Non-Linear Market Demand and Capital Accumulation in a Differential Oligopoly Game*

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March 13, 2000

Abstract

We investigate a differential oligopoly game where firms compete in a Cournot market whose demand function is always downward sloping but can take any degree of curvature. There exist two economically meaningful saddle points, one dictated by demand conditions, the other by the Ramsey rule. In steady state, optimal capital is non-decreasing in market size. Then we show that the socially efficient output is independent of the curvature of market demand. This entails that the welfare loss associated to the Cournot equilibrium decreases as market size increases.

JEL Classification: L13

Keywords: Cournot competition, curvature of market demand, capital accumulation, differential games, steady state

*We thank the seminar audience at the University of Trieste for comments and discussion. The usual disclaimer applies.

1 Introduction

The current literature on oligopoly theory usually adopts a static approach and a linear market demand, with either homogeneous or differentiated goods (for exhaustive surveys, see Tirole, 1988; Martin, 1993, *inter alia*). A relatively scanty attention has been devoted to the analysis of the effects of non-linear demand on firms' strategic behaviour. In this respect, a relevant exception is the work of Anderson and Engers (1992, 1994), where Cournot and Stackelberg equilibria are investigated under the perspective that market demand can be either convex or concave.[1] They connect the solution concept (Nash or Stackelberg) with firms' optimal commitments in terms of installed capital.

However, their approach is inherently static, in the sense that capital accumulation is instantaneous instead of taking place over time. Therefore, costs enter the generic firm's objective function only in the form of an exogenous capacity, so that the output levels chosen on the basis of strategic interaction can indeed be produced if capacity is at least as large as the equilibrium output determined by the intersection of reaction functions. Otherwise, firms operate at the boundary of capacity.

The alternative perspective of endogenising capital accumulation is typically taken by the literature on differential games, where we avail of a relatively small literature on oligopoly. However, the existing contributions examine intertemporal capital accumulation for production in models with linear market demand (Simaan and Takayama, 1978; Fershtman and Muller, 1984; Fershtman and Kamien, 1987; Cellini and Lambertini, 1998), or else investigate firms' R&D decision in innovation races where the market payoff associated to the successful innovation is a prize blackboxing market demand (e.g., Reinganum, 1982a).

Our aim consists in nesting the model introduced by Anderson and Engers (1992) into a differential game setup, so as to endogenise firms' capital commitment and link it explicitly to the curvature of market demand. We prove that the Cournot oligopoly produces multiple steady state equilibria; in particular, the economic solution is dictated either by demand conditions, or by the Ramsey rule of capital accumulation. We also illustrate the relationship holding, in steady state, between demand curvature and capital

[1] See also Lambertini (1996) for the analysis of the bearings of the curvature of market demand on the stability of implicit collusion either in prices or in quantities.

commitment. This can be summarised in the following terms. Suppose demand turns from linear into convex. If so, each firm's capital decreases. The opposite happens if we turn a linear demand into a concave one, and it is increasingly so as concavity increases. The intuition is that in the first case market size and firms' optimal outputs decrease, while in the second the opposite is true.

Then, we evaluate the behaviour of a social planner aiming at welfare maximisation, to find that the socially efficient output is independent of the curvature of market demand. This, in connection with the aforementioned results, entails that the welfare loss associated to the Cournot equilibrium decreases as market size increases.

The remainder of the paper is structured as follows. The setup is laid out in section 2. Section 3 describes the Cournot oligopoly. Section 4 contains the analysis of the social optimum. Concluding remarks are in section 5.

2 The model

We borrow the demand structure from Anderson and Engers (1992, 1994). The market is served by n firms selling a homogeneous product over time $t \in [0, \infty)$. The market demand function is defined as follows:

$$Q(t) = A - (p(t))^{\alpha} \ , \ \alpha > 0 \, . \tag{1}$$

The above function is always downward sloping, and can be either convex ($\alpha \in (0,1)$) or concave ($\alpha > 1$). Firms are quantity-setters, the inverse demand function being:

$$p(t) = (A - Q(t))^{\frac{1}{\alpha}} \ , \tag{2}$$

where $Q(t) = \sum_{i=1}^{n} q_i(t)$, and $q_i(t)$ is the individual output of firm i at time t. Production requires physical capital k, accumulating over time to create capacity. At any t, the output level is $y_i(t) = f(k_i(t))$, with $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f" \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$.

A reasonable assumption is that $q_i(t) \leq y_i(t)$, that is, the level of sales is at most equal to the quantity produced. Excess output is reintroduced into the production process yielding accumulation of capacity according to the following process:

$$\frac{\partial k_i(t)}{\partial t} = f(k_i(t)) - q_i(t) - \delta k_i(t), \tag{3}$$

where δ denotes the rate of depreciation of capital. In order to simplify further the analysis, suppose that unit variable cost is constant and equal to zero. The cost of capital is represented by the opportunity cost of intertemporal relocation of unsold output. Firm i's instantaneous profits i are

$$\pi_i(t) = p(t)q_i(t). \tag{4}$$

Firm i maximizes the discounted flow of its profits:

$$J_i = \int_0^{\infty} e^{-\rho t}\pi_i(t)\, dt \tag{5}$$

under the constraint (3) imposed by the dynamics of the state variable $k_i(t)$. Notice that the state variable does not enter directly the objective function.

For future reference, we first outline the features of the demand function (2) in terms of the elasticity of demand w.r.t. price, $\varepsilon_{Q,p}$. The price elasticity of demand can be written as follows:

$$|\varepsilon_{Q,p}| = -\frac{\partial Q(\alpha)}{\partial p\left(Q\left(\alpha\right)\right)} \cdot \frac{p\left(Q\left(\alpha\right)\right)}{Q(\alpha)} = \frac{\alpha p^{\alpha}}{A - p^{\alpha}}\,. \tag{6}$$

3 The Cournot equilibrium

In solving the quantity-setting game between profit-seeking agents, we shall focus upon a single firm. The relevant objective function of firm i is:

$$J_i = \int_0^{\infty} e^{-\rho t}\, q_i(t) \cdot \left[A - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{\alpha}}\, dt \tag{7}$$

where $Q_{-i}(t) = \sum_{j \neq i} q_j(t)$ is the total output of the $n-1$ rivals of firm i at time t. The function (7) must be maximised w.r.t. $q_i(t)$, under (3). The corresponding Hamiltonian function is:

$$\mathcal{H}(t) = e^{-\rho t} \cdot \left\{q_i(t) \cdot \left[A - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{\alpha}} + \lambda_i(t)\left[f(k_i(t)) - q_i(t) - \delta k_i(t)\right]\right\}, \tag{8}$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, and $\mu_i(t)$ is the co-state variable associated to $k_i(t)$. We adopt the open-loop Nash equilibrium as the solution concept.[2]

[2] The limitations affecting the open-loop solution are well known (see Kydland, 1977; Fudenberg and Tirole, 1991, pp. 520-36, *inter alia*). In line of principle, the closed-loop solution would be preferable. However, the form of the instantaneous payoff used in this setting does not allow us to pursue the closed-loop equilibrium. For a more detailed discussion of this issue, see Reinganum (1982b), Mehlmann (1988) and Başar and Olsder (1995^2).

The necessary and sufficient conditions for a path to be optimal are:

$$\frac{\partial\mathcal{H}(t)}{\partial q_i(t)} = \left[A - q_i(t) - \sum_{j\neq i} q_j(t)\right]^{\frac{1}{\alpha}} + \tag{9}$$

$$-\left\{ q_i(t)\cdot\frac{\left[A - q_i(t) - \sum_{j\neq i} q_j(t)\right]^{\frac{1-\alpha}{\alpha}}}{\alpha} + \lambda_i(t)\right\} = 0;$$

$$-\frac{\partial\mathcal{H}(t)}{\partial k_i(t)} = \frac{\partial\mu_i(t)}{\partial t} \Rightarrow \frac{\partial\lambda_i(t)}{\partial t} = \left[\rho + \delta - f'(k_i(t))\right]\lambda_i(t); \tag{10}$$

$$\lim_{t\to\infty} \mu_i(t)\cdot k_i(t) = 0\,. \tag{11}$$

Condition (9) implicitly defines the reaction function of firm i to the rivals' output decisions. Rewrite (9) as follows:

$$\left[A - q_i(t) - \sum_{j\neq i} q_j(t)\right]^{\frac{1}{\alpha}} - \left\{\frac{q_i(t)}{\alpha}\cdot\frac{\left[A - q_i(t) - \sum_{j\neq i} q_j(t)\right]^{\frac{1}{\alpha}}}{A - q_i(t) - \sum_{j\neq i} q_j(t)} + \lambda_i(t)\right\} = 0 \tag{12}$$

that is:

$$\left[A - q_i(t) - \sum_{j\neq i} q_j(t)\right]^{\frac{1}{\alpha}}\cdot\left[1 - \frac{q_i(t)}{\alpha\left(A - q_i(t) - \sum_{j\neq i} q_j(t)\right)}\right] - \lambda_i(t) = 0 \tag{13}$$

In order to simplify calculations and to obtain an analytical solution, we adopt the following assumption, based on firms' *ex ante* symmetry:

$$\sum_{j\neq i} q_j(t) = (n-1)q_i(t) \tag{14}$$

Thanks to symmetry, in the remainder we drop the indication the identity of the firm and rewrite the FOC as follows:

$$\left[A - nq_i(t)\right]^{\frac{1}{\alpha}}\cdot\left\{\frac{\alpha\left[A - nq_i(t)\right] - q_i(t)}{\alpha\left[A - nq_i(t)\right]}\right\} - \lambda_i(t) = 0 \tag{15}$$

where we can write $[A - nq_i(t)]^{\frac{1}{\alpha}} = p(t)$, with $q \leq A/n$. Therefore:

$$p(t) \cdot \left\{ \frac{\alpha [A - nq_i(t)] - q_i(t)}{\alpha [A - nq_i(t)]} \right\} - \lambda_i(t) = 0 \tag{16}$$

that is:

$$\frac{p(t)\alpha [A - nq_i(t)] - p(t)q_i(t) - \alpha [A - nq_i(t)] \lambda_i(t)}{\alpha [A - nq_i(t)]} = 0 \tag{17}$$

from which we get:

$$A [p(t) - \lambda_i(t)] \alpha - q_i(t) \{p(t) + n\alpha [p(t) - \lambda_i(t)]\} = 0 \tag{18}$$

Then, we obtain the symmetric per-firm output:

$$q^*(t) = \frac{A [p(t) - \lambda(t)] \alpha}{(1 + n) p(t) - n\lambda(t)\alpha} \tag{19}$$

which can be rewritten in several equivalent ways, e.g.:

$$\lambda(t) = p(t) - \frac{q^*(t) (p(t))^{1-\alpha}}{\alpha} . \tag{20}$$

The above discussion, in particular (19-20), produces the following result, which needs no further proof:

Lemma 1 *In equilibrium, the following necessarily holds:*

$$\frac{\partial q^*(t)}{\partial t} = 0 \Rightarrow \frac{\partial p(t)}{\partial t} = 0 \Rightarrow \frac{\partial \lambda(t)}{\partial t} = 0$$

and

$$\frac{\partial p(t)}{\partial t} = 0 \Rightarrow \frac{\partial q^*(t)}{\partial t} = 0 \Rightarrow \frac{\partial \lambda(t)}{\partial t} = 0 .$$

Equation (20) can be differentiated w.r.t. time to obtain:

$$\frac{\partial \lambda(t)}{\partial t} = 0 \Rightarrow \frac{\partial p(t)}{\partial t} - \frac{(p(t))^{1-\alpha}}{\alpha} \cdot \frac{\partial q^*(t)}{\partial t} - \frac{(1-\alpha) q^*(t) (p(t))^{-\alpha}}{\alpha} \cdot \frac{\partial p(t)}{\partial t} = 0 . \tag{21}$$

Using the symmetry assumption (14) and differentiating the direct demand function (1) w.r.t. time, we get:

$$\frac{\partial q^*(t)}{\partial t} = -\frac{\alpha\,(p(t))^{\alpha-1}}{n} \cdot \frac{\partial p(t)}{\partial t} \tag{22}$$

which can be plugged into (21) to yield:

$$\frac{\partial \lambda(t)}{\partial t} = 0 \Rightarrow \frac{\partial p(t)}{\partial t}\left(1+\frac{1}{n}\right) - \frac{(1-\alpha)\,q^*(t)\,(p(t))^{-\alpha}}{\alpha} \cdot \frac{\partial p(t)}{\partial t} = 0 \tag{23}$$

from which we derive the following:

Lemma 2 *The condition* $\dfrac{\partial \lambda_i(t)}{\partial t} = 0$ *is satisfied when*

$$\textit{either } \frac{\partial p(t)}{\partial t} = 0 \textit{ or } q^*(t) = \frac{A\alpha(n+1)}{n(\alpha n+1)}$$

or

$$\left\{\frac{\partial p(t)}{\partial t} = 0 \textit{ and } q^*(t) = \frac{A\alpha(n+1)}{n(\alpha n+1)}\right\},$$

provided $A \neq nq^*(t)$*. The solution* $q^*(t) = \dfrac{A\alpha(n+1)}{n(\alpha n+1)}$ *is relevant only for* $\alpha \in (0,1)$*.*

However, the condition $q^*(t) = A\alpha(n+1)/\left[n(\alpha n+1)\right]$ is relevant only for $\alpha \in (0,1)$, as $q^*(t) = \dfrac{A\alpha(n+1)}{n(\alpha n+1)} > \dfrac{A}{n}$ for $\alpha > 1$, the condition would imply $p(t) < 0$. See also below.

Now rewrite (9) as follows:

$$q^*(t) = \alpha\left[p(t) - \lambda(t)\right](p(t))^{\alpha-1}\ . \tag{24}$$

The above expression can be differentiated w.r.t. time:

$$\begin{aligned} \frac{dq^*(t)}{dt} &= \alpha\left\{\left[\frac{dp(t)}{dt} - \frac{d\lambda(t)}{dt}\right](p(t))^{\alpha-1} + \right. \\ &\quad \left. (\alpha-1)\,(p(t))^{\alpha-2}\left[p(t)-\lambda(t)\right]\frac{dp(t)}{dt}\right\} \end{aligned} \tag{25}$$

or

$$\frac{1}{\alpha} \cdot \frac{dq^*(t)}{dt} = (p(t))^{\alpha-1} \left\{ \frac{p(t) + (\alpha-1)\left[p(t) - \lambda(t)\right]}{p(t)} \right\} \frac{dp(t)}{dt} + \tag{26}$$

$$-\frac{d\lambda(t)}{dt} (p(t))^{\alpha-1} \ .$$

Since $p(t) - \lambda(t) = q^*(t)\,(p(t))^{1-\alpha}/\alpha$, (26) rewrites as:

$$\frac{1}{\alpha} \cdot \frac{dq^*(t)}{dt} = (p(t))^{\alpha-1} \left\{ \left[1 + \frac{\alpha-1}{\alpha} \cdot q^*(t)\,(p(t))^{-\alpha}\right] \frac{dp(t)}{dt} - \frac{d\lambda(t)}{dt} \right\} . \tag{27}$$

Now, using the following information:

$$\begin{cases} \frac{dp(t)}{dt} = -\frac{n(p(t))^{1-\alpha}}{\alpha} \cdot \frac{\partial q^*(t)}{\partial t} \\ \frac{\partial \lambda(t)}{\partial t} = \left[\rho + \delta - f'(k(t))\right] \lambda(t) \\ \lambda(t) = p(t) - \frac{q^*(t)(p(t))^{1-\alpha}}{\alpha} \end{cases} \tag{28}$$

we obtain:

$$\frac{1}{\alpha} \cdot \frac{dq^*(t)}{dt} = -\frac{n}{\alpha} \left[1 + \frac{\alpha-1}{\alpha} \cdot q^*(t)\,(p(t))^{-\alpha}\right] \frac{dq(t)}{dt} + \tag{29}$$

$$-\,(p(t))^{\alpha-1} \left[\rho + \delta - f'(k(t))\right] \left[p(t) - \frac{q^*(t)\,(p(t))^{1-\alpha}}{\alpha}\right]$$

that is,

$$\frac{dq^*(t)}{dt} \cdot \left\{ \frac{1}{\alpha} + \frac{n}{\alpha} \left[1 + \frac{\alpha-1}{\alpha} \cdot q^*(t)\,(p(t))^{-\alpha}\right] \right\} = \tag{30}$$

$$= -\,(p(t))^{\alpha-1} \left[\rho + \delta - f'(k(t))\right] \left[p(t) - \frac{q^*(t)\,(p(t))^{1-\alpha}}{\alpha}\right] .$$

Define

$$\frac{1}{\alpha} + \frac{n}{\alpha} \left[1 + \frac{\alpha-1}{\alpha} \cdot q^*(t)\,(p(t))^{-\alpha}\right] \equiv \beta \tag{31}$$

with

$$\beta \begin{array}{c} > 0 \\ = 0 \\ < 0 \end{array} \ \text{if } q^*(t) \begin{array}{c} < \frac{A\alpha(n+1)}{n(\alpha n+1)} \\ = \frac{A\alpha(n+1)}{n(\alpha n+1)} \\ > \frac{A\alpha(n+1)}{n(\alpha n+1)} \end{array} \tag{32}$$

so that (30) simplifies as follows:

$$\frac{dq^*(t)}{dt} = -\frac{1}{\beta}(p(t))^{\alpha-1}\left[\rho+\delta-f'(k(t))\right]\left[p(t)-\frac{q^*(t)\,(p(t))^{1-\alpha}}{\alpha}\right] . \qquad (33)$$

Then, notice that for all $\alpha > 1$, we have $\beta > 0$ if $n > 1/\alpha$. Otherwise, the sign of β is ambiguous.

We are now in a position to state what follows:

Theorem 1 *The steady state requirement, $dq^*(t)/dt = 0$ is satisfied if*

$$\left.\begin{array}{c} f'(k(t)) = \rho+\delta \\ p(t) = \dfrac{q^*(t)\,(p(t))^{1-\alpha}}{\alpha} \Rightarrow q^*(t) = \dfrac{A\alpha}{1+\alpha n} < \dfrac{A}{n} \end{array}\right\} \forall\, \alpha > 0$$

and

$$p(t) = 0 \Rightarrow q^*(t) = \frac{A}{n}\ ,\ \forall\, \alpha \geq 1\ .$$

If $\alpha \in (0,1)$, $q^*(t) = A/n$ does not represent a solution to $dq/dt = 0$.

Proof. The proof largely relies on Lemma 2 and the above discussion. To complete it, just observe that, for all $\alpha \geq 1$, we have

$$\frac{A\alpha}{1+\alpha n} < \frac{A}{n} < \frac{A\alpha(n+1)}{n(\alpha n+1)} \qquad (34)$$

so that $\beta > 0$ everywhere, while for all $\alpha \in (0,1)$,

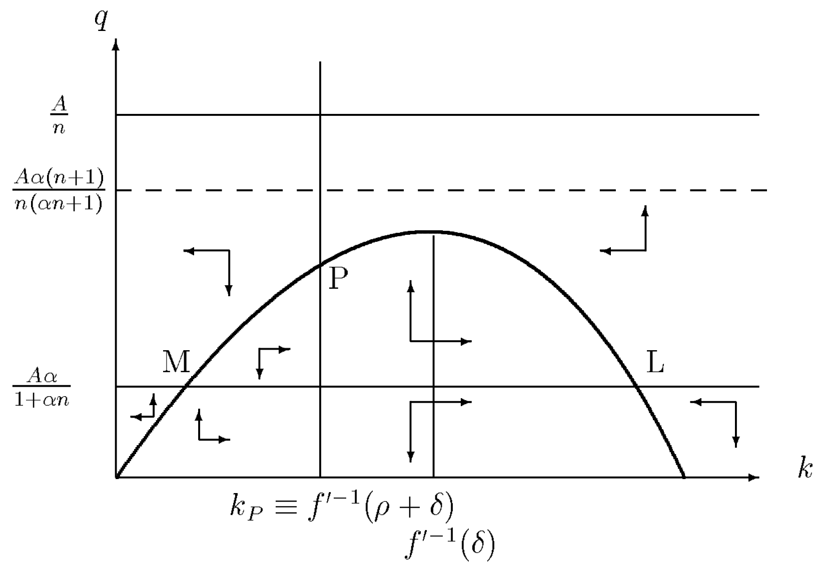
$$\frac{A\alpha}{1+\alpha n} < \frac{A\alpha(n+1)}{n(\alpha n+1)} < \frac{A}{n} \qquad (35)$$

so that $\beta > 0$, when evaluated in the steady state, where surely $\alpha > 1/n$. ■

We are now able to draw a phase diagram in the space $\{k, q\}$, in order to characterise the steady state equilibrium. For the sake of simplicity, consider first the case $\alpha \in (0,1)$. The locus $\dot{q} \equiv dq/dt = 0$ is given by $q = A\alpha/(1+\alpha n)$ and $f'(k) = \rho+\delta$ in figure 1. Notice that the horizontal locus $q = A\alpha/(1+\alpha n)$ denotes the usual equilibrium solution we are well accustomed with from the existing literature dealing with static market games (see Anderson and Engers, 1992, 1994; Lambertini, 1996). The two loci partition the space $\{k, q\}$

into four regions, where the dynamics of q is given by (33), as summarised by the vertical arrows. The locus $\dot{k} \equiv dk/dt = 0$ as well as the dynamics of k, depicted by horizontal arrows, derive from (3). Steady states, denoted by M, L along the horizontal arm, and P along the vertical one, are identified by intersections between loci.

Figure 1: Cournot competition, $\alpha < 1$ (and $\beta > 0$ in ss)



It is worth noting that the situation illustrated in figure 1 is only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the horizontal locus $q = A\alpha/(1 + \alpha n)$ shifts upwards as A and/or α increase. Therefore, we obtain one out of five possible regimes:

[1] There exist three steady state points, with $k_M < k_P < k_L$ (this is the situation depicted in figure 1). M is a saddle point; P is an unstable focus. L is again a saddle point, with the horizontal line as the stable arm.

[2] There exist two steady state points, with $k_M = k_P < k_L$. Here, M coincides with P, so that we have only two steady states which are both saddle points. In $M = P$, the saddle path approaches the saddle point from the left only, while in L the stable arm is again the horizontal line.

[3] There exist three steady state points, with $k_P < k_M < k_L$.Here, P is a saddle; M is an unstable focus; L is a saddle point, as in regimes [1] and [2].

[4] There exist two steady state points, with $k_P < k_M = k_L$. Here, points M and L coincide. P remains a saddle, while $M = L$ is a saddle whose converging arm proceeds from the right along the horizontal line.

[5] There exists a unique steady state point, corresponding to P. Here, there exists a unique steady state point, P, which is also a saddle point.

An intuitive explanation may be given as follows. The vertical locus $f'(k) = \rho + \delta$ identifies a constraint on optimal capital embodying firms' intertemporal preferences, i.e., their common discount rate. Accordingly, maximum output level in steady state would be that corresponding to the capacity such that $f'(k) = \delta$. Yet, a positive discounting (that is, impatience) induces producers to install a smaller steady state capacity, much the same as it happens in the well known Ramsey model. On these grounds, define this level of k as the *optimal capital constraint*, and label it as $\widehat{k}$. When the reservation price A is very large (or α is large, or n is low), points M and L either do not exist (regime [5]) or fall to the right of P (regimes [2], [3], and [4]). Under these circumstances, the capital constraint is operative and firms choose the capital accumulation corresponding to P. As we will see below, this is fully consistent with the dynamic properties of the steady state points.

Notice that, since both steady state points located along the horizontal locus entail the same levels of sales, point L is surely inefficient in that it requires a higher amount of capital. Point M, as already mentioned above, corresponds to the optimal quantity emerging from the static version of the game. It is hardly the case of emphasising that this solution encompasses both monopoly (when $n = 1$) and perfect competition (as, in the limit, $n \rightarrow \infty$). In M, marginal instantaneous profit is nil.

We can sum up the above discussion as follows. The unique efficient and non-unstable steady state point is P if $k_P < k_M$, while it is M if the opposite inequality holds. Such a point is always a saddle. Individual equilibrium output is $q^*_{ss} = A\alpha/(1+\alpha n)$ (where subscript ss stands for *steady state*) if the equilibrium is identified by point M, or the level corresponding to the optimal capital constraint $\widehat{k}$ if the equilibrium is identified by point P. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and firms' decisions in each period are solely driven by the unconstrained maximisation of single-period profits. It is apparent that, in the present setting, firms always operate at full capacity. The possibility for firms to choose capacity strategically has been extensively debated in static settings, modelled either as one-shot games (see Levitan and Shubik, 1972; Kreps and Scheinkman, 1983; Davidson and Deneckere, 1986; Osborne and Pitchick, 1986), or as repeated games (see Brock and Scheinkman, 1985; Benoit and Krishna, 1987; Davidson and Deneckere, 1990). However, this literature envisages the possibility for firms to choose capacity in order to affect the equilibrium behaviour at the market stage, which in our model would correspond to the steady state in M. From the above discussion, we know that such a decision never arises in the differential game, where the endogenous boundary to capacity accumulation is given by the Ramsey rule, which, by definition, does not appear in the static games on capacity constraints.

When optimal output is q^*_{ss}, the steady state price is

$$p^*_{ss} = \left(\frac{A}{1+\alpha n}\right)^{\frac{1}{\alpha}} . \tag{36}$$

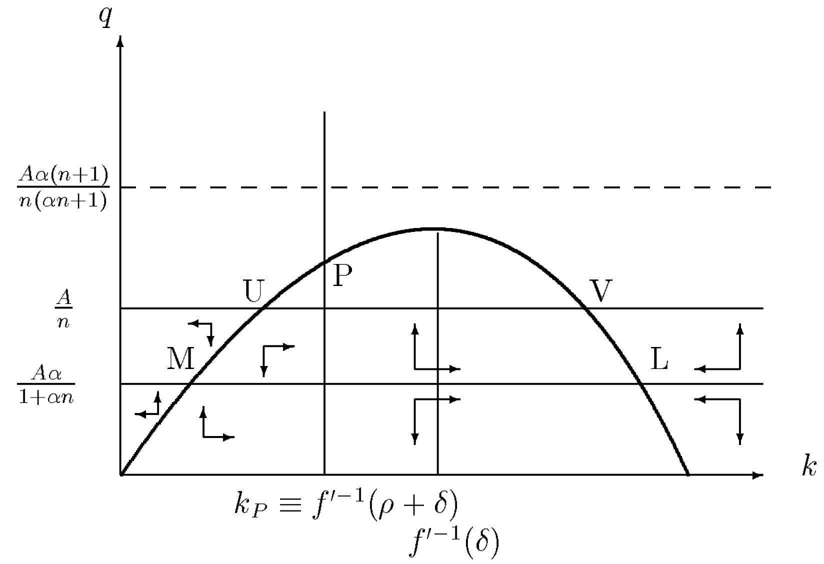
The per-firm instantaneous profits in steady state are

$$\pi^*_{ss} = \alpha \cdot \left(\frac{A}{1+\alpha n}\right)^{\frac{1+\alpha}{\alpha}} \tag{37}$$

while they are $\pi^*_{ss} = \widehat{k}\left(A - n\widehat{k}\right)^{\frac{1+\alpha}{\alpha}}$ if optimal output is $\widehat{k}$.

Now consider the case $\alpha > 1$. Here, there exists the additional horizontal arm given by $q = \dfrac{A}{n} > \dfrac{A\alpha}{1+\alpha n}$. The overall situation is depicted in figure 2.

Figure 2: Cournot competition, $\alpha > 1$ (and $\beta > 0$ always)



Points M, P and L are characterised as in the previous case (where $\alpha \in (0, 1)$). The features of points U and V can be quickly summarised as follows. From the direction of arrows in figure 2, it appears that point U is completely unstable, while point V is clearly inefficient, and can be disregarded. Notice finally that U portrays a situation where Cournot players would indeed behave as perfect competitors. This also appears in the first order condition of the static game (see, e.g., Lambertini, 1996, p. 331), where $q = A/n$ is a minimum. Steady state profits per-firm are as above.

It is worth stressing that the foregoing analysis encompasses the settings examined by Fershtman and Kamien (1984) and Cellini and Lambertini (1998). Fershtman and Kamien (1984) consider an homogenous-good duopoly where firm always sell a quantity equal to the installed capacity at time t. Therefore, in their paper the steady state at $q = \dfrac{A\alpha}{1 + \alpha n}$ does not appear. Cellini and Lambertini (1998) extend Fershtman and Kamien's model, to account for the equilibrium dictated by demand parameters. Although allowing for product differentiation, they confine to a linear demand setup.

Theorem 1 produces the following relevant corollaries:

Corollary 1 *In the steady state at M, the elasticity of demand w.r.t. price is constant and it is* $\left|\varepsilon^*_{Q_{ss},p_{ss}}\right| = \dfrac{1}{n}$ *for all* $\alpha > 0$.

Proof. To prove the above Corollary, plug the steady state output of the overall population of firms, $Q^*_{ss} = nq^*_{ss} = nA\alpha/(1+\alpha n)$ into (6) to obtain:

$$\left|\varepsilon^*_{Q_{ss}p_{ss}}\right| = \frac{1}{n} \, . \tag{38}$$

■

The explanation for this result is intuitive. In general, the pricing behaviour of a Cournot oligopoly is described by:

$$p(Q)\left(1 + \frac{s_i}{|\varepsilon|}\right) = c_i^{'}(q_i) \tag{39}$$

where $s_i \equiv q_i/Q$ and $c_i^{'}(q_i)$ define, respectively, the market share and marginal cost of firm i. (see Novshek, 1980). In our setting, $c_i^{'}(q_i)$ is constant, so that optimal per-firm output is chosen so as to determine the price as a constant markup over marginal cost.

Corollary 2 *As the number of firms tends to infinity, the steady state equilibrium at point M reproduces perfect competition, for all* $\alpha > 0$.

Proof. To prove the above statement, recall that $Q^*_{ss} = nq^*_{ss} = \dfrac{nA\alpha}{1+\alpha n}$, and check that

$$\lim_{n\to\infty} \frac{nA\alpha}{1+\alpha n} = A \, . \tag{40}$$

■

Corollary 3 *The steady state capital is everywhere non-decreasing in* α.

Proof. To prove the above result, it suffices to observe that, given the assumptions about technology, then in general $\partial q(t)/\partial k(t) > 0$ to the left of point P, and

$$\text{sign}\left\{\frac{\partial k^*(t)}{\partial\alpha}\right\} = \text{sign}\left\{\frac{\partial q^*(t)}{\partial\alpha}\right\}$$

in the same range, where clearly $\partial q^*(t)/\partial\alpha > 0$. This holds for all $k^*(t)$ as determined by point M. If M coincides with P, then the optimal capital endowment is given by the Ramsey rule. This argument implies the Corollary. ■

The above entails that the dynamic description of a Cournot oligopoly with non-linear demand allows us to endogenise and characterise, to some extent, production costs in the form of intertemporal accumulation of capital. This has to be contrasted with the static approach to the same market, where accounting for firms' size in the form of installed capital gives rise to corner solutions in output levels, since in the static model there is no endogenous optimization describing the rational choice of capacity.

4 The social optimum

The solution of the planner's problem can be quickly dealt with, as its analysis is largely analogous to but simpler than the oligopoly equilibrium. First of all, notice that, as operative unit production cost is constant (and nil), and technology is concave (or equivalently, there are decreasing returns to scale, since we have assumed $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f" \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$), the planner finds it optimal to decentralise production in n firms (or plants).

The social planner maximises social welfare, defined as the sum of producer and consumer surplus:

$$SW(t) = \Pi(t) + CS(t) \tag{41}$$

where

$$\Pi(t) = \sum_i \pi_i(t) \tag{42}$$

with $\pi_i(t) = q_i(t)\,(A - Q(t))^{\frac{1}{\alpha}}$, and

$$CS(t) = \int_0^{Q(t)} (A - s(t))^{\frac{1}{\alpha}}\, ds(t) = \frac{\alpha}{1+\alpha}\left[A^{\frac{\alpha+1}{\alpha}} - (A - Q(t))^{\frac{1+\alpha}{\alpha}}\right]. \tag{43}$$

Technicalities are largely analogous to the case of the Cournot oligopoly. Therefore, we confine to the characterisation of the planner's solution in terms of outputs and capital endowments.

The steady state solutions for the planner are $f'(k_i(t)) = \rho + \delta$ and $q_{ss}^{SP}(t) = A/n$, where the superscript SP stands for *social planning*. Notice that the demand-driven solution corresponds to an overall output $Q_{ss}^{SP}(t) = A$. The latter is the perfectly competitive output for the whole market, at which $p(t) = 0$. It is then trivial to prove that this also coincides with the steady state of the Bertrand market game.[3]

Notice that the above argument concerning capacity-constrained competition extends to this case as well. Under either Bertrand competition or social planning, firms operate at full capacity in steady state along both arms ($f'(k_i(t)) = \rho + \delta$ and $q_{ss}^{SP}(t) = A/n$). Under no circumstances they would find it rational to limit capacity in order to play a Cournot equilibrium *à la* Kreps and Scheinkman (1983).

In the steady state given by $nq_{ss}^{SP}(t) = A$, we have:

$$SW_{ss}^{SP} = CS_{ss}^{SP} = \frac{\alpha A^{\frac{1+\alpha}{\alpha}}}{1+\alpha} \tag{44}$$

while obviously $\Pi_{ss}^{SP} = 0$.

Finally, we can assess the welfare distortion associated to the steady state Cournot equilibrium where $Q_{ss}^{*} = nq_{ss}^{*}$, compared to the above social optimum. We obtain the following:

Proposition 1 *The welfare distortion due to Cournot competition is decreasing both in α and in n.*

Proof. Consider that the welfare distortion, i.e., $SW_{ss}^{SP} - SW_{ss}^{*}$ is proportional to the difference between the output level of the planner and the overall steady state production of the Cournot firms:

$$Q_{ss}^{SP} - Q_{ss}^{*} = \frac{A}{1+\alpha n}\ , \tag{45}$$

which is everywhere decreasing both in α and (obviously) in the number of firms operating in the Cournot setting, n. This implies the Proposition. ■

The fact that the welfare loss associated to oligopoly is decreasing in the number of Cournot agents is not surprising at all. The intuition behind

[3] This is due to the assumption of product homogeneity. With differentiated products (as in Cellini and Lambertini, 1998), equilibrium outputs under both social planning and Bertrand competition would depend upon α.

an analogous effect, associated with an increase in α, can be immediately interpreted as follows. Any increase in α entails that the area between the demand function and the axes (p and Q) becomes larger. The same holds for each individual output q_{ss}^{*}. Thus, increasing the size of the market translates into increasing the toughness of competition and welfare. Given the socially efficient output at $Q_{ss}^{SP}(t) = A$, the foregoing argument implies that the Cournot welfare loss must decrease as α becomes higher.

If the planner operates with a single firm, the picture modifies as follows. Expression (42) becomes:

$$\Pi(t) = Q(t)\left(A - Q(t)\right)^{\frac{1}{\alpha}} \tag{46}$$

The relevant Hamiltonian is then:

$$\mathcal{H}(t) = e^{-\rho t} \cdot \{\Pi(t) + CS(t) + \lambda(t)\left[f(k(t)) - Q(t) - \delta k(t)\right]\} \tag{47}$$

where again $\lambda(t) = \mu(t)e^{\rho t}$, and $\mu(t)$ is the co-state variable associated to $k(t)$. For the sake of brevity, we can confine attention to the following FOCs:

$$\frac{\partial \mathcal{H}(t)}{\partial Q(t)} = \frac{\left(A - Q(t)\right)^{\frac{1-\alpha}{\alpha}}\left[2A\alpha - Q(t)(1+2\alpha)\right] - \lambda(t)}{\alpha e^{\rho t}} = 0 \tag{48}$$

and

$$-\frac{\partial \mathcal{H}(t)}{\partial k(t)} = \frac{\partial \mu(t)}{\partial t} \Rightarrow \frac{\partial \lambda(t)}{\partial t} = \left[\rho + \delta - f'(k(t))\right]\lambda(t). \tag{49}$$

From (48) we easily obtain:

$$\text{sign}\left\{\frac{dQ(t)}{dt}\right\} = \text{sign}\left\{\alpha^2\left(A - Q(t)\right)^{\frac{2\alpha-1}{\alpha}} \cdot \lambda'(t)\right\} \tag{50}$$

from which we derive the steady state solutions for the planner, $f'(k(t)) = \rho+\delta$ and $Q_{ss}^{SP}(t) = A$. Notice that, given decreasing returns to scale, although the demand-driven solution remains the same as above, with $q_{ss}^{SP}(t) = A/n$, we would obviously observe $f'(k_i(t)) > f'(k(t))$ both in the demand-driven equilibrium and in the Ramsey equilibrium. As a result, we have:

Proposition 2 *The Cournot outcome coincides with social planning at the Ramsey equilibrium, if and only if the number of plants is the same in the two settings.*

5 Concluding remarks

We have taken a differential game approach in order to study how market demand - in particular, the curvature of the demand function - affects firms' behaviour concerning the accumulation of capacity over time, in a Cournot oligopoly.

The main results can be summarised as follows. First, there are configurations of parameters (i.e., sufficiently high discount and depreciation rates, and/or a sufficnetly concave demand function) where capacity in steady state is dictated by the pure capacity accumulation rule *à la* Ramsey. In such cases, the long-run equilibrium does not replicate the optimum of the static problem, the reason being that firms are very impatient or capital depreciates too fast.. As a consequence, the capacity required to suatin the "market optimum" which we are accustomed with from the static analysis is too expensive, in terms of the discounted value of the investement. The same argument applies as α becomes increasingly high, i.e., demand becomes increasingly concave.

On the contrary, in the parameter range where the steady state replicates the static solution, we obtain that the curvature of demand affects not only the output (as it happens in the static game), but also the optimal capacity. In particular, the link is positive: the higher is α, the larger is the accumulated capacity in the steady state equilibrium.

As to the social optimum, when the steady state occurs in correspondence of the Ramsey equilibrium, then it coincides with the Cournot equilibrium if the planner operates with the same n firms playing the noncooperative market game. Therefore, social welfare in steady state coincides in the two regimes. Otherwise, this coincidence disappears. Otherwise, when the steady state replicates the market equilibrium, oligopoly is inefficient and the deadweight loss due to Cournot behaviour decreases in α, the reason being that the Cournot market approaches perfect competition as α increases, while the socially efficient output is unaffected by the curvature of demand.

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