RISK MEASURE MODELLING

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ABSTRACT

As a phenomenon, risk represents a latent quantity of money or equivalent values needed as a guarantee. We would like to model in some essential way the approach to potential loss caused by various agents. If the interest focuses on security, it is necessary to determine a limit.

The aim of this paper is to refer to relevant literature and show how measure theory can be built as a mathematical discipline into economic theory providing thereby risk managers with a tool by means of which they will be able to link mathematical and economic thought.

Key words: measure, risk measures, moments, random variable, probability, approximation,variance, decision making, risk function

INTRODUCTION

From a statistical point of view, a decision-making problem can be considered as a game played by two players. One player is reality and the other is a statistician, whereby the reality condition denoted by q is unknown to a statistician.

Let us denote the set of all reality conditions (parameter space) by Ω.

A statistician (operator) takes an action (decision) a, if he/she finds out that the reality condition is q.

Let A be a set of all actions or decisions.

The result of observation is a random variable[1] X for which law of probability $f(x,\text{q})$ depends on an unknown parameter q.

If the random variable X takes the value x, the operator makes a decision $a = d(x)$, whereby $d(X)$ is the decision function.

[1] Sarapa, N., Teorija vjerojatnosti, Školska knjiga, Zagreb, 1988

The decision-making procedure itself consists of the following:

a) We define the set of all possible values θ might take in the problem under consideration.
b) We define the set of all possible actions or decisions which might be made.
c) We define the decision function $a = d(X_1, X_2, ..., X_n)$ of a random sample $\{X_1, X_2, ..., X_n\}$

In this game the operator (statistician) will have either profit or loss, depending on the decision that will depend on a and θ.

For the purpose of quantitative measuring let us introduce a loss function $L(a,\theta) = L(d(X_1, X_2, ..., X_n),\theta)$ as a numeric function which associates number $L(a,\theta) = L(d(x_1, x_2, ..., x_n),\theta)$ representing loss to every decision $a = d(x_1, x_2, ..., x_n)$ from A and every parameter value θ from Ω.

A correct decision is a decision for which loss is equal to zero.

Clearly, loss function $L(a,\theta)$ is a random variable for which the expected value $E(L(a,\theta))$ represents a risk obtained by a decision a when the reality condition is θ.

On the basis of the aforementioned, let us introduce a risk function $R(d,\theta)$ as the expected value of the loss function $R(d,\theta) = E\left[L(d(x_1, x_2, ..., x_n),\theta)\right]$.

Two cases might occur:

1) If X is a continuous random variable, then

$$R(d,\theta) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} L(d(x_1, x_2, ..., x_n),\theta) f(x_1,\theta)...f(x_n,\theta)dx_1...dx_n$$

2) If X is a discrete random variable, then

$$R(d,\theta) = \sum_{x_1}\sum_{x_2}...\sum_{x_n} L(d(x_1, x_2, ..., x_n),\theta) f(x_1,\theta)...f(x_n,\theta).$$

Furthermore, two decision functions d_1 and d_2 may be compared by means of corresponding risk functions $R(d_1,\theta)$ and $R(d_2,\theta)$. The decision function with the risk function taking a less value is a better one.

In order to be able to select a decision function, it is natural to use maximum values of the risk function.

BAYESIAN APPROACH

Bayesian approach is simple and therefore interesting from the point of view of exploitability in the decision-making theory.

The reality condition is considered to be a random variable θ with the law of probability $\pi(x,\theta)$ which is a result of operator's (statistician's) personal conviction referring to the condition of reality.

Since θ is a random variable, risk function $R(d,\theta)$ will also be a random variable, whereby the associated law of probability reads $f(x,\theta)=\pi(\theta)f(x|\theta)$.

In this case the expected risk function value with respect to the law of probability of the a priori parameter θ defines a new function (Bayes risk)

$r(d,\pi)=E\left[R(d,\theta)\right]$.

According to (1) and (2) we have

I) $r(d,\pi)=\int\limits_{-\infty}^{\infty}R(d,\theta)\pi(\theta)d(\theta)$ and

II) $r(d,\pi)=\sum\limits_{i=1}^{n}R(d,\theta_i)\pi(\theta_i)$, respectively, depending on the random variable being continuous or discrete.

In this paper we will implement this approach into risk theory with special attention being paid to risk measure, by which security strategies will be optimised.

Naturally, security function depends on many variables and their bounds and the problem of risk is located on the finite time interval $[0,t]$.

An interesting definition of risk was given in 1989 by Castagnoli, not assuming market integrity, but defining risk as a future unacceptable value in the interval $[0,t]$.

Let us assume that D is a set of acceptable situations, X a random variable of the observed situations, and i the feedback instrument.

Risk will be measured as minimum additional capital C, which should be invested into the project in order to have the value of the new situation C and +X acceptable.

Risk measure can be taken as mapping

$m:D\rightarrow R,\quad m(X)=\inf\{C:Ci+X\in D\}$, whereby we accept all laws from the mathematical measure theory2, such as nonnegativity, subadditivity, translation, etc.

Since we deal with approximation, variance can be used, a mathematical notion

[2] P.R. Halmos, Measure Theory, Van Nostrand, Princeton, New Jersey, 1963

frequently used in statistics:

$m(X) = -E(X) + k\text{s}_p(X)$ or $m(X) = -E(X) + \text{s}_p((E(X) - X))$, where E is mathematical expectancy.

Clearly, measures defined in this way are not subadditive3.

Probability distribution cannot be found easily, so that we encounter a definition problem by means of family p of possible situations. Let us take risk measure as expected loss from the most unfavourable situation

$m(X) = \sup\left\{E_p\left(\frac{-X}{i}\right) : P \in \text{p}\right\}$, least upper bound4.

Let us mention some examples of measures of the previously mentioned type:

- average surplus function (P. Embrechts-C. Kluppenberg-T. Mikosch, 1997)
  $V(f) = E(|X - f| : X > f)$
- retarded measure (Wirch-Hardy, 1999), which introduces a concave function5 $d : [0,1] \to [0,1]$ with properties $d(0) = 0,\ d(1) = 1$

$$E_D[X] = \int_0^\infty d\left[P(X > x)\right]dx$$

A general risk theory is studied well in (C. Fishburn, 1977) using much of the stochastic theory and utility theory.

With respect to the situation f risk is defined as the measure

$m(F) = \int_{-\infty}^{f} \text{j}\ (f - x)dF(x)$, where $\text{j}\ (y) \geq 0,\ y \geq 0,\ \text{j}\ (0) = 0$, the domain is bounded with f, and F is a distribution function6.

Average risk usefulness model exists if and only if there exists a real function H such that distribution F is more favourable than G, i.e. if and only if

$$H(E(F), m(F)) > H(E(G), m(G)).$$

[3] H.L. Royden, Real Analysis, Macmillan, New York, 1968
[4] See Appendix A and Appendix B.
[5] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, University Press, Warszaw, 1985
[6] N. Sarapa, Teorija vjerojatnosti, Školska knjiga, Zagreb, 1988

Dynamic approach to risk measures

We have seen how risk is measured in one period. However, very often insurance contracts, investments, etc. require a longer period of time, so that time must be clearly taken into account when defining risk and modelling risk measure.

Let $[0,t]$ be a finite time interval and C capital with due time t.

In case of a complete market, whereby there is no arbitration, and there is an interested party who cannot invest the whole amount immediately, i.e. in time t=0, so that

$C(0):=E\left[\frac{C}{S_0(t)}\right]$, which guarantees certain protection.

$S_0(T)$ is the price of a non-risky investment at the market, E expectancy of the bounded risk, and C is a risk that has to be measured.

A very interesting proposal was given in 1999 by J. Cvitanić and I. Karatzes.

If the model depends on probability distribution, strategies g should be selected aiming at the decrease of expected probabilities of net losses;

$m_C(X)=\inf_{g(\cdot)\in G(x)} E\left[\frac{C-X^{x,g}(t)}{S_0(t)}\right]$, whereby x is the initial

capital, and $G(x)$ a set of acceptable situations (strategies).

This measure is agreeing provided that investment strategy X and liability C are proportional.

This stochastic problem is solved under certain circumstances.

Uncertainty is determined through family $(P_a)_{a\in A}$ strategies.

For the purpose of controlling risk, we determine the interval of possible measures with bounds:

$m\ (X)=\sup_{a\in A}\inf_{g(\cdot)\in G(x)} E\left[\left(\frac{C-X^{x,g}(t)}{S_0(t)}\right)\right]$, whthe most unfavourable strategy (lower

bound) and $M\ (X)=\inf_{g(\cdot)\in G(x)}\sup_{a\in A} E\left[\left(\frac{C-X^{x,g}(t)}{S_0(t)}\right)\right]$, upper bound of the worst strategy.

Being familiar with laws of mathematical analysis enables us to divide interval $[0,t]$ into subintervals $[n,n+1]$, where n comes from the set of natural numbers and take into account that $\vee X_{n+1}$ is a change of portfolio values in the interval $[n,n+1]$.

Thus, time interval $[0,t]$ does not make a unit any more, but it is divided into subintervals, so that every subinterval is considered separately as a partition Π

$$m_{\Pi_n}\left(\vee X_{n+1} \middle| F_n\right) = \sup\left\{E_p\left[-\frac{\vee X_{n+1}}{i_n}\middle|\vee x_n\right] : P \in \Pi\right\},$$

whereby i_n is a feedback instrument.

If $m_{\Pi_n} > 0$, $\max\{m_\Pi - C_n, 0\}$ ensures alleviation of losses to the investor.

If $m_{\Pi_n} < 0$, $\min\{-m_{\Pi_n}, C_n\}$ gives a possibility of withdrawal from the account, putting up with the expected loss.

It can be easily seen that by means of real series (a_n, b_n) the risk measure interval is given,

$$a_n = \inf_{F^k \in \Pi^k}\left\{E_{F^k}\left[-\frac{\vee X_{n+1}}{i_n}\middle|\vee x_n\right] : k \in K\right\}$$

$$b_n = \sup_{F^k \in \Pi^k}\left\{E_{F^k}\left[-\frac{\vee X_{n+1}}{i_n}\middle|\vee x_n\right] : k \in K\right\}$$

$$\left[\min\{a_n - C_n, 0\}, \max\{b_n - C_n, 0\}\right]$$

for which there holds everything given in the following appendices.

Appendix A

Let $(a_n)_{n\in N}$ be a real series and S a set of all its accumulation points (S may also be an empty set). Element L of the set R_∞ is called the upper limit (limes superior, upper accumulation point) of the series $(a_n)_{n\in N}$, and it is determined in the following way

$$L = \begin{cases} +\infty, \text{ if the series } (a_n)_{n\in N} \text{ is unbounded above;} \\ -\infty, \text{ if the series } (a_n)_{n\in N} \text{ is bounded above and the set S is empty (i.e. if } \lim\limits_{n\to+\infty} a_n = -\infty\text{);} \\ \sup S, \text{ if the series } (a_n)_{n\in N} \text{ is bounded above and the set S is not empty.} \end{cases}$$

The lower limit (limes inferior, lower accumulation point) of the series $(a_n)_{n\in N}$ is defined analogously.

The upper limit of the series $(a_n)_{n\in N}$ is denoted by $\limsup_{n\to+\infty} a_n$, and the lower limit of the same series by $\liminf_{n\to+\infty} a_n$.

The upper and the lower limit are unambiguously determined (as elements of the set R_∞) for every real series and each of them is a series accumulation point in case the series is finite.

Hence, if $\limsup_{n\to+\infty} a_n \in R$, we have $\limsup_{n\to+\infty} a_n = \max S$, and if $\liminf_{n\to+\infty} a_n \in R$, then $\liminf_{n\to+\infty} a_n = \min S$.

In case a real series $(a_n)_{n\in N}$ is unbounded above i.e. below, the symbol $+\infty$, i.e. $-\infty$ is often referred to as its accumulation point. If this is taken into consideration, then the sequence of the set R naturally expanded to the set R_∞, the upper and the lower limit can be defined with no bounds in the following way:

$$\limsup_{n\to+\infty} a_n \overset{def}{=} \sup S \ , \qquad \liminf_{n\to+\infty} a_n \overset{def}{=} \inf S \ ,$$

but also in this way:

$$\limsup_{n\to+\infty} a_n \overset{def}{=} \max S \ , \qquad \liminf_{n\to+\infty} a_n \overset{def}{=} \min S \ .$$

Appendix B

1. Let a_n and b_n ($n \in N$) be real series. If for n great enough $a_n \le b_n$, then

$$\liminf_{n\to+\infty} a_n \le \liminf_{n\to+\infty} b_n \quad \text{and} \quad \limsup_{n\to+\infty} a_n \le \limsup_{n\to+\infty} b_n$$

2. Let $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ be two real series. Excluding the cases with meaningless notions, the following inequalities hold:

$$\liminf_{n\to+\infty} a_n + \limsup_{n\to+\infty} b_n \le \limsup_{n\to+\infty}(a_n + b_n) \le \limsup_{n\to+\infty} a_n + \limsup_{n\to+\infty} b_n,$$
$$\liminf_{n\to+\infty} a_n + \liminf_{n\to+\infty} b_n \le \liminf_{n\to+\infty}(a_n + b_n) \le \liminf_{n\to+\infty} a_n + \limsup_{n\to+\infty} b_n.$$

3. If $a_n \geq 0$, $b_n \geq 0$ for n great enough, then, with the same restriction as before, the following inequalities hold:

$$\liminf_{n\to+\infty} a_n \cdot \limsup_{n\to+\infty} b_n \leq \limsup_{n\to+\infty}(a_n \cdot b_n) \leq \limsup_{n\to+\infty} a_n \cdot \limsup_{n\to+\infty} b_n,$$
$$\liminf_{n\to+\infty} a_n \cdot \liminf_{n\to+\infty} b_n \leq \liminf_{n\to+\infty}(a_n \cdot b_n) \leq \liminf_{n\to+\infty} a_n \cdot \limsup_{n\to+\infty} b_n.$$

4. For every real series $(a_n)_{n\in N}$

$$\liminf_{n\to+\infty}(-a_n) = -\limsup_{n\to+\infty} a_n, \quad \limsup_{n\to+\infty}(-a_n) = -\liminf_{n\to+\infty} a_n,$$

If $a_n > 0 \ (n = 1, 2, ...)$, then

$$\liminf_{n\to+\infty} \frac{1}{a_n} = \frac{1}{\limsup_{n\to+\infty} a_n}, \quad \limsup_{n\to+\infty} \frac{1}{a_n} = \frac{1}{\liminf_{n\to+\infty} a_n},$$

where, in addition to the previously adopted convention $\frac{1}{+\infty} = 0$, it is taken that $\frac{1}{0} = +\infty$.

5. Let $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ be two real series. If

$b_n \to +\infty \ (n \to +\infty)$ and $b_n \uparrow$ for n great enough, then

$$\liminf_{n\to+\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \leq \liminf_{n\to+\infty} \frac{a_n}{b_n} \leq \limsup_{n\to+\infty} \frac{a_n}{b_n} \leq \limsup_{n\to+\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

INSTEAD OF A CONCLUSION

This paper partially gives a mathematical set of instruments to managers who can link a mathematical to an economic thought.
Risk measure assessment (interval) is given explicitly.

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