Price vs Quantity in a Repeated Di¤erentiated Duopoly[1]

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Abstract

We investigate the choice of market variable, price or quantity, of an optimal implicit cartel. If the discount factor is high, the cartel can realize the monopoly profit in both cases. Otherwise, it is optimal for the cartel to rely on quantities in the collusive phase if goods are substitutes and prices if goods are complements. The reason is that this minimizes the gains from deviations from collusive play.

KEYWORDS: partial collusion, product differentiation

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1 Introduction

A recurrent theme in industrial organization is whether …rms choose quantities, as envisioned by Cournot, or prices, as envisioned by Bertrand. This lead Singh and Vives (1984) to investigate the equilibrium choices of a differentiated doupoly, where each …rm can choose between setting a price or a quantity (but not both). Singh and Vives show that …rms choose quantities if goods are substitutes, while they choose prices if goods are complements. The purpose of the present paper is to extend the analysis of Singh and Vives to the case of tacit collusion.

The question we pose is whether …rms participating in an optimizing cartel, which try to maximize pro…ts but has to rely on tacit collusion, will use quantities or prices. Since the members of the cartel cannot write binding contracts they have to agree on self enforcing contracts, i.e. strategies which can be sustained in a subgame perfect equilibrium. There are two …rms producing di¤erentiated products, but otherwise the …rms are identical. As is well known, repeated games have many and very divergent equilibria (see e.g. Fudenberg and Maskin, 1986). In oligopoly theory, researchers have typically focussed on equilibria which are undominated in the set of equilibria. If …rms are symmetric, attention has been driven upon the symmetric subgame perfect equilibrium which gives the highest pro…t to …rms.[1] In this paper

[1] See, for instance, Rotemberg and Saloner (1986), Green and Porter (1984), Abreu, Pearce and Stachetti (1990) and Bernheim and Whinston (1990), or chapter 6 in Tirole (1988) for a survey.

we will take the same approach. So the question can be reformulated as follows: do …rms set prices or quantities in the symmetric subgame perfect equilibrium which give them the highest pro…t?

As is well known from the theory of repeated games (see Abreu, 1988), any subgame perfect equilibrium payo¤ can be realized in a so called simple equilibrium consisting of a normal (collusive) phase and a punishment phase for each of the …rms. We study such equilibria. It is also well known that the worse the punishment phase is, the higher payo¤ can be realized in the normal phase. Although very strong punishments can be part of a subgame perfect equilibrium, one may doubt the viability of such punishments (for further discussion of this see e.g. Farrell and Maskin, 1989). We therefore investigate two kinds of equilibria: equilibria involving optimal (very strong) punishments, and equilibria involving punishments consisting of reversion to the one-shot Nash equilibrium.

In a one-shot game the choice of price or quantity is …nal and commits the …rm for the rest of the game. In a repeated game, this is not necessarily so. In principle, the choice of market variable can commit the …rm for any number of periods. However, it is hard to think of a commitment technology, which can commit a …rm to set a price (or a quantity) for all future. In this paper, therefore, we will assume that the choice of market variable in a period only commits the …rm for that period, but not for subsequent periods. When setting a price, the …rm commits to selling as much as consumers will demand at the price, this corresponds to o¤ering a horizontal supply curve. When setting a quantity the …rm commits to selling this quantity at whatever price clears the market, i.e. a vertical supply curve. This is as in Singh and Vives (1984), they speak of a "quantity contract" and "price contract" between a …rm and its customers. In principle one could imagine other contracts, corresponding to di¤erent supply curves but we will not consider this here.

An optimizing cartel will aim at the highest possible pro…t, ideally the monopoly pro…t. A given pro…t can be realized both when …rms choose prices and when they choose quantities. Therefore, for an optimizing cartel, the crucial feature in the choice of market variables is not the pro…t in a period, but the pro…tability of a deviation. Say that the goods are very close substitutes and both …rms choose the monopoly price so each …rm gets half of the monopoly pro…t. If a …rm wants to deviate from collusive play, it can undercut the other …rm by a small amount and gain (almost) the whole market and obtain (almost) the whole monopoly pro…t. When goods are close substitutes, price setting makes deviations very pro…table, the more so the higher product substitutability is. If, on the other hand, …rms each set a quantity equal to half the monopoly production, they also obtain the monopoly pro…t. But now a deviator can never gain the whole market. The cheated …rm will sell its quantity regardless of the price. Thus when goods are close substitutes, a deviation is less tempting if the …rms set quantities than if they set prices.

For a given punishment, it therefore follows that when goods are close substitutes the smallest discount factor needed to sustain full collusion on the monopoly outcome is smaller when …rms choose quantities in the collusive phase.

We show that when goods are substitutes then, for a range of intermediate discount factors, an implicit cartel can realize the monopoly pro…t only if it relies on quantities. Similarly, if the discount factor is so low that the monopoly pro…t cannot be sustained in a subgame perfect equilibrium, we show that the highest pro…t which can be sustained if the …rms choose quantities is higher than if they choose prices

This holds true when goods are substitutes. When goods are complements, the reverse is true. If the discount factor is not very high, then the highest pro…t which can be sustained in a subgame perfect equilibrium is higher if the …rms choose prices. If the discount factor is very high, the choice of market variable does not matter. The discounted value of future losses due to a punishment is then so high that they are su¢cient to deter deviations even when the deviation pro…t is large.

Hence, for moderate discount factors, an optimizing cartel will choose to compete in quantities if goods are substitutes and choose to compete in prices if goods are complements. This is true regardless of the particular punishment phase involved: optimal or reversion to the one-shot Nash equilibrium.

Our results could be seen as vindicating those of Singh and Vives. However, the mechanism behind the results is di¤erent. In Singh and Vives' model, the choice of market variable is made non-cooperatively by the …rms who try to maximize short run pro…ts. In the repeated game, the optimal implicit cartel maximizes long run profits relying on tacit collusion. The choice of market variable is therefore guided by the consequences for the deviation profits: they should be minimized.

We also briefly consider the choice of market variable in the punishment phase of trigger-strategy equilibria with Nash-punishment. Here, the results of Singh and Vives directly give that firms choose quantities in the punishment phase when goods are substitutes and prices when goods are complements. With optimal punishments, things are more involved, the equilibrium strategies are presumably non-stationary and we cannot characterize the choice of market variable in the punishment phase.

The first to study the choice of market variable in a repeated duopoly was Deneckere (1983, 1984). He analyzed trigger-strategy equilibria à la Friedman (1971), and calculated the smallest discount factor necessary for sustaining collusion on the monopoly outcome for two firms committed to be price setters in all periods as well as two firms committed to be quantity setters in all periods. Deneckere found that when goods are substitutes the crucial discount factor is lower for quantity setting firms than for price setting firms, except when goods are very close substitutes. The opposite is true when goods are complements. Deneckere interpreted this as a cartel is more stable if it competes in quantities when goods are substitutes and more stable if it competes in prices when goods are complements. Majerus (1988) and Rothschild (1992) asked similar questions in slightly different settings (see also Albæk and Lambertini (1998a) for a discussion of Rotschild (1992)).

Lambertini (1997) and Albæk and Lambertini (1998b) assume that firms independently and non-cooperatively choose market variable once and for all in a meta-game, which takes place before the repeated game takes place. The payoff to the firms in the meta-game is not profit, rather each firm is assumed be interested in choosing the market variable which minimizes the discount factor necessary for sustaining collusion on the monopoly profit in the subsequent repeated game. In order for this to be a well specified game, the authors also calculate the lowest discount factors compatible with firms realizing monopoly profits in a subgame perfect equilibrium when one firm is a price setter and the other is a quantity setter. These papers show that the meta-game may have the form of a prisoners' dilemma, and hence that the non-cooperative choice of the market variable be inefficient - relative to the payoffs of the meta-game. Firms choose to be price setters in the meta-game, although cartel stability (in the sense of Deneckere) is higher if they chose to be quantity setters.

Compared to this literature, our paper differs in several aspects. Contrary to Lambertini and Albæk-Lambertini, we insist that a firms payoff is the total sum of discounted profits. There is no meta-game construction in the paper. Secondly, we do not assume that firms are able to commit to a particular market variable for all future, the choice of market variable only commits the firm for one period. An important implication is that the choice of market variable may be different in the normal and the punishment phase. Thirdly, we are able to say what happens when firms are unable to collude on monopoly outputs or prices, but still able to collude at some intermediate level.

The organization of the paper is as follows: section 2 describes the stage game, section 3 the repeated game. Section 4 features trigger strategy equilibria with Nash punishment, while optimal punishment equilibria are treated in section 5. Some concluding remarks are offered in section 6.

2 The stage game

There are infinitely many periods $t = 0, \ldots, \infty$. In each period, the economy is a symmetric, simplified, version of the economy in Singh and Vives (1984)[2]. There are two symmetric firms, producing differentiated goods, $i = 1, 2$ respectively. They are faced with inverse demand functions[3]

$$p_i = 1 - q_i - \gamma q_j \tag{1}$$

where q_i and p_i are the quantity and price respectively of good of firm i and $j \neq i$. We only consider non-negative quantities and assume $-1 < \gamma < 1$. Goods are substitutes if $0 < \gamma < 1$ and complements if $-1 < \gamma < 0$. When

[2] The same version is used in Lambertini (1997). A qualitatively equivalent formulation is also in Deneckere (1983, 1984).

[3] As is well known, these inverse demand functions can be rationalized as follows. A continuum of consumers all have an indirect utility function per period

$$q_1 + q_2 - (q_1^2 + 2\gamma q_1 q_2 + q_2^2)/2 - \sum_{i=1} p_i q_i,$$

where $-1 < \gamma < 1$. Each consumer maximizes utility by choosing q_1, q_2 given prices, p_1, p_2. See Spence (1976) and Dixit (1979).

non-negative; direct demands are

$$q_i = \frac{1}{1+\gamma} - \frac{1}{1-\gamma^2}p_i + \frac{\gamma}{1-\gamma^2}p_j \tag{2}$$

Firms have constant marginal costs, which we normalize to zero. Alternatively, one could interpret the model as one where a positive (constant) marginal cost already has been subtracted in the price, which should then be interpreted as a net price. Therefore negative prices could be sensible. However, there will be a lower bound for prices given by minus the marginal cost if we assume that the firm is not willing to pay consumers for taking its product. For simplicity we let the bound be zero, and consider only non-negative prices. The per period profit of firm i is

$$\pi_i = p_i q_i$$

In each period there are two stages. In the first stage, each firm decides which market variable, MV; to use, either price, PR; or quantity, QY. The choices commit the firms for the period, but not for subsequent periods. In the second stage, each of them chooses the value, σ; of the market variable selected in the first stage, i.e. a price, p; or a quantity, q: When setting a price, the firm commits to selling as much as consumers will demand at the price, as long as the demand is non-negative, this corresponds to offering a horizontal supply curve. When setting a quantity the firm commits to selling this quantity at whatever (non-negative) price clears the market, corresponding to a vertical supply curve. If this quantity can only be sold at a negative price, the firm only "sells" the amount consumers are willing to take at zero price.

This is as in Singh and Vives (1984). One could justify the two kinds of behavior by alluding to a "price-contract" or a "quantity-contract" between firms and their customers. In principle, one could imagine other contracts, quantity rebates etc., but we will not consider this here. Notice that with this formulation, the profits to the firms first accrue in the second stage.

If the firms choose (σ_i, σ_j), firm i's profit is $\pi_i(\sigma_i, \sigma_j)$. In the sequel we will need the expressions when both firms set quantities and when they both set prices. We first consider quantities. Using (1), and (2), we write the profit as follows[4],

$$
\pi_1(q_1, q_2) =
\begin{cases}
\pi^C(q_1, q_2) = q_1 - q_1^2 - \gamma q_2 q_1 & \text{if } 1 - q_1 - \gamma q_2 \geq 0,\ 1 - q_2 - \gamma q_1 \geq 0 \\
\tilde{\pi}^C(q_1, q_2) = q_1 - (1 - \gamma^2) q_1^2 - \gamma q_1 & \text{if } 1 - q_1 - \gamma q_2 \geq 0,\ 1 - q_2 - \gamma q_1 \leq 0 \\
0 & \text{if } 1 - q_1 - \gamma q_2 \leq 0
\end{cases}
\tag{3}
$$

In (3), π^C is the ordinary Cournot profit function, which is only valid if the implied prices are positive at the quantities involved. This gives the restrictions $p_1(q_1, q_2) \geq 0$, which is equivalent to $1 - q_1 - \gamma q_2 \geq 0$ and $p_2(q_1, q_2) \geq 0$ corresponding to $1 - q_2 - \gamma q_1 \geq 0$.

For $q_1 = \frac{1 - q_2}{\gamma}$, $p_2(q_1, q_2)$ is zero. Suppose we are in the case of substitutes, $\gamma > 0$. If firm one increases its production further then $p_2(q_1, q_2) < 0$. This means that consumers are only willing to take the amount q_2 if the price

[4] With a slightly abused notation.

is negative. However, firm 2 is not willing to pay consumers for taking its product, it is willing to supply q_2 at any non-negative price. Market clearing then forces firm 2's price to zero, and the quantity demanded by the consumers is $\bar{q}_2 \equiv 1 - \gamma q_1$. Looking then at firm 1's price as a function of firm 1's supply and the amount of q_2 traded, we get

$$p_1(q_1; \bar{q}_2) = 1 - q_1 - \gamma \bar{q}_2 = 1 - q_1 - \gamma(1 - \gamma q_1)$$

Inserting into firm 1's profit function we get the expression π^C. Although firm 1's profit function is patched together by two different parts, it is concave, since both parts are concave and $\left.\frac{\partial \pi^C}{\partial q_1}\right|_{q_1=\frac{1-q_2}{\gamma}} < \left.\frac{\partial \pi^C}{\partial q_1}\right|_{q_1=\frac{1-q_2}{\gamma}}$.

2.1 Reaction functions, quantities

We can now find the best reply of a firm. Suppose firm 2 has chosen a quantity, q. The best quantity for firm 1 solves

$$\max_{q_1} \pi_1(q_1; q)$$

we will denote it $RC_1(q)$[5]; the associated profit is denoted $\pi_1^{DC}(q)$. We have the following Lemma:

Lemma 1 Consider quantity setting (Cournot behavior) and assume $q \in [0; 1]$.

a. Suppose $\gamma < 0$. Then, $RC_1(q) = \frac{1 - \gamma q}{2}$ and $\pi_1^{DC}(q) = \frac{(1 - \gamma q)^2}{4}$.

b. Suppose $\gamma > 0$.

[5] R for best response, C for Cournot.

i. If $q \leq \dfrac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$; then $RC_1(q) = \dfrac{1-\gamma q}{2}$ and $\pi_1^{DC}(q) = \dfrac{(1-\gamma q)^2}{4}$

ii. If $q > \dfrac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$; then $RC_1(q) = \dfrac{1}{2(1+\gamma)}$ and $\pi_1^{DC}(q) = \dfrac{1-\gamma}{4(1+\gamma)}$

Proof. Suppose $\gamma < 0$: From (3) we see that $\pi_1 = \hat{\pi}_1^C$ iff $1 - q - \gamma q_1 < 0$ or $\dfrac{1-q}{\gamma} > q_1$: Since $\gamma < 0$; this implies that $q_1 < 0$; which is impossible. Hence $\pi = \tilde{\pi}^C$ for all q_1: The result follows from maximization of $\tilde{\pi}^C$ w.r.t. q_1:

Suppose therefore that $\gamma > 0$: Again from (3) we have that $p_2 \geq 0$ iff $q_1 \leq \dfrac{1-q}{\gamma}$; in which case $\pi = \tilde{\pi}^C$ and $p_2 < 0$ for $q_1 > \dfrac{1-q}{\gamma}$ in which case $\pi = \hat{\pi}^C$: Maximization of $\tilde{\pi}^C$ w.r.t. q_1 gives $q_1 = \dfrac{1-\gamma q}{2}$:

If $\dfrac{1-\gamma q}{2} > \dfrac{1-q}{\gamma}$; which is equivalent to $q > \dfrac{2-\gamma}{2-\gamma^2}$; then $\tilde{\pi}^C$ is increasing at $\dfrac{1-q}{\gamma}$: Now look at $\hat{\pi}^C$: Maximizing $\hat{\pi}^C$ w.r.t. q_1 gives the expressions in b:ii: of the Lemma. Furthermore, the best response (from maximizing $\hat{\pi}^C$) $\dfrac{1}{2(1+\gamma)} > \dfrac{1-q}{\gamma}$; for $q > \dfrac{2+\gamma}{2(1+\gamma)}$: As $\dfrac{2-\gamma}{2-\gamma^2} > \dfrac{2+\gamma}{2(1+\gamma)}$; and we assume $q > \dfrac{2-\gamma}{2-\gamma^2}$ we have $q > \dfrac{2+\gamma}{2(1+\gamma)}$, so indeed $\dfrac{1}{2(1+\gamma)} > \dfrac{1-q}{\gamma}$ and $\hat{\pi}^C$ is the relevant part of the profit function.

If $\dfrac{1-\gamma q}{2} < \dfrac{1-q}{\gamma}$; which is equivalent to $q < \dfrac{2-\gamma}{2-\gamma^2}$: Then $\tilde{\pi}^C$ is decreasing at the cut-off point $\dfrac{1-q}{\gamma}$: However, it may be that the optimal profit is nevertheless obtained at $q > \dfrac{1-q}{\gamma}$: The maximal profit for $q < \dfrac{1-q}{\gamma}$ is given by $\max_q \tilde{\pi}^C = \dfrac{(1-\gamma q)^2}{4}$; the maximal profit for $q > \dfrac{1-q}{\gamma}$ is given by $\max_q \hat{\pi}^C = \dfrac{1-\gamma}{4(1+\gamma)}$: We find that $\dfrac{1-\gamma}{4(1+\gamma)} \geq \dfrac{(1-\gamma q)^2}{4}$ if and only if $q \geq \frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$ (or q is negative, which is irrelevant). It is easily checked that $\frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)} < \frac{2-\gamma}{2-\gamma^2}$. Hence, for q fulfilling $\frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)} \leq q \leq \frac{2-\gamma}{2-\gamma^2}$, the best response and profit are as given in b.ii. We also have that for $q \leq \frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$, the best response and profit are as given in b.i.

This completes the proof of the Lemma. ■

We notice that when goods are substitutes $(0 < \gamma < 1)$, then quantities are strategic substitutes, as long as the reaction function is the "normal" reaction function, where the price of the other firm is positive. When goods are complements, quantities are strategic complements.

As also noted by Singh and Vives, it does not matter whether firm 1 chooses a best reply in quantities or prices, the profit will be the same as long as 2 sets the quantity q. Firm 1 chooses the best point along the residual demand curve, whether this is done by choosing a price or a quantity is irrelevant. Hence $\pi^{DC}(q)$ gives the deviation profit to firm 1, regardless of whether it has chosen to set prices or quantities in the first stage, as long as firm 2 sets a quantity (see also Deneckere, 1983, 1984).

The quantity of each firm in the Cournot equilibrium is $q^{CN} = \frac{1}{2+\gamma}$ and the corresponding profit level is $\pi^{CN} = \frac{1}{(2+\gamma)^2}$.

2.2 Reaction functions, prices

Suppose firms are price setters. The profit function of firm 1 is:

$$\pi_1(p_1, p_2) =$$

$$\begin{cases} \widehat{\pi}^B(p_1, p_2) = \left(\dfrac{1}{1+\gamma} - \dfrac{p_1}{1-\gamma^2} + \dfrac{\gamma p_2}{1-\gamma^2} \right) p_1 & \text{if } p_1 \leq 1-\gamma+\gamma p_2;\ p_2 \leq 1-\gamma+\gamma p_1 \\ \widetilde{\pi}_1^B(p_1, p_2) = (1-p_1)p_1 & \text{if } p_1 \leq 1-\gamma+\gamma p_2;\ p_2 \geq 1-\gamma+\gamma p_1 \\ 0 & \text{if } p_1 \geq 1-\gamma+\gamma p_2 \end{cases} \tag{4}$$

Here the condition $p_1 \leq 1-\gamma+\gamma p_2$ ensures that firm 1's quantity is non-negative and $p_2 \leq 1-\gamma+\gamma p_1$ ensures that q_2 is non-negative, as is clear from (2). $\widehat{\pi}^B$ is the standard profit function when the involved quantities are non-negative, $\widetilde{\pi}^B$ corresponds to the case where firm 2's price is so high, that it sells nothing (and everything is as if firm 1 were a monopolist). There are similar expressions for firm 2.

If firm two sets the price p, firm 1's best reply is the price which solves

$$\max_{p_1} \pi_1(p_1, p)$$

We denote this price $RB_1(p)$ and the associated profit $\pi_1^{DB}(p)$. In the sequel, we will only be interested in prices for which quantities are non-negative when both firms set the price. Using (2), we see that this imply that $p \leq 1$.

Lemma 2 Consider price setting (Bertrand behavior) and assume $p \in [0, 1]$.

a. Suppose $\gamma < 0$. Then, $RB_1(p) = \dfrac{1-\gamma(1-p)}{2}$ and $\pi_1^{DB}(p) = \dfrac{[1-\gamma(1-p)]^2}{4(1-\gamma^2)}$.

b. Suppose $\gamma > 0$.

i. If $p \leq \frac{2-\gamma-\gamma^2}{2-\gamma^2}$, then $RB_1(p) = \frac{1-\gamma(1-p)}{2}$ and $\pi_1^{DB}(p) = \frac{[1-\gamma(1-p)]^2}{4(1-\gamma^2)}$.

ii. If $p \in \left(\frac{2-\gamma-\gamma^2}{2-\gamma^2}, 1\right]$, then $RB_1(p) = \frac{p-1+\gamma}{\gamma}$ and $\pi_1^{DB}(p) = \frac{(1-p)(p-1+\gamma)}{\gamma^2}$.

Proof. Suppose $\gamma < 0$. From (4) we have that $\tilde{\pi}^B$ is the relevant function for $p \leq 1-\gamma+\gamma p_1$, which is equivalent to $p_1 \leq \frac{p-1+\gamma}{\gamma}$. For $\gamma < 0$ and $p \in [0,1]$, $\frac{p-1+\gamma}{\gamma} > 1$, hence $\tilde{\pi}^B$ is relevant for all $p_1 \in [0,1]$. The result follows from maximization of $\tilde{\pi}^B$ w.r.t. p_1.

Now suppose $\gamma > 0$. Again from (4) we have that $\pi = \tilde{\pi}^B$ for $p \leq 1-\gamma+\gamma p_1$ which is equivalent to $p_1 \geq \frac{p-1+\gamma}{\gamma}$, and $\pi = \hat{\pi}$ for $p_1 < \frac{p-1+\gamma}{\gamma}$.

Maximizing $\tilde{\pi}^B$ w.r.t. p_1 yields $p_1 = \frac{1-\gamma(1-p)}{2}$. If

$$\frac{1-\gamma(1-p)}{2} > \frac{p-1+\gamma}{\gamma} \iff p < \frac{2-\gamma-\gamma^2}{2-\gamma^2}$$

then $\tilde{\pi}^B$ is increasing at $\frac{p-1+\gamma}{\gamma}$.

Maximizing $\hat{\pi}^B$ w.r.t. p_1 yields $p_1 = \frac{1}{2}$, which is larger than $\frac{p-1+\gamma}{\gamma}$ iff $p < \frac{2-\gamma}{2}$, which is fulfilled since $\frac{2-\gamma}{2} > 1$ for $\gamma \in]0,1[$ and we assume that $p \leq 1$. Hence, $\hat{\pi}^B$ is increasing at $\frac{p-1+\gamma}{\gamma}$. Since $\hat{\pi}^B = \tilde{\pi}^B$ at $\frac{p-1+\gamma}{\gamma}$, we conclude that the global optimum is attained in the optimum of $\tilde{\pi}^B$. This proves b.i of the Lemma.

If instead $p > \frac{2-\gamma-\gamma^2}{2-\gamma^2}$, then $\frac{1-\gamma(1-p)}{2} < \frac{p-1+\gamma}{\gamma}$ and $\tilde{\pi}^B$ is decreasing at the cut-off point $\frac{p-1+\gamma}{\gamma}$. Hence, the optimal price is less than or equal to $\frac{p-1+\gamma}{\gamma}$, where $\hat{\pi}^B$ is the relevant profit function. Maximizing

π^B, yields $p_1 = 1/2$. However, as we showed above, $1/2 > \frac{p - 1 + \gamma}{\gamma}$ for all $p \leq 1$, so π^B is increasing at $\frac{p - 1 + \gamma}{\gamma}$, and the optimal price is $p_1 = \frac{p - 1 + \gamma}{\gamma}$. This proves b.ii. of the Lemma. We also need to check that the profits are non-negative at the optimal solutions, but this is trivial. ■

We may notice that, as long at the reaction function is the "normal", where the quantity of the other firm is positive, then prices are strategic substitutes when goods are complements and strategic complements when goods are substitutes.

Using the formula found in a and b.i. of the Lemma, the Bertrand equilibrium price is $p^{BN} = \frac{1-\gamma}{2-\gamma}$, and the associated profit is $\pi^{BN} = \frac{1-\gamma}{(2-\gamma)^2(1+\gamma)}$. It is easily checked that indeed $p^{BN} = \frac{1-\gamma}{2-\gamma} \leq \frac{2-\gamma-\gamma^2}{2-\gamma^2}$ for $\gamma < 1$.

Let π^{QPN} (π^{PQN}) be the Nash equilibrium profit to the quantity (price) setter in the game where the firms have chosen different market variables. Singh and Vives (1984) show that the following relations then hold:

$$\text{If } 0 < \gamma < 1 \text{ then } \pi^{CN} > \pi^{QPN} > \pi^{BN} > \pi^{PQN} \qquad (5)$$
$$\text{If } -1 < \gamma < 0 \text{ then } \pi^{BN} > \pi^{PQN} > \pi^{CN} > \pi^{QPN}$$

These relations imply that if there is only one period, then the subgame perfect equilibrium of the two-stage game is unique. If $0 < \gamma < 1$, it is a dominant strategy for both firms to choose quantity as the market variable; if $-1 < \gamma < 0$, it is a dominant strategy for both firms to choose price as the market variable (Singh and Vives (1984), proposition 2).

3 Deviation profits

We will be interested in symmetric equilibria, where the firms get the same profit. Let $\pi^Q(q)$ be the profit to each firm if they both choose the quantity q; and $\pi^B(p)$ be the profit if they both choose the price p. Using (1) and (2) they are respectively

$$\pi^Q(q) = q - (1+\gamma)\,q^2 \tag{6}$$

$$\begin{aligned}\pi^B(p) &= \frac{1}{1+\gamma}p - \left(\frac{1}{1-\gamma^2} - \frac{\gamma}{1-\gamma^2}\right)p^2 \\ &= \frac{1}{1+\gamma}\left(p - p^2\right)\end{aligned} \tag{7}$$

The monopoly price, quantity per firm and profit per firm are

$$p^m = \frac{1}{2};\ q^m = \frac{1}{2}\frac{1}{1+\gamma};\ \pi^m = \frac{1}{4}\frac{1}{1+\gamma}. \tag{8}$$

A given profit level can be obtained either by setting prices or quantities. In each case, we can calculate the deviation profit associated with this level of prices or quantities. For a given profit level, π; we would like to know whether the deviation profit to a firm is smaller or larger if the firms choose quantities rather than prices. If they should obtain this profit level by setting quantities, they should each choose a quantity, $q(\pi)$; solving

$$\pi = q - (1+\gamma)\,q^2 \tag{9}$$

This equation has two roots

$$q = \frac{1+\sqrt{1-4(1+\gamma)\pi}}{2\,(1+\gamma)} \text{ and } q = \frac{1-\sqrt{1-4(1+\gamma)\pi}}{2\,(1+\gamma)} \tag{10}$$

The square root is well defined and less than one, since $\pi \leq \pi^m = \frac{1}{4}\frac{1}{1+\gamma}$. As $-1 < \gamma$; the second root is the smaller of the two. As we will see in the sequel, the firms will be interested in minimizing the deviation profits. Therefore, the relevant root is the root with lower deviation profit. From Lemma 1 a and b.i; it is clear that, if $\gamma < 0$ or $\gamma > 0$ and $q \leq \frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$; the deviation profit is $\frac{(1-\gamma q)^2}{4}$. For $\gamma < 0$; this deviation profit increases in q, while decreases in q for $\gamma > 0$. Hence for $\gamma < 0$; the second (lower) root gives the smallest deviation profit and is relevant. For $\gamma > 0$; the opposite is true if indeed $q \leq \frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$. Inserting the Cournot profit in the first root and evaluating, we get the Cournot production:

$$q = \frac{1+\sqrt{1-4(1+\gamma)\frac{1}{(2+\gamma)^2}}}{2(1+\gamma)} = \frac{1}{2+\gamma};$$

which is smaller than $\frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$. Since the root is decreasing in the profit level, it is less than $\frac{1+\gamma-\sqrt{1-\gamma^2}}{\gamma(1+\gamma)}$ for profit levels above the Cournot profit. We conclude that, for $\gamma > 0$; the first root gives rise to the smaller deviation profits and therefore is the relevant one. Hence we have

$$q(\pi) = \begin{cases} \frac{1-\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)} & \text{for } -1 < \gamma < 0 \\ \frac{1+\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)} & \text{for } 0 < \gamma < 1 \end{cases} \tag{11}$$

In fact, the above is very intuitive. When $\gamma > 0$; there is a negative externality from choosing a larger production and the Cournot production is larger than the monopoly production. The lowest deviation profits obtains when production is high corresponding to the first root. When $\gamma < 0$, the externality from choosing a larger production is positive and the Cournot production is smaller than the monopoly production. The lowest deviation profits then obtain when the production is low corresponding to the second root.

The deviation profit is $\pi^{DC}(q(\pi))$: We can summarize the above discussion in

Lemma 3 Consider quantity setting (Cournot behavior). For a given profit level π the deviation profit is given by

$$\pi^{DC}(q(\pi)) = \begin{cases} \dfrac{\left(1-\gamma\dfrac{1+\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)}\right)^2}{4} & \text{if } \gamma > 0 \\[2ex] \dfrac{\left(1-\gamma\dfrac{1-\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)}\right)^2}{4} & \text{if } \gamma < 0 \end{cases} \tag{12}$$

Now consider the case where firms set prices. The price which gives profit level π is $p(\pi)$ which solves

$$\pi = \frac{1}{1+\gamma}\left(p - p^2\right):$$

There are two roots

$$p = \frac{1-\sqrt{1-4(1+\gamma)\pi}}{2} \text{ and } p = \frac{1+\sqrt{1-4(1+\gamma)\pi}}{2}: \tag{13}$$

Again firms will chose the price level, which minimizes the deviation profit. From Lemma 2, we see that if $\gamma < 0$ or $\gamma > 0$ and the price is not too high $\left(p \le \dfrac{2-\gamma-\gamma^2}{2-\gamma^2}\right)$; the deviation profit is $\pi_1^{DB}(p) = \dfrac{(1-\gamma(1-p))^2}{4(1-\gamma^2)}$.

For $\gamma < 0$, $\frac{(1-\gamma(1-p))^2}{4(1-\gamma^2)}$ is decreasing in p, so the deviation profit is smallest when p is high, and the relevant root is the second (large) root.

When $\gamma > 0$, $\frac{(1-\gamma(1-p))^2}{4(1-\gamma^2)}$ is increasing in p, so for p below $\frac{2-\gamma-\gamma^2}{2-\gamma^2}$ the first (lower) root is relevant. The first root is lower than $\frac{2-\gamma-\gamma^2}{2-\gamma^2}$ iff

$$\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2} \le \frac{2-\gamma-\gamma^2}{2-\gamma^2}$$

or

$$1-2\frac{2-\gamma-\gamma^2}{2-\gamma^2} \le \sqrt{1-4(1+\gamma)\pi} \qquad (14)$$

The right hand side is positive for profit levels π below the monopoly profit $\pi^m = \frac{1}{4(1+\gamma)}$. For $\gamma \le \sqrt{3}-1$, the left hand side is negative. Hence the inequality is fulfilled for all relevant profit levels if $\gamma \le \sqrt{3}-1$.

For $\sqrt{3}-1 \le \gamma < 1$, the left hand side of (14) is positive. The right hand side is larger than the left hand side if the profit level is sufficiently small. Solving (14), we see that it is equivalent to

$$\pi \le \pi^* \equiv \frac{1-\left(1-2\frac{2-\gamma-\gamma^2}{2-\gamma^2}\right)^2}{4(1+\gamma)}. \qquad (15)$$

For $\pi \ge \pi^*$, the small root in (17), i.e., $\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}$, is larger than $\frac{2-\gamma-\gamma^2}{2-\gamma^2}$. Evidently, so is the larger root, so from Lemma 2, the deviation profit equals $\frac{(1-p)(p-1+\gamma)}{\gamma^2}$. Notice, this deviation profit is positive as $p \ge \frac{2-\gamma-\gamma^2}{2-\gamma^2}$. We claim that the deviation profit evaluated at the small root in (17) is smaller than evaluated at the large root. The claim is equivalent to

$$\frac{\left(1-\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}\right)\left(\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}-1+\gamma\right)}{\gamma^2} \leq \frac{\left(1-\frac{1+\sqrt{1-4(1+\gamma)\pi}}{2}\right)\left(\frac{1+\sqrt{1-4(1+\gamma)\pi}}{2}-1+\gamma\right)}{\gamma^2}$$

which is fulfilled iff

$$\frac{\left(1-\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}\right)}{\left(1-\frac{1+\sqrt{1-4(1+\gamma)\pi}}{2}\right)} \leq \frac{\left(\frac{1+\sqrt{1-4(1+\gamma)\pi}}{2}-1+\gamma\right)}{\left(\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}-1+\gamma\right)} \tag{16}$$

(remember that all parenthesizes are positive as the deviation profit is positive in the range we are considering now). Condition (16) is clearly fulfilled, the left hand side is less than one, while the right hand side is larger than one. Hence we know that the price the firms use to obtain the profit level π, $p(\pi)$, equals the smaller root $\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}$ and the deviation profit is given by

$$\pi^{DB}(p(\pi)) = \frac{\left(1-\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}\right)\left(\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}-1+\gamma\right)}{\gamma^2}$$

when $\sqrt{3}-1 \leq \gamma < 1$, and $\pi \geq \pi^*$. For later reference we state our result about $p(\pi)$:

$$p(\pi) = \begin{cases} \dfrac{1 + \sqrt{1 - 4(1+\gamma)\pi}}{2} & \text{for } -1 < \gamma < 0 \\[2ex] \dfrac{1 - \sqrt{1 - 4(1+\gamma)\pi}}{2} & \text{for } 0 < \gamma < 1 \end{cases} \tag{17}$$

We summarize this discussion in Lemma 4 below.

Lemma 4 Consider price setting (Bertrand behavior). For a given level of profits π; the deviation profit is given as follows

1. If $\gamma < 0$; then

$$\pi^{DB}(p(\pi)) = \frac{\left[1 - \gamma\left(1 - \dfrac{1 + \sqrt{1 - 4(1+\gamma)\pi}}{2}\right)\right]^2}{4(1 - \gamma^2)} \tag{18}$$

2. If $0 < \gamma \leq \sqrt{3} - 1$; or $\sqrt{3} - 1 \leq \gamma < 1$, and $\pi \leq \pi^*$; where π^* is given in (15), then

$$\pi^{DB}(p(\pi)) = \frac{\left[1 - \gamma\left(1 - \dfrac{1 - \sqrt{1 - 4(1+\gamma)\pi}}{2}\right)\right]^2}{4(1 - \gamma^2)} \tag{19}$$

3. If $\sqrt{3} - 1 \leq \gamma < 1$ and $\pi > \pi^*$; then

$$\pi^{DB}(p(\pi)) = \frac{\left(1 - \dfrac{1 - \sqrt{1 - 4(1+\gamma)\pi}}{2}\right)\left(\dfrac{1 - \sqrt{1 - 4(1+\gamma)\pi}}{2} - 1 + \gamma\right)}{\gamma^2} \tag{20}$$

As noted above, a given profit π can be obtained by setting prices and by setting quantities. In each case, there will be a particular deviation profit, which we have derived above. Therefore, we are now in a position to compare these deviation profits. As is clear from the avobe Lemmata, the comparison depends on γ.

If goods are complements $-1 < \gamma < 0$, equations (12) and (18) yield

$$
\begin{aligned}
&\pi^{DC}(q(\pi)) - \pi^{DB}(p(\pi)) \\
&= \frac{\left(1-\gamma\dfrac{1-\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)}\right)^2}{4} - \frac{(1-\gamma)\left(1-\left(\dfrac{1+\sqrt{1-4(1+\gamma)\pi}}{2}\right)\right)^2}{4(1-\gamma^2)} \\
&= \frac{1}{4}\left(2\pi\gamma + 2\pi\gamma^2 - \gamma - \sqrt{(1-4\pi-4\pi\gamma)}\right)\frac{\gamma^2}{(1-\gamma^2)(1+\gamma)}
\end{aligned}
$$

This expression has the same sign as the sign of the parenthesis

$$\rho(\gamma;\pi) \equiv 2\gamma(1+\gamma)\pi - \gamma - \sqrt{1-4(1+\gamma)\pi} \tag{21}$$

Evaluated at $\pi = 0$, $\rho(\gamma;0) = -\gamma - 1 < 0$. Evaluated at the Bertrand profit $\pi^{BN} = \dfrac{1-\gamma}{(2-\gamma)^2(1+\gamma)}$ we get

$$
\begin{aligned}
\rho(\gamma;\pi^{BN}) &= 2\gamma(1+\gamma)\frac{1-\gamma}{(2-\gamma)^2(1+\gamma)} - \gamma - \sqrt{1-4(1+\gamma)\frac{1-\gamma}{(2-\gamma)^2(1+\gamma)}} \\
&= 2\gamma\frac{1-\gamma}{(2-\gamma)^2} - \gamma - \sqrt{1-4\frac{1-\gamma}{(2-\gamma)^2}}
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
&\operatorname{sign}\left\{2\gamma\frac{1-\gamma}{(2-\gamma)^2} - \gamma - \sqrt{1-4\frac{1-\gamma}{(2-\gamma)^2}}\right\} \\
&= \operatorname{sign}\left\{2\gamma(1-\gamma) - \gamma(2-\gamma)^2 - (2-\gamma)^2\sqrt{1-4\frac{1-\gamma}{(2-\gamma)^2}}\right\} \\
&= \operatorname{sign}\left\{\gamma^2 - \gamma^3\right\} > 0 \text{ for } -1 < \gamma < 0.
\end{aligned}
$$

So

$$\rho(\gamma, \pi^{BN}) > 0$$

Furthermore, we have

$$\frac{\partial \rho(\gamma, \pi)}{\partial \pi} = 2\frac{(1+\gamma)\left(1+\gamma\sqrt{(1-4(1+\gamma)\pi}\right)}{\sqrt{(1-(1+\gamma)\pi}} > 0$$

Since $\rho(\gamma, \pi^{BN}) > 0$, $\frac{\partial \rho(\gamma, \pi)}{\partial \pi} > 0$ implies that $\rho(\gamma, \pi)$ is positive for all profit levels $\pi \in [\pi^{BN}, \pi^{m}]$. To summarize the above, we state for later reference:

$$\pi^{DC}(q(\pi)) > \pi^{DB}(p(\pi)) \text{ for } -1 < \gamma < 0 \text{ and } \pi^{BN} \leq \pi \leq \pi^{m}. \tag{22}$$

Now consider $\gamma > 0$. As is clear from Lemma 4, we have to distinguish according to whether $\gamma \gtreqless \sqrt{3} - 1$ and whether $\pi \gtreqless \pi^{*}$ as given in (15).

First, we consider the case where $0 < \gamma \leq \sqrt{3} - 1$ or where $\sqrt{3} - 1 < \gamma < 1$ and $\pi \leq \pi^{*}$. From (12) and Lemma 4 we have that

$$\begin{aligned}
&\pi^{DC}(q(\pi)) - \pi^{DB}(p(\pi)) \\
&= \frac{\left(1-\gamma\frac{1+\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)}\right)^{2}}{4} - \frac{\left(1-\gamma\left(1-\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}\right)\right)^{2}}{4(1-\gamma^{2})} \\
&= \frac{1}{4}\left(2\pi\gamma(1+\gamma) - \gamma + \sqrt{(1-4\pi-4\pi\gamma)}\right)\frac{\gamma^{2}}{(1-\gamma^{2})(1+\gamma)}
\end{aligned}$$

This expression has the same sign as the sign of the parenthesis

$$\xi(\gamma, \pi) \equiv 2\pi\gamma(1+\gamma) - \gamma + \sqrt{(1-4\pi(1+\gamma)} \tag{23}$$

which is positive evaluated at $\pi = 0$. Evaluated at the Cournot profit $\pi^{CN} =$

$\frac{1}{(2+\gamma)^2}$; we get:

$$\begin{aligned}
\xi(\gamma;\pi^{CN}) &\equiv 2\frac{1}{(2+\gamma)^2}\gamma(1+\gamma)-\gamma+\sqrt{\left(1-4\frac{1}{(2+\gamma)^2}(1+\gamma)\right)} \\
&= -\gamma^2\frac{1+\gamma}{(2+\gamma)^2}<0.
\end{aligned}$$

Furthermore,

$$\frac{\partial\xi(\gamma;\pi)}{\partial\pi}=2\frac{\gamma\sqrt{1-4\pi(1+\gamma)}+\gamma^2\sqrt{1-4\pi(1+\gamma)}-1-\gamma}{\sqrt{1-4\pi(1+\gamma)}}<0 \text{ for } 0<\gamma<1.$$

We thus have that $\xi(\gamma;\pi)$ is negative for all profit levels in-between π^{CN} and π^m. To summarize:

$$\pi^{DC}(q(\pi))<\pi^{DB}(p(\pi))\begin{cases}\text{for } 0<\gamma\leq\sqrt{3}-1 \text{ and } \pi^{CN}\leq\pi\leq\pi^m\\ \text{for } \sqrt{3}-1<\gamma<1 \text{ and } \pi\leq\pi^*\end{cases}\tag{24}$$

Finally, we need to consider the case where $\sqrt{3}-1<\gamma<1$ and the profit is high, $\pi>\pi^*$. Using (12) and Lemma 4 we get:

$$\begin{aligned}
\pi^{DC}(q(\pi))-\pi^{DB}(p(\pi)) &= \frac{\left(1-\gamma\frac{1+\sqrt{1-4(1+\gamma)\pi}}{2(1+\gamma)}\right)^2}{4}+ \\
&-\frac{\left(1-\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}\right)\left(\frac{1-\sqrt{1-4(1+\gamma)\pi}}{2}-1+\gamma\right)}{\gamma^2}
\end{aligned}\tag{25}$$

Now define

$$k\equiv\sqrt{1-4(1+\gamma)\pi}$$

then k is decreasing in π. The expression in (25) above can then be written

$$\zeta(k) \equiv \frac{\left(1-\gamma\frac{1+k}{2(1+\gamma)}\right)^2}{4} - \frac{\left(1-\frac{1-k}{2}\right)\left(\frac{1-k}{2}-1+\gamma\right)}{\gamma^2}$$

which is a second degree polynomium in k, as can be seen from the following rewrite

$$\zeta(k) = \frac{1}{16(1+\gamma)^2\gamma^2}\cdot\left(\left(\gamma^4+4+8\gamma+4\gamma^2\right)k^2+\left(8+8\gamma-8\gamma^2-12\gamma^3-2\gamma^4\right)k-8\gamma^2-4\gamma^3+\gamma^4+4\right)$$

There are two real roots, k_1 and k_2:

$$\begin{aligned} k_2 &= \frac{1}{(4\gamma^2+\gamma^4+4+8\gamma)}\left(\gamma^4+4\gamma^2+6\gamma^3-4-4\gamma+4\sqrt{(\gamma^4+3\gamma^5+3\gamma^6+\gamma^7)}\right) \\ k_1 &= \frac{1}{(4\gamma^2+\gamma^4+4+8\gamma)}\left(\gamma^4+4\gamma^2+6\gamma^3-4-4\gamma-4\sqrt{(\gamma^4+3\gamma^5+3\gamma^6+\gamma^7)}\right) \end{aligned}$$

As the coefficient to the squared term, k^2, is positive, we know that $\zeta(k)$ is positive for $k < k_1$ and $k > k_2$ and negative for $k_1 < k < k_2$.

In the range $\gamma \in [\sqrt{3}-1, 1]$, $k_1 < 0 < k_2$. Now

$$k(\pi^{CN}) = \sqrt{1-4(1+\gamma)\frac{1}{(2+\gamma)^2}} = \frac{\gamma}{2+\gamma} > 0$$

and

$$\begin{aligned} k(\pi^{*}) &= \sqrt{1-4(1+\gamma)\frac{1-\left(1-2\frac{2-\gamma-\gamma^2}{2-\gamma^2}\right)^2}{4(1+\gamma)}} \\ &= 1-2\frac{2-\gamma-\gamma^2}{2-\gamma^2} \geq 0 \text{ for } \gamma \in [\sqrt{3}-1, 1] \end{aligned}$$

while

$$k(\pi^{m}) = \sqrt{1-4(1+\gamma)\frac{1}{4(1+\gamma)}} = 0$$

We are only interested in $\pi \in [\max[\pi^*, \pi^{CN}], \pi^m]$. Since $k(\pi)$ is decreasing in π, this implies that we are interested in $k \in [0, \min[k(\pi^*), k(\pi^{CN})]]$. Now consider the following equation.

$$k(\pi^{CN}) = k_2 \quad \Leftrightarrow$$

$$\frac{\gamma}{2+\gamma} = \frac{\gamma^4 + 4\gamma^2 + 6\gamma^3 - 4 - 4\gamma + 4\sqrt{(\gamma^4 + 3\gamma^5 + 3\gamma^6 + \gamma^7)}}{4\gamma^2 + \gamma^4 + 4 + 8\gamma}.$$

It has two solutions

$$\begin{aligned} \gamma_1 &= \frac{1}{6}\sqrt[3]{\left(100 + 12\sqrt{69}\right)} + \frac{2}{3\sqrt[3]{\left(100 + 12\sqrt{69}\right)}} - \frac{1}{3} \approx .75488 \\ \gamma_2 &= -1 \end{aligned}$$

For $\gamma > \gamma_1$, $k(\pi^{CN}) < k_2$. Hence in this range, $\zeta(k) < 0$ for $k \leq k(\pi^{CN})$. Accordingly we have

$$\zeta(k(\pi)) < 0 \text{ for } \pi > \pi^{CN} \text{ and } \gamma \in [\gamma_1, 1] \tag{26}$$

Then consider the equation

$$k(\pi^*) = k_2 \quad \Leftrightarrow$$

$$1 - 2\frac{2 - \gamma - \gamma^2}{2 - \gamma^2} = \frac{\gamma^4 + 4\gamma^2 + 6\gamma^3 - 4 - 4\gamma + 4\sqrt{(\gamma^4 + 3\gamma^5 + 3\gamma^6 + \gamma^7)}}{4\gamma^2 + \gamma^4 + 4 + 8\gamma}.$$

In the range $\gamma \in [\sqrt{3} - 1, 1]$, this equation has a unique solution, $\nu \approx .94697$. For $\gamma < \nu$, $k(\pi^*) < k_2$. Hence in this range, $k < k(\pi^*)$ implies that $\zeta(k) < 0$. We therefore have

$$\zeta(k(\pi)) < 0 \text{ for } \pi > \pi^* \text{ and } \gamma \in [\sqrt{3} - 1, \nu]. \tag{27}$$

Using (26) and (27), we can now conclude

$$\zeta(k(\pi)) < 0 \text{ for } \pi \in [\max[\pi^{CN}, \pi^{*}], \pi^{m}] \text{ and } \gamma \in [\sqrt{3} - 1, 1]. \tag{28}$$

Finally, this yields

$$\pi^{DC}(q(\pi)) < \pi^{DB}(p(\pi)) \quad \text{for } \pi \in [\max[\pi^{CN}, \pi^{*}], \pi^{m}] \text{ and } \gamma \in [\sqrt{3} - 1, 1]. \tag{29}$$

The following proposition summarizes the results of equations (22), (24) and (29).

Proposition 5 For a given profit level π, we have the following relations between the deviation profits:

1. $\pi^{DC}(q(\pi)) > \pi^{DB}(p(\pi))$ for $-1 < \gamma < 0$ and $\pi^{BN} \leq \pi \leq \pi^{m}$
2. $\pi^{DC}(q(\pi)) < \pi^{DB}(p(\pi))$ for $0 < \gamma < 1$ and $\pi^{CN} \leq \pi \leq \pi^{m}$

4 The repeated game

Both firms seek to maximize the discounted sum of profits. They have the same discount factor δ, where $0 < \delta < 1$. Discounting occurs between periods, but not between the two stages of a period. At time t, at the beginning of stage 1, the history h_t^1 of the game consists of the market variables chosen by each firm in the previous periods' first stages, $MV_{i\tau}$, as well as the values chosen in the second stages, $\sigma_{i\tau}$, so $h_t^1 = (MV_{10}, MV_{20}, \sigma_{10}, \sigma_{20}, \ldots,$

$MV_{1t-1}; MV_{2t-1}; \sigma_{1t-1}; \sigma_{2t-1})$. In the second stage of period t; the history, h_t^2; consists of h_t^1 as well as the chosen market variables in the first stage of period t : $h_t^2 = (h_t^1; MV_{1t}; MV_{2t})$: A (pure) strategy for a firm is a sequence of functions mapping histories into the relevant actions. For firm i; a strategy is $(\theta_{it}^1; \theta_{it}^2)_{t=0}^{\infty}$, where for each t : $\theta_{it}^1 : h_t^1 \mapsto \{PR; QY\}$; and $\theta_{it}^2 : h_t^2 \mapsto \mathbb{R}$:

We will study subgame perfect equilibria of this repeated game. In each period and at each stage, the pair of continuation strategies from that point on should form a Nash equilibrium.

As is well known from Abreu (1986, 1988), it is without loss of generality to restrict attention to simple strategies, which consists of a normal phase and a punishment phase. We will study two kinds of equilibria in simple strategies: 1. trigger strategy equilibria à la Friedman (1971) where the punishment phase consists of reversion to the one shot Nash equilibrium in all future periods, and 2. equilibria with optimal symmetric punishment schemes à la Abreu (1986, 1988), Abreu, Pearce and Stacchetti (1986).

5 Nash punishment

We will focus on subgame perfect equilibria where the firms receive the same payoff in the normal phase. In this section, the focus is on the best such equilibrium, where the punishment phase consists of reversion to the one shot Nash equilibrium in all future.

First notice that (5) directly gives that if goods are substitutes ($0 < \gamma < 1$); then the one shot Nash equilibrium involves firms choosing quantities and subsequently playing the Cournot equilibrium. If goods are complements $(-1 < \gamma < 0)$; then they choose prices and play the Bertrand equilibrium. Let π^N denote the per period profit of the punishment phase.

We divide into two cases, first where the discount factor is so large that the monopoly profit can be realized in each period. Secondly, we look at the case of a moderate discount factor, where the firms have to settle on a profit level smaller than the monopoly profit. Clearly, if the discount factor is very close to one, then the monopoly profit can be sustained in a subgame perfect equilibrium, regardless of whether the firms choose prices or quantities. For a lower discount factor, this may not be possible. For each case, quantities and prices, there is a crucial smallest discount factor, which allows the firms to sustain the monopoly profit in a subgame perfect equilibrium. We will now derive these crucial discount factors.

Consider first the case of quantities. The trigger strategy equilibrium looks like this:

I. If $t = 0$ or both firms have chosen QY and q^m in all previous periods, choose QY in the first stage and q^m in the second stage.

II. If there is an earlier period $t' < t$ where at least one firm has chosen PR in the first stage or something different from q^m in the second stage, or if at least one of the firms have chosen PR in the first stage of this period, choose $QY; q^{CN}$ $\left(PR; p^{BN}\right)$ from now on and in all future if $0 < \gamma < 1$; (if $-1 < \gamma < 0$):

Since the punishment phase (II) consists of infinite repetition of the one shot Nash equilibrium, we just need to check that strategies are optimal for each firm in the normal phase. If a firm adheres to the strategy, it receives π^m in all periods. Clearly, if a firm wants to deviate, it should do it in the second stage of a period. If it deviates in the first stage, it is punished already in the second stage, so the deviation profit will be smaller, than if it waits until the second stage. The best deviation consists of choosing the best reply, which will give the deviation profit $\pi^{DC}(q^m)$ in the period of deviation, and π^N in all future. Hence, the condition that a firm will not deviate is

$$\frac{1}{1-\delta}\pi^m \geq \pi^{DC}(q^m) + \frac{\delta}{1-\delta}\pi^N$$

which is fulfilled if and only if

$$\delta \geq \delta^Q \equiv \frac{\pi^{DC}(q^m) - \pi^m}{\pi^{DC}(q^m) - \pi^N} \tag{30}$$

Notice that, although we have not explicitly written it, δ^Q is a function of γ. Then consider the case where firms set prices in the normal phase. The non-deviation constraint becomes

$$\frac{1}{1-\delta}\pi^m \geq \pi^{DB}(p^m) + \frac{\delta}{1-\delta}\pi^N$$

or

$$\delta \geq \delta^P \equiv \frac{\pi^{DB}(p^m) - \pi^m}{\pi^{DB}(p^m) - \pi^N} \tag{31}$$

As $\pi^m > \pi^N$; the fraction $\frac{x-\pi^m}{x-\pi^N}$ is increasing in x: Hence we have that

$$\delta^Q < \delta^P \quad \Leftrightarrow \quad \pi^{DC}(q^m) < \pi^{BC}(p^m) \tag{32}$$

It then directly follows from Proposition 5 that if goods are complements $(-1 < \gamma < 0)$, then $\delta^P < \delta^Q$, and if goods are substitutes $(0 < \gamma < 1)$, then $\delta^Q < \delta^P$.

Hence,

$$\delta^Q < \delta^P \text{ if and only if } 0 < \gamma < 1$$
$$\delta^Q > \delta^P \text{ if and only if } -1 < \gamma < 0$$

We see that if goods are substitutes $(0 < \gamma < 1)$, then there is a non-empty range of discount factors, $[\delta^Q; \delta^P]$ where the firms can realize the monopoly profit, if they choose quantities while this is not possible if they choose prices. Hence, in this range a profit maximizing implicit cartel will let the firms choose quantities. When goods are substitutes they will also choose quantities in the punishment phase, as we discussed above. When goods are complements, on the other hand, there is a non-empty range of discount factors $[\delta^P; \delta^Q]$ for which firms only can realize the monopoly profit by choosing prices, so in this range the cartel chooses prices. For very high discount factors, the firms can realize the monopoly profit whether they choose prices or quantities. The result resembles the result of Deneckere (1983,1984), but there is a difference. In Deneckere, firms are committed to either prices or quantities in all periods and phases of the repeated game, this means that, for given γ, the discount factor for quantities is calculated with quantities in the punishment phase, while discount factor for prices is calculated with prices in the punishment phase. Thus, for given γ, the punishment profit differs in the two cases. In our game, on the other hand, there is no commitment, so it is not necessarily the case that the market variable is the same in the two phases. This implies that, for given γ; the two discount factors are calculated with the same punishment profit. Hence, although the relative ranking is the same as if we had proceeded like Deneckere, the exact values of the discount factors are different.

What happens when the discount factor is not so high that the monopoly profit can be realized? Recall that, in the one shot Nash equilibrium, firms choose quantities as market variable and the profit is the Cournot profit, π^{CN}; if $0 < \gamma < 1$. A profit-maximizing cartel will at least get the one shot Nash equilibrium profit as average profit, hence the equilibrium average profit, π; fulfills $\pi \geq \pi^{CN}$ if $0 < \gamma < 1$: Similarly, if $-1 < \gamma < 0$; firms choose prices in the one shot Nash equilibrium and the profit is the Bertrand profit π^{BN}: A cartel will at least get this profit, hence the equilibrium average profit fulfills $\pi \geq \pi^{BN}$ when $-1 < \gamma < 0$:

Given the discount factor, δ; the implicit cartel will aim at the highest average profit level, π; where the non-deviation constraint is not violated. Thus, if firms cannot get the monopoly profit, then the constraint will be binding, and this is true in each period. Furthermore, the profit will be the same and equal to the average profit in each period. To see this, suppose that there are two periods where the profit is lower in the first. Then, the average profit can be increased in the first period by dropping the first period action and choosing the actions prescribed for all subsequent periods one period earlier. If, on the other hand, the equilibrium profit is higher in the first period than in the second, the profit of the second period can be increased to the profit of the first period by repeating the actions of the first period. This will not cause firms to deviate in the first period, since it will increase the normal phase profit and thus lessen the deviation constraint in the first period.

Suppose the firms set quantities. To obtain the average profit level π, the firms choose the quantity $q(\pi)$ as given by (11). If a firm wants to deviate from $q(\pi)$, it will receive the deviation profit $\pi^{DC}(q(\pi))$. Therefore, the no-deviation constraint associated with the highest profit level, π, which can be sustained becomes:

$$\frac{1}{1-\delta}\pi \geq \pi^{DC}(q(\pi)) + \frac{\delta}{1-\delta}\pi^{N}. \tag{33}$$

The highest possible profit level attainable when firms set quantities solves this condition with equality. Similarly, if firms set prices, the best profit level, π', is the solution to the following non-deviation constraint:

$$\frac{1}{1-\delta}\pi' \geq \pi^{DB}(p(\pi')) + \frac{\delta}{1-\delta}\pi^{N}. \tag{34}$$

Consider the profit level π, which solves equation (33)[6]. If goods are substitutes ($0 < \gamma < 1$), then Proposition 5, 2. directly imply that at this π the right hand side of equation (34) is larger than the left hand side. Hence, if this π should be obtained by setting prices the firms want to deviate from collusive play. Conversely, because of Proposition 5, 2, at the largest π' which solves (34) with equality, (33) holds with strict inequality. We conclude that,

[6] If there are several solutions, pick the largest.

if goods are substitutes, the cartel can obtain a higher profit by choosing quantities rather than prices. When goods are complements $(-1 < \gamma < 0)$; Proposition 5, 1, directly gives the opposite conclusion. If π' solves (34), then (33) is violated at this π'; so the cartel can obtain a higher profit by setting prices.

We can summarize the discussion:

Theorem 6 Given γ: There exist discount factors δ^Q and δ^P which depend on γ where $0 < \delta^Q; \delta^P < 1$ such that the following is true for the optimal trigger-strategy equilibria with Nash-punishment.

1. If $\delta > \max[\delta^Q; \delta^P]$; the implicit cartel is indifferent between choosing prices or quantities in the normal phase. Firms receive the monopoly profit.

2. If goods are substitutes $(0 < \gamma < 1)$; then $\delta^Q < \delta^P$: If $\delta < \delta^P$; firms set quantities in the normal phase. If $\delta \in [\delta^Q; \delta^P]$; firms receive the monopoly profit; if $\delta < \delta^Q$; they receive less. For all $\delta \in (0; 1)$, firms set quantities in the punishment phase.

3. If goods are complements $(-1 < \gamma < 0)$; then $\delta^P < \delta^Q$: If $\delta < \delta^Q$; firms set prices in the normal phase. If $\delta \in [\delta^P; \delta^Q]$ firms receive the monopoly profit; if $\delta < \delta^P$; they receive less. For all $\delta \in (0; 1)$, firms set prices in the punishment phase.

Qualitatively, the results for the normal phase obtained above carry over to the case where the punishment is the optimal symmetric punishment. The arguments do not depend on the particular punishment phase, the size of π^N does not enter the arguments. Let $\pi^L(\delta)$ be the lowest average profit which can be sustained in a symmetric subgame perfect equilibrium, when the firms' discount factor is δ. From the results of Abreu (1986, 1988) it is clear that such an equilibrium exists. Similarly, let $\pi^H(\delta)$ be the highest average profit which can be sustained in a symmetric subgame perfect equilibrium. This profit can be obtained in a simple equilibrium, where the punishment phase is as severe as possible (given the equilibrium is symmetric), which means that it gives the firms an average profit of $\pi^L(\delta)$. The same arguments as above show that if the discount factor is high, the monopoly profit can be realized regardless of the choice of market variable. There are crucial discount factors δ^{QO} (O for optimal) and δ^{PO} below which the choice of market variable is important for the profit the cartel can realize. Without further proof, we state for completeness:

Theorem 7 *Given γ. There exist discount factors δ^{QO} and δ^{PO} which depend on γ where $0 < \delta^{QO}, \delta^{PO} < 1$ such that the following is true for the optimal trigger-strategy equilibria with optimal punishment.*

1. *If $\delta > \max[\delta^{QO}, \delta^{PO}]$, the implicit cartel is indifferent between choosing prices or quantities in the normal phase. Firms receive the monopoly profit.*

2. *If goods are substitutes $(0 < \gamma < 1)$, then $\delta^{QO} < \delta^{PO}$. If $\delta < \delta^{PO}$, firms set quantities in the normal phase. If $\delta \in [\delta^{QO}, \delta^{PO}]$, firms receive the monopoly profit; if $\delta < \delta^{Q}$, they receive less.*

3. If goods are complements ($-1 < \gamma < 0$), then $\delta^{PO} < \delta^{QO}$. If $\delta < \delta^{QO}$, firms set prices in the normal phase. If $\delta \in [\delta^{PO}, \delta^{QO}]$ firms receive the monopoly profit; if $\delta < \delta^{PO}$, they receive less.

An interesting question, which is not easy to answer, is which market variable the firms use in the optimal punishment phase. Unfortunately, we have not been able to solve this question. A major obstacle is that presumably the optimal punishment phase is non-stationary.

6 Concluding Remarks

We have considered the choice of market variable of an optimizing implicit cartel, which has to rely on tacit collusion. The framework is similar to the framework of Singh and Vives (1984). Our results partly correspond to the results Singh and Vives found for the one shot game. If goods are substitutes firms compete in quantities, if goods are complements firms compete in prices. However, the mechanism behind the results are different. In the static setting of Singh and Vives, firms choose market variables non-cooperatively in order to maximize short run profits, in the repeated game the choice of market variable is guided by deviation profits, the optimizing cartel seeks to minimize deviation profits.

While we have used the framework of Singh and Vives in order to facilitate comparison, it is clear that our results hold more generally. The important feature is the size of the deviation profit. For a given profit level in the normal phase, the deviation pro…t depends on the market variable. The cartel will use the market variable which gives the smallest deviation pro…t. This is clearly also true in more general settings. In the framework of Singh and Vives this furthermore is linked to whether goods are substitutes of complements in demand.

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