arg\max R_{2L}\left(\gamma\tilde{q}_L, \gamma\tilde{q}_1\right) = \frac{4 - 7x - \delta(4-x)^3}{\delta(1-x)(4-x)} \equiv \tilde{q}_L^*$$

where $x \equiv q_L/q_1$ *and* $\delta \equiv cr/\bar{\theta}^2$.

Notice that in order to have $\tilde{q}_L \geq 0$ we must impose

$$\delta \leq \frac{4 - 7x}{(4-x)^3} \tag{8}$$

which in turn implies:

$$0 \leq x \leq \frac{4}{7} \approx 0.571\,43, \tag{9}$$

and correspondigly:

$$0 \leq \delta \leq \frac{1}{16} = 0.062\,5. \tag{10}$$

Moreover:

$$\frac{\partial\tilde{q}_L}{\partial x} = \frac{-8 - x - 7x(1-x) - \delta(x+2)(4-x)^3}{\delta(1-x)^2(4-x)^2} < 0$$

hence $\tilde{q}_L$ is a monotonically decreasing function in the relevant range.

3.2.2 Entry from above

Follower's profits if it enters with the high quality good are:

$$\begin{aligned} \Pi_{2H} &= R_H \int_{q_1+q_H}^{\infty} e^{-rt}dt - c\int_{q_1}^{q_1+q_H} e^{-rt}dt \\ & R_H \frac{e^{-t_2 r}}{r} - \frac{c}{r}\left(e^{-q_1 r} - e^{-(q_1+q_H)r}\right) \end{aligned}$$

which using (4) can be re-written as follows

$$\begin{aligned} R_{2H}\left(q_H, q_1\right) = \frac{4\bar{\theta}^2 q_H^2 (q_H - q_1)}{(4q_H - q_1)^2} \frac{e^{-(q_1+q_H)r}}{r} - \frac{c}{r}\left(e^{-q_1 r} - e^{-(q_1+q_H)r}\right) = \\ \frac{\bar{\theta}^2}{r}\left(\frac{4q_H^2(q_H - q_1)}{(4q_H - q_1)^2} e^{-rq_H} + \frac{c}{\bar{\theta}^2} e^{-rq_H} - \frac{c}{\bar{\theta}^2}\right) e^{-rq_1}. \end{aligned}$$

Then, setting $\gamma \equiv c/\bar{\theta}^2$, $\gamma \tilde{q}_L \equiv q_L$ and $\gamma \tilde{q}_1 \equiv q_1$ we obtain:

$$\frac{r}{c} R_{2H}\left(\gamma \tilde{q}_H, \gamma \tilde{q}_1\right) = \left(\left(\frac{4\tilde{q}_H^2(\tilde{q}_H - \tilde{q}_1)}{(4\tilde{q}_H - \tilde{q}_1)^2} + 1\right) e^{-\delta \tilde{q}_H} - 1\right) e^{-\delta \tilde{q}_1} \tag{11}$$

where $\delta \equiv \gamma r \equiv rc/\bar{\theta}^2$. Differentiating (11) for $\tilde{q}_H$ we obtain the first order condition:

$$-e^{-\delta(\tilde{q}_1+\tilde{q}_H)}.$$

$$\frac{\left((4\tilde{q}_H - \tilde{q}_1)^3 + 4\tilde{q}_H^2\left(\tilde{q}_H - \tilde{q}_1\right)\left(4\tilde{q}_H - \tilde{q}_1\right)\right)\delta - 4\tilde{q}_H\left(4\tilde{q}_H^2 - 3\tilde{q}_H\tilde{q}_1 + 2\tilde{q}_1^2\right)}{\left(4\tilde{q}_H - \tilde{q}_1\right)^3} = 0$$

If we set $\tilde{q}_1 = x\tilde{q}_H$, the numerator becomes:

$$\tilde{q}_H^3\left(4\delta\left(1-x\right)\left(4-x\right)\tilde{q}_H - 4\left(4 - 3x + 2x^2\right) + \delta\left(4-x\right)^3\right)$$

which is nought if:

$$\tilde{q}_H = \frac{4\left(4 - 3x + 2x^2\right) - \delta\left(4-x\right)^3}{4\delta\left(4-x\right)\left(1-x\right)}. \tag{12}$$

Hence, the following holds:

Remark 2 *The follower entering with the high quality chooses the following quality level:*

$$\arg\max R_{2H}\left(\gamma\tilde{q}_H, \gamma\tilde{q}_1\right) = \frac{4\left(4-3x+2x^2\right) - \delta\left(4-x\right)^3}{4\delta\left(4-x\right)\left(1-x\right)} \equiv \tilde{q}_H^* .$$

Notice that in order to have $\tilde{q}_H \geq 0$ we must impose:

$$0 \leq \delta \leq 4\frac{(4-3x+2x^2)}{(4-x)^3} \tag{13}$$

Moreover,

$$\frac{\partial \tilde{q}_H}{\partial x} = \frac{1}{4}\frac{4\left(8+x+7x\left(1-x\right)\right)-\left(x+2\right)\left(4-x\right)^3\delta}{\delta\left(1-x\right)^2\left(4-x\right)^2}$$

It is easy to check that:

$$\frac{\partial}{\partial x}\tilde{q}_H\left(x|\delta = 4\frac{(4-3x+2x^2)}{(4-x)^3}\right) = \frac{2x\left(x+5\right)}{\left(4-x\right)^2\left(1-x\right)\delta} > 0.$$

Hence, noticing that $\partial\tilde{q}_H/\partial x$ is decreasing in δ, we have that $\partial\tilde{q}_H/\partial x > 0$ in the relevant range. Therefore, $\tilde{q}_H\left(x\right)$ is monotonically increasing.

3.3 The Leader's Quality Choice

The leader has to take two choices on the quality level: one when it enters as a monopolist and the other when it has to cope with the entry of the competitor. On the basis of Assumption 1, the second level of quality cannot exceed the monopoly one. As usual, we start by analyzing the last quality choice, that when the follower enters.

3.3.1 The Quality in the Last Stage Game

As for the follower, we determine the conditions inducing the follower to enter either with a lower or with a higher quality than the leader's. We will define the two situations as entry from below and entry form above, and they will be analyzed in a sequel.

Entry from above First of all notice that once firm 2 has entered, firm 1 wishes to produce at the highest quality level in the product space. It is sufficient to compute the derivative of R_H with respect to q_H and check that it is always positive:

$$\frac{\partial}{\partial q_H}R_H = 4\bar{\theta}^2 q_H\frac{4q_H^2-3q_Hq_2+2q_2^2}{\left(4q_H-q_2\right)^3}$$

which is positive if: $4q_H^2 - 3q_H q_2 + 2q_2^2 > 0$. However:

$$4q_H^2 - 3q_H q_2 + 2q_2^2 \geq 4q_2^2 - 3q_2 q_2 + 2q_2^2 = 3q_2^2 \geq 0$$

where the first inequality is an implication of $q_H \geq q_2$. Since $q_1 \leq q_M$, where q_M is the quality level of monopolist's product, we can summarize the result in the following proposition.

Proposition 3 *If the leader enters with the high quality good, then it will produce a good of the same quality level after and before follower's entry.*

Entry from below After firm 2 entered the market, the leader's optimal quality level is $q_1 = 4q_H/7$, if it entered with a low quality. In fact:

$$\frac{\partial}{\partial q_L} R_L = \bar{\theta}^2 q_H^2 \frac{4q_H - 7q_1}{(4q_H - q_1)^3}$$

which implies the assertion.

Moreover, $q_1 = 4q_H/7$ implies $\tilde{q}_1 = 4\tilde{q}_H/7$, therefore if we substitute in the follower's first order condition we obtain:

$$C_{2H}\left(\tilde{q}_H, \frac{4}{7}\tilde{q}_H\right) = -\frac{288}{343}\left(+7\delta\tilde{q}_H + 48\delta - 14\right)\tilde{q}_H^3 = 0$$

whose solution is:

$$\tilde{q}_H = \frac{2}{7}\frac{7 - 24\delta}{\delta} \tag{14}$$

which is meaningful if and only if

$$\delta = \frac{c}{\bar{\theta}^2} r \leq \frac{7}{24} \simeq 0.29167 \tag{15}$$

Under the above condition we have:

$$\tilde{q}_1 = \frac{8}{49}\frac{7 - 24\delta}{\delta} \tag{16}$$

This discussion implies:

Proposition 4 *If the leader entered with the low quality good and if the quality chosen after the follower has entered is lower than that chosen in the monopoly phase, the equilibrium quality levels when the followers enters are:*

$$\tilde{q}_1 = \frac{8}{49}\frac{7 - 24\delta}{\delta}, \qquad \tilde{q}_2 = \frac{2}{7}\frac{7 - 24\delta}{\delta} \tag{17}$$

provided that: $\delta = cr/\bar{\theta}^2 \leq 7/24$.

3.3.2 Monopoly Phase

After having discussed the choices in the competition game, we have to describe what happens in the monopoly phase. As usual, we start by describing the price policy and then the choice of quality, distinguishing the entry with high and low quality respectively.

The Monopolist's Price In the monopoly phase, revenues are $R_M = p\left(\bar{\theta} - \ p/q_M\right)$, where q_M is the quality level chosen by firm 1 when monopolist. The first order conditions for the price is:

$$\frac{\bar{\theta}q_M - 2p}{q_M} = 0$$

and hence $p = \bar{\theta}q_M/2$. Substituting again in the profits, it yields:

$$R_M = \frac{1}{4}\bar{\theta}^2 q_M$$

Entry from above. The profit function of firm 1 when entering from above:

$$R_M \int_{q_M}^{q_M+q_L} e^{-rt}dt + R_H \int_{q_M+q_L}^{\infty} e^{-rt}dt - c\int_0^{q_M} e^{-rt}dt =$$

$$R_M \frac{e^{-rq_M} - e^{-(q_M+q_L)r}}{r} + R_H \frac{e^{-(q_M+q_L)r}}{r} - c\frac{\left(1 - e^{-rq_M}\right)}{r} =$$

$$\frac{q_M\bar{\theta}^2}{4}\frac{e^{-rq_M} - e^{-(q_M+q_L)r}}{r} + \frac{4\bar{\theta}^2 q_H^2(q_H - q_L)}{(4q_H - q_L)^2}\frac{e^{-(q_M+q_L)r}}{r} - c\frac{\left(1 - e^{-rq_M}\right)}{r}$$

which is equivalent to:

$$\frac{r}{\bar{\theta}^2}\Pi_{1H}\left(q_L, q_M\right) = 4\frac{q_1^2\left(q_1 - q_L\right)}{\left(4q_1 - q_L\right)^2}e^{-(q_M+q_L)r} + \\ \frac{1}{4}\left(1 - e^{-q_L r}\right)e^{-q_M r}q_M - \frac{c}{\bar{\theta}^2}\left(1 - e^{-q_M r}\right)$$

We already know from the above analysis that, when the follower enters, the leader will produce the highest quality and hence we have $q_1 = q_M$. Therefore, the leader maximizes:

$$\Pi_{MH}(q_L, q_1) = 4\frac{q_1^2(q_1 - q_L)}{(4q_1 - q_L)^2}e^{-(q_1+q_L)r} + \frac{1}{4}e^{-q_1 r}q_1 - \frac{1}{4}e^{-(q_1+q_L)r}q_1 + \gamma e^{-q_1 r} - \gamma$$

with $\gamma = c/\bar{\theta}^2$. Using the usual variable transformations, we obtain:

$$\Pi_{MH}(\gamma\tilde{q}_L, \gamma\tilde{q}_1) = \gamma\left(\frac{4\tilde{q}_1^2(\tilde{q}_1 - \tilde{q}_L)}{(4\tilde{q}_1 - \tilde{q}_L)^2}e^{-\delta\tilde{q}_L} - \frac{1}{4}\tilde{q}_1 e^{-\delta\tilde{q}_L} + \frac{1}{4}\tilde{q}_1 + 1\right)e^{-\delta\tilde{q}_1} - \gamma$$

where $\delta = \gamma r$. Then, defining:

$$\Pi_H(\tilde{q}_L, \tilde{q}_1, \delta) = \frac{1}{\gamma}\Pi_{MH}(\gamma\tilde{q}_L, \gamma\tilde{q}_1) + \gamma$$

and using $\tilde{q}_L = x\tilde{q}_1$, we obtain:

$$\Pi_H(x\tilde{q}_1, \tilde{q}_1, \delta) = \frac{1}{4}\left(\left(1 - \frac{8+x}{(4-x)^2}xe^{-\delta x\tilde{q}_1}\right)\tilde{q}_1 + 4\right)e^{-\delta\tilde{q}_1}$$

Using (7) and substituting in the profit of the monopolist we obtain the following expression:

$$\Pi_H\left(\frac{4 - 7x - \delta(4-x)^3}{\delta(1-x)(4-x)}, \frac{4 - 7x - \delta(4-x)^3}{\delta x(1-x)(4-x)}\right) \tag{18}$$

Therefore, the leader's problem is formally equivalent to mazimizing (18) with respect to x. Given restrictions $(8-10)$, we can carry out an exploration of the monopolist profit function in Figure 1, highlighting the existence of a global maximum for any given value of δ.

The monopolist's first order condition:

$$D_H(x, \delta) = \frac{\partial}{\partial x}\Pi_H\left(\frac{4 - 7x - \delta(4-x)^3}{\delta(1-x)(4-x)}, \frac{4 - 7x - \delta(4-x)^3}{\delta x(1-x)(4-x)}\right) = 0 \tag{19}$$

cannot be solved analytically. However, we can draw the implicit plot in Figure 2. The dotted line plots the locus $\delta = (4-7x)/(4-x)^3$. Accordingly, the only meaningful area is the one below the dotted line. The continuous line below the dotted one is the locus of the global maxima of the profit function, as established by comparing Figure 1 and 2.

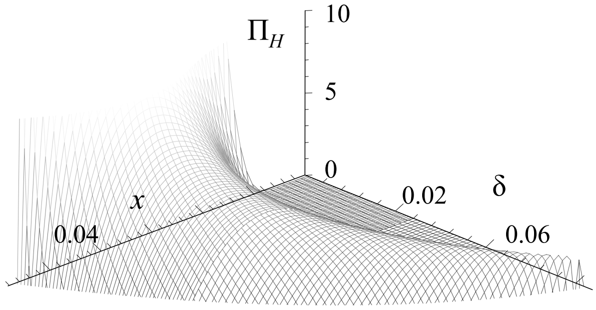

Figure 1: Profit of the leader when entering from above

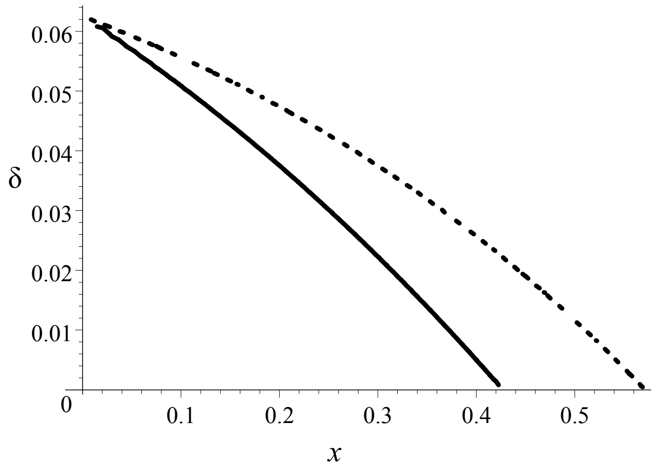

Figure 2: First order condition for the leader when entering from above

Entry from below The profit function of firm 1 when entering from below is:

$$R_M \int_{q_M}^{q_M+q_H} e^{-rt}dt + R_L \int_{q_M+q_H}^{\infty} e^{-rt}dt - c\int_0^{q_M} e^{-rt}dt =$$

$$R_M \frac{e^{-rq_M} - e^{-(q_M+q_H)r}}{r} + R_L \frac{e^{-(q_M+q_H)r}}{r} - c\frac{1-e^{-rq_M}}{r} =$$

$$\frac{q_M\bar{\theta}^2}{4}\frac{e^{-rq_M} - e^{-(q_M+q_H)r}}{r} + \bar{\theta}^2 \frac{q_H q_L (q_H - q_1)}{(4q_H - q_1)^2}\frac{e^{-(q_M+q_H)r}}{r} - c\frac{(1-e^{-rq_M})}{r}.$$

This is equivalent to:

$$\frac{r}{\bar{\theta}^2}\Pi_{ML}(q_M, q_H, q_L) =$$
$$\frac{q_H q_L (q_H - q_L)}{(4q_H - q_L)^2} e^{-(q_M+q_H)r} + \frac{1}{4}\left(1 - e^{-q_H r}\right) e^{-q_M r} q_M - \gamma\left(1 - e^{-q_M r}\right)$$

where $\gamma = c/\bar{\theta}^2$, as usual.

We have to distinguish two different cases. In the first one, $\delta \leq 7/24$ and therefore Proposition 4 holds. In the second one, $\delta > 7/24$. Let us start from the first case.

Case I: $\delta \leq 7/24$. Hence we can set:

$$q_1 = \gamma\tilde{q}_1 = \frac{8}{49}\frac{7-24r\gamma}{r}, \qquad q_2 = \gamma\tilde{q}_2 = \frac{2}{7}\frac{7-24r\gamma}{r}$$

which can be substituted in the profit function to yield:

$$\frac{1}{\gamma}\Pi_{ML}\left(\gamma\tilde{q}_M, \frac{2}{7}\frac{7-24r\gamma}{r}, \frac{8}{49}\frac{7-24r\gamma}{r}\right) =$$
$$\frac{1}{168}\left(\frac{(7-24\delta-42\delta\tilde{q}_M)\, e^{\left(-2+\frac{48}{7}\delta\right)}}{\delta} + 42(\tilde{q}_M + 4)\right) e^{-\delta\tilde{q}_M} - 1.$$

From the first order condition w.r.t. q_M, we obtain:

$$\tilde{q}_M = \frac{1}{42}\frac{(-49+24\delta)\, e^{-2+\frac{48}{7}\delta} + 42(1-4\delta)}{\delta\left(1 - e^{-2+\frac{48}{7}\delta}\right)}$$

Notice that above expression characterizes the leader's choice when entering with the low quality if $q_M \geq q_1$ or $\tilde{q}_M \left(= q_M/\gamma\right) \geq \tilde{q}_1 \left(= q_1/\gamma\right)$, that is, if:

$$\frac{1}{42}\frac{\left(-49+24\delta\right)e^{-2+\frac{48}{7}\delta}+42\left(1-4\delta\right)}{\delta\left(1-e^{-2+\frac{48}{7}\delta}\right)}-\frac{8}{49}\frac{7-24\delta}{\delta}\geq 0$$

which after some manipulation is equivalent to:

$$\frac{1}{294}\frac{7e^{-2+\frac{48}{7}\delta}+984e^{-2+\frac{48}{7}\delta}\delta+42+24\delta}{\delta\left(e^{-2+\frac{48}{7}\delta}-1\right)}\geq 0$$

Since the numerator is always positive, the above condition implies that the denominator should be positive, or equivalently that:

$$\delta \geq \frac{7}{24} \simeq 0.291\,67$$

and recalling Proposition 4 we know that the condition cannot be satisfied for a positive quality level. We summarize the above analysis in the following proposition:

Proposition 5 *Irrespective of whether the leader enters with the low or the high quality, the quality of the leader after the follower has entered the market is equal to that of the monopoly phase, i.e.,* $q_1 = q_M$*, if* $\delta \leq 7/24$*.*

Case II: $\delta > 7/24$. Now we analyze the situation where the monopolist's choice is binding in the duopoly phase. In such a case the leader's profits, after trivial transformation, become:

$$\Pi_{ML}\left(q_H, q_1\right) = \left(\left(\frac{q_H q_1 \left(q_H - q_1\right)}{\left(4q_H - q_1\right)^2} - \frac{1}{4}q_1\right)e^{-q_H r} + \frac{1}{4}q_1 + \gamma\right)e^{-q_1 r} - \gamma$$

and after the usual variable transformations:

$$\Pi_{ML}\left(\gamma\tilde{q}_H, \gamma\tilde{q}_1\right) = \gamma\left(\left(\tilde{q}_H\tilde{q}_1\frac{\tilde{q}_H - \tilde{q}_1}{\left(4\tilde{q}_H - \tilde{q}_1\right)^2} - \frac{1}{4}\tilde{q}_1\right)e^{-\delta\tilde{q}_H} + \frac{1}{4}\tilde{q}_1 + 1\right)e^{-\delta\tilde{q}_1} - \gamma$$

where again $\delta = \gamma r$. Defining:

$$\Pi_L\left(\tilde{q}_H, \tilde{q}_1, \delta\right) = \frac{1}{\gamma}\Pi_{ML}\left(\gamma\tilde{q}_H, \gamma\tilde{q}_1\right) + \gamma$$

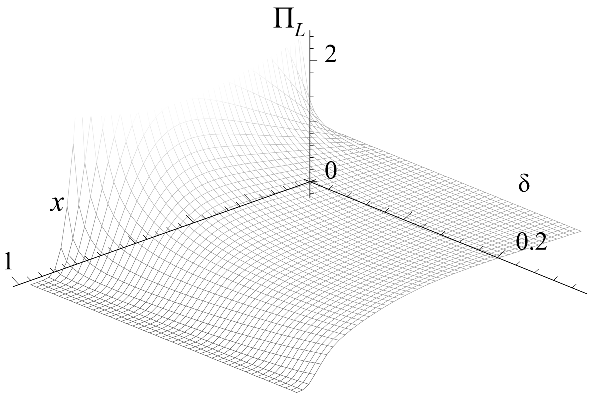

Figure 3: Leader's profit when entering from below

and setting $\tilde{q}_H = x\tilde{q}_1$, we obtain:

$$\Pi_L\left(\tilde{q}_H, x\tilde{q}_H, \delta\right) = \frac{1}{4}\left(\left(1 - \frac{12 - 4x + x^2}{(4-x)^2} e^{-\delta\tilde{q}_H}\right) x\tilde{q}_H + 4\right) e^{-\delta x\tilde{q}_H}$$

Recalling (12), the monopolist problem is equivalent to maximizing the following expression, with respect to x:

$$\Pi_L\left(\frac{4\left(4 - 3x + 2x^2\right) - \delta\left(4-x\right)^3}{4\delta\left(4-x\right)\left(1-x\right)}, x\frac{4\left(4 - 3x + 2x^2\right) - \delta\left(4-x\right)^3}{4\delta\left(4-x\right)\left(1-x\right)}\right) \tag{20}$$

Using restriction (13), we can produce a graphical exploration of the problem in Figure 3. It shows that the function has a unique global maximum for each value of δ.

Moreover, the first order condition is:

$$\begin{aligned} D_L\left(x,\delta\right) &= \\ \frac{\partial}{\partial x}\Pi_L\left(\frac{4\left(4 - 3x + 2x^2\right) - \delta\left(4-x\right)^3}{4\delta\left(4-x\right)\left(1-x\right)}, x\frac{4\left(4 - 3x + 2x^2\right) - \delta\left(4-x\right)^3}{4\delta\left(4-x\right)\left(1-x\right)}\right) &= 0 \end{aligned} \tag{21}$$

and it is not solvable analytically. However, its implicit plot is in Figure 4.

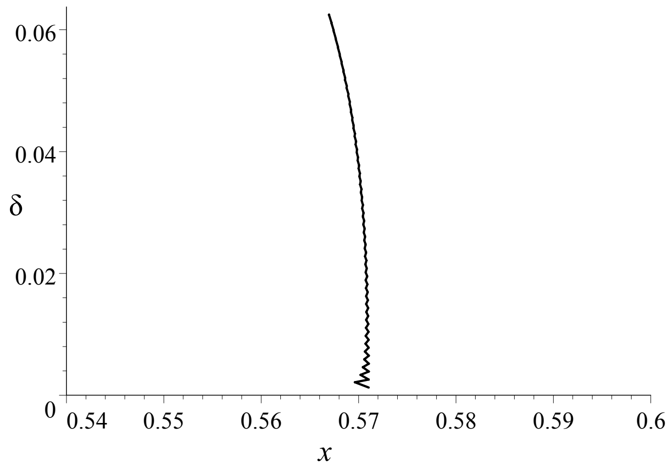

Figure 4: Leader's first order condition when entering from below

4 Is it Convenient to Enter the Market with a High-quality Product?

Now we can solve for the subgame perfect equilibriumof the whole game by determining whether the leader will enter with a high or a low quality. We first prove a preliminary result.

Proposition 6 *No equilibrium with the follower entering the market with a lower quality than the leader does exist if* $\delta = rc/\bar{\theta}^2 > 0.0625$.

Proof. It is a direct consequence of (10). ■

We are now in the position to prove the main Lemma of this section.

Lemma 7 *There exists a* $\bar{\delta}$ *such that, for* $\delta \in \left[0, \bar{\delta}\right)$ *there is no equilibrium with the follower entering the market and the leader producing the lower quality good, while for* $\delta \in \left(\bar{\delta}, 0.0625\right]$ *there exists no equilibrium with the follower entering the market and the leader producing the higher quality good. The value of* $\bar{\delta}$ *is approximately:* $\bar{\delta} = 0.0203125$.

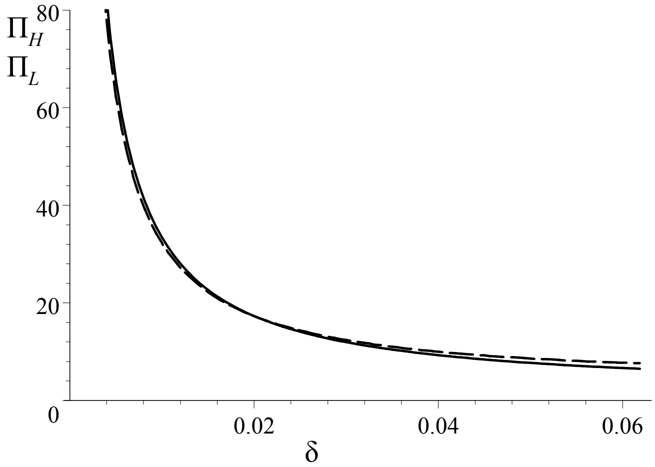

Figure 5: Profits of firm 1 when entering with high (solid) and when entering with low (dash) quality.

Proof. We solve numerically equations (19) and (21), finding the optimal x for the two problems for various values of δ. The computed values are reported in the Table 1-3 of the Appendix in columns denoted respectively by x_{HL} and x_{LH}. By using (7) we can compute $\tilde{q}_1$ and $\tilde{q}_L = x\tilde{q}_1$, the optimal values of transformed variables replacing q_L and q_1. By using (12) we can compute, instead, $\tilde{q}_H$ and $\tilde{q}_1 = x \cdot \tilde{q}_H$. Given the various level of qualities, the profits of the monopolist entering from above and entering from below can be computed and are drawn in Figure 5. It can be seen that the profit of the high quality monopolist are higher for lower level of δ and lower thereafter. The two curves cross at $\bar{\delta}$. ■

The two levels of the follower's profits, R_L when it chooses a lower quality than the leader's and R_H when it chooses a higher quality, are represented in the following two figures. The first one represents the two variables when the leader tries to enter with a higher quality than the follower and, as we can see, the best response for the follower consists in choosing a higher quality.

In the second Figure, instead, we represent the two follower's profit levels when the leader tries to enter with a low quality. In this case, we see that the response of the follower is consistent with the leader's strategy.

The above Lemma allows us to infer that, in our model, the leader enters with the high quality only if it can block the follower's entry. Therefore, here we may have only two types of equilibria. In the first one the leader invests in R&D in such a way to be able to maintain its monopoly position. In the second type of equilibrium, the leader enters with the low quality and then the follower enters with a higher quality. The following proposition will exclude the first outcome, that where the leader can have a monopoly power.

Proposition 8 *For δ sufficiently small, there exists no equilibrium where the leader succeeds in pre-emptying the market. In particular, $\delta \leq 1/16$ is a sufficient condition for the leader not to be able to pre-empt the market.*

Proof. In order to prove the Proposition, we must check that he follower can always enter with a lower quality for any choice of the leader, making positive profits. Recall that the optimal choice of the follower is expressed by (7), which is re-written for convenience:

$$\tilde{q}_L = \frac{4 - 7x - \delta (4 - x)^3}{\delta (1 - x)(4 - x)}$$

We know that x must satisfy inequality (8):

$$\delta \leq \frac{4 - 7x}{(4 - x)^3}$$

Recall also that profits for the follower entering with the low quality are:

$$\frac{r}{c} R_{2L} (\gamma \tilde{q}_L, \gamma \tilde{q}_1) = \left(\left(\frac{\tilde{q}_1 \tilde{q}_L (\tilde{q}_1 - \tilde{q}_L)}{(4\tilde{q}_1 - \tilde{q}_L)^2} + 1 \right) e^{-\delta \tilde{q}_L} - 1 \right) e^{-\delta \tilde{q}_1}$$

and using again the definition of $\tilde{q}_L$ and the fact that $\tilde{q}_L = x\tilde{q}_1$, profits can be re-written as:

$$\left(\frac{4 - 7x}{(4 - x)^3 \delta} e^{-\frac{4 - 7x - \delta(4 - x)^3}{(1 - x)(4 - x)}} - 1 \right) e^{-\delta \tilde{q}_1}$$

Notice that if (8) is satisfied as an equality, then the follower profits are nought, otherwise profits are positive for any value of x and δ. ■

The only possible equilibria left are those with the leader entering with the low quality and the follower responding with a higher one and the other where the opposite happens, depending on the value of the composite parameter δ. However, we still have to ascertain whether it is optimal for the follower to respond with a higher (lower) quality if the leader enters with a low (high) one. This is done in the following two propositions.

Proposition 9 *If* $\delta \in \left[0, \bar{\delta}\right)$ *the leader enters with a high quality and the follower will always respond with a lower one.*

Proof. This proof is conceptually similar to the previous one. On the basis of Lemma 7, we can compute q_1 and q_L. With the two levels of quality we can compute numerically Firm's 2 profit as from equation (6). Moreover, using the first order condition of the follower when entering from above (12), we can compute numerically the corresponding value of x, for any given q_1 and δ and hence $q_H = q_1/x$. Those values of x are reported in the tables of the Appendix in the column denoted as x_{HH}. Finally, we use q_1 and q_H to compute the follower's profit when deviating and entering with the high quality using (11). We provide here the graphical representation of the two levels of profit of the follower showing that the follower never deviates from the low quality. ■

Proposition 10 *If* $\delta \in \left(\bar{\delta}, \bar{\bar{\delta}}\right]$*, the leader enters with a low quality and the follower will always respond with a higher one, while for* $\left(\bar{\bar{\delta}}, 0.0625\right]$*, there is no equilibrium (in pure strategies) since the follower has an incentive to undercut the leader's quality.*

Proof. Relying on the proof of Lemma 7, we can compute q_1 and q_H. With the two levels of quality we can compute numerically Firm's 2 profit as from equation (11). Moreover, using the first order condition of the follower when entering from below (7), we can compute numerically the appropriate value of

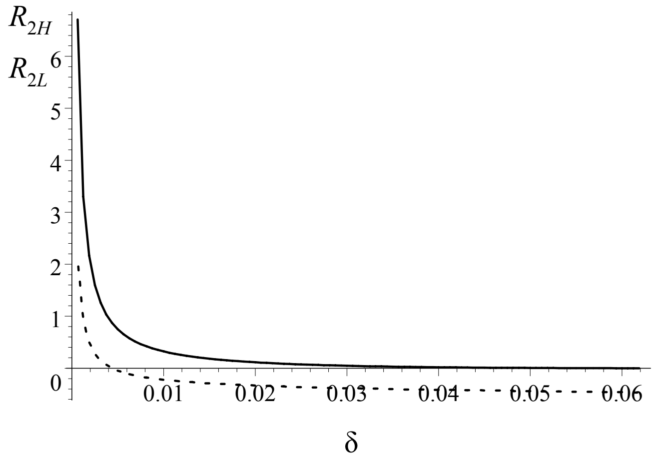

Figure 6: The leader enters with the high quality. Follower's profits when choosing the low one (solid) and the high one (dots).

x, for any given q_1 and δ, and hence $q_L = xq_1$. Those values of x are reported in the tables of the Appendix in the column denoted as x_{LL}. Finally, we use q_1 and q_L to compute the follower's profit when deviating and entering with the low quality using (6). We provide here the graphical representation of the two profit levels of the follower showing that the follower never deviates from the high quality, which shows that the follower's profit are higher when entering with the high quality, except fo very high values of δ. ■

A few remarks are now in order. First, a trivial one, refers to $\delta = \bar{\delta}$. For that value of δ both equilibria hold. Second, recall that $\delta = rc/\bar{\theta}^2$. The two Propositions 9 and 10 together imply that in the interval $\left[0, \bar{\bar{\delta}}\right]$, for low δ the leader will enter with high quality, while with high ones he will choose a low quality. That is, the leader will enter with the high quality for low levels of r and with the low quality for high levels of r, for given c and $\bar{\theta}$. Since a low r implies a high discount rate, the result has a very intuitive explanation: a patient monopolist will enter later in order to obtain a better qulity, while impatient ones will enter earlier, even at the cost of choosing a low quality.

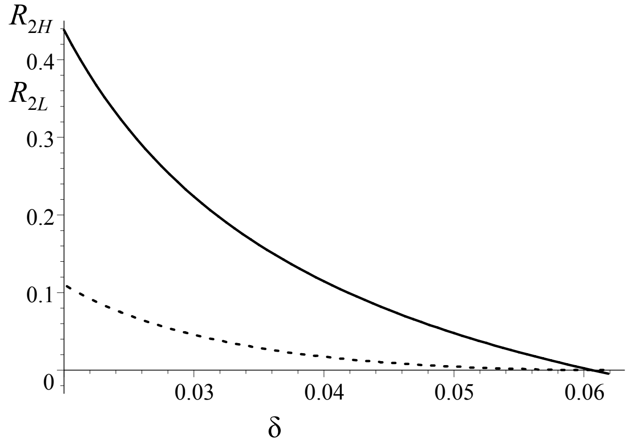

Figure 7: The leader enters with the low quality. Follower's profits when choosing the high one (solid) and the low one (dots).

It is also rather intuitive that c, the cost of investing in quality, has the same effects as r: a high c makes the monopolist impatient. On the contrary, the consumers' willingness to pay for quality, summarized by $\bar{\theta}$, has opposite effects, since the strategy of waiting for a higher quality has higher returns. Third, we should like to assess our results against those of Lehmann-Grube (1997) and Dutta *et al.* (1995), so as to evaluate how different assumptions about the time horizon and the technology affect the features of the subgame perfect equilibrium. Lehmann-Grube (1997) generalises the analisys conducted by Shaked and Sutton (1982, 1983) to account for a technology which is convex in the quality level, but remains in a single-period model where there esists no monopoly phase. This produces the result that surplus extraction is maximised when the firm locates at the top of the available quality spectrum.

In Dutta *et al.* (1995), it is assumed that (i) per-period operative duopoly profits are proportional to relative quality and are symmetric; (ii) adoption (entry) dates are endogenous, while (iii) the growth of quality over time is not endogenously determined by firms; (iv) unit production cost is flat w.r.t. quality; and (v) innovation costs are summarised by the waiting time before the adoption. In this setup, the authors find that a later entrant obtains larger profits than an earlier entrant, and no monopoly rent is dissipated at the subgame perfect equilibrium.

In our setting, the entry timing is endogenously linked to quality improvement, and the cost borne to supply superior qualities can be high enough to offset the advantage attached to serving rich customers. The interplay of these factors may entail that, in some relevant parameter ranges, all firms would prefer to enter early with an inferior quality rather than late with a superior one.

5 Concluding Remarks

We have investigated the bearings of R&D expenditures in continuous time over the entry process in a market for vertically differentiated goods.

We have shown that entering first and enjoying an *ad interim* monopoly rent may counterbalance the incentive towards the supply of high quality goods in duopoly after the entry of a second innovator. Indeed, we have proved that this is the only subgame perfect equilibrium in a large range of parameters.

The foregoing analysis shows that the established wisdom produced by previous literature in this field does not properly account for the role of time and its interaction with R&D technology in determining firms' incentives.

References

[1] Aoki, R. and T. Prusa, 1997, Sequential versus simultaneous choice with endogenous quality, *International Journal of Industrial Organization*, **15**, 103-121.

[2] Beath, J., Y. Katsoulacos and D. Ulph, 1987, Sequential product innovation and industry evolution, *Economic Journal*, **97**, 32-43.

[3] Choi, C.J. and H.S. Shin, 1992, A comment on a model of vertical product differentiation, *Journal of Industrial Economics*, **40**, 229-31.

[4] Donnenfeld, S. and S. Weber, 1992, Vertical product differentiation with entry, *International Journal of Industrial Organization*, **10**, 449-72.

[5] Donnenfeld, S. and S. Weber, 1995, Limit qualities and entry deterrence, *RAND Journal of Economics*, **26**, 113-30.

[6] Dutta, P.K., S. Lach and A. Rustichini, 1995, Better late than early: vertical differentiation in the adoption of a new technology, *Journal of Economics and Management Strategy*, **4**, 563-589.

[7] Gabszewicz, J.J. and J.-F. Thisse, 1979, Price competition, quality and income disparities, *Journal of Economic Theory*, **20**, 340-359.

[8] Gabszewicz, J.J. and J.-F. Thisse, 1980, Entry (and exit) in a differentiated industry, *Journal of Economic Theory*, **22**, 327-38.

[9] Lambertini, L., 1999, Endogenous timing and the choice of quality in a vertically differentiated duopoly, *Research in Economics* (*Ricerche Economiche*), **53**, 101-09.

[10] Lehmann-Grube, U., 1997, Strategic choice of quality when quality is costly: the persistence of the high-quality advantage, *RAND Journal of Economics*, **28**, 372-84.

[11] Levin, R.C., A.K. Klevorick, R.R. Nelson and S.G. Winter, 1987, Appropriating the returns from industrial research and developments, *Brookings Papers on Economic Activity*, 783-820.

[12] Mansfield, E., M. Schwartz and S. Wagner, 1981, Imitation costs and patents: an empirical study, *Economic Journal*, **91**, 907-918.

[13] Motta, M., 1992, Cooperative R&D and vertical product differentiation, *International Journal of Industrial Organization*, **10**, 643-61.

[14] Motta, M., 1993, Endogenous quality choice: price vs quantity competition, *Journal of Industrial Economics*, **41**, 113-32.

[15] Rosenkranz, S., 1995, Innovation and Cooperation under Vertical Product Differentiation, *International Journal of Industrial Organization*, **13**, 1-22.

[16] Rosenkranz, S., 1997, Quality improvements and the incentive to leapfrog, *International Journal of Industrial Organization*, **15**, 243-61.

[17] Shaked, A. and J. Sutton, 1982, Relaxing price competition through product differentiation, *Review of Economic Studies*, **49**, 3-13.

[18] Shaked, A. and J. Sutton, 1983, Natural oligopolies, *Econometrica*, **51**, 1469-83.

Appendix

Table 1: Numerical solutions for low δ's.

	Entry from below		Entry from above	
δ	x_{LH}	x_{LL}	x_{HL}	x_{HH}
0.000625	0.5710955777	0.4252061589	0.4237058956	0.2092704695
0.001250	0.5710989664	0.4215844866	0.4203553403	0.2074909267
0.001875	0.5711018893	0.4179537220	0.4169942492	0.2056948230
0.002500	0.5711043377	0.4143137629	0.4136224093	0.2038821871
0.003125	0.5711063032	0.4106645062	0.4102396054	0.2020530634
0.003750	0.5711077769	0.4070058473	0.4068456203	0.2002074979
0.004375	0.5711087500	0.4033376809	0.4034402348	0.1983455435
0.005000	0.5711092133	0.3996598999	0.4000232276	0.1964672625
0.005625	0.5711091574	0.3959723970	0.3965943754	0.1945727202
0.006250	0.5711085732	0.3922750625	0.3931534528	0.1926619948
0.006875	0.5711074507	0.3885677869	0.3897002322	0.1907351684
0.007500	0.5711057807	0.3848504580	0.3862344841	0.1887923325
0.008125	0.5711035528	0.3811229631	0.3827559768	0.1868335893
0.008750	0.5711007573	0.3773851884	0.3792644768	0.1848590436
0.009375	0.5710973838	0.3736370179	0.3757597481	0.1828688159
0.010000	0.5710934220	0.3698783352	0.3722415530	0.1808630310
0.010625	0.5710888610	0.3661090221	0.3687096516	0.1788418249
0.011250	0.5710836905	0.3623289590	0.3651638019	0.1768053412
0.011875	0.5710778990	0.3585380248	0.3616037600	0.1747537347
0.012500	0.5710714757	0.3547360970	0.3580292799	0.1726871712
0.013125	0.5710644090	0.3509230515	0.3544401135	0.1706058225
0.013750	0.5710566875	0.3470987630	0.3508360106	0.1685098810

0.014375	0.5710482994	0.3432631042	0.3472167191	0.1663995359
0.015000	0.5710392326	0.3394159465	0.3435819850	0.1642749982
0.015625	0.5710294752	0.3355571593	0.3399315520	0.1621364858
0.016250	0.5710190143	0.3316866107	0.3362651620	0.1599842274
0.016875	0.5710078378	0.3278041673	0.3325825548	0.1578184683
0.017500	0.5709959325	0.3239096932	0.3288834683	0.1556394605
0.018125	0.5709832856	0.3200030513	0.3251676384	0.1534474724
0.018750	0.5709698835	0.3160841028	0.3214347991	0.1512427825
0.019375	0.5709557124	0.3121527066	0.3176846822	0.1490256846
0.020000	0.5709407591	0.3082087198	0.3139170180	0.1467964828
0.020625	0.5709250091	0.3042519977	0.3101315343	0.1445554960

Table 2: Numerical solutions for intermediate δ's.

	Entry from below		Entry from above	
δ	x_{LH}	x_{LL}	x_{HL}	x_{HH}
0.021250	0.5709084481	0.3002823939	0.3063279576	0.1423030611
0.021875	0.5708910616	0.2962997593	0.3025060119	0.1400395220
0.022500	0.5708728346	0.2923039433	0.2986654198	0.1377652416
0.023125	0.5708537521	0.2882947929	0.2948059018	0.1354805970
0.023750	0.5708337985	0.2842721533	0.2909271764	0.1331859775
0.024375	0.5708129580	0.2802358667	0.2870289604	0.1308817922
0.025000	0.5707912150	0.2761857738	0.2831109687	0.1285684678
0.025625	0.5707685527	0.2721217130	0.2791729145	0.1262464329
0.026250	0.5707449547	0.2680435198	0.2752145090	0.1239161526
0.026875	0.5707204040	0.2639510277	0.2712354616	0.1215780901
0.027500	0.5706948832	0.2598440676	0.2672354799	0.1192327367
0.028125	0.5706683748	0.2557224678	0.2632142698	0.1168805932
0.028750	0.5706408607	0.2515860543	0.2591715353	0.1145221874
0.029375	0.5706123228	0.2474346499	0.2551069787	0.1121580586
0.030000	0.5705827422	0.2432680753	0.2510203005	0.1097887570
0.030625	0.5705521000	0.2390861482	0.2469111995	0.1074148657
0.031250	0.5705203767	0.2348886833	0.2427793726	0.1050369829
0.031875	0.5704875525	0.2306754927	0.2386245150	0.1026557079
0.032500	0.5704536073	0.2264463850	0.2344463204	0.1002716774
0.033125	0.5704185202	0.2222011666	0.2302444803	0.0978855505
0.033750	0.5703822707	0.2179396398	0.2260186850	0.0954979899
0.034375	0.5703448369	0.2136616045	0.2217686227	0.0931096840
0.035000	0.5703061971	0.2093668569	0.2174939800	0.0907213469

0.035625	0.5702663289	0.2050551901	0.2131944419	0.0883337068
0.036250	0.5702252095	0.2007263935	0.2088696914	0.0859475081
0.036875	0.5701828157	0.1963802531	0.2045194100	0.0835635313
0.037500	0.5701391237	0.1920165513	0.2001432774	0.0811825600
0.038125	0.5700941095	0.1876350669	0.1957409718	0.0788054078
0.038750	0.5700477480	0.1832355747	0.1913121692	0.0764329080
0.039375	0.5700000141	0.1788178457	0.1868565444	0.0740659098
0.040000	0.5699508820	0.1743816469	0.1823737701	0.0717052971
0.040625	0.5699003253	0.1699267414	0.1778635173	0.0693519571
0.041250	0.5698483171	0.1654528876	0.1733254556	0.0670068124

Table 3: Numerical solutions for high δ's.

	Entry from below		Entry from above	
δ	x_{LH}	x_{LL}	x_{HL}	x_{HH}
0.041875	0.5697948299	0.1609598402	0.1687592523	0.0646707925
0.042500	0.5697398357	0.1564473490	0.1641645734	0.0623448659
0.043125	0.5696833057	0.1519151595	0.1595410828	0.0600300129
0.043750	0.5696252105	0.1473630125	0.1548884429	0.0577272307
0.044375	0.5695655203	0.1427906439	0.1502063139	0.0554375448
0.045000	0.5695042043	0.1381977848	0.1454943546	0.0531619999
0.045625	0.5694412313	0.1335841613	0.1407522217	0.0509016549
0.046250	0.5693765691	0.1289494940	0.1359795701	0.0486575954
0.046875	0.5693101851	0.1242934986	0.1311760528	0.0464309355
0.047500	0.5692420458	0.1196158850	0.1263413209	0.0442227972
0.048125	0.5691721169	0.1149163575	0.1214750236	0.0420343272
0.048750	0.5691003633	0.1101946149	0.1165768081	0.0398666744
0.049375	0.5690267493	0.1054503495	0.1116463198	0.0377210423
0.050000	0.5689512381	0.1006832480	0.1066832018	0.0355986197
0.050625	0.5688737922	0.0958929905	0.1016870953	0.0335006319
0.051250	0.5687943732	0.0910792507	0.0966576396	0.0314283115
0.051875	0.5687129417	0.0862416955	0.0915944716	0.0293829044
0.052500	0.5686294573	0.0813799851	0.0864972263	0.0273656750
0.053125	0.5685438792	0.0764937725	0.0813655364	0.0253779143
0.053750	0.5684561647	0.0715827035	0.0761990324	0.0234209261
0.054375	0.5683662708	0.0666464162	0.0709973427	0.0214960158
0.055000	0.5682741531	0.0616845411	0.0657600932	0.0196044799
0.055625	0.5681797662	0.0566967009	0.0604869077	0.0177476705

0.056250	0.5680830636	0.0516825096	0.0551774074	0.0159268892
0.056875	0.5679839975	0.0466415730	0.0498312113	0.0141435302
0.057500	0.5678825190	0.0415734883	0.0444479358	0.0123989271
0.058125	0.5677785781	0.0364778434	0.0390271949	0.0106944486
0.058750	0.5676721233	0.0313542170	0.0335686000	0.0090314093
0.059375	0.5675631020	0.0262021781	0.0280717598	0.0074112368
0.060000	0.5674514598	0.0210212859	0.0225362806	0.0058352248
0.060625	0.5673371415	0.0158110892	0.0169617657	0.0043047257
0.061250	0.5672200899	0.0105711263	0.0113478159	0.0028211172
0.061875	0.5671002466	0.0053009246	0.0056940290	0.0013857253