Advertising and endogenous exit in a differentiated duopoly*

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Abstract

In this paper we consider a duopoly two-stage duopoly where firms first decide whether to invest in advertising and then compete in prices. Advertising has two effects: a market enlargement for both firms and a predatory gain for the investing firm only.

Both symmetric and asymmetric equilibria may arise. The two most interesting cases are a *coordination game* where both firms investing and non-investing are equilibria, and a *chicken game* where only one firm invests while the other is possibly driven (endogenously) out of the market. Our results suggest that product differentiation has an ambiguous impact on investment in advertising and that strong product substitutability may induce a coordination problem.

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1 Introduction

The role of advertising in the competition among firms has always represented an interesting issue. Advertising has been studied following different aspects of its nature. Advertising can be *informative*, given that it provides informations to consumers about the potential quality of a brand. Furthermore, by advertising, a firm reveals the features of its product and this tends to increase product differentiation. As Kaldor acknowledged:

> Advertising is a method of differentiating, in the eyes of the consumer, the products of one firm from those of its competitor; it is a method, therefore, of reducing the scope and effectiveness of price-competition by attaching a strong element of "goodwill" to each firm (Kaldor, 1950, p.14).

But advertising is also *persuasive*, given that the investing firm could aim at convincing the consumer that what he really wants is its particular variety. The dual role of advertising becomes then evident. On the one hand, advertising acts to shift firms' demand curves; while on the other hand, it makes a good more differentiated from the one produced by rivals. The study of this tension will be one of the main subject of this paper.

The economic literature has initially dealt with the negative impact of advertising with respect to welfare considerations. Advertising has in fact been considered as socially harmful. Kaldor (1950) himself recognized that advertising could have a "manipulative" effect that reduces competition by convincing consumers that two identical products are differentiated. On the other hand, Nelson (1970) and Demsetz (1979) acknowledged the beneficial function of advertising when it conveys the right information to consumers, whose searching costs then tend to decline.

Moreover, advertising could give rise to barriers to entry for newcomers that would need to spend a substantial amount of money to overcome the reputation of the incumbents. Many authors focused on the issue of strategic advertising as an instrument to deter entry (Bagwell and Ramey, 1988 and 1990). Schmalensee (1983) considered a duopoly two-stage Cournot model where an initial investment in advertising was able to deter the entry of new rivals. More recently, Ishigaki (2000) found that Schmalensee's results did not hold in a similar Bertrand setting.

In our study of advertising we will be particularly concerned with two aspects: first, we analyze the impact of advertising in the enlargement of a market for a 'non well-known' product; second, we explicitly deal with the predatory interaction that could characterize advertising games.

The first consideration comes from the fact that consumers might not be fully aware of the presence of certain types of products in the market. This is especially true for products belonging to the hi-tech sector. A firm that develops a 'novelty' must invest resources to explain which kind of product has become available. The creation of a new market, or the enlargement of an existing one, could represent nonetheless an advantage for a potential rival, which would benefit from an information spill-over that shifts the demand curve upward for all those kinds of goods. In the literature this has been often referred to as *cooperative* advertising (Friedman, 1983; Martin, 1993, ch. 6), even if firms do not necessarily cooperate in the profit maximization stage. The issue of advertising that increases the size of the market has been analyzed also in a dynamic setting (see Jorgensen, 1982 and Dockner et al., 2000, ch.11 for exhaustive surveys). Cellini and Lambertini (2002) consider a differential oligopoly game with differentiated goods where firms compete *à la* Cournot in the market phase and may finance advertising to enlarge their market shares. Furthermore, each firm's advertising effort produces a positive spill-over for the rival in terms of market enlargement.

The second consideration mentioned above refers to the conventional view that advertising also creates "brand loyalty" and "goodwill" that sticks to a determined brand. In particular, we focus on the *predatory* nature of advertising (Friedman, 1983; Martin, 1993, ch. 6). In fact, by engaging in advertising, a firm increases its own demand while at the same time it reduces the demand of the rival. An example is given by the use of comparative advertising, through which a firm compare the characteristics of its product with those of the competitors.[1] Crucially, and that is why we decided to deal with price competition and product differentiation, this is more likely to happen the higher the substitutability among the products. The degree of differentiation on the product market has a direct impact on advertising decisions, and this interaction could not be properly modeled in the standard quantity competition framework. Grossman and Shapiro (1984), for example, considered a differentiation duopoly model with price competition and showed that advertising is positively related to the degree of product differentiation. Other models dealing with such a relationship can be found in Butters (1977), Wolinsky (1984) and Von der Fehr and Stevik (1998).

As a consequence, in our analysis an investment in advertising will have two effects, that we will denote as a "market enlargement effect" and a "predatory effect".[2] The former captures the expansion of the market and represents an advantage for every operating firm, while the latter accounts for the individual incentive for each single firm to spend resources on advertising. As we will see, the relative strength of these two components will determine which outcome represent an equilibrium. Depending on the parameter values, both symmetric and asymmetric equilibria may possibly arise. Among them, two outcomes are of particular interest: a *coordination game* in which both investing and non-investing are simultaneously equilibria; a *chicken game* in which only one firm invests in equilibrium with the second one possibly driven (endogenously) out of the market. Furthermore, we will also provide some insight about the Pareto optimality (from firms' standpoint) of market outcomes that will enable us to identify *prisoner dilemma* situations. Particular attention will be paid to the role of product differentiation in determining the equilibrium level of advertising, as well as to shed some light on the problem of coordination.

This paper is organized as follows. In the following section we will introduce the analytic features of the model. Section 3 analyzes the second stage price game while in Section 4 we solve (backward) the first

[1] The use of comparative advertising progressively increased both in the United States and, more recently, also in the European Union. According to Muehling *et al.* (1990), in the United States around 40% of all advertising is comparative.

[2] For an alternative (dynamic) framework in which advertising is both cooperative and predatory, see Piga (1998).

stage advertising game. Section 5 provides a complete characterization of the equilibria of the game in terms of parameters and then turns to their economic interpretation. Section 6 finally provides conclusions and directions for further research.

2 The Model

Consider an industry composed of two symmetric firms that produce a differentiated good. They are engaged in the following two-stage game. In the first stage, each firm decides whether to devote resources to advertising or not, while in the second stage they compete in prices. In particular, firms can only decide to advertise or not, while the strategy set of each firm for the second stage price game is the entire $\Re^{+}$.[3]

Marginal costs are supposed to be zero and there are no fixed costs in production. When a firm engages in advertising it modifies its own demand, as well as that of its rival, while incurring a fixed cost that we normalize to one. We restrict our attention to subgame perfect Nash equilibria.

The demand structure turns out to be extremely important in our analysis. As we want to deal with both product differentiation and price competition, a natural starting point is the linear demand function:[4]

$$q_i = a - bp_i + c\,(p_{-i} - p_i) = a - (b + c)\,p_i + c\,p_{-i} \tag{1}$$

The parameter a stands the market size, while b is meant to represent the surplus of the own price over the cross price effect. The parameter c is an (inverse) measure of product differentiation; the higher c, the higher the substitutability between the products, given the stronger impact of a price difference.

As we mentioned in the introduction, the kind of advertising we are interested in is such that demand curves shift outward (informative advertising), thus enlarging the market for that product. Suppose that a new kind of product becomes available, or that such product is not very well-known. When a firm advertises its own good, it provides also general information about that kind of product. This turns out to be beneficial also for a potential rival that produces a similar good. A good example can be traced in the DVD market expansion boosted by Sony's massive advertising campaign. This positive spill-over, that we call "market enlargement effect", gives then an advantage to all firms as sellers of that type of product, and could be modeled with a symmetric shift of the parameter a in the demand function of our two firms.

On the other hand, by advertising, each firm also creates "brand loyalty" and "goodwill" for its own product, thus drawing consumers away from rivals. This leads to a kind of 'stealing' process that we call, coherently with the existing literature, "predatory effect". Crucially, the lower is the degree of product

[3] Clearly, it would be preferable to use a more sophisticated set of alternatives for advertising decisions. However, as many other forms of investment, advertising can have a discrete nature in the sense that it is sometimes more important to decide whether to invest or not rather than the exact amount to be spent on it. Furthermore, as we will see afterwards, our simple binary assumption will allow for a complete characterization of all possible equilibria of the game.

[4] The proposed linear demand function is consistent with utility maximizing consumers with quadratic utility functions (see Shubik and Levitan 1980).

differentiation (high values of c), the higher will be the impact of this second effect. In fact, as long as products are perceived as highly substitutes, firms have a strong incentive to attach an element of differentiation on their own good through advertising. In order to capture this "predatory effect", we make the hypothesis that a firm that does advertising receives a demand gain $c\alpha$, while imposing at the same time an equivalent demand cut $-c\alpha$ on its rival.

Although our two firms are *a priori* identical, the game could have asymmetric equilibria. In particular, we have to deal with the possibility that firm i sets a price lower or equal to a limit price p_i^l, pushing the other firm (endogenously) out of the market. This clearly raises the problem of defining the demand received by the remaining firm. Starting from equation (1), the solution we adopt is to define demand in the limit pricing domain in such a way that continuity is preserved for all admissible price strategies. This leads to the following demand system:[5]

$$q_i(p_i, p_{-i}, I_i, I_{-i}) = \left\{ \begin{array}{ll} \max\left\{a(I_i, I_{-i}) - b\,p_i + c\,[p_{-i} - p_i + \alpha_i(I_i) - \alpha_{-i}(I_{-i})],\, 0\right\}, & if\ p_i > p_i^l \\ \max\left\{2a(I_i, I_{-i}) - b\,p_i - b\varphi(p_i),\, 0\right\}, & if\ \ p_i \leq p_i^l \end{array} \right\} \tag{2}$$

where:

$$I_i = \{0, 1\}\ \ for\ i = 1, 2$$

$$a(I_i, I_{-i}) = a(I_i + I_{-i}) = \left\{ \begin{array}{cl} \underline{a} & if\ I_i + I_{-i} = 0 \\ \underline{a} + \gamma & if\ I_i + I_{-i} = 1 \\ \underline{a} + 3\gamma/2 & if\ I_i + I_{-i} = 2 \end{array} \right\}$$

$$\alpha_i = \left\{ \begin{array}{cl} 0 & if\, I_i = 0 \\ \alpha & if\ I_i = 1 \end{array} \right\}\ for\ i = 1, 2$$

$$\varphi(p_i) = \max\left\{ \frac{a(I_i + I_{-i}) + c[\alpha_{-i}(I_{-i}) - \alpha_i(I_i)]}{b + c} + \frac{c}{b + c} p_i,\ 0 \right\}$$

$$\underline{a}, b, \alpha, \gamma, c > 0$$

The binary variable I_i represents advertising strategies: firm i could either advertise ($I_i = 1$) or not ($I_i = 0$). The "market enlargement" effect induced by advertising is captured by $a(I_1, I_2)$ and depends upon total investment: $I_1 + I_2$. If no firm advertises, then $a(I_1 + I_2)$ is stuck to a basic level $\underline{a}$. If only

[5] See Appendix A.1 for further details. It is important to stress that the requirement of continuity for all admissible prices leads to a 'unique' definition of demand in the limit price domain.

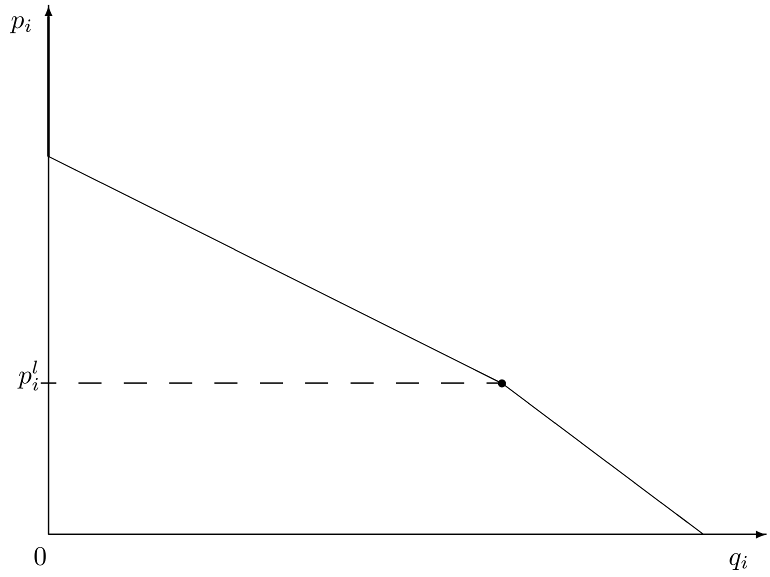

Figure 1 : The demand function

one firm advertises, then $a(I_1 + I_2)$ increases to $\underline{a} + \gamma$, while if the other does the same the new marginal increase is just $(1/2)\gamma$. This series of diminishing increments accounts for the fact that there cannot be unlimited expansion of the market.[6] The "predatory effect" is instead parameterized by α_i, that could be either zero or α. If only one firm advertises then, as long as the other one is actually on the market $(p_i > p_i^l)$, its demand increases by $c\alpha$ while the demand of the rival decreases by the same amount. As argued in the previous section, the magnitude of the "predatory effect" is in fact positively related to product substitutability.

On the other hand, if prices and advertising strategies are such that only firm i makes positive sells $(p_i \leq p_i^l)$, its demand depends only on p_i in such a way that continuity in prices is guaranteed.

3 The second stage price game

In equilibrium, there can obviously be just two possibilities: either the two firms sell a positive amount of goods, or just one of them receives a positive demand while the other has a zero output. As we already pointed out, our demand system (2) is continuous for all $p_1, p_2 \in [0, \infty)$ and it is clearly monotone decreasing (increasing) in each firm own (cross) price whenever a firm's demand is positive. Anyway, first-order conditions alone do not suffice to characterize Nash Equilibria in the price game because demand functions have kinks. In fact, demands are just piece-wise linear in both own and cross price, and their slope changes in view of limit pricing. However, in Appendix A.1 we show that this change is "well-behaved" in the sense that the slope is lower (in absolute value) when only one firm sells on the market. Figure 1 shows the demand of firm i in case a positive limit price p_i^l exists.

This change in price responsiveness comes from the fact that, when both firms are active on the market,

[6] In particular, we adopt the geometric series $a(n) = \underline{a} + \sum_i^n \frac{1}{i}\gamma$.

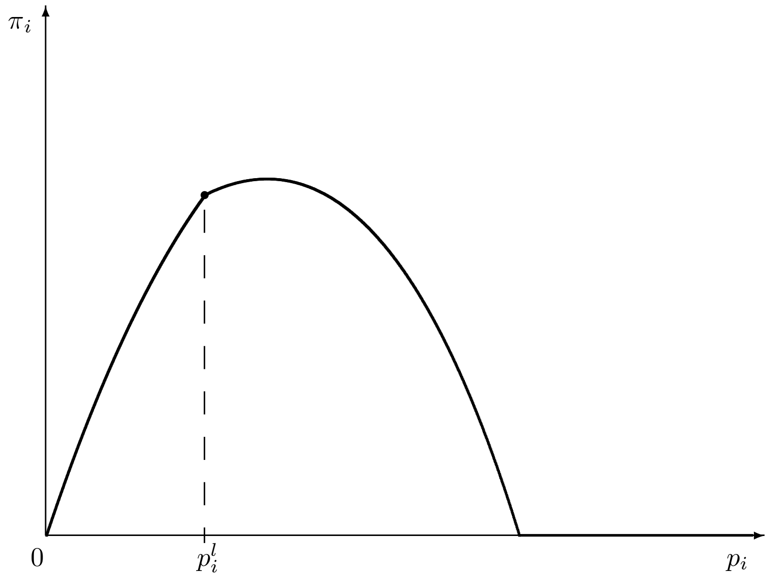

Figure 2 : The profit function

a price reduction by one firm induces not only new customers to buy the product, but also usual clients of the rival to drift to the price-reducing firm. This has a very useful implication for firms' profit functions which, by linearity of demands and absence of variable costs, turn out to be strictly concave whenever quantities are positive. Therefore, we can state the following:

Claim 1: *If a NE in the second stage price game with both firms making positive sells exists, then first-order conditions are necessary and sufficient to identify it.*

In Figure 2 we represent an example of how the profit function of firm i looks like when a positive limit price p_i^l exists.

As we will see afterwards, for equilibria in which only one firm is active on the market, first-order conditions will be of a little help since one typically faces corner solutions. Let us now analyze in details the 3 possible outcomes arising as a consequence of different advertising decisions.

3.1 Case A: None invests

We start by considering the symmetric case where no firm invests in advertising. If both firms receive positive demands, then the demand curves of each firm are (with $I_1 = I_2 = 0$):

$$q_1 = \underline{a} - b\,p_1 + c\,(p_2 - p_1) \quad \text{and} \quad q_2 = \underline{a} - b\,p_2 + c\,(p_1 - p_2).$$

Since we assume that marginal costs are zero, and there are no fixed costs in production, profits are:

$$\pi_1 = p_1 q_1 = p_1\left[\underline{a} - b\,p_1 + c\,(p_2 - p_1)\right] \quad \text{and} \quad \pi_2 = p_2 q_2 = p_2\left[\underline{a} - b\,p_2 + c\,(p_1 - p_2)\right].$$

Profits are quadratic in each firm's own price, and by first-order conditions we get equilibrium prices:

$$p_1^A = p_2^A = \frac{\underline{a}}{2b+c} > 0.$$

The demands corresponding to these prices are always positive (so that this NE is always acceptable) and the equilibrium profits (obtained using equilibrium prices p_1^A and p_2^A) are:

$$\pi_1^A = \pi_1(p_1^A, p_2^A) = \pi_2^A = \pi_2(p_1^A, p_2^A) = \frac{\underline{a}^2\,(b+c)}{(2b+c)^2} > 0 \tag{3}$$

According to Claim 1, the pair $\{p_1^A, p_2^A\}$ thus represents the unique "SPE" characterized by both firms making positive sells. On the other hand, it is straightforward to check that each firm can always find here, whatever the other does, a strictly positive price such that it receives some demand and makes strictly positive profits. This clearly means that there is no room for equilibria with just one active firm, i.e.:

Lemma 1: *In the subgame where no firm invests, there is a unique Nash equilibrium in pure strategies given by* $\{p_1^A, p_2^A\}$.

3.2 Case B: Only one firm invests

We now examine the case where only one firm invests in advertising. Without loss of generality, we assume that firm 1 invests while firm 2 does not: $I_1 = 1$ and $I_2 = 0$. In case of positive sells for both firms, the demand curves are given by:

$$q_1 = \underline{a} + \gamma - b\,p_1 + c\,(p_2 - p_1 + \alpha) \quad \text{and} \quad q_2 = \underline{a} + \gamma - b\,p_2 + c\,(p_1 - p_2 - \alpha).$$

Compared to the previous case, where none of them invested in advertising, both firms enjoy here an increase in demand equal to γ due to the market enlargement effect. However, due to the predatory effect, firm 1 receives an additional gain $c\alpha$, while imposing a penalty $-c\alpha$ to the rival. Profits are now given by:

$$\pi_1 = p_1 q_1 - 1 = p_1\left[\underline{a} + \gamma - b\,p_1 + c\,(p_2 - p_1 + \alpha)\right] - 1 \quad \text{and} \quad \pi_2 = p_2 q_2 = p_2\left[\underline{a} + \gamma - b\,p_2 + c\,(p_1 - p_2 - \alpha)\right]$$

By first-order conditions we get equilibrium prices:

$$p_{1Ac}^B = \frac{(\underline{a}+\gamma)\,(2b+3c) + c\alpha(2b+c)}{(2b+c)\,(2b+3c)} > 0 \tag{4}$$

$$p_{2Ac}^B = \begin{cases} \frac{(\underline{a}+\gamma)(2b+3c) - c\alpha(2b+c)}{(2b+c)(2b+3c)} & if \quad \alpha < \alpha_a \\ 0 & otherwise \end{cases} \tag{5}$$

where the subscript Ac indicates that both firms are active on the market. However, this equilibrium is not always acceptable because the equilibrium price p^B_{2Ac} can be negative as well as the corresponding demand for firm 2. One can easily check that both p^B_{2Ac} and $q^B_2(p^B_{1Ac}, p^B_{2Ac})$ are positive iff $\alpha < \alpha_a = \frac{(2b+3c)\,(\underline{a}+\gamma)}{(2b+c)\,c}$. Following Claim 1, when such a condition on α is satisfied, the pair of strategies $\{p^B_{1Ac}, p^B_{2Ac}\}$ is the unique NE characterized by both firms making positive sells. The associated equilibrium profits are:

$$\begin{aligned}\pi^B_{1Ac} &= \frac{1}{(2b+c)^2\,(2b+3c)^2}\cdot \\ &\left\{\left(\underline{a}^2+\gamma^2\right)(b+c)\,(2b+3c)^2+(2b+c)^2\left[c^2\alpha^2(b+c)-(2b+3c)^2\right]+\right. \\ &\left.+2\,(b+c)\,(2b+3c)\left[c\alpha\,(2b+c)\,(\gamma+\underline{a})+\underline{a}\gamma\,(2b+3c)\right]\right\}\end{aligned} \tag{6}$$

$$\pi^B_{2Ac} = \frac{(b+c)\left[(\underline{a}+\gamma)\,(2b+3c)-c\alpha\,(2b+c)\right]^2}{(2b+c)^2\,(2b+3c)^2} \tag{7}$$

We should now turn to the study of equilibria characterized by just one active firm. It is easy to check that only firm 1 can always find, whatever the other does, a strictly positive price such that it still receives some demand. Therefore, it is possible that firm 2, which does not advertise its product, finds itself out of business in equilibrium. Furthermore, we can prove that (without loss of generality) one could simply focus on equilibria in which $p_2 = 0$:

Lemma 2: *Consider the subgame where just firm 1 invests. If $\{p^*_1, p^*_2\}$ is a NE with $p^*_1, p^*_2 > 0$ and $q_2(p^*_1, p^*_2) = 0$, then also $\{p^*_1, p_2\}$ is, for any $p_2 \in [0, p^*_2]$, a NE with $q_2(p^*_1, p_2) = 0$. Furthermore, $\pi_1(p^*_1, p^*_2) = \pi_1(p^*_1, p_2)$ and $\pi_2(p^*_1, p^*_2) = \pi_2(p^*_1, p_2) = 0$ for any such $p_2 \in [0, p^*_2]$.*

Proof. Suppose that $\{p^*_1, p^*_2\}$ is a NE with $p^*_1, p^*_2 > 0$, and $q_2(p^*_1, p^*_2) = 0$. For p^*_2 to be a best reply to p^*_1, there should not exist any $p_2 > 0$ such that $q_2(p^*_1, p_2) > 0$. By continuity of our demand system, this implies that also $q_2(p^*_1, 0)$ cannot be positive, and so all $p_2 \in [0, p^*_2]$ are certainly best replies to p^*_1. On the other hand, for any $p_2 \in [0, p^*_2]$, we have that $q_2(p^*_1, p_2) = 0$ and so firm 1's demand does not certainly depend on such p_2 for prices lower or equal than p^*_1, i.e. $q_1(p_1, p_2) = q_1(p_1, p^*_2) = q_1(p_1)\ \forall p_1 \in [0, p^*_1]$, while for prices $p_1 \in (p^*_1, \infty)$ it satisfies the inequality $q_1(p_1, p_2) \leq q_1(p_1, p^*_2)$ that comes from the fact that demand is non-decreasing in the cross price. Being p^*_1 a best reply to p^*_2 we have $q_1(p^*_1, p^*_2) \geq q_1(p_1, p^*_2)$ $\forall p_1$, and using the previous relations we obtain that, for any $p_2 \in [0, p^*_2]$, $q_1(p^*_1, p_2) \geq q_1(p_1, p_2)\ \forall p_1$ so that p^*_1 is also a best reply to any such p_2 and in particular to $p_2 = 0$. ■

Lemma 2 actually means that, whenever firm 1 pushes firm 2 out of the market, we are in the same situation (in terms of equilibrium price p_1 and payoffs) as if firm 2 charges a zero price.

In order to study equilibria with only firm 1 on the market, we should first figure out how its profit function looks like. Indicating with $p^B_{1Dt} = \alpha - \frac{\underline{a}+\gamma}{c}$ the limit price p^l_1 corresponding to $p_2 = 0$ we have that, depending on its price p_1, firm 1 could find itself in the domain in which both firm sell something

($p_1 > p^B_{1Dt}$), or in the domain in which it is the only active firm ($p_1 \le p^B_{1Dt}$). The two domains correspond to two different analytical expressions of the demand function.

If firm 1 prices above the limit price p^B_{1Dt}, its demand (for $p_2 = 0$) will be given by:

$$q_1 = \underline{a} + \gamma - b\,p_1 + c\,(-p_1 + \alpha) \tag{8}$$

with profits:

$$\pi_1 = p_1 q_1 - 1 = p_1\left[\underline{a} + \gamma - b\,p_1 + c\,(-p_1 + \alpha)\right] - 1. \tag{9}$$

By first-order conditions we get the unique maximum:

$$\hat{p}^B_1 = \frac{\underline{a} + \gamma + c\alpha}{2(b+c)}.$$

Nonetheless, this solution rests on the hypothesis that $q_2 > 0$, which has to be checked. If firm 1 instead prices below the limit price p^B_{1Dt}, its demand (for $p_2 = 0$) will be given by:

$$q_1 = 2(\underline{a} + \gamma) - b\,p_1 \tag{10}$$

and profits:

$$\pi_1 = p_1 q_1 - 1 = p_1\left[2(\underline{a} + \gamma) - b\,p_1\right] - 1. \tag{11}$$

By first-order conditions we get the unique maximum:

$$p^B_{1Mp} = \frac{\underline{a} + \gamma}{b} > 0.$$

It is easy to check that, for $\alpha = \alpha_a$, $\hat{p}^B_1 = p^B_{1Dt} > 0$ while, for $\alpha > \alpha_a$ ($\alpha < \alpha_a$), $\hat{p}^B_1$ is strictly lower (higher) than p^B_{1Dt} and they are still both positive.

In case of $\alpha > \alpha_a$, this means that for prices bigger than p^B_{1Dt} ($\ge \hat{p}^B_1$), the 'true' demand firm 1 faces is given by (8) and so profits, given by the concave parabola (9), are decreasing in this range of prices precisely because we are to the right of $\hat{p}^B_1$. We can thus exclude all prices $p_1 > p^B_{1Dt}$ from equilibrium. If firm 1 instead charges a price lower or equal than p^B_{1Dt}, its 'true' demand is given by (10), with relative profits given by (11) which is again a concave parabola in p_1 with a unique maximum p^B_{1Mp}. There are consequently 2 possible scenarios, represented respectively in Figures 3a and Figure 3b, referring to firm 2 being out of the market:

- When $\alpha_a \leq \alpha < \alpha_b = \frac{(b+c)(a+\gamma)}{bc}$, we have $p_{1Mp}^B > p_{1Dt}^B$. Consequently, for prices higher than p_{1Dt}^B, firm 1's profit corresponds to the decreasing branch of the parabola (9), while in the other case it corresponds to the increasing branch of the parabola (11). The two parabolas touch each other at $p_1 = p_{1Dt}^B$, that is the unique maximum.

- When $\alpha \geq \alpha_b$, we have $p_{1Mp}^B \leq p_{1Dt}^B$. Consequently, for prices higher than p_{1Dt}^B firm 1's profit corresponds again to the decreasing branch of the parabola (9), while in the other case it corresponds to the decreasing branch of the parabola (11). The two parabolas touch each other at $p_1 = p_{1Dt}^B$, and the unique maximum is reached for $p_1 = p_{1Mp}^B$.

Obviously, $p_2 = 0$ is a best reply to limit prices p_{1Dt}^B and p_{1Mp}^B and so all the conditions needed in order to have a Nash Equilibrium are satisfied.

In case $\alpha < \alpha_a$ instead, we have that for prices bigger than p_{1Dt}^B ($< \hat{p}_1^B$), the 'true' demand firm 1 faces is still given by (8) but profits, represented by the concave parabola (9), are increasing in this range of prices because we are now to the left of $\hat{p}_1^B$, which represents the unique maximum. Consequently, as limit pricing is never a best strategy for firm 1, we can exclude equilibria with only one active firm whenever $\alpha < \alpha_a$. We can thus summarize the above discussion with:

Lemma 3: *in the subgame when only firm one invests, there is a "unique" equilibrium (in terms of payoffs and price p_1) in pure strategies given by*

1. $\left\{p_{1Ac}^B, p_{2Ac}^B\right\}$ *when* $\alpha < \alpha_a$;
2. $\left\{p_{1Dt}^B, 0\right\}$ *when* $\alpha_a \leq \alpha < \alpha_b$;
3. $\left\{p_{1Mp}^B, 0\right\}$ *when* $\alpha \geq \alpha_b$.

Results of Lemma 3 are actually quite intuitive. If the predatory effect is sufficiently small ($\alpha < \alpha_a$), then both firms makes positive sells in equilibrium. Anyway, beyond the critical value α_a, the advertising firm finds it convenient to charge a limit price such that its competitor is (endogenously) squeezed out of the market.[7] In particular, if $\alpha_a \leq \alpha < \alpha_b$ then firm 1 charges the highest limit price, while if the predatory effect is really strong ($\alpha \geq \alpha_b$), firm 1 is able to take the all market by setting a kind of "monopoly" price p_{1Mp}^B. It is interesting to note that each firm's best reply is continuous with respect to α (as well as with respect to the other parameters), and the same applies (due to the continuity of demand) to equilibrium profits.

For future reference, we write the equilibrium profits of firms in the three subcases considered:

- $\alpha < \alpha_a \Longrightarrow p_1^B = p_{1Ac}^B$, $p_2^B = p_{2Ac}^B$ and equilibrium profits $\pi_1^B, \pi_2^B$are given by 6) and (7);

- $\alpha_a \leq \alpha < \alpha_b \Longrightarrow p_1^B = p_{1Dt}^B$, $p_2^B = 0$ and equilibrium profits are:

[7] Amir (2000) found conditions leading to endogenous exit in a two-period symmetric Cournot duopoly with R&D returns to process innovation.

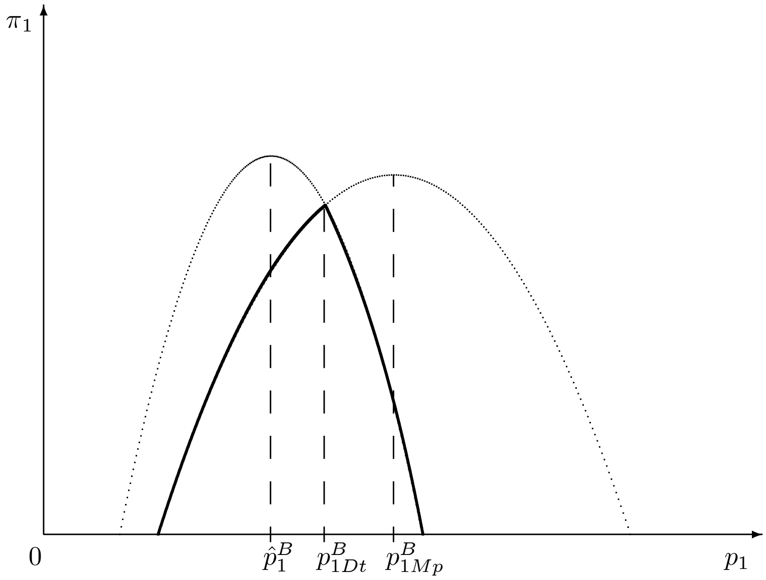

Figure 3a : Firm 1's profit function for $\alpha_a < \alpha \leq \alpha_b$ (bold line).

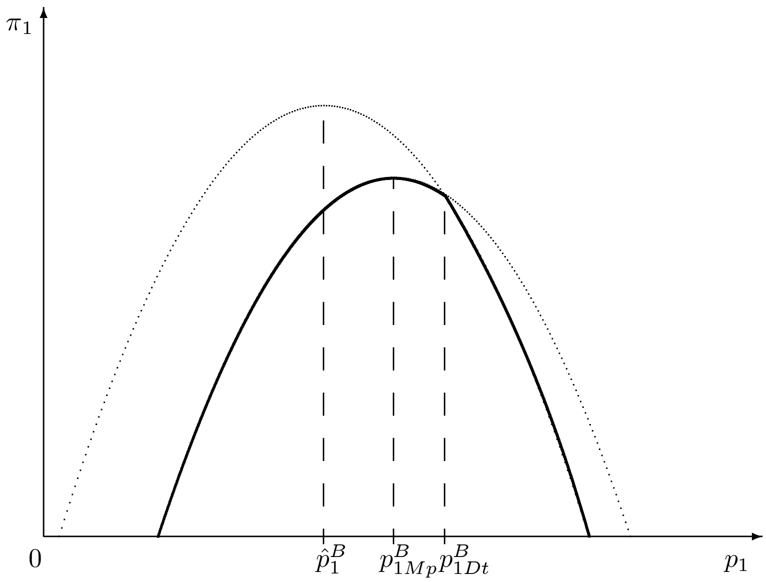

Figure 3b : Firm 1's profit function for $\alpha > \alpha_b$ (bold line).

$\pi_1^B = \pi_{1Dt}^B = \frac{(\alpha c-\gamma-\underline{a})[2c(\underline{a}+\gamma)-b(\alpha c-\gamma-\underline{a})]}{c^2} - 1,\ \pi_2^B = \pi_{2Dt}^B = 0;$

- $\alpha \geq \alpha_b \Longrightarrow p_1^B = p_{1Mp}^B,\ p_2^B = 0$ and equilibrium profits are:

$$\pi_1^B = \pi_{1Mp}^B = \frac{(\underline{a}+\gamma)^2}{b} - 1,\ \pi_2^B = \pi_{2Mp}^B = 0. \tag{12}$$

Obviously, due to the symmetric structure of the game, in the case where only firm 2 invests in advertising, we obtain the reversed equilibrium prices and payoffs.

3.3 Case C: Both firms invest

We finally consider the (symmetric) case where both firms decide to invests in advertising. The demand curves of each firm are (with $I_1 = I_2 = 1$):

$$q_1 = \underline{a} + \frac{3}{2}\gamma - b\,p_1 + c\,(p_2 - p_1) \quad \text{and} \quad q_2 = \underline{a} + \frac{3}{2}\gamma - b\,p_2 + c\,(p_1 - p_2).$$

Now only the market enlargement effect appears, while the strategic effect is reciprocally cancelled out by the investment of the two firms. Profits are given by:

$$\pi_1 = p_1 q_1 = p_1\left[\underline{a} + \frac{3}{2}\gamma - b\,p_1 + c\,(p_2 - p_1)\right] - 1 \quad \text{and} \quad \pi_2 = p_2 q_2 = p_2\left[\underline{a} + \frac{3}{2}\gamma - b\,p_2 + c\,(p_1 - p_2)\right] - 1.$$

By first-order conditions we get equilibrium prices:

$$p_1^C = p_2^C = \frac{2\underline{a} + 3\gamma}{4b + 2c} > 0$$

The corresponding demands are always positive (so that this NE is always acceptable) and the equilibrium profits are:

$$\pi_1^C = \pi_2^C = \frac{(b+c)\,(2\underline{a}+3\gamma)^2}{4\,(2b+c)^2} - 1 \tag{13}$$

Following again Claim 1, the pair $\{p_1^C, p_2^C\}$ thus represents the unique subgame Nash equilibrium characterized by both firms making positive sells. Furthermore, it is easy to check that we are here in the same situation as for case A, and so:

Lemma 4: *In the subgame when both firm invest, there is a unique Nash equilibrium in pure strategies given by* $\{p_1^C, p_2^C\}$.

4 The advertising game

In the last section, we have dealt with equilibrium profits associated to the three possible cases arising in the price game. For every parameter value, we can identify a unique (in term of payoffs) NE of the corresponding subgame.[8] Now, given the binary nature of the advertising choice, we can solve backward the first stage with a simple 2x2 matrix containing equilibrium payoffs from the second stage. It is important to stress the role of the uniqueness in equilibrium payoffs. It is in fact such feature that makes it possible to have a unique representation of the matrix, that we show in Table 1:

firm 2

		0	1
firm 1	0	$\pi_1^A = \pi_2^A$	$\pi_2^B \quad \pi_1^B$
	1	$\pi_1^B \quad \pi_2^B$	$\pi_1^C = \pi_2^C$

Table 1

Due to the symmetric structure of the above representation, it can be easily established that at least one "SPE" will always exist.

Lemma 5: *For every given value of the parameters considered* ($\underline{a}, b, c, \gamma, \alpha$)*, there exists at least one "SPE" in the reduced form of the game.*

Proof. The proof can be just given by contradiction. Suppose $(0,0)$ is not a SPE, then (i) $\pi_i^B > \pi_i^A$. Assume now that also $(1,1)$ is not a "SPE", hence (ii) $\pi_2^B > \pi_i^C$. But when (i) and (ii) hold simultaneously then $(1,0)$ and $(0,1)$ are "SPE" and this contradicts the claim that no "SPE" exists. Hence we always have at least one "SPE". ■

To begin the study of "SPE" for our game, we first analyze the payoffs appearing in the principal diagonal. This will shed light on the Pareto efficiency of the NE from firms' standpoint as well as on the qualitative nature of the game. One may easily check that

$$\pi_i^C \geq \pi_i^A \text{ iff } \gamma \geq \gamma_1 \tag{14}$$

where $\gamma_1 = \frac{2}{3}\left[\frac{\sqrt{\underline{a}^2(b+c)+(2b+c)^2}}{\sqrt{b+c}} - \underline{a}\right] > 0$. Obviously, both firms gain in investing when the enlargement of the market due to advertising is big enough. On the contrary, when γ is low, both firms would prefer not to spend resources on advertising. Interestingly, it may be the case that the two firms invest in equilibrium while it would have been better not to invest, or the other way round, thus giving rise to *prisoner dilemma* outcomes.

The simple structure of the game is such that we can quite easily characterize all possible situations. We can in fact encounter just four outcomes. Omitting cases of weak inequalities, we already know that $(0,0)$ is a "SPE" iff $\pi_i^A > \pi_1^B$, while $(1,1)$ is a "SPE" iff $\pi_i^C > \pi_2^B$, and thus combining the two we get:

[8] From now on, if not elsewhere specified, we refer to uniqueness in terms of payoffs.

1. When only $\pi_i^A > \pi_1^B$ holds, then $(0,0)$ is the unique "SPE" of the game. The two firms do not invest in advertising and, depending on the value of γ, we could possibly obtain a *prisoner dilemma* game.

2. When only $\pi_i^C > \pi_2^B$ holds, then $(1,1)$ turns out to be the unique "SPE" of the game. Again, depending on γ, we may have or not a *prisoner dilemma*.

3. If both conditions hold together, we obtain a *coordination game* with two "SPE" along the principal diagonal.

4. Lastly, if both these conditions are not satisfied, we get a *chicken game* with two asymmetric "SPE" along the secondary diagonal characterized by only one firm investing in advertising.

To link equilibrium profits with the parameters of the model, we have to consider three different expressions associated to π_1^B, depending on the value taken by α. We can give necessary and sufficient conditions on (α, γ) for $(0,0)$ to be a "SPE":

Proposition 1 $(0,0)$ *is a "SPE" for sufficiently low combinations between the values of* α *and the ones of* γ*. In particular: (i) when* $\alpha \geq \alpha_b$*, we need* $\gamma \leq \gamma_2$*; (ii) when* $\alpha_a \leq \alpha < \alpha_b$*, we need either* $\gamma \leq \gamma_2$ *or, if* $\gamma > \gamma_2$*, then* $\alpha \leq \alpha_c (< \alpha_b)$ *; (iii) when* $0 < \alpha < \alpha_a$*, we need either* $\gamma \leq \gamma_3(> \gamma_2)$ *or, if* $\gamma > \gamma_3$*, then* $\alpha \leq \alpha_d \, (< \alpha_a)$*. When* $\gamma \geq \gamma_4 (> \gamma_3)$*,* $(0,0)$ *is never an equilibrium, independently of* α*. Moreover, this Nash Equilibrium, when it exists, turns out to be Pareto dominant from firms' standpoint for sufficiently low values of* γ *(*$\gamma < \gamma_1$*), otherwise the game is of a prisoner dilemma type.*

Proof. see Appendix A.2. ■

The dashed area in Figure 4 indicates those values that sustain $(0,0)$ as a "SPE" in the (α, γ) space.[9] As one can see, both firms decide not to invest when the combination of the market size effect and the strategic predatory effect is weak enough. There is, indeed, a certain degree of substitution in the two effects. A strong predatory gain α could be compensated with a weakening of the market enlargement in order for $(0,0)$ to be a "SPE". However, if γ is big enough ($\gamma \geq \gamma_4$), then, whatever α is, $(0,0)$ is never a "SPE". Furthermore, when it exists as an equilibrium, $(0,0)$ is Pareto dominant for firms only when the market size effect is sufficiently weak ($\gamma < \gamma_1$).

Interestingly, a *prisoner dilemma* (indicated by the portion of the dashed area on the right of γ_1) arises when there are quite good perspectives of enlarging the market, but the predatory gain is limited. One firm alone has no advantage to invest since it does not steal that much from the other which can, by contrast, enjoy the enlargement of the market due to advertising without paying any cost for it. Although both firms would be better off by investing, they thus refrain from doing so.

Let us now consider the equilibrium $(1,1)$. By evaluating π_i^C *vs* π_2^B, and taking into account the restrictions on both profit functions, we can conclude that in the parameter space (α, γ):

[9] Figure 4 has been depicted using $c = 1$, $b = 1$, and $\underline{a} = 0.3$.

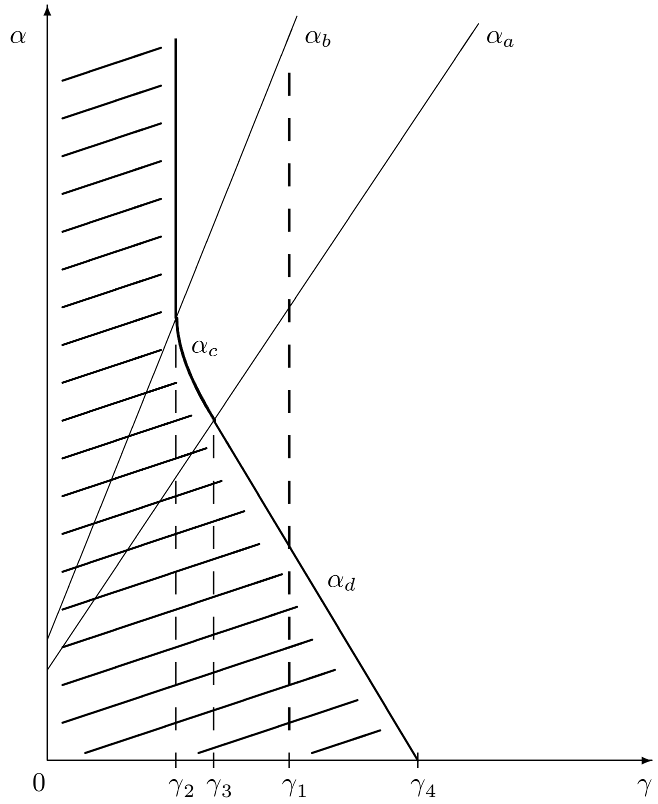

Figura 4 : The equilibrium (0,0)

Proposition 2 $(1,1)$ *is a "SPE" for sufficiently high combinations between the values of* α *and those of* γ*. In particular, we need at least that* $\gamma \geq \gamma_5$ *and either* $\alpha \geq \alpha_a$*, or, when* $\alpha < \alpha_a$*,* $\alpha \geq \alpha_e (< \alpha_a)$*. When* $\gamma \leq \gamma_5$*,* $(1,1)$ *is never an equilibrium, independently of* α*. On the contrary, if* $\gamma \geq \gamma_6 (> \gamma_5)$*,* $(1,1)$ *is always a "SPE". Such a solution represents a Pareto dominant strategy for firms for sufficiently high values of* γ *(*$\gamma > \gamma_1$*), otherwise it gives rise to a prisoner dilemma.*

Proof. see Appendix A.3. ■

The dotted area in Figure 5 describes our equilibrium conditions in the (α, γ) space.[10] Contrary to before, both firms invest in equilibrium when the combination of the two effects is strong enough. There is, again, a certain degree of substitution between α and γ. When the predatory effect is weak, $(1,1)$ constitutes a "SPE" of the game only if the market expansion translates into a considerable increase in firms' profits. However, if γ is big enough ($\gamma > \gamma_6$), then, whatever is α, $(1,1)$ is always a "SPE". Furthermore, $(1,1)$ is Pareto dominant from firms' standpoint whenever $\gamma > \gamma_1$. Conforming to intuition, a *prisoner dilemma* situation (indicated by the portion of the dotted area on the left of γ_1) still arises, but its nature is the mirror image of the previous case. Crucially, the predatory gain here needs to be strong enough. Firm would in fact be better off without doing advertising because the market expansions possibilities are quite limited ($\gamma > \gamma_1$). However, they advertise in equilibrium because they are fully aware of the substantial gain (loss) of being the only advertising (non-advertising) firm.

[10] Figure 5 has also been drawn using $c = 1$, $b = 1$, and $\underline{a} = 0.3$.

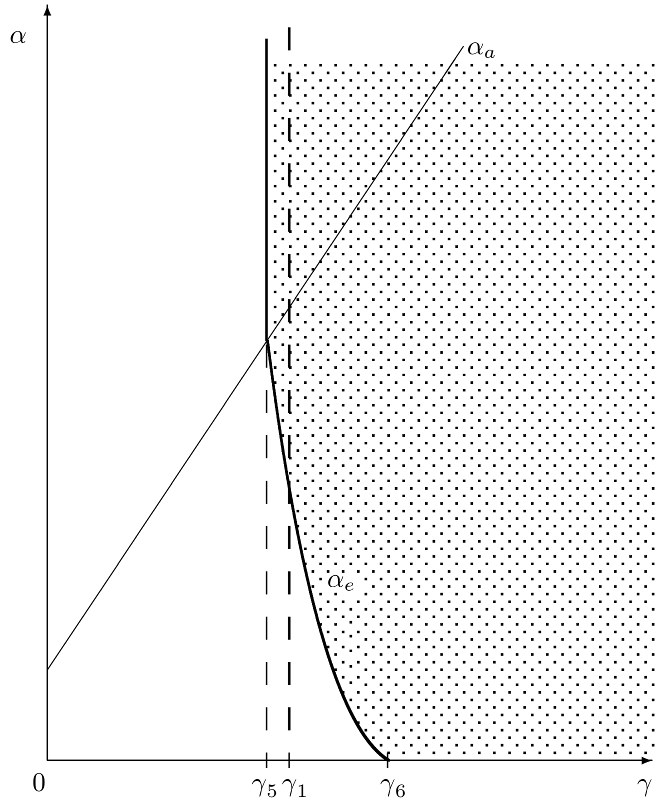

Figura 5 : The equilibrium (1,1)

Combining Proposition 1 and 2, we can fully characterize the four possible outcomes of the model in terms of the parameters. In the next section we will give some insights on how these equilibria configurations react to changes in parameters as well as their underlying economic interpretation.

5 Further results and economic interpretations

In the previous section we gave necessary and sufficient conditions on the two-dimensional parametric space (α, γ) such that $(0,0)$ and $(1,1)$ are "SPE". Following Propositions 1 and 2, we reasonably expect to find that, when there are small incentives for firms to advertise (i.e. low values of α and γ), $(0,0)$ is the only equilibrium of the game. By contrast, for sufficiently high values of α and γ, we expect $(1,1)$ to be the only outcome. Now, what is not clear is what happens in intermediate situations. Both a *coordination* and a *chicken game* will be possible, but conditions leading to each outcome still remain unknown at this stage.

In order to shed some light on the forces underpinning the game, we resort to comparative statics analysis. This task turns out to be extremely difficult because equilibria are characterized in the two-dimensional space (α, γ) and we need to consider simultaneously all the different threshold values appearing in Propositions 1 and 2. After tedious calculations, one can show that:

$$\gamma_6 > \gamma_1 > \gamma_5 \quad \text{and} \quad \gamma_4 > \gamma_1 > \gamma_3 > \gamma_2.$$

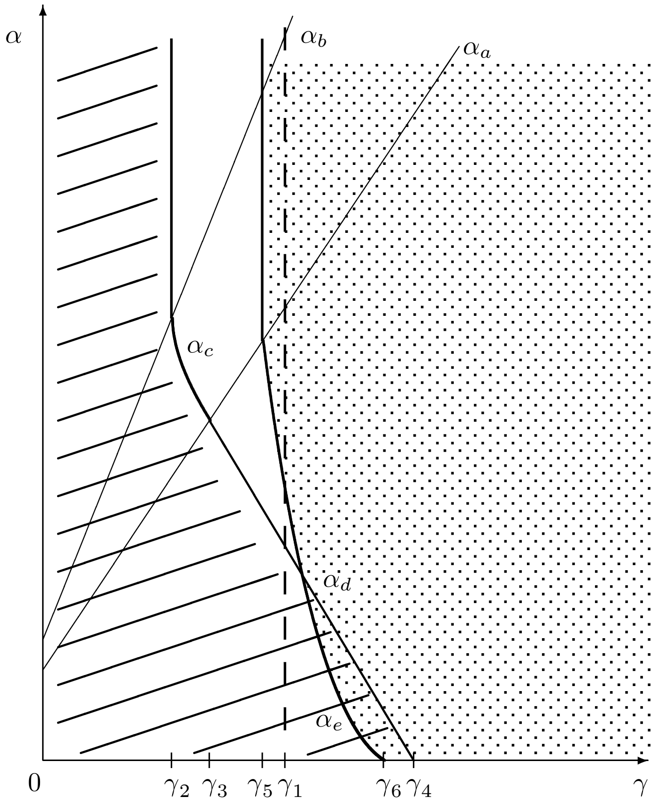

Figure 6 : Analysis of equilibria: First case

Unfortunately, we cannot directly order γ_6 *vs* γ_4 and γ_5 *vs* γ_3, γ_2, but we use the initial size of the market, parameterized by $\underline{a}$, to discriminate between such threshold values of γ. This will allow for a complete characterization of the equilibria. In particular, depending on the relative position of $\underline{a}$ with respect to two critical values, $\underline{a}_1$ and $\underline{a}_2$, with $0 < \underline{a}_1 < \underline{a}_2$, we get the following results:[11]

Lemma 6: *(i) When $\underline{a} < \underline{a}_1$, then $\gamma_4 > \gamma_6$ and $\gamma_5 > \gamma_3$; (ii) when $\underline{a}_1 < \underline{a} < \underline{a}_2$, then $\gamma_4 < \gamma_6$ and $\gamma_5 > \gamma_3$; (iii) when $\underline{a} > \underline{a}_2$, then $\gamma_4 < \gamma_6$ and $\gamma_5, \gamma_2, \gamma_3 < 0$.*

Using the above results, we can sketch a complete rank of the threshold values of γ. Depending on the value taken by $\underline{a}$, three different situations will appear:

1. $\underline{a} < \underline{a}_1 \Longrightarrow \ \gamma_4 > \gamma_6 > \gamma_1 > \gamma_5 > \gamma_3 > \gamma_2 > 0$;
2. $\underline{a}_1 < \underline{a} < \underline{a}_2 \Longrightarrow \ \gamma_6 > \gamma_4 > \gamma_1 > \gamma_5 > \gamma_3 > \gamma_2 > 0$;
3. $\underline{a}_2 < \underline{a} \Longrightarrow \ \gamma_6 > \gamma_4 > \gamma_1 > 0 > \gamma_5, \gamma_2, \gamma_3$.

Figure 6 describes the whole situation in the first case (i.e. $\underline{a} < \underline{a}_1$)[12]. We can now clearly identify both a *chicken game* and a *coordination game*. In the white area, neither the conditions of Proposition 1 nor those of Proposition 2 hold, and so $(1, 0)$ and $(0, 1)$ are the (unique) equilibria. This scenario appears

[11] Calculations are available upon request.

[12] Figure 6 merges figures 4 and 5. It has in fact been drawn using $c = 1$, $b = 1$, and $\underline{a} = 0.3 < \underline{a}_1 = 0.43$.

when the possibility of enlarging the market is neither too limited nor too excessive (otherwise either $(0,0)$ or $(1,1)$ would respectively be the only outcomes), and the strategic predatory gain is sufficient for the investing firm to profitably cover the investment costs. Moreover, as we know from Lemma 3, if $\alpha < \alpha_a$, then the firm which is not investing is still selling a positive amount of product, while for $\alpha > \alpha_a$ it is endogenously driven out of the market. In particular, when $\alpha_a \leq \alpha < \alpha_b$ the investing firm finds it convenient to set a limit price, while above α_b it has such a big advantage that it can charge a kind of monopoly price.

On the other hand, where the dotted and dashed areas overlap, we have an interesting *coordination game*. The simple structure of the game allows us to treat the (relatively) unaddressed issue of coordination in advertising decisions in a fairly straightforward way. Both $(0,0)$ and $(1,1)$ can be simultaneously equilibria and this happens again for intermediate values of γ, but now coupled with a weak strategic effect. When α is small, a firm that invests alone must bear all costs of advertising, while its gain comes almost entirely from the enlargement of the market. At the same time, the other firm gains more or less the same, without paying anything for it. If the return on advertising is quite good (in terms of γ), then the non-investing firm could find it profitable to devote resources to adverting too. On the contrary, if this return is not too high, then the other firm can reasonably reconsider its investment decision. Therefore, for intermediate values of γ and low α, both kinds of deviations are plausible and we have a problem of coordination. Interestingly, as the figure shows, when such a problem arises the Pareto optimal equilibrium for our firms is the one with investment.

Figure 7 and 8 depict respectively the all set of equilibrium conditions for the cases where $\underline{a}_1 < \underline{a} < \underline{a}_2$ and $\underline{a}_2 < \underline{a}$.[13] There are two main differences with respect to Figure 6. First, the dashed area shrinks indicating that the equilibrium $(0,0)$ is less and less likely to occur. This is due the fact that, when the initial size of the market $\underline{a}$ increases, then firms are, *ceteris paribus*, more capable to cover the fixed costs of advertising. This reasonably makes firms more willing to invest in advertising. Second, the coordination game disappears. This happens for the same reasons that cause the dashed area to reduce. Rising $\underline{a}$, it is less likely that a firm cannot cover the fixed cost of advertising, even if it invests alone.

Further intuitions could be drawn by the relative dimension of c, the parameter which measures the degree of substitutability between the products of the two firms. Actually, we can alternatively rephrase Lemma 6 in term of c. As a function of c, both $\underline{a}_1$ and $\underline{a}_2$ vary from values close to zero to infinity and their first derivatives are strictly positive. Therefore, for a given $\underline{a}$, we can always find values of c sufficiently high to have $\underline{a} < \underline{a}_1$, and then decreasing c we pass to the other two situations $\underline{a}_1 < \underline{a} < \underline{a}_2$ and $\underline{a}_2 < \underline{a}$.

For high values of c, we are then more likely to happen in a situation like the one depicted in Figure 6, where the dashed area expands while the dotted one shrinks with respect to Figures 7 and 8. Intuitively, when products are close substitutes (high c), then competition in prices turns out to be very fierce. Consequently, equilibrium profits decrease (and at the limit they tend to zero) and firms are then more

[13] Figure 7 and 8 have been drawn still taking $c = 1$, and $b = 1$, but while the former refers to $\underline{a} = 1$ (which is in between $\underline{a}_1 = 0.43$ and $\underline{a}_2 = 2.12$), the second uses $\underline{a} = 2.5 > \underline{a}_2 = 2.12$.

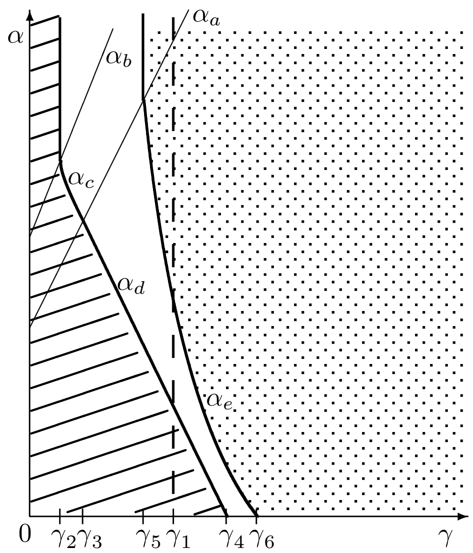

Figure 7 : Analysis of equilibria: Second case

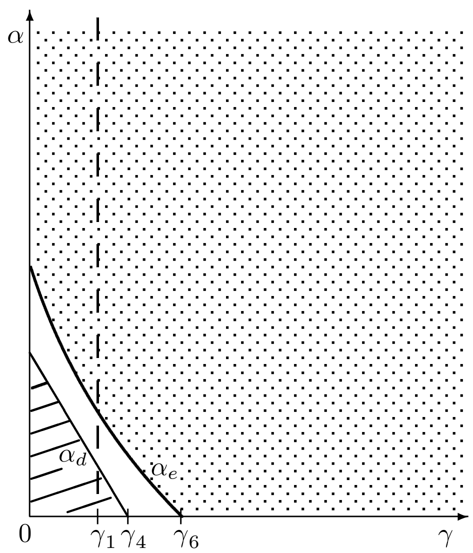

Figure 8 : Analysis of equilibria: Third case

reluctant to advertise given that such an activity requires a fixed cost. This behavior is certainly consistent with the findings of Grossman and Shapiro (1984) *inter alia*.

Turning to the asymmetric situation where just one firm advertises, the net effect of a change in c is instead quite ambiguous. Indeed, the impact of the strategic effect on demands, given in equation (2) by $c\alpha$, would be stronger, giving a relative advantage to the investing firm. On the other hand, the increase in competition in the goods market lowers profits, dampening the incentive to advertise. Consequently, the size of the white area may either increase or decrease. In our simulations, it actually increases from Figure 8 to 7, while decreasing when moving from 7 to 6. As products become more differentiated, it is not necessarily the case that a firm finds it profitable to invest in advertising if the other does not. Taking into account asymmetric outcomes thus leads to discover this somehow counter-intuitive relation between the equilibrium level of advertising and the degree of product differentiation. A similar ambiguous relation has been highlighted, although in a different framework, by von der Fehr and Stevik (1998).

Finally, coming back to the *coordination game,* we can observe that it arises only for high values of c. In other words, coordination becomes an issue when the degree of interdependence among agents, captured here by price competition, is strong enough. This indeed conforms with intuition, but we can further prove that in such a case investing is the best choice:

Proposition 3 *The coordination game could arise only for low values of* $\underline{a}$ *(*$\underline{a} < \underline{a}_1$*) or, equivalently, when product are highly substitutes (big c). Furthermore, when both* $(0, 0)$ *and* $(1, 1)$ *are "*SPE*", then the latter is always Pareto dominant from firms' standpoint.*

Proof. see Appendix A.4. ■

6 Conclusions

In this paper we considered a two-stage duopoly model with differentiated products where firms decide whether to invest in advertising or not and then they compete in prices. We focused on two main effects of advertising: a market enlargement effect and a predatory effect. Particular attention has been paid to the specific role of product differentiation in advertising decisions as well as to the assessment of their Pareto optimality from firms' standpoint.

Depending on the values taken by the parameters, both symmetric and asymmetric equilibria appeared. Among them, two outcomes are of particular interest: a *coordination game* in which both investing and non-investing are simultaneously equilibria; a *chicken game* in which only one firm invests in equilibrium with the second one possibly driven (endogenously) out of the market.

The *coordination game* arises only when products are strongly substitutes, suggesting that coordination matters when the degree of interdependence among agents, captured by price competition, is sufficiently high. Interestingly, when such a problem of coordination appears, the investment strategy leads in our model to the Pareto optimum.

Turning to the *chicken game*, we actually found two very interesting results. First, there exists a parameter region that supports a limit pricing behavior by the investor with the rival being *endogenously* squeezed out of the market. Furthermore, if the predatory advantage for the investing firm is strong enough, then it is able to take the entire market just by setting a kind of monopoly price. Second, in this asymmetric case the impact of product substitutability on the advertising efforts is ambiguous. This result contradicts the common view of a positive relationship between product differentiation and the equilibrium level of advertising and comes from the interplay between two opposite forces at work in the asymmetric equilibrium: the predatory gain that depends negatively on differentiation, and the strength of price competition that is instead relaxed by a decrease in product substitutability.

Starting from a very simple framework, we obtained quite interesting results. However, one of the main limitations of our approach stands in the use of a binary strategy set for investment decisions. Ideally, it would be better to use a more sophisticated relationship between investment in advertising and demand changes. On the other hand, as many other forms of investment, advertising has a strong discrete nature in the sense that it is sometimes more important to decide whether to invest or not rather than the exact amount to be spent on it. Furthermore, our plain structure turns out to be extremely flexible in the sense that it allows us to treat a large number of parameters (without resorting to normalization) and to completely characterize the game. Situations like the *coordination game* or the *chicken game* would in fact have been hardly treated in a continuous framework.

The same forces at work in the present paper could also be translated into a dynamic setting. Every firm could in fact be endowed with a stock of advertising that summarizes the effects of past advertising efforts. As a consequence, apart form current advertising, both the enlargement of the market and the consumers' shift from one firm to the other would depend on the stock of accumulated goodwill.

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A Appendix

A.1 The demand structure

Let's start by considering demand functions q_i and q_{-i} for our firms as given by equation (1). This analytic formulation is clearly meaningful as long as price strategies are such that the implied q_i and q_{-i} are non negative. Our goal here is to show how the demand system (2) can be obtained from equation (1) using continuity arguments. Consider the limit case in which p_i and p_{-i} are such that q_{-i}, as computed from (1), exactly equals zero. Solving the equation $q_{-i} = a(I_i, I_{-i}) - b\,p_{-i} + c\,[p_i - p_{-i} + \alpha_{-i}(I_{-i}) - \alpha_i(I_i)] = 0$ for p_{-i} and plugging the solution into the equation of q_i one gets (after rearranging terms):

$$q_i = 2a(I_i, I_{-i}) - b\,p_i - b\varphi(p_i) \tag{A1}$$

which, as we argued, is precisely the demand of firm i when the other firm gets zero sells ($p_i \leq p_i^l$). Equation (A1) is certainly correct for any couple of prices p_i and p_{-i} such that $q_{-i} = 0$ in (1). What remains to prove is that this is true for all prices p_i and p_{-i} that leads firm $-i$ to be out of the market, that is for lower p_i and greater p_{-i}.

Consider for example a higher p_{-i}. Since firm $-i$ is already out of the market, it cannot certainly hope to ameliorate its position by increasing the price. Demands should thus be invariant to this increases in p_{-i} and, by continuity, q_i equals (A1), which is in fact a function of p_i only. On the other hand, if firm i charges a price lower then before, then firm $-i$ is again out of the market, and we will actually have $q_{-i} < 0$ in (1). Following the above reasoning, demand of firm i should not, as long as q_{-i} computed with (1) is non-positive, depend on p_{-i}. Everything thus works as if firm $-i$ was charging a new price p_{-i} that would make q_{-i} exactly equal to zero, leading us back to formulation (A1). Finally, since the price p_{-i} solution to the equation $q_{-i} = 0$ cannot be negative, we have $\varphi(p_i) = \max\left\{\frac{a(I_i+I_{-i})+c[\alpha_{-i}(I_{-i})-\alpha_i(I_i)]}{b+c} + \frac{c}{b+c}p_i,\, 0\right\}$.

Now let's turn to price responsiveness of our demand system (2). As long as firm $-i$ is on the market, the appropriate demand curve is $q_i = a(I_i, I_{-i}) - b\,p_i + c\,[p_{-i} - p_i + \alpha_i(I_i) - \alpha_{-i}(I_{-i})]$ and its derivative with respect to p_i is simply $-(b + c)$. If price p_i now goes below the limit price p_i^l, then the right demand function is (A1) and its slope can be either $-\frac{b(b+2c)}{b+c} > -(b + c)$ or, if p_i is so low to hit the constrain $\varphi(p_i) = 0$, it amounts to $-b > -\frac{b(b+2c)}{b+c}$. As a conclusion, when p_i decreases demand q_i becomes less and less sensitive to price changes.

Finally, the limit price p_i^l is simply the non-negative solution (if it exist) to the equation $q_{-i} = a(I_i, I_{-i}) - b\,p_{-i} + c\,[p_i - p_{-i} + \alpha_{-i}(I_{-i}) - \alpha_i(I_i)] = 0$ with respect to p_i. This solution to turns out to be:

$$p_i^l = \alpha_i(I_i) - \alpha_{-i}(I_{-i}) + \frac{-a(I_i, I_{-i}) + (b + c)\,p_{-i}}{c},$$

which depends on p_{-i} (as expected), and can possibly be negative meaning that a limit price does not exist.

A.2 Proof of Proposition 1

The necessary and sufficient condition for $(0,0)$ to be an equilibrium is that none of the two firms has an incentive to advertise alone. The profit accruing to our firms in case of no investments ($I_1 = I_2 = 0$) is simply equal to $\pi_1^A = \pi_2^A = \frac{\underline{a}^2(b+c)}{(2b+c)^2} > 0$. By symmetry, we can consider indifferently the deviation of one of the two firms. Suppose that firm 1 deviates ($I_1 = 1$) and invests in advertising; its equilibrium profits in the second stage price game is then that of case B. As we have seen, although this payoff is (for each and every given value of the parameters) unique, its analytic expression changes in the parameters space and we actually have three cases. When $\alpha < \alpha_a$, profits of firm 1 are given by π_{1Ac}^B, while for $\alpha_a \leq \alpha < \alpha_b$ we have that firm 1 gets π_{1Dt}^B. Finally, for $\alpha \geq \alpha_b$, firm 1 receives π_{1Mp}^B. For each of the three cases, we should thus compare the payoff that firm 1 gets when invests with the one that it gets without undertaking advertising. As long as the latter is greater or equal to the former, (0,0) will be a "SPE" of the reduced form of the game.

We begin with the case where $\alpha \geq \alpha_b$. Here, we need to compare $\pi_1^A = \frac{\underline{a}^2(b+c)}{(2b+c)^2}$ with $\pi_{1Mp}^B = \frac{(\underline{a}+\gamma)^2}{b} - 1$ and, as long as $\pi_1^A \geq \pi_{1Mp}^B$, (0,0) will be an equilibrium. The equation $\pi_{1Mp}^B - \pi_1^A = 0$ is a convex parabola in γ with a negative (uninteresting) real root and a possibly positive real root $\gamma_2 = \frac{\sqrt{b}+\sqrt{\underline{a}^2(b+c)+(2b+c)^2}}{2b+c} - \underline{a}$. Therefore, since $\gamma \in (0, \infty)$, the necessary and sufficient condition we need is simply $\gamma \leq \gamma_2$. Clearly, if γ_2 turns out to be negative, there is no acceptable value of γ that makes (0,0) an equilibrium in such a case ($\alpha \geq \alpha_b$).

The second situation is characterized by $\alpha_a \leq \alpha < \alpha_b$. Now, the relevant profit to compare with π_1^A is given by $\pi_{1Dt}^B = \frac{(\alpha c - \gamma - \underline{a})[2c(\underline{a}+\gamma) - b(\alpha c - \gamma - \underline{a})]}{c^2} - 1$. Contrary to before, the equilibrium payoff π_{1Dt}^B now depends on α, reflecting the fact that firm 1 is not in the condition to be a "real" monopolist anymore. In fact, it is now convenient to charge the highest possible limit price, and so firm 1 is still sensitive to the extent of the "strategic effect" α. The equation $\pi_{1Dt}^B - \pi_1^A = 0$ is a concave parabola in α with two (possibly complex conjugate) roots. Now, since we have a concave parabola, if the two roots are actually complex conjugate we have that (0,0) is an equilibrium because $\pi_{1Dt}^B - \pi_1^A < 0$ for any α. After tedious calculations, it turns out that this happens iff $\gamma < \gamma_2$. On the other hand, if $\gamma \geq \gamma_2$ then the two roots are real, and in particular they both coincide with α_b for $\gamma = \gamma_2$ (consistently with the findings of the previously analyzed case where $\alpha \geq \alpha_b$). As we deal here with a concave parabola, we are interested in external solutions of our equation $\pi_{1Dt}^B - \pi_1^A = 0$ that have to be compatible with the interval of analysis ($\alpha_a \leq \alpha < \alpha_b$). Since the difference between one of these root and α_b is increasing in γ (as revealed by the sign of the first derivative) we can neglect it because, whenever this root is a real number ($\gamma \geq \gamma_2$), it is greater or equal to α_b and so it lays out of the interval we are analyzing. The other (smaller) root $\alpha_c = \frac{\frac{(b+c)(\underline{a}+\gamma)}{c} - \frac{\sqrt{\underline{a}^2(3b^2+3bc+c^2)+2\underline{a}(2b+c)^2\gamma-(2b+c)^2(b-\gamma^2)}}{2b+c}}{b}$ is decreasing in γ instead (with respect to α_b) and reaches α_a for $\gamma = \gamma_3 = \frac{\sqrt{\underline{a}^2(b+c)+(2b+c)^2}}{2\sqrt{b+c}} - \underline{a} > \gamma_2$. Consequently, if $\gamma \geq \gamma_2$, the condition $\pi^B_{1Dt} - \pi^A_1 \leq 0$ is equivalent to $\alpha \leq \alpha_c$.

Finally, we have the third scenario characterized by $0 < \alpha < \alpha_a$. Here we have to compare the usual π^A_1 with π^B_{1Ac}, whose analytic expression is given by equation (6). The profit difference $\pi^B_{1Ac} - \pi^A_1 = 0$ is now a convex parabola in α with a negative (uninteresting) real root and a (possibly) positive real one, that is always greater than the other, given by $\alpha_d = \frac{(2b+3c)\left[\sqrt{b+c}\sqrt{\underline{a}^2(b+c)+(2b+c)^2} - (b+c)(\underline{a}+\gamma)\right]}{c(b+c)(2b+c)}$. Equilibrium involves here those internal solutions that are compatible with the interval of analysis ($0 < \alpha < \alpha_a$). It is easy to check that α_d is a decreasing function of γ and that $\alpha_d = \alpha_a$ when $\gamma = \gamma_3$. Consequently, when $\gamma \leq \gamma_3$, all $\alpha \in (0, \alpha_a)$ are solutions to the inequality $\pi^B_{1Ac} - \pi^A_1 \leq 0$ and so (0,0) is certainly an equilibrium. On the other hand, when $\gamma > \gamma_3$, then $\alpha_d < \alpha_a$ and we need $\alpha \leq \alpha_d$ for a deviation to be unprofitable. Furthermore, since $\alpha_d = 0$ when γ equals $\gamma_4 = \frac{\sqrt{\underline{a}^2(b+c)+(2b+c)^2}}{\sqrt{b+c}} - \underline{a} > \gamma_3$, we have that $(0,0)$ cannot be an equilibrium for $\gamma \geq \gamma_4$ because no positive internal solution α exists for our equation $\pi^B_{1Ac} - \pi^A_1 = 0$.

A.3 Proof of Proposition 2

As before, the necessary and sufficient condition for $(1,1)$ to be an equilibrium requires that each firm takes no advantage in reconsidering its investment decision. Whenever both firms invest in advertising ($I_1 = I_2 = 1$), profits are simply equal to $\pi^C_1 = \pi^C_2 = \frac{(b+c)(2\underline{a}+3\gamma)^2}{4(2b+c)^2} - 1$ and they are non-negative iff $\gamma \geq \gamma_5 = \frac{2}{3}\left(\frac{2b+c}{\sqrt{b+c}} - \underline{a}\right)$. By symmetry, one knows that it is indifferent to consider the deviation of one of the two firms. Imagine that firm 2 deviates ($I_2 = 0$), then its equilibrium profits in the second stage price game is that of case B. As we have seen, although this payoff is unique, its analytic expression changes in the parameters space. However, here we just have two scenarios. When $0 < \alpha < \alpha_a$, profits of firm 2 are given by $\pi^B_2 = \pi^B_{2Ac} = \frac{(b+c)[(\underline{a}+\gamma)(2b+3c)-c\alpha(2b+c)]^2}{(2b+c)^2(2b+3c)^2}$ while, for both $\alpha_a \leq \alpha < \alpha_b$ and $\alpha \geq \alpha_b$, firm 2 gets zero profits and so these two cases collapse in the interval $\alpha \geq \alpha_a$.

Let us begin with the last case, where $\alpha \geq \alpha_a$. Here, we just need to compare π^C_2 with $\pi^B_2 = 0$, and so the equilibrium condition $\pi^C_2 \geq \pi^B_2$ only amounts to require that π^C_2 is non-negative, i.e. that $\gamma \geq \gamma_5$.

The other case ($0 < \alpha < \alpha_a$) turns out to be more cumbersome. Relevant profits are given by π^B_{2Ac} (which is now a strictly positive number) and π^C_2. The equation $\pi^B_{2Ac} - \pi^C_2 = 0$ is a convex parabola in α with two (possibly complex conjugate) roots. Now, since we have a convex parabola, if the two roots are actually complex conjugate we have that (1,1) is never an equilibrium because $\pi^B_{2Ac} - \pi^C_2 > 0$ for any α. After tedious calculations, it turns out that this happens iff $\gamma < \gamma_5$. On the other hand, if $\gamma \geq \gamma_5$ then the two roots are real, and in particular they both coincide with α_a for $\gamma = \gamma_5$ (consistently with the findings of the previously analyzed case where $\alpha \geq \alpha_a$). As we deal here with a convex parabola, we are interested in internal solutions of our equation $\pi^B_{2Ac} - \pi^C_2 = 0$ which are compatible with the interval of analysis ($0 < \alpha < \alpha_a$). As long as $\gamma \geq \gamma_5$, a close inspection at the first derivative (with respect γ) of the difference between one of these root and α_a reveals that this derivative is positive, so that we can forget about it. This in fact means that, whenever this root is a real number ($\gamma \geq \gamma_5$), it is greater or equal to α_a and so out of the interval we are analyzing. Concerning the other (smaller) root, $\alpha_e = \frac{2c(b+c)(2b+c)(2b+3c)(\underline{a}+\gamma)-\sqrt{c^2(b+c)(2b+c)^2(2b+3c)^2[4\underline{a}^2(b+c)-4(2b+c)^2+12\underline{a}(b+c)\gamma+9(b+c)\gamma^2]}}{2c^2(b+c)(2b+c)^2}$, the quantity $\alpha_e - \alpha_a$ is on the contrary decreasing in γ and in particular α_e reaches 0 for $\gamma = \gamma_6 = \frac{2}{5}\left(\frac{\sqrt{\underline{a}^2(b+c)+5(2b+c)^2}}{\sqrt{b+c}} - \underline{a}\right) > \gamma_5$. Consequently, the equilibrium condition $\pi^B_{2Ac} - \pi^C_2 \leq 0$ is satisfied by the (internal) solution $\alpha \geq \alpha_e$. In particular, when $\gamma \geq \gamma_6$, then $(1,1)$ is always an equilibrium because no positive internal solution α exists for our equation $\pi^B_{2Ac} - \pi^C_2 = 0$.

A.4 Proof of Proposition 3

Let us consider the situation in which $\underline{a} < \underline{a}_1$, that yields the ranking $0 < \gamma_2 < \gamma_3 < \gamma_5 < \gamma_1 < \gamma_6 < \gamma_4$. Starting from the interval $0 < \gamma < \gamma_5$, we know from Proposition 2 that $(1,1)$ will never be a "SPE". We rule out this situation given that we look for intervals where *both* $(0,0)$ *and* $(1,1)$ hold simultaneously as equilibria of the game. Let us take the interval $\gamma_5 < \gamma < \gamma_4$; in this case $(0,0)$ is a "SPE" (Proposition 1) if $\alpha \leq \alpha_d$, while $(1,1)$ requires $\alpha \geq \alpha_e$. From Appendix A.2 we know that both α_d and α_e are decreasing functions of γ. We also know that α_d starts from $\gamma = \gamma_3$ and reaches 0 in $\gamma = \gamma_4$, while α_e starts from $\gamma = \gamma_5$ and reaches 0 in $\gamma = \gamma_6$. Given the above ranking of γ, it is then obvious that the two curves will cross. It is in fact possible to demonstrate that the two curves meet twice. However, one of these two roots can be neglected because it would require negative values for the parameter α. In the admissable region of parameters, thus, α_d meets α_e just once in $\gamma_7 = \frac{8(b+c)\sqrt{\underline{a}^2(b+c)+(2b+c)^2}-10\underline{a}(b+c)^{3/2}-2\sqrt{(b+c)^2\left[13\underline{a}^2(b+c)+2(2b+c)^2-12\underline{a}\sqrt{b+c}\sqrt{\underline{a}^2(b+c)+(2b+c)^2}\right]}}{7(b+c)^{3/2}}$. Furthermore, it is possible to rank also this last threshold value of γ and we find that $\gamma_1 < \gamma_7 < \gamma_6$. Hence, $\alpha_e > \alpha_d$ for $\gamma_5 < \gamma < \gamma_7$ and $\alpha_e < \alpha_d$ for $\gamma_7 < \gamma < \gamma_4$, as we can also see in Figure 6. It follows as a consequence that a coordination game appears in the region of parameters where both $\gamma_7 < \gamma < \gamma_4$ and $\alpha_e < \alpha < \alpha_d$. In the remaining interval considered, $\gamma_4 < \gamma$, only $(1,1)$ can be a "SPE" given that we know from Proposition 6 that $(0,0)$ is never an equilibrium of the game.

To complete our demonstration we only need to prove that the coordination game does not emerge when we consider higher values of $\underline{a}$, i.e $\underline{a}_1 < \underline{a}$. Let us first examine the case where $\underline{a}_1 < \underline{a} < \underline{a}_2$. The main variation with respect to the previous case is that $\gamma_4 < \gamma_6$. In the two 'lateral' intervals $0 < \gamma < \gamma_5$ and $\gamma_4 < \gamma$, as before, a coordination game will never arise. Furthermore, when we consider the 'intermediate' interval $\gamma_5 < \gamma < \gamma_4$, we do not find anymore, at least in the admissable region of parameters, the cross between α_d and α_e. This is obvious given that γ_4 and γ_6 are inversely positioned with respect to the previous situation. Here, when the two curves exist, $\alpha_e > \alpha_d$ for every given value of γ, as we can see in Figure 7. It is not possible then to find a region where $\alpha > \alpha_e$ and $\alpha < \alpha_d$ and, in turn, to sustain at the same time $(0,0)$ and $(1,1)$ as "SPE". The same reasoning applies to the interval in which $\underline{a}_2 < \underline{a}$, with the only difference that α_d and α_e are only partly represented given that they start for negative values of γ, as one can find in Figure 8.

We have then proved the first part of Proposition 3, showing that a coordination game only appears for $\underline{a} < \underline{a}_1$. In particular, this happens when $\gamma_7 < \gamma < \gamma_4$ and $\alpha_e < \alpha < \alpha_d$. The second part of Proposition 3 can be easily proved given that $(1,1)$ and $(0,0)$ are both "SPE" only for $\gamma_7 < \gamma < \gamma_4$ and we demonstrated before that $\gamma_1 < \gamma_7$. Remembering that the threshold value for Pareto efficiency is γ_1 (see14), a coordination game could emerge only in a region where $(1,1)$ Pareto dominates $(0,0)$.

The last part of this proof deals with the possibility of using c instead of $\underline{a}$ to discern the case where the coordination game could arise. Unfortunately, a complete characterization of the game is not obtainable anymore because we cannot find values of c that rank the threshold values of γ. We consider then the limit values for γ_4 and γ_6 when c going to infinity. The former goes to infinity, while the latter tend to a finite number. For high values of c, it becomes hence clear that $\gamma_4 > \gamma_6$ and we come back to the situation where α_d and α_e cross, giving thus rise to the possibility that a coordination game exists.

There is moreover another consideration that reinforces the previous result. It is easy to prove that $\frac{\partial \underline{a}_1}{\partial c} > 0$ and $\frac{\partial \underline{a}_2}{\partial c} > 0$. Hence, when c increases, it becomes more likely to happen in the region where $\underline{a}_1 < \underline{a}$, i.e. in the region where the coordination game could come out as a result of the game.