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Correlated liquidity shocks, financial contagion and asset price dynamics

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Abstract

Recent literature shows how the destabilising effect of portfolio insurance activity on the price of the underlying asset depends on the liquidity of the asset market. We build a simple model where market timers shift capital around asset markets in order to exploit gains from temporary excess-volatility of asset prices. In this way, market timers increase the liquidity of asset markets reducing the excess volatility, while they increase the cross-market correlation, whereas long-ranged financial contagion eventually occurs. We show how liquidity of asset markets, cross-market correlation and excess volatility of asset prices depend on structural parameters of asset markets.

Keywords: Correlated liquidity shocks, financial contagion, asset price dynamics, endogenous liquidity

J.E.L.: G10, G11

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1 Introduction

It has been pointed out in the recent literature that portfolio insurance activity has a destabilising effect on the price of the underlying if the market of the underlying is only finitely liquid[1]. The extent of this destabilising effect, which is visible through an increased volatility of the asset price, depends on the liquidity of the market. If the market is perfectly liquid, then the asset price dynamics are independent from the portfolio insurance activity, which is the standard assumption in the Black and Scholes world, while if the market is only finitely liquid, this is no longer true. It is so crucial to understand what determines the liquidity of the asset market.

Following the ideas of Grossman (1988) we assume that the liquidity of the single asset market depends partly on the willingness of liquidity providers to furnish capital. Important liquidity providers are market timers. A deviation of an asset price from its normal level leads to arbitrage possibilities for market timers. Consider, for example, the case where the asset price is lower than its normal level. In this case, market timers have an incentive to invest capital in this asset market, furnishing liquidity. If we assume that market timers strategies have to be self-financing, then they have to acquire liquidity, disinvesting capital from other asset markets. As a consequence, the deviation of the asset price from its fundamentals is reduced. Thus, market timers have a stabilising effect on the asset price but at the same time they create correlated liquidity shocks and as a consequence other asset markets suffer a deviation of prices from their normal level.

The destabilising effect of program traders on asset prices has received much attention in the recent literature. In particular, the destabilising effect of dynamic hedging strategies on the underlying asset has been studied, for example, by Schönbucher and Wilmott (2000) and Frey and Stremme (1997). These authors take liquidity as exogenous, and study the positive feedback effects of dynamic hedging strategies on the price of the underlying. The destabilising effect of program traders on other asset markets has been studied by Gennotte and Leland (1990). The latter authors show how market crashes can emerge through the action of program traders. Further, Kodres and Pritsker (2001) and Pritsker (2000) rationalises, among others, financial contagion through correlated liquidity shocks. While we will focus in this paper on correlated liquidity shocks, there exists many other models which identify different channels of financial contagion[2].

The aim of this paper is to study in a unified way the relationship between liquidity of asset markets and the correlation and excess volatility of asset prices.

Given that an asset market is only finitely liquid, the destabilising effect of program traders in the case of either liquidity shocks or shocks to fundamentals leads to an excess volatility of asset price. Thus, potential gains for market timers arise. Notice that if the market is perfectly liquid, even if there are

[1] See for example Frey and Stremme (1997) and Schönbucher and Wilmott (2000).

[2] See for example Kyle and Xiong (2001), Kodres and Pritsker (2001), Lagunoff and Schreft (2001) and their references.

program traders operating on markets, there will be no excess volatility and no financial contagion due to correlated liquidity shocks can occur. Thus, we are looking for an equilibrium where asset markets remain finitely liquid.

We are going to assume that market timers face, in each period of time participation costs. Thus, we will derive the number of market timers operating in equilibrium in each asset market. In this way we will show how the liquidity of asset markets is determined endogenously. In particular, we will show how the liquidity of asset markets, and also the cross-market correlation and the excess volatility of asset prices depend on structural parameters characterising the single asset markets.

The remaining part of the paper is organised as follows. In Section 2 we describe the structure of the model. In Section 3 we give a simple example where closed form solutions for the cross-market correlation and excess volatility of asset prices can be derived. In Section 4 we give some numerical results and derive some implications of the model. Section 5 concludes.

2 The model

The model is a simple extension of Grossman (1988) in a multiple asset setting. We are going to assume that there are $i = 1, ..., n$ types of risky assets in our economy and a riskless bound yielding zero net interest. Each asset is characterised by fundamental values F^i, and by an asset price S^i, for each $i = 1, ..., n$. We assume that all assets share the same stationary state dynamics of fundamentals. We are going to assume that there are four types of traders in the economy: business traders, program traders, market timers and market makers.

Business traders face a standard portfolio selection problem. In order to focus just on the effects of liquidity shocks, we will assume that there are $g^i = g$ of these traders in each market i, each investing just in one risky asset or in the riskless bound. In this way we exclude the possibility of financial contagion through cross-market portfolio rebalancing undertaken by these agents. In each period of time, these agents receive a noisy signal of the realisation of the fundamentals and consequently rebalance optimally their portfolio.

Program traders use dynamic hedging strategies. These latter are convex functions of the price of the underlying, and require to sell the risky asset if its price decreases, while they require to buy the asset if the price of the underlying increases. Thus, these strategies have a destabilising effect on the asset price. We will assume that program traders observe the fundamentals of the risky asset, and change demand according to the realisation of the fundamentals and the current price of the asset. We are interested in the stationary state behaviour of the economy. Assuming that there is a continuous influx and outflux of program traders from the markets, we will use a stationary state demand of program traders[3]. We will assume further that for each risky asset i there are $f^i = f$ program traders in each asset market i.

[3]See for example Frey and Stremme (1997).

There are m market timers in the economy, m to be determined endogenously. Market timers invest and disinvest capital in order to exploit gains from the temporary deviations of the asset prices from their fundamentals. We are going to assume that the strategies of the market timers have to be self-financing, i.e. if they want to invest capital in a given asset market, then they have to acquire liquidity through a disinvestment of capital in other asset markets. If a market timer observes that the price of an asset is lower than its normal price, then he will disinvest capital from other asset markets in order to acquire liquidity and to be able to buy the former asset. Symmetrically, we assume that if a market timer observes that the price of an asset is higher than its normal price, then he will sell assets, and invest the acquired liquidity in other asset markets.

We will assume that market makers furnish liquidity such that at the end of each period of time, asset prices are equal to their normal counterpart, i.e. fundamental values.

In each period of time t, the fundamentals of asset i are observed by market timers $\aleph_t^i$, where $\left|\aleph_t^i\right| = N_t^i \geqslant 1$, and further, in order to keep the model analytically tractable, we assume that each market timer j observes the fundamentals of only two risky assets. The whole economy can be thought of as a collection of vertices representing the asset markets, while the local connections between the asset markets represent the market timers. In this way asset markets are locally correlated since they have common market timers. We are going to neglect higher order correlations in the interaction between asset markets. In particular, we are going to study the model neglecting higher order correlations through N. In other words, we are going to solve the model, assuming that on each asset market there are on average $N_t^e > 1$ market timers, where the average has been taken over all asset markets. Further, we are going to neglect higher order correlations in the interaction structure. In particular, we are going to neglect cycles in the linkages between asset markets. We assume also that market timers can invest and disinvest capital only once in each period of time.

The interaction structure we are going to consider has a tree structure, where the root of the tree is given by the asset market whose fundamentals changed, and the direction of the connections indicates the diffusion of the liquidity shocks. Thus, if an asset market i is hit by a liquidity shock, then the shock can be propagated further on to other N_t^e asset markets, and so on (see Figure 1). This latter structure can also be considered as a first order approximation to more complex structures.

We introduce in this structure a distance between asset markets, defined as the number of common market timers, i.e. direct connections, needed to pass from the asset market where the shock to the fundamentals occurred (root of the tree) to another asset market which is directly or indirectly linked to this. Starting from the asset market i whose fundamentals changed, we have that the number of asset markets at distance u directly linked to the former are given by $N_t^e \left(N_t^e - 1\right)^{u-1}$. Given that there are n asset markets in the economy, and N_t^e is the average number of market timers operating on each market, the

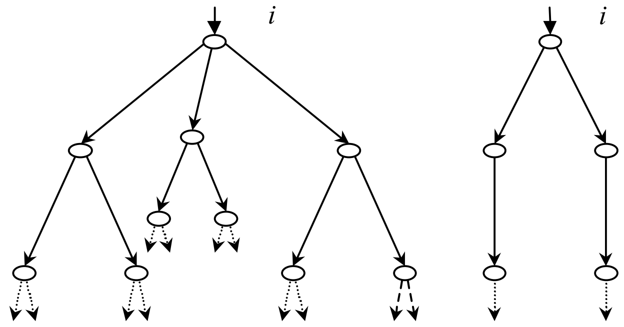

Figure 1: Interaction structure, LHS $N_t^e = 3$, RHS $N_t^e = 2$.

maximum distance $U = \max(u)$ is defined implicitly as $n = N_t^e \left(N_t^e - 1\right)^{U-1}$. We are interested in the case where $n \to \infty$ while N_t^e remains finite.

We collect market timers observing the same two assets in one single market timer, and thus we assume that each market timer observes at least one different asset.

Each time period Δt consists of five sub-periods. In the first part, market timers decide whether to operate or not, facing in the former case participation costs c^i per unit of time Δt. In the second part an asset $i \in \{1, ..., n\}$ will be chosen at random. Further, F^i realises, where with probability $p_+^i = p_+$ and $p_-^i = p_-$ the fundamentals of asset i increase and decrease, respectively. Thus, all fundamentals but the one of asset i remain unchanged. In the third part business traders receive a noisy signal of the realisation of the fundamentals while each market timer observes the realisation of the fundamentals of the assets. In the fourth part portfolio selection and rebalancing occurs. If the fundamentals of asset i increased (decreased), then the price of the asset will increase (decrease), and in particular, through the additional positive (negative) demand of program traders, it will we be larger (lower) than the normal level. Thus, market timers make profits by rebalancing their portfolio, i.e. selling (buying) the asset where the shock to the fundamentals occurred and buying (selling) assets in correlated asset markets. Each market timer will choose the capital to disinvest or invest in other markets in order to exploit gains in market i. In this way market timers transfer a liquidity shock to correlated asset markets. These liquidity shocks lead to a deviation of the asset price from its fundamentals in correlated asset markets, and as a consequence, to exploitable gains for other market timers. In this way liquidity shocks can be propagated further on. Equilibrium prices are set once all buy and sell orders are set. In the fifth part market makers supply the liquidity such that asset prices return to their normal level, i.e. given only by their fundamentals. See Figure 2.

We are first going to describe the cross-market correlation and the expected deviation of the asset price within each small time period Δt. After this, we

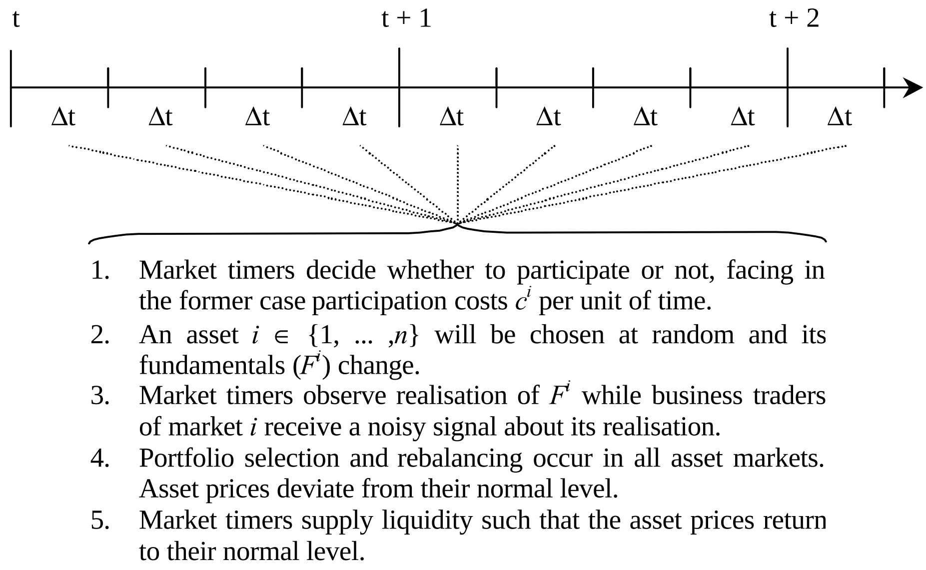

Figure 2: The timing of the game

will pass to continuous time taking the limit of $\Delta t \rightarrow 0$ and characterise the excess volatility of asset prices.

3 Market structure

Consider first the business traders. As pointed out in the previous Section, we assume that each business trader invests his wealth either in a riskless bound B, yielding zero net interest, or in one of the $i = 1, ..., n$ risky assets. There are g business traders investing in each asset i. Further, we assume that business traders receive a noisy signal of the realisation of the fundamentals of the asset, where the noise variable is normally distributed. Each of these traders faces the following maximisation problem

$$\max E\left[U\left(W_{t+\Delta t}\right)\right]$$
$$s.t.\ W_{t+\Delta t} = B + x_t S^i_{t+\Delta t}$$
$$S^i_t x_t + B = W_t$$

where S^i_t is the price of asset i at time t. Assuming that the utility function is exponential and given that the signal about the realisation of the fundamentals is normally distributed, we have that the demand of asset i for each business trader is given by

$$D_t^i\left(S_t^i\right) = \frac{E\left(S_{t+\Delta t}^i\right) - S_t^i}{a\sigma_I^2}$$

where σ_I is the volatility of the signal while a indicates the risk aversion. Assuming that the supply of each asset i is equal to zero, the market clearing condition implies that

$$\alpha\left(E\left(S_{t+\Delta t}^i\right) - S_t^i\right) = 0$$

where $\alpha = \frac{g}{a\sigma_I^2}$. Notice that $\alpha \to \infty$ as the number of business traders diverges towards infinity. In the long run we have that $S_t^i = E\left(S_t^i\right) = S_t^{i*} = e^{F_t^i}$, where F_t^i indicates the fundamentals of asset i and S_t^{i*} the normal price of the asset. Thus, in the long run, the asset price follows its fundamentals.

We are going to study how the price of asset i changes if its fundamentals change. Consider, for example, a decrease in the fundamentals, $F_{t+\Delta t}^{i\prime} = F_t^i - \Delta F$.[4] From the market clearing condition we have that the normal asset price decreases $E\left(S_{t+\Delta t}^{i*}\right) = S_{t+\Delta t}^{i*} = e^{F_t^i - \Delta t} < e^{F_t^i} = S_t^{i*}$. Thus, if there are no program traders, the equilibrium price S_t^i switches immediately towards its normal level.

Let us now introduce program traders. We assume that these agents buy the asset if the price of the asset increases, while they sell the asset if the price decreases. Assume, for simplicity, that the stationary state aggregate demand of program traders is given by[5]

$$\Delta^i(y) = f\begin{cases} -b(y)^2 & if \quad y < 0 \\ b(y)^2 & if \quad y > 0 \\ 0 & if \quad y = 0 \end{cases}$$

In the absence of market timers, the market clearing condition for asset i will be

$$\alpha\left(E\left(S_{t+\Delta t}^i\right) - S_t^i\right) - \beta\left(S_t^i - e^{F_t^i}\right)^2 = 0 \tag{1}$$

where $\beta = fb$. Taking a second order approximation to (1) around $\Delta F = 0$ we have that the equilibrium price is

$$S_t^i = e^{F_t^i - \Delta F} - \frac{\beta}{\alpha}\left(e^{F_t^i - \Delta F}\Delta F\right)^2 < S_{t+\Delta t}^{i*} \tag{2}$$

[4] We will assume that the size of the change in the fundamentals is the same for each asset, i.e. $\Delta F^i = \Delta F$, for each $i = 1...n$.

[5] A more general demand function would be $fDy + \Delta^i(y)$. In order to keep the model analytically tractable, we assume that $D \to 0$. In this way we focus on cases where the destabilising effect of program traders is lowest.

Thus, since $S_t^i < S_{t+\Delta t}^{i*}$, there are arbitrage opportunities for market timers. The size of the deviation of the asset price from its normal level depends on the value of the parameter $\frac{\beta}{\alpha}$. In particular, the larger is α or the lower is β, the lower will be the deviation of the asset price from its fundamentals. In particular, if the number of business traders diverges towards infinity, i.e. $\alpha \to \infty$, then the deviation of the asset price from its fundamentals will be negligible. In other words, if $\alpha \to \infty$, then the excess demand (or supply) will be eliminated by the business traders.

Now we are going to introduce in this framework market timers $j \in \aleph^i$ observing the fundamentals of asset i. These latter have to determine how much capital to withdraw from other asset markets in order to acquire the liquidity necessary to buy assets i. Let us define total demand of market timers observing fundamentals of asset i as $Q_t^i = \sum_{j \in \aleph_t^i} q_t^j$, where q_t^j is the demand of market timer j. Taking for the moment being Q_t^i as given, the market clearing condition becomes

$$\alpha \left(E \left(S_{t+\Delta t}^i \right) - S_t^i \right) - \beta \left(S_t^i - e^{F_t^i} \right)^2 + Q_t^i = 0$$

Also in this case we take a second order approximation of the equilibrium price around $\Delta F = 0$, and the asset price S_t^i becomes[6]

$$S_t^i = e^{F_t^i - \Delta F} + \frac{1}{\alpha} Q_t^i - \frac{\beta}{\alpha} \left(e^{F_t^i - \Delta F} \Delta F \right)^2 \tag{3}$$

3.1 Liquidity shocks

Asset markets $k \in \aleph_t^i$ receive a liquidity shock of size l_t^j, to be determined further on. This liquidity shock leads to a deviation of the asset price S_t^k from its normal level. Thus, potential gains for market timers arise.

The market clearing condition for asset market k is

$$\alpha \left(E \left(S_t^k \right) - S_t^k \right) - \beta \left(S_t^k - e^{F_t^k} \right)^2 - l_t^j = 0 \tag{4}$$

Taking Taylor expansion up to the second order of (4) and taking into account that the liquidity shock is of order $(\Delta F)^2$, and that higher order terms are negligible, we obtain an equilibrium price of

$$S_t^k = e^{F_t^k} - \frac{1}{\alpha} l_t^j$$

A negative liquidity shock decreases the asset price and leads to a deviation of the price from its normal level. The size of this deviation depends on the number of program traders and on the degree of risk aversion. A decrease in α, ceteris paribus, increases the deviation of the asset price from its normal level.

[6] As we will see next Q^i is of order $(\Delta F)^2$ and so higher order terms of Q^i can be neglected.

Market timers operating in this market will withdraw capital from neighbouring asset markets $v \in \aleph_t^k$. Thus, taking into account the demand of market timers, the market clearing condition will be

$$\alpha \left(E\left(S_t^k\right) - S_t^k\right) - \beta\left(S_t^k - e^{F_t^k}\right)^2 - l_t^j + Q_t^k = 0$$

and the corresponding equilibrium asset price is given by

$$S_t^k = e^{F_t^k} + \frac{1}{\alpha}\left(Q_t^k - l_t^j\right)$$

Demand of market timers operating on asset market k will lead to other liquidity shocks, leading to a further propagation of these latter. See Figure 3.

3.2 Optimal strategies for market timers

In characterising the solution of the problem of the market timers and their effect on asset prices, we will make use of the following definitions:

Definition 1 q_t^u *indicates the optimal quantity of capital invested in the asset market we are considering, given that a negative or positive shock to the fundamentals occurred in an asset market which is at distance u from the one we are considering.*

Definition 2 *We define* $\Delta S_{F,t}^u = S_t - E\left(S_{t+\Delta t}\right)$ *the deviation of the asset price from its fundamentals, given that a shock to the fundamentals occurred to an asset market which is at distance u from the one we are considering.*

Consider first asset market i, where a negative shock to the fundamentals occurred. Market timers operating on this market have to determine the amount of capital to disinvest from a neighbouring asset market k, taking into account the gain they make in asset market i (since they buy the asset at a price lower than the normal one) and the loss they make in asset market k (since they sell the asset at a price lower than the normal one). This maximisation problem is subject to the constraint that the portfolio rebalancing strategy has to be self-financing, i.e. the capital invested in the former market has to be acquired from the latter asset market. The problem can be stated as follows

$$\begin{array}{l} \max_{q^i}\left\{q^i\left[E\left(S_{t+\Delta t}^i\right) - S_t^i\right] - l^i\left(e^{F_t^k} - S_t^k\right)\right\} \\ s.t.\ q^i S_t^i = l^i S_t^k \end{array} \tag{5}$$

where S_t^i and S_t^k are equilibrium prices where all buy and sell-orders have been made. Using the first order condition of problem (5) we obtain the following demand of asset i

$$q_t^0 = \frac{1}{d_t'}\left(\beta\left(S_t^{*i}\Delta F\right)^2 + \left(N_t^e - 1\right)\frac{S_t^i}{S_t^k} q_t^1\right) \tag{6}$$

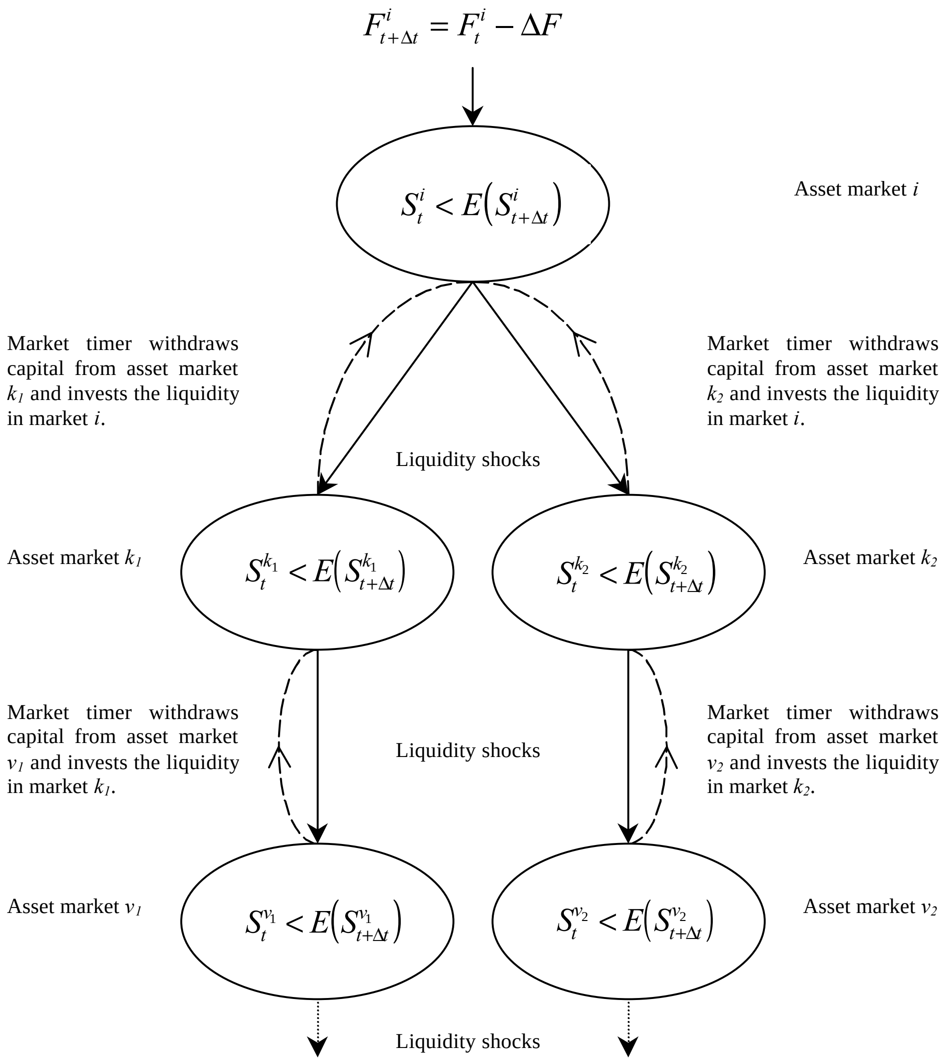

Figure 3: Correlated liquidity shocks and excess deviation of asset prices, $N^e = 2$.

where $d'_t = 1 + 2\left(\frac{S^i_t}{S^k_t}\right)^2 + N^e_t$ and q^1_t indicates the demand of market timers operating on asset market $k \in \aleph^i_t$.

Now consider the problem of a representative market timer operating on asset market k and r. The problem can be stated as follows:

$$\begin{array}{l} \max_{q^k} \left\{ q^k \left(e^{F^k_t} - S^k_t \right) - l^k \left(e^{F^r_t} - S^r_t \right) \right\} \\ s.t.\ q^k S^k_t = l^k S^r_t \end{array} \tag{7}$$

From the first order condition of problem (7) we obtain the following

$$q^1_t = \frac{1}{d''_t} \left(\frac{S^i_t}{S^k_t} q^0_t + \left(N^e_t - 1\right) \frac{S^k_t}{S^r_t} q^2_t \right) \tag{8}$$

where $d''_t = 2\left(\frac{S^k_t}{S^r_t}\right)^2 + N^e_t$ and q^2_t indicates the demand of market timers operating on asset market $r \in \aleph^k_t$.

We will make the following assumption about the distribution of the relative asset prices:

Assumption 1 *There exists the following moments of the distribution of relative prices* $\varepsilon^{i,j}_t = \frac{S^i_t}{S^j_t}$: $E\left(\varepsilon^{i,j}_t\right) = \mu_{\varepsilon,t}$, $E\left(\left(\varepsilon^{i,j}_t\right)^2\right) = s_{\varepsilon,t}$, $E\left(\left(\varepsilon^{i,j}_t\right)^3\right) = \omega_{\varepsilon,t}$, $E\left(\left(\varepsilon^{i,j}_t\right)^4\right) = \nu_{\varepsilon,t}$.

In Section 3.4 we will explicitly calculate the moments of the distribution of relative prices.

The problem of the market timers has to be solved recursively. We can state the following proposition which characterises the average behaviour of the equilibrium.

Proposition 1 *The average behaviour of the equilibrium is characterised as follows*

$$q^u_t = \frac{1}{1 + \lambda_t \left[1 + d_t \left(\mu_{\varepsilon,t} - 1\right)\right]} \left(\lambda_t\right)^{u+1} \beta \left(S^{*i}_{t+\Delta t} \Delta F\right)^2$$

$$\Delta S^0_{F,t} = \left(\frac{\lambda_t}{1 + \lambda_t \left[1 + d_t \left(\mu_{\varepsilon,t} - 1\right)\right]} N^e_t - 1 \right) \frac{\beta}{\alpha} \left(S^{*i}_{t+\Delta t} \Delta F\right)^2$$

$$\Delta S^u_{F,t} = \frac{\left(N^e_t - 1\right)\lambda_t - \mu_{\varepsilon,t}}{1 + \lambda_t \left[1 + d_t \left(\mu_{\varepsilon,t} - 1\right)\right]} \left(\lambda_t\right)^u \frac{\beta}{\alpha} \left(S^{*i}_{t+\Delta t} \Delta F\right)^2$$

for each $u > 0$, *where* $\lambda_t = \frac{d_t - d_t\sqrt{1 - 4\frac{N^e_t - 1}{d^2_t}}}{2\left(N^e_t - 1\right)}$ *and* $d_t = \frac{2 s_{\varepsilon,t} + N^e_t}{\mu_{\varepsilon,t}}$.

Proof. It is easy to see that all the problems for the markets but the first one are characterised by the general average solution

$$q_t^u = \frac{1}{d_t}\left(q_t^{u-1} + (N_t^e - 1)\, q_t^{u+1}\right) \tag{9}$$

where $d_t = \frac{2s_{\varepsilon,t}+N_t^e}{\mu_{\varepsilon,t}}$. (9) is a second order difference equation. The solution to (9) is given by $q_t^u = (\lambda_t)^u$, where

$$\lambda_{t,(1,2)} = \frac{d_t \pm d_t\sqrt{1 - 4\frac{N_t^e - 1}{d_t^2}}}{2\,(N_t^e - 1)} \tag{10}$$

The general solution to (9) will be

$$q_t^u = C_{t,1}\,(\lambda_{t,1})^u + C_{t,2}\,(\lambda_{t,2})^u \tag{11}$$

where $C_{t,1}$ and $C_{t,2}$ are constants. (11) has to be true for all $u >> 0$. We can observe that $\lambda_{t,1} > 1$, and in particular, as long as $C_{t,1} > 0$, $(N_t^e - 1)\, q^1 > q^0$ for N_t^e sufficiently large, which is not an admissable solution. Thus, we obtain a meaningful solution only if $C_{t,1} = 0$. From this we obtain that $q_t^{u+1} = \lambda_t q_t^u$, for each $u > 0$, where $\lambda_{t,2} = \lambda_t$. Further, inserting this solution into (6) we obtain the initial condition that

$$q_t^0 = \frac{\lambda_t}{1 + \lambda_t\left[1 + d_t\left(\mu_{\varepsilon,t} - 1\right)\right]}\beta\left(S_{t+\Delta t}^{*i}\Delta F\right)^2$$

which concludes the proof. ■

It is easy to verify that the following results hold:

Lemma 1 *For $N_t^e > 1$*

1. $0 < \lambda_t < \frac{\mu_{\varepsilon,t}}{1+2s_{\varepsilon,t}}$, $\frac{\mu_{\varepsilon,t}}{1+2s_{\varepsilon,t}} < \lambda_t N_t^e < \mu_{\varepsilon,t}$ *and* $0 < \lambda_t\,(N_t^e - 1) < \mu_{\varepsilon,t}$.
2. $\lim_{N_t^e \to \infty} \lambda_t = 0$ *and, for* $\frac{N_t^e - 1}{d_t^2}$ *sufficiently small,* $\frac{\partial}{\partial N_t^e}\lambda_t < 0$.
3. $\lim_{N_t^e \to \infty} \lambda_t N_t^e = \lim_{N_t^e \to \infty} \lambda_t\,(N_t^e - 1) = \mu_{\varepsilon,t}$ *and, for* $\frac{N_t^e - 1}{d_t^2}$ *sufficiently small,* $\frac{\partial}{\partial N_t^e}\lambda_t N_t^e > 0$.

Proof. The limits can be calculated straightforwardly. In order to proof that λ_t is a decreasing, while $\lambda_t N_t^e$ is an increasing function of N_t^e, respectively, we take a first order approximation to $\sqrt{1 - 4\frac{N_t^e - 1}{d_t^2}}$, for $\frac{N_t^e - 1}{d_t^2} \to 0$. Thus, $\lambda_t \approx \frac{1}{d_t} = \frac{\mu_{\varepsilon,t}}{N_t^e + 2s_{\varepsilon,t}}$, and it is easy to see that $\frac{\partial}{\partial N_t^e}\lambda_t < 0$, while $\frac{\partial}{\partial N_t^e}\lambda_t N_t^e > 0$. ■

From Proposition 1 and Lemma 1 we observe that as the number of market timers working on each market diverges towards infinity, the quantity of capital invested by each market timer and also the deviation of asset prices from their fundamentals converge towards zero, i.e. $q_t^u \underset{N_t^e \to \infty}{\to} 0$ and $\Delta S_{F,t}^u \underset{N_t^e \to \infty}{\to} 0$, for each $u \geq 0$. Thus, the more are the market timers operating on each asset market, the more liquid are the asset markets and the lower is the excess deviation of the asset prices from their fundamentals.

3.3 Cross-market correlation

We are interested in the cross-market correlation, which can be defined as follows.

Definition 3 *The cross-market correlation at distance $u \geq 1$, is defined as $\Gamma_t(u) = \frac{\Delta S_{F,t}(u)}{\Delta S_{F,t}(0)}$, where*

$$\begin{aligned} \Delta S_{F,t}(u) &= \sum_{j=1}^{N_t^e (N_t^e - 1)^{u-1}} \Delta S_{F,t}^u \quad \text{for each } u > 0 \\ \Delta S_{F,t}(0) &= \Delta S_{F,t}^0 \quad \text{for } u = 0 \end{aligned}$$

$\Delta S_{F,t}(u)$ indicates the sum of the deviations of the asset prices from their fundamentals which are at distance u from the market where the shock to the fundamentals occurred. $\Gamma_t(u) \in [0, \infty)$ measures the total average susceptibility of asset prices at distance u from the asset market where the shock to the fundamentals occurred. If $\lim_{u \to \infty} \Gamma_t(u) \to 0$, then the total average susceptibility becomes vanishing small as u diverges towards infinity, while if $\lim_{u \to \infty} \Gamma_t(u) \to \infty$ then the total average susceptibility diverges towards infinity as u diverges towards infinity. Thus, in the former limit we have that the cross-market correlation is vanishing small, while in the latter case we observe long-ranged financial contagion.

Proposition 2 *The cross-market correlation at distance u is given by*

$$\Gamma_t(u) \approx C_t e^{-z_t(N_t^e) u}$$

where $z_t(N_t^e) = 1 - \lambda_t (N_t^e - 1)$ and $C_t = \frac{(N_t^e - 1)\lambda_t - \mu_{\varepsilon,t}}{(N_t^e - 1)\lambda_t - \lambda_t d_t (\mu_{\varepsilon,t} - 1) - 1} \lambda_t N_t^e$.

Proof. Using definition 3.3 we have that

$$\Delta S_{F,t}(u) = \lambda_t N_t^e [\lambda_t (N_t^e - 1)]^{u-1} \frac{(N_t^e - 1)\lambda_t - \mu_{\varepsilon,t}}{1 + \lambda_t [1 + d_t (\mu_{\varepsilon,t} - 1)]} \frac{\beta}{\alpha} \left(S_{t+\Delta t}^{*i} \Delta F\right)^2 \tag{12}$$

(12) can be rewritten as follows

$$\Delta S_{F,t}(u) \approx \lambda_t N_t^e e^{-z_t u} \frac{\left(N_t^e - 1\right)\lambda_t - \mu_{\varepsilon,t}}{1 + \lambda_t\left[1 + d_t\left(\mu_{\varepsilon,t} - 1\right)\right]} \frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i} \Delta F\right)^2 \tag{13}$$

where $z_t = 1 - \lambda_t\left(N_t^e - 1\right)$. ■

Lemma 2

1. $\lim\limits_{N_t^e \to \infty} z_t\left(N_t^e\right) = 1 - \mu_{\varepsilon,t}$ *and, for* $\frac{N_t^e - 1}{d_t^2}$ *sufficiently small,* $\frac{\partial}{\partial N_t^e} z_t\left(N_t^e\right) < 0$.

2. $\lim\limits_{N_t^e \to \infty} C_t = \mu_{\varepsilon,t} \frac{\mu_{\varepsilon,t}^2 - 1 - 2s_{\varepsilon,t}}{\mu_{\varepsilon,t} - 1 - 2s_{\varepsilon,t}}$ *and, for* $\frac{N_t^e - 1}{d_t^2}$ *sufficiently small,* $\frac{\partial}{\partial N_t^e} C_t > 0$.

Proof. The limits can be calculated straightforwardly. $\frac{\partial}{\partial N_t^e} z_t\left(N_t^e\right) < 0$ follows from Lemma 1. In order to proof that $\frac{\partial}{\partial N_t^e} C_t > 0$ we take a first order approximation to $\sqrt{1 - 4\frac{N_t^e - 1}{d_t^2}}$, for $\frac{N_t^e - 1}{d_t^2} \to 0$ and consequently it is easy to see that $\frac{\partial}{\partial N_t^e} C_t = \frac{\partial}{\partial N_t^e} \lambda_t N_t^e > 0$. ■

For $\mu_{\varepsilon,t} = 1$, we have that $z_t\left(N_t^e\right) > 0$ and as a consequence the cross-market correlation decreases in an exponential way with the distance u. From Lemma 2 we observe that for $N_t^e \to \infty$, $C_t \to 1$, while $z_t\left(N_t^e\right) \to 0$. Thus, the lower is N_t^e, the faster the cross-market correlation decreases down to zero as u increases, while as N_t^e increases, the cross-market correlation increases.

For $\mu_{\varepsilon,t} > 1$, we have that there exists an $\bar{N}_t\left(\mu_{\varepsilon,t}, s_{\varepsilon,t}\right)$ such that $z_t\left(\bar{N}_t\right) = 0$ and where $z_t\left(N_t^e\right) > 0$, for each $N_t^e < \bar{N}_t$ and $z_t\left(N_t^e\right) < 0$, for each $N_t^e > \bar{N}_t$. In Figure 3 we give examples of the function z_t.

We can treat the two cases in a unified way, if we define $\bar{N}_t\left(\mu_{\varepsilon,t}, s_{\varepsilon,t}\right) = \infty$, for each $\mu_{\varepsilon,t} \leqslant 1$. The following Proposition characterises the behaviour of the cross-market correlation.

Proposition 3 *a) If* $N_t^e < \bar{N}_t\left(\mu_{\varepsilon,t}, s_{\varepsilon,t}\right)$, *then the cross-market correlation decreases exponentially with the distance* u *at a rate* $z_t\left(N_t^e\right)$. *Further, this rate is decreasing in* N_t^e, *and bounded from below by* 0. *b) If* $N_t^e > \bar{N}_t\left(\mu_{\varepsilon,t}, s_{\varepsilon,t}\right)$, *then the cross-market correlation increases exponentially with the distance* u *at a rate* $-z_t\left(N_t^e\right) > 0$. *Further, this rate is increasing in* N_t^e *and upper bounded by* $\mu_{\varepsilon,t} - 1$. *c) If* $N_t^e = \bar{N}_t\left(\mu_{\varepsilon,t}, s_{\varepsilon,t}\right)$, *then the cross-market correlation remains non-negligible and further it is independent from the distance* u.

Proof. The proposition is a direct consequence of Lemma 3 and Proposition 2. ■

Part b) of Proposition 3 becomes relevant if $\mu_{\varepsilon,t} > 1$, for some $t > t'$. In this case, if N_t^e is sufficiently large, a single, small shock to the fundamentals

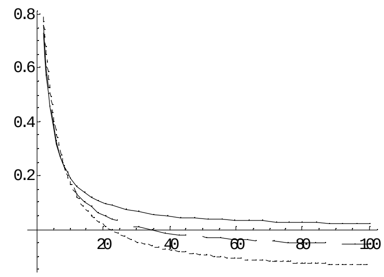

Figure 4: z as a function of N^e; $\mu_{\varepsilon,t} = e^{\sigma_F^2 t}$, $s_{\varepsilon,t} = e^{4\sigma_F^2 t}$ and $\sigma_F = 0.2$; Solidline: $t = 0$, dashed line: $t = 2$; dots: $t = 4$.

of an asset can lead to a long ranged financial contagion. The intuition for this result is the following. Consider the case where $\mu_{\varepsilon,t} > 1$. The larger is $\mu_{\varepsilon,t}$, the larger is the quantity of capital which has to be withdrawn from neighbouring (correlated) asset markets in order to exploit the gains from the temporary deviations of asset prices. Consequently, this leads to a larger deviation of the asset price from its normal level in these correlated asset markets, i.e. to a stronger propagation mechanism.

3.4 Asset price dynamics

In the previous Section we concentrated on the relationship between the average number of market timers operating on the asset markets (N_t^e) and the cross-market correlation between asset prices. In this Section we will see how N_t^e affects the liquidity of asset markets and consequently the excess volatility of asset prices.

The expected deviation of asset price i from its normal level and the second moment of this deviation are given by

$$E\left(\Delta S_{F,t}^i\right) = \left(\tilde{p}_- + \tilde{p}_+\right) \sum_{u=0}^{\infty} E\left(\widetilde{\Delta S}_{F,t}^i(u)\right)$$

$$E\left(\left(\Delta S_{F,t}^i\right)^2\right) = \left(\tilde{p}_- + \tilde{p}_+\right) \sum_{u=0}^{\infty} E\left(\left(\widetilde{\Delta S}_{F,t}^i(u)\right)^2\right)$$

where

$$\sum_{u=0}^{\infty} E\left(\widetilde{\Delta S}_{F,t}^i(u)\right) = -\frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i}\Delta F\right)^2 \left\{\left(1 - \frac{\lambda_t N_t^e}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\right) + s_{\varepsilon,t}\frac{\mu_{\varepsilon,t}-(N_t^e-1)\lambda_t}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\lambda_t N_t^e\left[1+\lambda_t\left(N_t^e-1\right)+\ldots+\left(\lambda_t\right)^{U-1}\left(N_t^e-1\right)^{U-1}\right]\right\} \tag{14}$$

$$\sum_{u=0}^{\infty} E\left(\left(\widetilde{\Delta S}_{F,t}^{i}(u)\right)^2\right) = \left[\frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i}\Delta F\right)^2\right]^2 \left\{\left(1-\frac{\lambda_t N_t^e}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\right)^2 + \nu_{\varepsilon,t}\left(\frac{\mu_{\varepsilon,t}-(N_t^e-1)\lambda_t}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\lambda_t\right)^2 N_t^e\left[1+(\lambda_t)^2\left(N_t^e-1\right)+...+(\lambda_t)^{2(U-1)}\left(N_t^e-1\right)^{U-1}\right]\right. \tag{15}$$

The excess volatility of the asset price is defined as σ_{S_F}, where $\sigma_{S_F}^2 = E\left(\left(\Delta S_{F,t}^i\right)^2\right) - \left(E\left(\Delta S_{F,t}^i\right)\right)^2$.

We can now describe the dynamics of the deviation of the asset price from its fundamentals. Given that an asset market has been chosen, with probability p and $1-p$ its fundamentals increase and decrease, respectively. Further we assume that each asset market is chosen with probability $\frac{1}{n^2}$. Thus, we assume that in each period of time, with probability $\frac{1}{n}$ one asset market, chosen at random, will be hit by a shock (either negative or positive) to its fundamentals, while with probability $1-\frac{1}{n}$ no asset market will be hit by a shock to the fundamentals. In other words, we have that $\tilde{p}_+ + \tilde{p}_- = \frac{1}{n^2}$. In this way we introduce implicitly a time scale of order $dt = \frac{1}{n^2}$.

Proposition 4 *For a time scale $dt = \frac{1}{n^2}$, the diffusion process describing the dynamics of the fundamentals F^i and of the asset price S^i will be, respectively*

$$dF_t^i = \mu dt + \sigma_F dw_t^i \tag{16}$$

$$dS_t^i = S_t^i\left(\mu + \frac{1}{2}\sigma_F^2\right)dt + \sigma_F S_t^i dw_t^i \tag{17}$$

where dw_t^i is a Wiener process, i.i.d. across $i = 1, ..., n$, μ and σ_F are the drift and the volatility of the fundamentals of the asset. The excess volatility of the asset price will be

$$\sigma_{S,t} = \frac{\beta}{\alpha}\sigma_F^2\left(S_t^i\right)^2\vartheta_t \tag{18}$$

where $\vartheta_t^2 = \left\{1-\frac{\lambda_t N_t^e}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\right\}^2 + \nu_{\varepsilon,t}\left\{\frac{\mu_{\varepsilon,t}-(N_t^e-1)\lambda_t}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\right\}^2\frac{(\lambda_t)^2 N_t^e}{1-(\lambda_t)^2(N_t^e-1)}$.

Proof. See Appendix. ■

Lemma 3 $\lim_{N_t^e\to\infty}\vartheta_t^2 = 0$. *Further, for $\frac{N_t^e-1}{d_t^2}$ sufficiently small, there exists a $\tilde{N}_t\left(s_{\varepsilon,t},\nu_{\varepsilon,t}\right)$ such that for each $N_t^e > \tilde{N}_t\left(s_{\varepsilon,t},\nu_{\varepsilon,t}\right)$, ϑ_t^2 is a decreasing function of N_t^e.*

Proof. For small values of $\frac{N_t^e-1}{d_t^2}$ we have that $\lambda_t \approx \frac{1}{d_t} = \frac{\mu_{\varepsilon,t}}{N_t^e+2s_{\varepsilon,t}}$. Thus, we have that $\vartheta_t^2 = \vartheta_{1,t}^2\vartheta_{2,t}^2$ where

$$\vartheta_{1,t}^2 = \left(1-\frac{\lambda_t N_t^e}{\mu_{\varepsilon,t}+\lambda_t}\right)^2$$

$$\vartheta_{2,t}^2 = 1 + \nu_{\varepsilon,t} \frac{(\lambda_t)^2 N_t^e}{1 - (\lambda_t)^2 (N_t^e - 1)}$$

Taking the first derivative of ϑ_t^2 with respect to N_t^e we obtain $\frac{\partial}{\partial N_t^e}\vartheta_t^2 = \frac{\partial \vartheta_{1,t}^2}{\partial N_t^e}\vartheta_{2,t}^2 + \frac{\partial \vartheta_{2,t}^2}{\partial N_t^e}\vartheta_{1,t}^2$, where

$$\frac{\partial}{\partial N_t^e}\vartheta_{1,t}^2 = -\left(1 - \frac{\lambda_t N_t^e}{\mu_{\varepsilon,t} + \lambda_t}\right) \frac{\frac{\partial \lambda_t N_t^e}{\partial N_t^e}\left(\mu_{\varepsilon,t} + \lambda_t\right) - \lambda_t N_t^e \frac{\partial \lambda_t}{\partial N_t^e}}{\left(\mu_{\varepsilon,t} + \lambda_t\right)^2} < 0 \tag{19}$$

$$\frac{\partial}{\partial N_t^e}\vartheta_{2,t}^2 = \nu_{\varepsilon,t}\mu_{\varepsilon,t}^2 \frac{(N_t^e + 2s_{\varepsilon,t})(2s_{\varepsilon,t} - N_t^e) + \mu_{\varepsilon,t}^2}{\left[(N_t^e + 2s_{\varepsilon,t})^2 - \mu_{\varepsilon,t}^2 (N_t^e - 1)\right]^2} \tag{20}$$

It is easy to see that (19) is negative, while (20) has an ambiguous sign. For N_t^e sufficiently large, the sign of (20) is unambiguously negative. ■

From Proposition 4 and Lemma 3 we have that, given that N_t^e is sufficiently large, an increase in N_t^e increases the liquidity of the asset markets, reducing the excess volatility of the asset price. On the other side, an increase in $\frac{\beta}{\alpha}$, increases the excess volatility of the asset price. In Section 4 we will see how $\frac{\beta}{\alpha}$ affects the excess volatility in a indirect (liquidity) and a direct way, respectively.

Next we state a result about the moments of the distribution of relative asset prices.

Lemma 4 *The moments of the distribution of relative asset prices* $\varepsilon_t^{i,j} = \frac{S_t^i}{S_t^j}$ *are given as follows:* $E\left(\varepsilon_t^{i,j}\right) = \mu_{\varepsilon,0}e^{\sigma_F^2 t}$, $E\left(\left(\varepsilon_t^{i,j}\right)^2\right) = s_{\varepsilon,0}e^{4\sigma_F^2 t}$, $E\left(\left(\varepsilon_t^{i,j}\right)^3\right) = \omega_{\varepsilon,0}e^{9\sigma_F^2 t}$, $E\left(\left(\varepsilon_t^{i,j}\right)^4\right) = \nu_{\varepsilon,0}e^{16\sigma_F^2 t}$.

Proof. Integrating (17) we obtain the following $S_t^i = S_0^i e^{\mu t + \sigma_F w_t^i - \sigma_F^2 \frac{t}{2}}$. Since the fundamentals of all assets share the same drift and volatility we have that the relative asset price is given by $\varepsilon_t^{i,j} = \varepsilon_0^{i,j} e^{\sigma_F\left(w_t^i - w_t^j\right)}$, where the moments are calculated straightforwardly. ■

From Lemma 4 we observe that the first four moments of the distribution of relative asset prices are exponentially increasing in time. We will see in Section 4 the importance of this result for the cross-market correlation and the excess volatility of asset prices.

In the next Section we will calculate the average number of market timers operating on each asset market.

3.5 Number of market timers

We have still to calculate the average number of market timers operating on each asset market. We have seen in the previous Section that the number of market timers influences the liquidity of the single asset markets and also the cross-market correlation of asset prices.

The problem of the market timers is to decide, in each period of time, whether to participate or not. We assume that in each period of time, market timers face participation costs per unit of time, i.e. $c\left(S_t^i\right) dt$. Potential market timers calculate the expected gain from participating, knowing the stochastic process of the fundamentals of the assets F, and decide to participate if the expected gains from participating are at least as large as the costs.

The expected gains from operating on a generic asset market i are

$$E\left(\Pi_t^i\left(m^i, N_t^e\right)\right) = \pi_t^{i,0} + \sum_{j\in\aleph_t^i} \pi_t^{j,1} + \sum_{j\in\aleph_t^i} \sum_{k\in\left\{\aleph_t^j - j\right\}} \pi_t^{k,1} + ...$$

Using the assumption that $\frac{\Delta F}{n} = \sigma_F \sqrt{dt}$ and setting the time scale $dt = \frac{1}{n^2}$ we can state the following Proposition.

Definition 4 *The number of market timers operating on each market, in each period of time, is given by the solution of the following problem*

$$N_t^i\left(N_t^e\right) = \arg\min_{m^i} \left\{E\left[\Pi_t^i\left(m^i, N_t^e\right)\right] \geqslant c\left(S_t^i\right)\right\}$$

$$N_t^e = \frac{1}{n}\sum_{i=1}^{n} N_t^i\left(N_t^e\right)$$

where

$$E\left(\Pi_t^i\left(N_t^i, N_t^e\right)\right) = \frac{\beta^2}{\alpha}\left(\sigma_F^2\right)^2\left(S_t^i\right)^4\left\{\lambda_t \frac{1-\lambda_t[N_t^e-1+s_{\varepsilon,t}-d_t(\mu_\varepsilon-1)]+(\lambda_t)^2(N_t^e-1)\mu_{\varepsilon,t}}{\left\{1+\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)+N_t^i-N_t^e\right]\lambda_t\right\}^2} + \right.$$
$$\left. + \left(\frac{\lambda_t}{1+\lambda_t\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]}\right)^2 \lambda_t \frac{\mu_{\varepsilon,t}-\lambda_t(N_t^e-1+s_{\varepsilon,t})+(\lambda_t)^2(N_t^e-1)\mu_{\varepsilon,t}}{\left(1+\frac{N_t^i-N_t^e}{\mu_{\varepsilon,t}}\lambda_t\right)^2}\nu_{\varepsilon,t}\frac{N^i}{1-(\lambda_t)^2(N_t^e-1)}\right\} \quad (21)$$

For the following we take a first order approximation to the equilibrium defined in Definition 4. In Proposition 5 we state the condition characterising the average number of market timers operating on each asset market and further we state sufficient conditions for a solution of the same.

Proposition 5 *Consider an economy with n assets, where $n \to \infty$ and where one asset price has been normalised to one. a) The condition characterising the* *first order approximation to the average number of market timers operating in equilibrium is the following*

$$E\left(\Pi_t\left(N_t^e\right)\right)=E\left(c\left(S_t^i\right)\right) \tag{22}$$

where

$$\begin{gathered} E\left(\Pi_t\left(N_t^e\right)\right)=\frac{\lambda_t}{\left\{1+\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)\right]\lambda_t\right\}^2}\nu_{\varepsilon,t}\frac{\beta^2}{\alpha}\left(\sigma_F^2\right)^2\times \\ \left\{1-\lambda_t\left[N_t^e-1+s_{\varepsilon,t}-d_t\left(\mu_\varepsilon-1\right)\right]+\lambda_t^2\left(N_t^e-1\right)\mu_{\varepsilon,t}+\right. \\ \left.+\left[\mu_{\varepsilon,t}-\lambda_t\left(N_t^e-1+s_{\varepsilon,t}\right)+\lambda_t^2\left(N_t^e-1\right)\mu_{\varepsilon,t}\right]\nu_{\varepsilon,t}\frac{\lambda_t^2N_t^e}{1-\lambda_t^2\left(N_t^e-1\right)}\right\} \end{gathered} \tag{23}$$

b) A sufficient condition for the solution of (22) is that

$$E\left(c\left(S_t^i\right)\right)\leqslant\frac{\beta^2}{\alpha}\left(\sigma_F^2\right)^2\frac{\nu_{\varepsilon,t}}{4\left(1+s_{\varepsilon,t}\right)}\left(1+\frac{\nu_{\varepsilon,t}\mu_{\varepsilon,t}^2}{\left(1+2s_{\varepsilon,t}\right)^2}\right)$$

Proof. Consider first part a) of the Proposition. For each asset market i we have that

$$E\left(\Pi_t^i\left(N_t^i,N_t^e\right)\right)\geqslant c\left(S_t^i\right) \tag{24}$$

where N_t^i is a solution of the problem stated in Definition 4. Now multiply both sides of (24) by $\left\{1+\left[1+d_t\left(\mu_{\varepsilon,t}-1\right)+N_t^i-N_t^e\right]\lambda_t\right\}^2\left(1+\frac{N_t^i-N_t^e}{\mu_{\varepsilon,t}}\lambda_t\right)^2$ and sum all inequalities over i and divide both sides by n. Taking the limit where $n\to\infty$ and further, taking a first order approximation where $N_t^i\to N_t^e$ and neglecting higher order correlations between S_t^i and N_t^i we obtain (22), Consider now part b) of the Proposition. Since the support for N_t^e is bounded from below by 1, while it is unbounded from above, we can take the limit

$$\lim_{N_t^e\to\infty}E\left(\Pi_t\left(N_t^e\right)\right)=0$$

$$\lim_{N_t^e\to 1}E\left(\Pi_t\left(N_t^e\right)\right)=\frac{\beta^2}{\alpha}\left(\sigma_F^2\right)^2\frac{\nu_{\varepsilon,t}}{4\left(1+s_{\varepsilon,t}\right)}\left(1+\frac{\nu_{\varepsilon,t}\mu_{\varepsilon,t}^2}{\left(1+2s_{\varepsilon,t}\right)^2}\right)>0$$

Since $E\left(\Pi_t\left(N_t^e\right)\right)$ is a continuous function, at least one solution to (22) exists. ■

In the following Lemma we characterise further the properties of the equilibrium (22).

Lemma 5 $\lim_{N_t^e\to\infty}E\left(\Pi_t\left(N_t^e\right)\right)=0$. *Further, for* $\frac{N_t^e-1}{d_t^2}$ *sufficiently small, there exists a* $\hat{N}_t\left(\mu_{\varepsilon,t},s_{\varepsilon,t},\nu_{\varepsilon,t}\right)$ *such that for each* $N_t^e>\hat{N}_t\left(\mu_{\varepsilon,t},s_{\varepsilon,t},\nu_{\varepsilon,t}\right)$, $E\left(\Pi_t\left(N_t^e\right)\right)$ *is a decreasing function of* N_t^e.

Proof. The limit can be calculated straightforwardly. *For* $\frac{N_t^e-1}{d_t^2}$ *sufficiently small,*

$$E\left(\Pi_t\left(N_t^e\right)\right)=\nu_{\varepsilon,t}\frac{\beta^2}{\alpha}\left(\sigma_F^2\right)^2\Pi_{1,t}\left(N_t^e\right)\Pi_{2,t}\left(N_t^e\right)\Pi_{3,t}\left(N_t^e\right)$$

where

$$\Pi_{1,t}\left(N_t^e\right)=\frac{\lambda_t}{\left(\mu_{\varepsilon,t}+\lambda_t\right)^2}$$

$$\Pi_{2,t}\left(N_t^e\right)=\left(\mu_{\varepsilon,t}-\lambda_t\left(N_t^e-1+s_{\varepsilon,t}\right)+\lambda_t^2\left(N_t^e-1\right)\mu_{\varepsilon,t}\right)$$

$$\Pi_{3,t}\left(N_t^e\right)=1+\nu_{\varepsilon,t}\frac{\left(\lambda_t\right)^2N_t^e}{1-\left(\lambda_t\right)^2\left(N_t^e-1\right)}$$

where

$$\frac{\partial}{\partial N_t^e}\Pi_{1,t}\left(N_t^e\right)=\frac{\partial\lambda_t}{\partial N_t^e}\frac{\mu_{\varepsilon,t}-\lambda_t}{\left(\mu_{\varepsilon,t}+\lambda_t\right)^3}<0$$

$$\frac{\partial}{\partial N_t^e}\Pi_{2,t}\left(N_t^e\right)=-\mu_{\varepsilon,t}\frac{\left(1+s_{\varepsilon,t}\right)\left(1-2\mu_{\varepsilon,t}\right)+\mu_{\varepsilon,t}^2N_t^e}{\left(N_t^e+2s_{\varepsilon,t}\right)^2}$$

$$\frac{\partial}{\partial N_t^e}\Pi_{3,t}\left(N_t^e\right)=\nu_{\varepsilon,t}\mu_{\varepsilon,t}^2\frac{\left(N_t^e+2s_{\varepsilon,t}\right)\left(2s_{\varepsilon,t}-N_t^e\right)+\mu_{\varepsilon,t}^2}{\left[\left(N_t^e+2s_{\varepsilon,t}\right)^2-\mu_{\varepsilon,t}^2\left(N_t^e-1\right)\right]^2}$$

Notice that $\frac{\partial}{\partial N_t^e}\Pi_{1,t}\left(N_t^e\right)$ is unambiguously negative, while $\frac{\partial}{\partial N_t^e}\Pi_{2,t}\left(N_t^e\right)$ and $\frac{\partial}{\partial N_t^e}\Pi_{3,t}\left(N_t^e\right)$ are negative only if N_t^e is sufficiently large. ■

4 Discussion and Simulation results

In this Section we put together the results seen in the previous Sections. In particular, we are interested in how cross-market correlation and volatility of asset prices change as structural parameters, such as α, β and c change.

We can summarise the previous results as follows: the larger is N^e a) the larger is the cross-market correlation (Lemma 2 and Proposition 3) and b) the lower is the excess volatility of asset prices. Thus, market timers have a stabilising effect on asset prices, i.e. they increase the liquidity of asset markets, while they increase the cross-market correlation. From Proposition 5 and Lemma 5 we observe also that the larger are the participation costs for market timers, the lower will be N^e. Consequently, the larger are the participation costs for market timers, the larger will be the excess volatility but the lower will be the cross-market correlation.

From Proposition 5 and Lemma 5 we observe that the larger is $\frac{\beta^2}{\alpha}$ the larger are the number of market timers. Thus, the larger is $\frac{\beta^2}{\alpha}$ the larger is N^e and as a consequence the larger is the cross-market correlation. On the other side, β (α) has an ambiguous effect on the excess volatility. In particular, a direct and an indirect effect can be distinguished: an increase (decrease) in β (α) increases directly the excess volatility (direct effect), while it decreases indirectly the excess volatility through an increase in N^e (liquidity effect).

We provide a numerical solution for the model. We normalise one asset price to one. The average excess volatility (18) is given by

$$\sigma_{S,t} = V_t S_t^2$$

where $V_t = \sigma_F^2 \frac{\beta}{\alpha} \vartheta_t$ and the average expected profits for each market timer are given by $E\left(\Pi_t\left(N_t^e\right)\right)$ (23).

In order to simplify the analysis, we assume that at time 0 all asset prices are the same, i.e. $\mu_{\varepsilon,0} = s_{\varepsilon,0} = \omega_{\varepsilon,0} = \nu_{\varepsilon,0} = 1$. We made simulations using $\sigma_F = 0.2$. In the Figures 4-15 we provide some simulation results. Notice that in the case where $c = 0.001$ and $c = 0.002$ the participation costs are a constant fraction of the reference asset price, while in the case where $c = 0.001S^2$ and $c = 0.002S^2$ the participation costs are related to the volatility of the relative asset prices.

From Lemma 4 we know that the moments of the distribution of relative asset prices are increasing in time. From the Figures below we observe that this leads to an increase in the expected gain of market timers and thus more and more market timers enter the market as time goes on. Consequently, the excess volatility of asset prices decreases while the cross-market correlation increases. Further, we observe the stabilising function of business traders - destabilising function of program traders. In particular, the more are the business traders (i.e. the larger is α), the lower is the excess volatility of asset prices and the lower is the cross-market correlation. On the other side, the more are the program traders (i.e. the larger is β), the larger is the excess volatility and the larger is the cross-market correlation. Thus, regarding the excess volatility, the direct effect dominates the indirect (liquidity effect).

From the Figures below we observe that long- ranged financial contagion is possible. This latter effect is due to the continuous increase in $\mu_{\varepsilon,t}$: as this latter variable increases, a larger quantity of capital has on average to be withdrawn from correlated asset markets in order to exploit the gains and thus, the propagation mechanism is becoming stronger. While the cross-market correlation does not directly depend on α, β and c, it depends indirectly on these variables through N_t^e. In particular, we observe from these Figures that the cross-market correlation can be reduced if either α or c are increased, or β is decreased. In this way, long ranged financial contagion can be avoided.

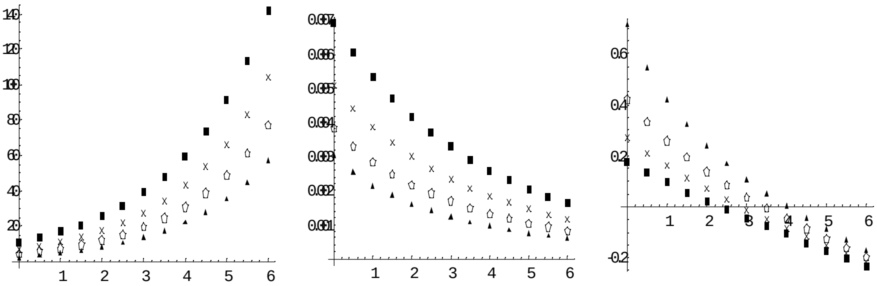

Figure 4: $c = 0.001$; $\beta = 50$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

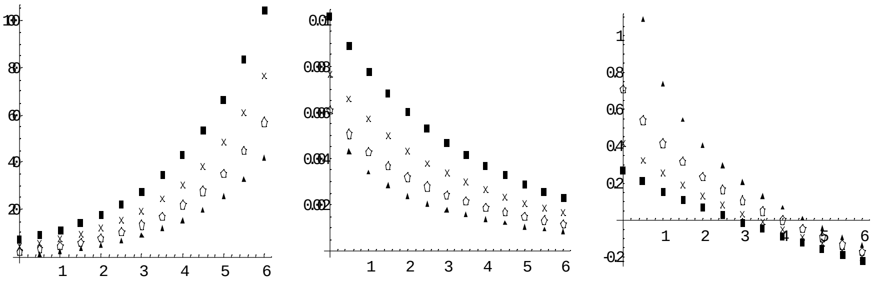

Figure 5: $c = 0.002$; $\beta = 50$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

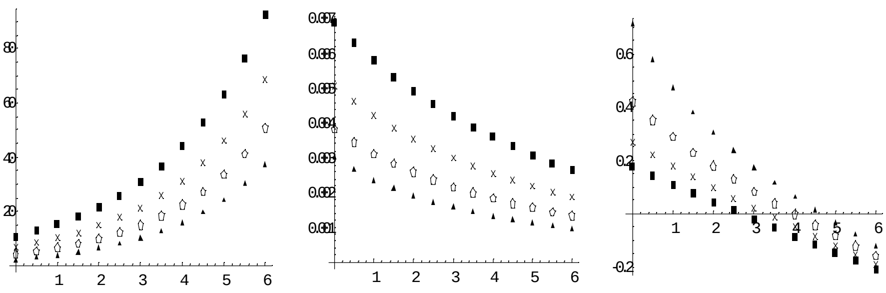

Figure 6: $c = 0.001S^2$; $\beta = 50$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t

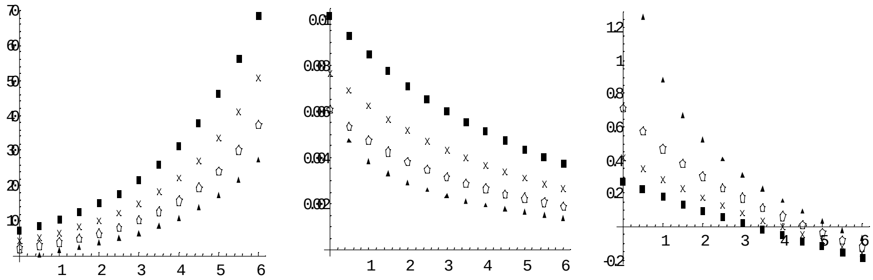

Figure 7: $c = 0.002S^2$; $\beta = 50$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

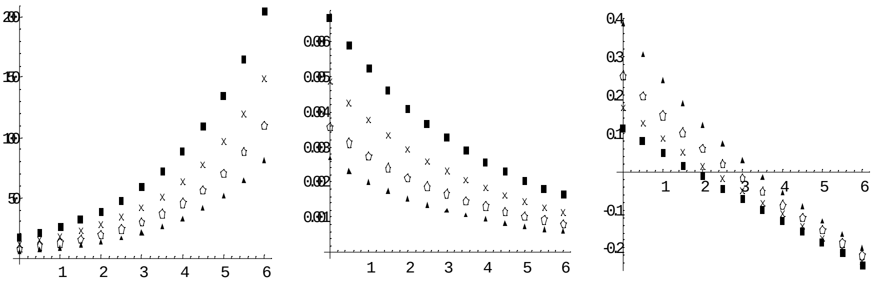

Figure 8: $c = 0.001$; $\beta = 75$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

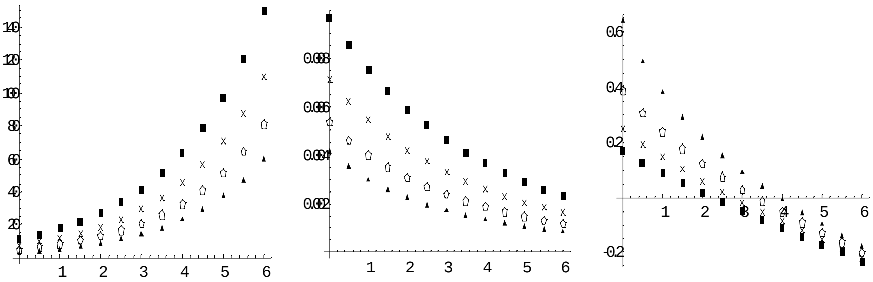

Figure 9: $c = 0.002$; $\beta = 75$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

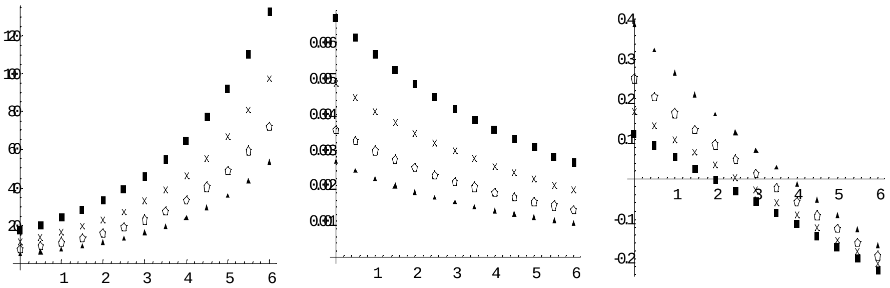

Figure 10: $c = 0.001S^2$; $\beta = 75$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

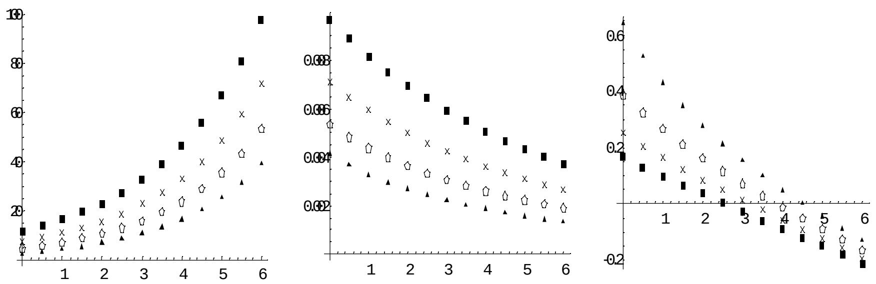

Figure 11: $c = 0.002S^2$; $\beta = 75$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

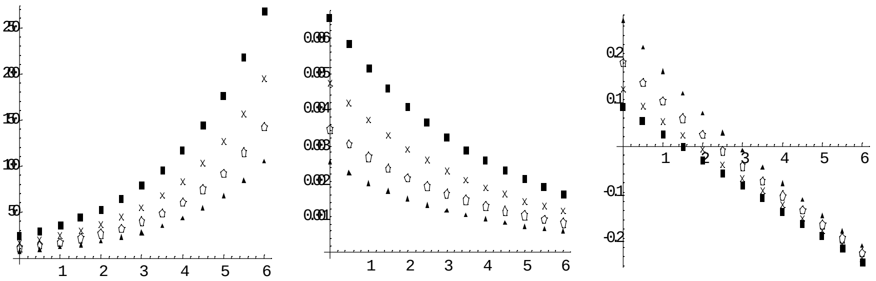

Figure 12: $c = 0.001$; $\beta = 100$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t

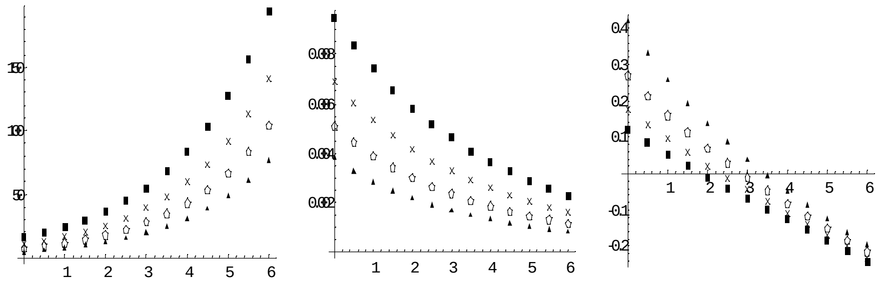

Figure 13: $c = 0.002$; $\beta = 100$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

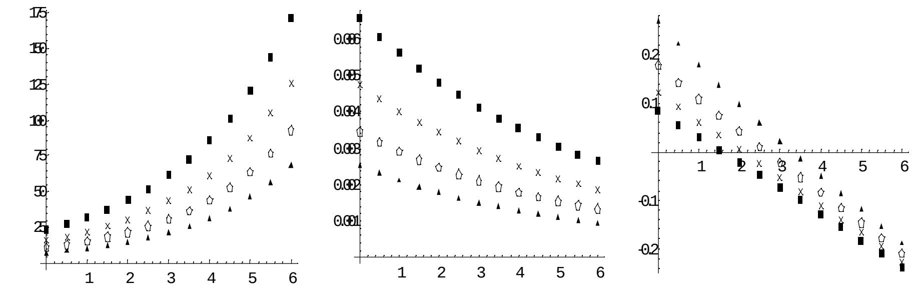

Figure 14: $c = 0.001S^2$; $\beta = 100$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t

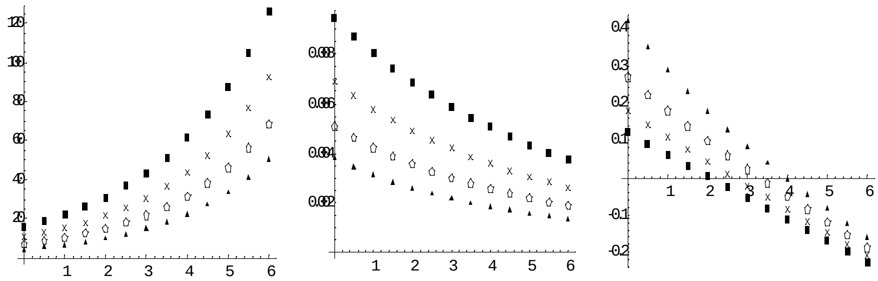

Figure 15: $c = 0.002S^2$; $\beta = 100$, $\alpha = 50$ (blackboxes), $\alpha = 100$ (stars), $\alpha = 200$ (polygon) and $\alpha = 400$ (triangle); LHS: N^e as a function of t; Center: V_t as a function of t; RHS: z as a function of t.

5 Conclusion

In this paper we tried to address the relationships between liquidity of asset markets and the cross-market correlation and excess volatility of asset prices. We proposed a simple model where market timers shift capital around asset markets in order to exploit temporary deviations of asset prices from their fundamental values. The number of market timer operating on each asset market influences the liquidity of the asset market, and consequently the excess volatility of asset prices as well as the cross-market correlation between asset markets. We derived an equilibrium for the number of market timers operating on each market in each period of time. In this way, liquidity of each asset market is determined endogenously, and depends on structural parameters characterising the asset markets.

Appendix

Proof of Proposition 4. Let us first derive the discrete dynamics, and after this pass to continuous time dynamics.

(14) and (15) simplify to the following

$$E\left(\Delta S_{F,t}^{i}\right)=\begin{cases}\left(\tilde{p}_{+}-\tilde{p}_{-}\right)\frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i}\Delta F\right)^{2}\rho_{t} & for \quad N_{t}^{e}<\bar{N}_{t}\\ \left(\tilde{p}_{+}-\tilde{p}_{-}\right)\frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i}\Delta F\right)^{2}\tilde{\rho}_{t}\left(U\right) & for \quad N_{t}^{e}>\bar{N}_{t}\end{cases} \tag{25}$$

$$E\left(\left(\Delta S_{F,t}^{i}\right)^{2}\right)=\left(\tilde{p}_{+}+\tilde{p}_{-}\right)\left(\frac{\beta}{\alpha}\right)^{2}\left(S_{t+\Delta t}^{*i}\Delta F\right)^{4}\vartheta_{t}^{2}$$

where

$$\rho_{t}=1-\frac{\lambda_{t}N_{t}^{e}}{1+\lambda_{t}\left[1+d_{t}\left(\mu_{\varepsilon,t}-1\right)\right]}\left(1-\frac{\mu_{\varepsilon,t}-\left(N_{t}^{e}-1\right)\lambda_{t}}{1-\left(N_{t}^{e}-1\right)\lambda_{t}}s_{\varepsilon,t}\right)$$

$$\tilde{\rho}_{t}\left(U\right)=1-\frac{1}{1+\lambda_{t}\left[1+d_{t}\left(\mu_{\varepsilon,t}-1\right)\right]}\left(\lambda_{t}N_{t}^{e}-\frac{\mu_{\varepsilon,t}-\left(N_{t}^{e}-1\right)\lambda_{t}}{\left(N_{t}^{e}-1\right)\lambda_{t}-1}s_{\varepsilon,t}n\left(\lambda_{t}\right)^{U}\right)$$

where $U=\max\left(u\right)$, which is defined implicitly as $n=N_{t}^{e}\left(N_{t}^{e}-1\right)^{U-1}$. For $N_{t}^{e}<<n$, we have that U is large and so $\left(\lambda_{t}\right)^{U}$ is negligible small.

Using the assumptions about the timing stated above, we obtain the following moments for the change of the deviation of the asset price from its fundamentals

$$E\left(\Delta S_{F,t}^{i}\right)=\frac{2p-1}{n}\frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i}\right)^{2}\frac{\left(\Delta F\right)^{2}}{n}\rho_{t}^{*} \tag{26}$$

$$Var\left(\Delta S_{F,t}^{i}\right)=\left[\vartheta_t^2-\left(\frac{2p-1}{n}\rho_t^*\right)^2\right]\left[\frac{\beta}{\alpha}\left(S_{t+\Delta t}^{*i}\right)^2\frac{(\Delta F)^2}{n}\right]^2 \tag{27}$$

where $\rho_t^*\in\{\rho_t,\tilde{\rho}_t(U)\}$. Further, the mean and variance of the change in the fundamentals of a single asset are given by

$$E\left(\Delta F^i\right)=\frac{2p-1}{n}\frac{\Delta F}{n} \tag{28}$$

$$Var\left(\Delta F^i\right)=\left(\frac{\Delta F}{n}\right)^2\left[1-\left(\frac{2p-1}{n}\right)^2\right] \tag{29}$$

In order to pass from discrete to continuous time, we will make the assumptions that the size of the change in the fundamentals is $\frac{\Delta F}{n}=\sigma_F\sqrt{dt}$ and that the probability of receiving a positive shock p is given by $p=\frac{1}{2}\left(1+n\frac{\mu}{\sigma_F}\sqrt{dt}\right)$. Inserting these expressions in (26), (27), (28) and (29) we obtain the following results

$$E\left(\Delta F^i\right)=\mu dt \tag{30}$$

$$Var\left(\Delta F^i\right)=\sigma_F^2 dt\left[1-\left(\frac{\mu}{\sigma_F}\right)^2 dt\right]\underset{dt\to 0}{\to}\sigma_F^2 dt \tag{31}$$

$$E\left(\Delta S_{F,t}^{i}\right)=\frac{\beta}{\alpha}\left(S_t^{*i}\right)^2\mu\sigma_F dtn\sqrt{dt}$$

$$Var\left(\Delta S_{F,t}^{i}\right)=\left[\vartheta_t^2-\left(\frac{\mu}{\sigma_F}\rho_t^*\right)^2 dt\right]\left(\frac{\beta}{\alpha}\right)^2\left(S_t^{*i}\right)^4\sigma_F^2 dt\sigma_F^2 n^2 dt$$

From (30) and (31) we obtain (16) stated in Proposition 4. Now, setting a time scale of $dt=\frac{1}{n^2}$ we obtain

$$E\left(\Delta S_{F,t}^{i}\right)=\frac{\beta}{\alpha}\left(S_t^{*i}\right)^2\rho_t^*\sigma_F\mu dt \tag{32}$$

$$\begin{aligned}Var\left(\Delta S_{F,t}^{i}\right)=&\left(\vartheta_t^2-\left(\tfrac{\mu}{\sigma_F}\rho_t^*\right)^2 dt\right)\left(\tfrac{\beta}{\alpha}\right)^2\left(S_{t+\Delta t}^{*i}\right)^4\left(\sigma_F^2\right)^2\underset{dt\to 0}{\to}\\&\left(\tfrac{\beta}{\alpha}\right)^2\left(S_t^{*i}\right)^4\left(\sigma_F^2\right)^2\vartheta_t^2 dt\end{aligned} \tag{33}$$

where this latter limit holds since $(\lambda_t)^{2U}$ becomes vanishing small as $dt\to 0$ while N_t^e remains finite. ■

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