Bargaining with Uncertain Value Distributions*

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Abstract

This paper studies a bargaining model in which the seller is uncertain about which distribution the buyer's values are drawn from. The distribution of the buyer's values is fixed across periods, while the buyer's values are drawn independently from the distribution each period. In the classical model of repeated bargaining where the buyer's value is drawn from a commonly known distribution and fixed across periods, the high-value buyer has a strong incentive to conceal his value, and the seller loses most of her bargaining power. An important question is whether adding a layer of uncertainty makes the high-value buyer more willing to accept high-price offers and improves the seller's revenue. We find this to be the case as long as the seller's ex ante beliefs are sufficiently optimistic.

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1 Introduction

One important question in the literature of repeated bargaining is how players' private information is revealed over time, and related to that, how economic surplus is distributed between the bargaining parties. In order to examine these questions, we focus on a framework commonly used in the previous literature: a buyer (denoted as he) and a seller (denoted as she) bargain over multiple periods, with the buyer having private information; in each period, the seller proposes a take-it-or-leave-it offer and the buyer decides whether to accept or reject the offer. In the classical models a common assumption is that the buyer's private information is his value, which is fixed across periods, and the distribution of the buyer's value is common knowledge. Under this assumption, the seller has a large disadvantage and loses most of her monopoly power. This stark result is established in both cases when the seller sells or rents a durable good.

The literature on the Coase conjecture finds that if the durable-good monopolist sells over time and can quickly lower prices, the seller can hardly achieve profits greater than the lowest buyer valuation and the buyer obtains the entire surplus from trade in excess of his lowest valuation (Coase 1972, Fudenberg et al. 1985).[1] When the monopolist bargains over renting the durable good to a buyer with private value, Hart and Tirole (1988) show that the seller always offers a low price until the end of the game if the horizon is long enough.[2] Intuitively, when the time horizon is long, the high-value buyer type will not accept any price rejected by the low-value buyer type, in order to avoid being charged with a high price in all later periods. So the seller is not able to price discriminate and she charges a low price to both low-value and high-value types, until close to the end of the horizon. Therefore, if the durable-good monopolist rents the durable good, the seller is again caught in an unfavorable position. Notice that in Hart and Tirole the buyer is *non-anonymous*,[3] that is, the seller is able to identify the buyer's previous action, and consequently the buyer is concerned with how his action today will affect his payoff tomorrow. When the buyer's value is fixed, revealing the current value means revealing all the private information on future values, so the high-value type has a strong incentive to hide his value.

In this paper, we examine a two-period rental model (equivalent to the case that a seller repeatedly charges to sell a perishable good or provide service to a buyer) where the buyer has private information not only about his valuation when each period comes about but also about the distribution from which his values are drawn. Our model is different from Hart and Tirole's rental model in two ways. First, we introduce an additional layer of uncertainty on the buyer's value

[1] This result holds under the assumption that the seller's marginal cost is lower than the buyer's lowest value, which is called the "gap" case in the literature. Failures of the Coase conjecture are found when the lowest buyer valuation does not exceed the seller's cost, which is referred as the "no-gap" case in the literature (Gul et al. 1986, Ausubel and Deneckere 1989).

[2] Hart and Tirole (1988) examine the sale model and the rental model in three cases: (1) where the parties can commit themselves to a contract once and for all; (2) where the parties can only write short-term contracts which rule within a period, but cannot commit themselves between periods; (3) where parties can write a long-term contract which rules across periods, but cannot commit themselves not to renegotiate this contract by mutual agreement. The rental model without commitment as in this paper is part of the analysis in Hart and Tirole.

[3] Bulow (1982) argues that the durable-good monopolist may be better off when renting the durable good rather than selling it, if the buyer is anonymous.

distribution. The distribution may be either good or bad. Both distributions can draw high value or low value, with the good distribution generating a high value with a higher probability. The buyer privately observes the distribution at the beginning of the game. But the seller only knows the ex ante probability of the two distributions. Second, the buyer's value is drawn from one of the two distributions independently across periods at the beginning of each period. Since the seller does not know which distribution the buyer's values are drawn from, the buyer's value is correlated across time periods from the seller's perspective.

The purpose of the paper is to ask whether the seller can improve her standing by introducing this second layer of uncertainty about the distribution of the buyer's values. On one hand, our model maintains the buyer's strategic considerations across periods, which makes the problem still interesting and close to many real life examples where the bargaining parties are involved in a long-term relationship. On the other hand, we are able to examine whether allowing the buyer's value to be redrawn provides a leeway to solve the problem of the durable-good monopolist, without assuming the buyer is anonymous.

The assumption that the buyer's value distribution is uncertain can be illustrated in the following example. Imagine that a construction company rents big machines from a monopolist every time when a new project begins. The value of using the machines depends on the quality of the project, which depends on both the construction company's technology and some random effects. The construction company may have a superior technology or an inferior technology, and the superior technology may generate a project of high quality with a higher probability. Both the technology and the quality of the project are the construction company's private information.

The main result we find is that the seller is indeed better off when she has sufficiently optimistic ex ante beliefs about the favorable distribution, compared to a two-period version of Hart and Tirole' (1988) model with the same ex ante probability of a high-value buyer type. The unique equilibrium outcome is for the seller to offer a high price and for the buyer type with a high value to accept the offer in each period. When the seller has a moderate ex ante belief, the buyer does not always truthfully reveal his value, mixed strategy is involved and there exist multiple equilibria. The seller's revenue, however, can still be higher than that in Hart and Tirole (1988). Sufficient conditions for the seller to be better off are provided.

Two other papers also examine a rental model in which a non-anonymous buyer's value randomly changes over time.[4] Kennan (2001) analyzes infinitely repeated contract negotiations where the buyer has persistent (but not permanent) private information. The buyer's value is assumed to change according to a two-state Markov chain. Kennan (2001) focuses on the cyclic screening equilibria in which several pooling offers in sequence make the seller more and more optimistic and the seller makes an aggressive screening offer eventually.

[4]Several other papers also allow the buyer's valuations to vary over time. Sobel (1991) shows that when there is a constant flow of new buyers, a Folk theorem holds in a sale model. Blume (1990) examines a sale model where the low buyer type's value varies over time and the high buyer type's value stays fixed and demonstrates that both uniqueness and Coase conjecture may fail to hold. Blume (1998) and Battaglini (2005) study long-term contracting. Biehl (2001) analyzes a durable-goods model with anonymous buyers. Lemke (2004) presents a dynamic bargaining model in which actions in the last period affect the buyer's expected future value.

Loginova and Taylor (2008) investigate a two-period model where the monopolist employs price experimentations to learn the permanent demand parameter of the buyer. Although we have benefited a lot from reading their paper, the two papers were developed independently and differ in several aspects. First, Loginova and Taylor (2008) assume that the value distribution is represented by λ, which is a continuous random variable distributed on $[0, 1]$. In this paper, we assume that the value distribution may either be favorable or unfavorable. Second, our major concern is whether introducing the additional layer of uncertainty on the buyer's value distribution can improve the seller's revenue, and if so, under what condition. We keep our model simple so that we can completely characterize the equilibria and compare the seller's revenue with that in Hart and Tirole (1988), in which the buyer's value distribution is common knowledge. Finally, Loginova and Taylor (2008) assume that there is no discounting, which makes some results different from ours. For instance, they find that when all the low-value types accept an offer less than the low value, the seller never offers a first-period price that yields her valuable information about the buyer's permanent demand parament λ. In our model, the seller offers a price that yields valuable information if the discount rate is low enough or the seller's prior is high enough.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 presents some intermediate results. Section 4 presents the set of equilibria. Section 5 compares the seller's revenue in this model with that in Hart and Tirole (1988). Section 6 concludes. All proofs are in Appendix A. Appendix B provides discussion on equilibrium concept.

2 The Model

One buyer and one seller bargain over renting a durable good in two periods $t = 1, 2$. The seller's cost is assumed to be 0. The buyer has a positive value, v_t, in consuming the good in each period t. The buyer's value v_t is drawn from one of two distributions in each period: the bad distribution B or the good distribution G. For a given distribution d, v_t equals h with probability q^d and equals l with probability $1 - q^d$. Assume that $0 < q^B < q^G < 1$ and $0 < l < h$, i.e., the good distribution G has a higher probability of generating a high value h. The buyer knows which one of the two distributions his values are actually drawn from as well as his current and former values. However, the seller only knows that the buyer's value is drawn each period from one of these two distributions. The ex ante probability is α for the G distribution and $1 - \alpha$ for the B distribution.

At the beginning of the game, the buyer privately observes the realization of distribution d, which will be fixed throughout the game. At the beginning of each period t, the buyer's valuation v_t is drawn from the realized distribution independently across time periods. After the buyer privately observes v_t, the seller offers a price $p_t \in \mathbb{R}$, and then the buyer chooses an action $a_t \in \{0, 1\}$, where $a_t = 1$ means acceptance and $a_t = 0$ means rejection.

Both the seller and the buyer are assumed to be risk-neutral. If the buyer accepts the seller's offer in period t, the buyer's payoff is $v_t - p_t$ and the seller's payoff is p_t in period t. They both gain nothing in period t if p_t is rejected. The two players share a common discount factor δ, and both of them maximize the discounted present value of expected payoffs.

Let $\theta_1 = (d, v_1)$ denote as the buyer's type in period 1 and $\theta_2 = (d, v_1, v_2)$ as the buyer's type in period 2. Since we will focus on the buyer's first period behavior later, it is helpful to notice that there are four buyer types in period 1: (G, l), (B, l), (G, h), and (B, h). Denote h_t^s as the history observed by the seller before she announces p_t and h_t^b as the history observed by the buyer before he chooses a_t. Specifically, $h_1^s = \varnothing$, $h_1^b = (\theta_1, p_1)$, $h_2^s = (p_1, a_1)$ and $h_2^b = (\theta_2, p_1, a_1, p_2)$. A behavioral strategy for the seller, σ^s, assigns probability (or density) $\sigma^s(p_t \mid h_t^s)$ to p_t given any history h_t^s for $t = 1, 2$. A behavioral strategy for the buyer, σ^b, assigns probability $\sigma^b(a_t \mid h_t^b)$ to a_t given any history h_t^b for $t = 1, 2$. For convenience, let $\sigma^b(h_t^b) \equiv \sigma^b(a_t = 1 \mid h_t^b)$ denote the probability that the buyer accepts p_t given history h_t^b, since the buyer can only choose to accept or reject an offer. Finally, let $\gamma(h_t^s)$ denote the probability that the seller's belief assigns to the G distribution at the beginning of period t given history h_t^s. Notice that $\gamma(p_1, 0)$ and $\gamma(p_1, 1)$ denote the seller's belief of $d = G$ given that p_1 is rejected and accepted respectively.

The equilibrium concept used is strong Perfect Bayesian equilibrium.[5] Bayes' rule is used to update the seller's belief conditional on reaching any price p_1, even if p_1 is off the equilibrium path. We also employ a refinement which is a variant of criterion D_1 in the signalling game (Cho and Kreps 1987, Banks and Sobel 1987). In Appendix B, we formally define criterion D_1.

3 Preliminary Results

3.1 The Second-Period Equilibrium Strategies

We start the analysis from the second (last) period. Similar as in the previous literature, the equilibrium strategies in the last period are simple: the buyer accepts p_2 if and only if p_2 does not exceed v_2; the seller either offers $p_2 = l$ or $p_2 = h$, depending on whether her belief of $v_2 = h$ is less or greater than the cutoff belief l/h. Given the seller's belief of $d = G$ is γ, her belief of $v_2 = h$ is $q^G\gamma + q^B(1-\gamma)$, so she offers $p_2 = l$ ($p_2 = h$) if her belief of $d = G$ is less (greater) than γ^*, where γ^* satisfies the equation $q^G\gamma^* + q^B(1-\gamma^*) = l/h$. Lemma 1 formally states the discussion above, in which $x(h_2^s)$ denotes the probability that the seller offers $p_2 = l$ following history h_2^s. In order to make the problem interesting, we assume $q^B < l/h < q^G$ throughout the paper.[6]

Lemma 1 *In any PBE, the buyer's strategy in the second period is*

$$\sigma^b(h_2^b) = \begin{cases} 1, & \text{if } p_2 \le v_2; \\ 0, & \text{if } p_2 > v_2, \end{cases}$$

and the seller's strategy in the second period is

$$x(h_2^s) = \begin{cases} 1, & \text{if } \gamma(h_2^s) < \gamma^*; \\ 0, & \text{if } \gamma(h_2^s) > \gamma^*; \\ \in [0,1], & \text{if } \gamma(h_2^s) = \gamma^*, \end{cases}$$

[5] For the consideration of efficiency, we require the buyer's strategy be left continuous at the cutoff prices where the buyer is indifferent between two actions, that is, the behavioral strategy following the cutoff price p_1 is the same as the behavioral strategy following $p_1 - \epsilon$.

[6] The seller always offers $p_2 = h$ if q^B is greater than l/h, and always offers $p_2 = l$ if q^G is smaller than l/h, regardless of her belief of the buyer's value distribution.

where $\gamma^* = (l/h - q^B)/(q^G - q^B)$.

3.2 Cutoff Values

Recall that there are four buyer types in the first period: (G, l), (G, h), (B, l), and (B, h). Next we define the cutoff value for each buyer type, at which the buyer type is indifferent between acceptance and rejection. Remember that q^d is the probability for the buyer type to draw an h value, $x(p_1, 1)$ and $x(p_1, 0)$ is the probability for the seller to offer $p_2 = l$ after acceptance and rejection of p_1 respectively. So the buyer's expected payoff from accepting p_1 is $v_1 - p_1 + \delta q^d x(p_1, 1)(h - l)$, and the buyer's expected payoff from rejecting p_1 is $\delta q^d x(p_1, 0)(h - l)$. By comparing these two payoffs, the buyer type (d, v_1) accepts p_1 with probability one if p_1 is smaller than $v_1 + \delta q^d [x(p_1, 1) - x(p_1, 0)](h - l)$ and rejects p_1 with probability one if it is greater than $v_1 + \delta q^d [x(p_1, 1) - x(p_1, 0)](h - l)$. Define $C(d, v_1) \equiv v_1 + \delta q^d [x(p_1, 1) - x(p_1, 0)](h - l)$ as the *Cutoff Value* for buyer type $\theta_1 = (d, v_1)$ given $x(p_1, 0)$ and $x(p_1, 1)$.

Lemma 2 *In any PBE, the probability for buyer type $\theta_1 = (d, v_1)$ to accept p_1 is*

$$\sigma^b(\theta_1, p_1) = \begin{cases} 1, & \text{if } p_1 < C(d, v_1); \\ 0, & \text{if } p_1 > C(d, v_1); \\ \in [0, 1], & \text{if } p_1 = C(d, v_1). \end{cases}$$

By definition the buyer's cutoff value depends on his type (d, v_1) as well as the seller's second-period strategy. Figure 1 below describes the order of all buyer types' cutoff values regarding different strategies the seller may use in the second period. We see that the buyer types with $v_1 = l$ always have a smaller cutoff value than types with $v_1 = h$, regardless of the value distribution d and the seller's strategy in the second period. However, the order of the cutoff values of two buyer types who have the same v_1 but draw from different distributions, for instance (B, l) and (G, l), depends on the seller's strategy in the second period. If the seller offers $p_2 = l$ with a higher probability when p_1 is accepted (i.e., $x(p_1, 0) < x(p_1, 1)$), type (G, l) has a larger cutoff value than type (B, l). On the contrary, if the seller offers $p_2 = l$ with a larger probability when p_1 is rejected than accepted (i.e., $x(p_1, 0) > x(p_1, 1)$), type (G, l) has a smaller cutoff value than type (B, l). The basic intuition is that, type (G, l) has a larger probability of generating an h value in the second period and a larger expected payoff if $p_2 = l$ than type (B, l), so he is more willing to take the action that will induce the seller to offer $p_2 = l$.

3.3 Screening

In this subsection we discuss some preliminary observations on what kind of screening of buyer types is possible/impossible in equilibrium. The first observation (Lemma 3) is that the seller can never offer a first-period price which separates one buyer type, say (G, h), from the other three types, say (G, l), (B, h) and (B, l) in equilibrium. Therefore, in this two-period model, the seller can never learn the buyer's distribution for sure. It is possible, however, for the seller to offer a p_1

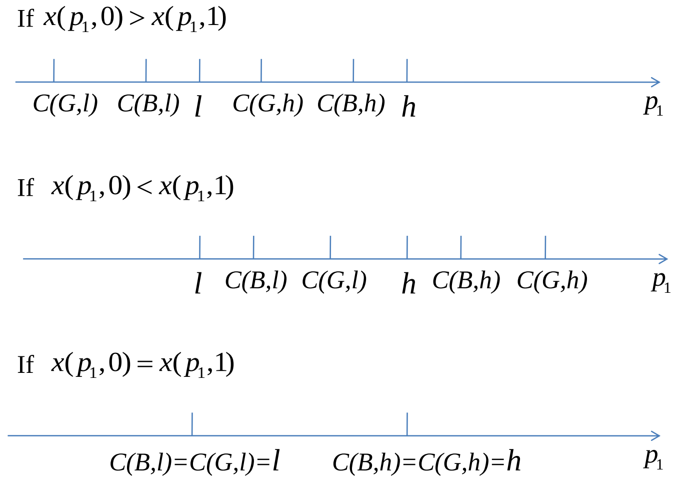

Figure 1: Order of Cutoffs

which is accepted by h-value types and rejected by l-value types. Observation 2 (Lemma 4) shows that the seller is more optimistic when p_1 is accepted than rejected. Furthermore, observation 3 (Lemma 5) says that the difference in the seller's posterior belief affects the seller's second-period strategy only when she has a moderate prior belief. In that case, the seller offers different p_2 conditional on whether p_1 is accepted or rejected. When the seller has an extreme prior belief, her second-period strategy is not influenced by the first-period outcome.

Lemma 3 *No first-period offer can screen one buyer type from the other three types in equilibrium.*

From Figure 1 it is obvious that no p_1 in equilibrium can screen one buyer type, for instance (B, h), from the other three types (G, l), (G, h) and (B, l). To see that, suppose the seller's strategy is to offer $p_2 = l$ if p_1 is rejected and $p_2 = h$ if p_1 is accepted, i.e., $x(p_1, 0) = 1 > x(p_1, 1) = 0$. Given the seller's strategy, only p_1 between the cutoff value of type (G, h) and (B, h) may separate type (B, h) from the other three types, as shown in the first case of Figure 1. This, however, cannot happen in equilibrium, since only type (B, h) will accept such p_1, and then the seller should offer $p_2 = l$ after acceptance of p_1, which is contradictory to the proposed strategy. Using the same reasoning, we can easily show that the seller cannot screen any single buyer type out.

The key point here is that the buyer types from the G distribution have a larger incentive to take the action that induces the seller to offer a low price in the second period, the seller however becomes extremely optimistic if that action is taken by a single type drawing from the G distribution. Thus it reaches a contradiction and cannot happen in equilibrium. The same intuition holds for the case that the seller cannot separate a single type drawing for the B distribution.

Importantly, Lemma 3 implies that the seller is not able to learn the buyer's distribution for sure. Then the question is whether the seller is able to learn at all about the buyer's type. It turns out that partial screening may still happen in equilibrium. We find that when that happens, the seller is always more optimistic when p_1 is accepted than rejected. Consider the case that p_1

separates the l-value buyer types, i.e. type (G, l) and (B, l), from the h-value types, i.e. type (G, h) and (B, h). Since the cutoff values of l-value types are always smaller than those of h-value types, a price can screen the l-value types from the h-value types only if the l-value types reject the offer and the h-value types accept the offer. Given that the G distribution has a higher probability of generating an h value, the seller must be more optimistic after acceptance than rejection of p_1. This result extends even when some buyer type employs a mixed strategy between accepting and rejecting p_1. Consequently, shown by the next lemma, if p_1 is both accepted and rejected by some buyer types, the probability for the seller to offer $p_2 = l$ after rejection of p_1 must be at least as high as after acceptance of p_1.

Lemma 4 *Let $\Psi(p_1, a_1)$ denote the probability that action a_1 is taken in the continuation game following p_1. If $\Psi(p_1, 1) \in (0, 1)$ for a given p_1 in a PBE, then $x(p_1, 0) \geq x(p_1, 1)$.*

We show above that the seller cannot screen one single buyer type from the other types, so it is impossible for her to learn the buyer's distribution for sure. However, learning still occurs when p_1 separates the l-value types from the h-value types. Compared to the prior belief, the seller becomes more optimistic when p_1 is accepted and more pessimistic when p_1 is rejected. Different from Hart and Tirole, in which the buyer's value is the private information, in this model the seller's belief about the distribution from which the buyer's values are drawn changes gradually even if she learns the buyer's first-period value. Therefore, the seller's posterior beliefs conditional on acceptance and rejection of p_1 may not be different enough for her to offer a different p_2. Intuitively, when the seller's prior belief is very extreme, her posterior beliefs after acceptance and rejection of p_1 may still be both beyond or below the cutoff belief γ^*, and the seller offers the same p_2 no matter whether p_1 is accepted or rejected. On contrast, when her prior belief is in a more intermediate range, her posterior belief may be different enough for her to offer a different p_2. The following analysis addresses this formally.

Define functions

$$\overline{\gamma}(\alpha) \equiv \frac{\alpha q^G}{\alpha q^G + (1-\alpha) q^B},$$

and

$$\underline{\gamma}(\alpha) \equiv \frac{\alpha(1-q^G)}{\alpha(1-q^G) + (1-\alpha)(1-q^B)}.$$

Based on Bayes' rule, $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ are the seller's posterior beliefs of the G distribution conditional on $v_1 = h$ and $v_1 = l$ respectively. Define $\widetilde{\alpha} \equiv \overline{\gamma}^{-1}(\gamma^*)$ and $\widehat{\alpha} \equiv \underline{\gamma}^{-1}(\gamma^*)$.[7]

Figure 2 plots $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ as functions of the seller's ex ante belief α, choosing $q^B = 0.4$, $q^G = 0.8$, and $l/h = 0.6$. The curve $\underline{\gamma}(\alpha)$ is below the 45^0 line since the seller becomes more pessimistic when conditional on $v_1 = l$. On the contrary, the curve $\overline{\gamma}(\alpha)$ is above the 45^0 line since the seller becomes more optimistic conditional on $v_1 = h$. Furthermore, when the seller has an *extreme* ex ante belief (α smaller than $\widetilde{\alpha}$ or greater than $\widehat{\alpha}$), her posterior beliefs conditional on

[7]Both $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ are continuous and increasing in α; $\underline{\gamma}(\alpha) < \alpha < \overline{\gamma}(\alpha)$ for $\alpha \in (0, 1)$; $\underline{\gamma}(\alpha) = \overline{\gamma}(\alpha) = \alpha$ for $\alpha \in \{0, 1\}$. $\widetilde{\alpha}$ and $\widehat{\alpha}$ are well-defined and $\widetilde{\alpha} < \gamma^* < \widehat{\alpha}$.

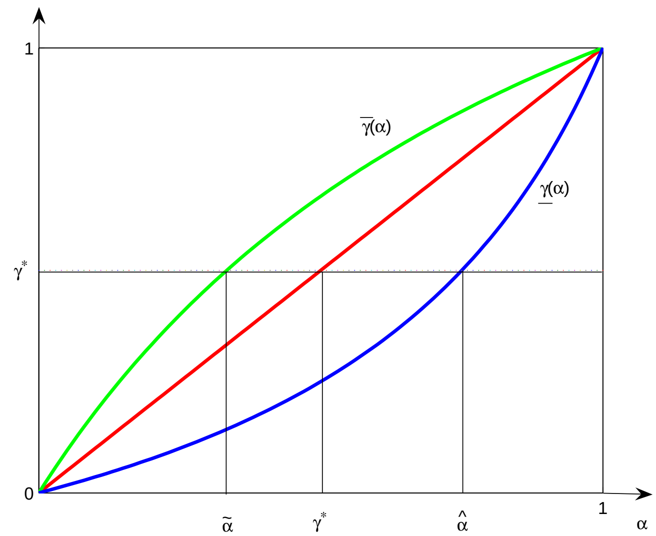

Figure 2: $\overline{\gamma}(\alpha)$ and $\underline{\gamma}(\alpha)$ with $q^B = 0.4$, $q^G = 0.8$ and $l/h = 0.6$

$v_1 = h$ and $v_1 = l$ are both below or above γ^*. In that case the seller offers the same price even if p_1 screens l-value types from h-value types. However, when the seller has a *moderate* ex ante belief (α between $\widetilde{\alpha}$ and $\widehat{\alpha}$), her posterior belief is above γ^* conditional on $v_1 = h$ and below γ^* conditional on $v_1 = l$. So the seller offers different p_2 when the buyer's behavior suggests $v_1 = l$ or $v_1 = h$. This leads to the following lemma.

Lemma 5 *If $\Psi(p_1, 1) \in (0, 1)$ for a given p_1 in a PBE, then*
(i) $x(p_1, 0) > x(p_1, 1) \Rightarrow \alpha \in [\widetilde{\alpha}, \widehat{\alpha}]$;
(ii) $\alpha \in (0, \widetilde{\alpha}) \cup (\widehat{\alpha}, 1) \Rightarrow x(p_1, 0) = x(p_1, 1)$.

According to the seller's ex ante belief of the G distribution, we define a seller *Pessimistic* if $0 < \alpha < \widetilde{\alpha}$, *Moderately Pessimistic* if $\widetilde{\alpha} < \alpha < \gamma^*$, *Moderately Optimistic* if $\gamma^* < \alpha < \widehat{\alpha}$, and *Optimistic* if $\widehat{\alpha} < \alpha < 1$. As in the previous literature, the knife-edge cases are omitted.

4 The Equilibria

In this section we present the equilibria of the game for each range of the seller's ex ante belief as defined above. We call an equilibrium pooling if p_1 on the equilibrium path is accepted with probability one, and an equilibrium semi-separating if p_1 is both accepted and rejected with a positive probability.

4.1 Seller with Extreme Ex Ante Beliefs

Given the second part of Lemma 5, when the seller has an extreme prior, she will offer the same price in the second period regardless whether p_1 is accepted or rejected. Anticipating that, a buyer has no incentives to strategically choose whether to accept or reject p_1 in the first period, that is, all buyer types truthfully reveal their value and accept p_1 if $p_1 \leq v_1$. Given the buyer's strategy, the seller can perfectly distinguish l-value buyer types from h-value types. Due to the seller's extreme prior belief, her second-period strategy will be independent of the buyer's acceptance/rejection decision in the first period.

Proposition 1 (Extreme Seller) *When the seller is pessimistic* $(0 < \alpha < \widetilde{\alpha})$*, there is a unique* D_1 *equilibrium outcome in which the seller offers* $p_t = l$ *and all buyer types accept* p_t *for* $t = 1, 2$*; when the seller is optimistic* $(\widehat{\alpha} < \alpha < 1)$*, there is a unique* D_1 *equilibrium outcome in which the seller offers* $p_t = h$*, the buyer types with* $v_t = h$ *accept* p_t *and the buyer types with* $v_t = l$ *reject* p_t*, for* $t = 1, 2$*.*

The outcome for optimistic seller is of particular interest to us. In the model of Hart and Tirole (1988), in which the buyer's value is private information but the value distribution is common knowledge, it does not happen in any equilibrium that the h-value buyer accepts $p_1 = h$ with probability one, even if the seller has a very optimistic ex ante belief of the buyer's value. The intuition is that, the seller will offer $p_2 = l$ after rejection of p_1 if the h-value buyer accepts $p_1 = h$ with probability one, and then the h-value buyer has an incentive to deviate to reject $p_1 = h$. Here introducing the uncertainty about the buyer's value distribution improves the seller's revenue. We will discuss the comparison of expected revenue between our model and Hart and Tirole (1988) in more details in Section 5.

4.2 Seller with Moderate Ex Ante Beliefs

When the seller's ex ante belief is moderate, the buyer's strategy is quite different from when the seller has an extreme ex ante belief. First, the buyer does not always truthfully reveal his value. Recall from Figure 2, when the seller has a moderate prior, she offers $p_2 = h$ conditional on $v_1 = h$ and $p_2 = l$ conditional on $v_1 = l$. This gives the l-value buyer types an incentive to signal their current values. Therefore, the two l-value buyer types may reject $p_1 < l$, in order to be distinguished from the h-value types and get a low offer in the second period. Given that $x(p_1, 0) = 1 > x(p_1, 1) = 0$, the cutoff value of type (G, l) is smaller than that of type (B, l), i.e., type (G, l) rejects any p_1 if type (B, l) rejects it. Thus, the lowest price the l-value types may reject, denoted as $\underline{p}$, is derived from the largest payoff that type (B, l) is willing to give up now in order to get a low price next period. So we have $\underline{p} \equiv l - \delta q^B(h - l)$ and all buyer types accept $p_1 \leq \underline{p}$.

For $p_1 \in (\underline{p}, l]$ there are multiple equilibrium strategies. One equilibrium strategy, as described above, is for the l-value buyer types to reject p_1 and for the h-value buyer types to accept p_1, with the seller's second-period strategy being $x(p_1, 0) = 1 > x(p_1, 1) = 0$. Another equilibrium strategy, however, is for all buyer types to accept $p_1 \in (\underline{p}, l]$, with the seller's second-period strategy being independent of whether p_1 is accepted or rejected. This strategy can be supported by a consistent belief system if the seller assigns the same beliefs following the acceptance and rejection of p_1. Finally, if the seller adopts a mixed strategy in the second period, i.e., $0 < x(p_1, 0) - x(p_1, 1) < 1$,

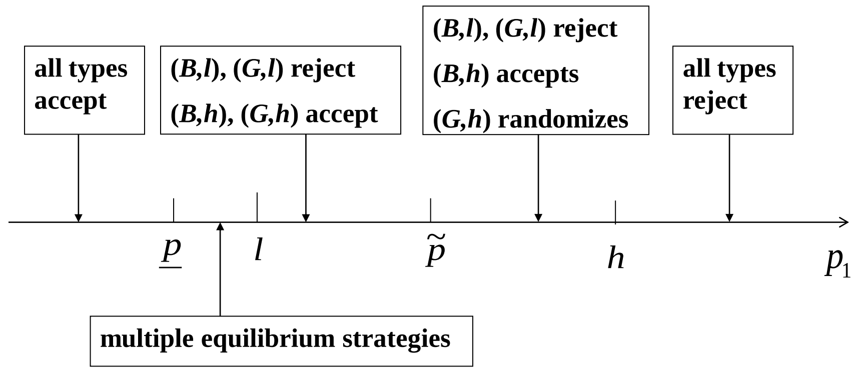

Figure 3: Buyer's Equilibrium Strategy when $\widetilde{\alpha} < \alpha < \widehat{\alpha}$

there also exists an equilibrium strategy in which the buyer type (G, l) rejects $p_1 \in (\underline{p}, l]$, the h-value buyer types accept $p_1 \in (\underline{p}, l]$, and buyer type (B, l) plays mixed strategy.

When p_1 exceeds l, the l-value types will reject the offer and the h-value types will accept it if it is relatively low. However, when p_1 approaches h, the h-value types do not accept p_1 with probability one, since the gain in the first period cannot compensate the loss from being offered with a high price in the second period. Similar to the strategy of l-value types for $p_1 \in (\underline{p}, l]$, the h-value types have an incentive to conceal their current value. In particular, buyer type (B, h) accepts p_1 and buyer type (G, h) plays a mixed strategy. Given $x(p_1, 0) > x(p_1, 1)$, the cutoff value of type (G, h) is smaller than that of type (B, h), i.e., type (B, h) accepts any p_1 if type (G, h) accepts it. Therefore, the cutoff price, denoted as $\widetilde{p}$, is derived from the incentive constraint of type (G, h), and type (G, h) is the one who plays mixed strategy. Thus, the highest p_1 both h-value types accept with probability one is $\widetilde{p} \equiv h - \delta q^G(h - l)$.

To see the intuition why type (G, h) plays mixed strategy, suppose both h-value types accept $p_1 \in (\widetilde{p}, h]$. Then the seller offers $p_2 = l$ after rejection and $p_2 = h$ after acceptance of p_1. Then buyer type (G, h) has an incentive to reject p_1 since $p_1 > \widetilde{p}$. But buyer type (B, h) should then accept p_1 since he gains a positive payoff in the first period and gets the low offer in the second period by revealing his distribution. Conditional on type (B, h) being the only type that accepts p_1, type (G, h) has an incentive to accept p_1 as well. Thus, the unique equilibrium strategy for type (G, h) is to play mixed strategy.

Finally, for $p_1 > h$, all buyer types reject p_1. Figure 3 and Lemma 6 summarize the buyer's strategy.

Lemma 6 *When the seller has a moderate prior belief ($\widetilde{\alpha} < \alpha < \widehat{\alpha}$), the buyer's strategy in a D_1 equilibrium is as follows:*

- *if $p_1 \leq \underline{p}$, all buyer types accept p_1;*
- *if $\underline{p} < p_1 \leq l$, there exist multiple equilibrium strategies: (1) all buyer types accept p_1; (2) type (B, l) and (G, l) reject p_1 and type (B, h) and (G, h) accept p_1; (3) type (G, l) rejects p_1,*

type (B,l) randomizes and type (G,h) and (B,h) accept p_1;

- *if $l < p_1 \leq \widetilde{p}$, type (B,l) and (G,l) reject p_1 and type (B,h) and (G,h) accept p_1;*
- *if $\widetilde{p} < p_1 \leq h$, type (B,l) and (G,l) reject p_1, type (G,h) randomizes and type (B,h) accepts p_1;*
- *if $p_1 > h$, all buyer types reject p_1.*

There are several points we find important about the equilibrium strategies presented above. First, although some buyer types strategically reject an offer less than their first-period value, all buyer types truthfully reject $p_1 > v_1$, and the buyer never incurs a loss in any period in equilibrium. This is because the seller always gets more optimistic when p_1 is accepted than rejected, so no buyer type has an incentive to accept an offer larger than his value.

Second, the first-period offer accepted by all buyer types may be less than the buyer's lowest value l. This feature is also found by Blume (1990), Kennan (2001) and Loginova and Taylor (2008). In all these models including ours, a buyer type with a low value in the current period has a positive probability of drawing a high value in the next period, so the low-value type may reject an offer less than but close to his value if rejection can induce the seller to offer a low price in the next period.

Finally, the equilibrium strategies in this paper are different from those in Kennan (2001). In Kennan (2001), the buyer's value changes according to a Markov process, so the seller's posterior belief becomes more optimistic when all buyer types accept a pooling offer, and the seller offers an aggressive screening offer following acceptance of several pooling offers when her posterior belief grows beyond some threshold. This pattern is described as a cyclic equilibrium. In our model, the seller's posterior belief is the same as her ex ante belief after acceptance of a pooling offer. So we do not expect that the same pattern as in the cyclic equilibrium emerges in this model, even in a longer horizon.

Next we discuss the seller's optimal p_1 and conclude by describing the equilibria of the game for moderately pessimistic and moderately optimistic seller respectively. All cases presented below in Proposition 2-5 arise for a non-negligible set of parameters.[8]

4.2.1 Moderately Pessimistic Seller ($\widetilde{\alpha} < \alpha < \gamma^*$)

Given the buyer's strategy, it is sufficient to consider the seller's payoff at the cutoff prices $p_1 \in \{\underline{p}, l, \widetilde{p}, h\}$, based on the buyer's strategy described in Lemma 6.

(1) $p_1 = \underline{p}$: The seller can always guarantee herself a payoff of U_1 by offering $p_1 = \underline{p}$ and $p_2 = l$, with p_1 and p_2 accepted by all buyer types. U_1 is the seller's lowest payoff from a pooling offer.

$$U_1 = \underline{p} + \delta l;$$

[8]This is proved using Mathematica. The program is available upon request.

(2) $p_1 = l$: Since there are multiple equilibrium strategies for the buyer following $p_1 \in (\underline{p}, l]$, the seller's payoff from offering $p_1 = l$ depends on which strategy the buyer is using. Suppose that all buyer types choose to accept $p_1 = l$, then the seller obtains the highest payoff from a pooling offer U_2 by offering $p_1 = p_2 = l$, with p_1 and p_2 accepted by all buyer types.

$$U_2 = l + \delta l;$$

If buyer type (G, l) rejects p_1, buyer type (B, l) randomizes, and buyer types (B, h) and (G, h) accept p_1, then the seller's payoff from offering $p_1 = l$ is

$$U_3 = [\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-q^B)(1-X^*)]l + \delta l,$$

where $X^* = 1 + \frac{q^B}{1-q^B} - \frac{\alpha q^G(1-\gamma^*)}{(1-\alpha)(1-q^B)\gamma^*}$ is the probability that buyer type (B, l) randomizes to reject p_1 in order to make the seller indifferent in offering $p_2 = l$ and $p_2 = h$ after acceptance of p_1,[9]

$$\gamma(p_1, 1) = \frac{\alpha q^G}{\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-X^*)(1-q^B)} = \gamma^*;$$

Finally, if both l-value buyer types choose to reject and both h-value buyer types choose to accept $p_1 = l$, then offering $p_1 = l$ is dominated by offering $p_1 = \widetilde{p}$.

(3) $p_1 = \widetilde{p}$: Buyer types (B, l) and (G, l) reject p_1, buyer types (B, h) and (G, h) accept p_1, and the seller offers $p_2 = l$ if p_1 is rejected and $p_2 = h$ if p_1 is accepted.

$$\begin{aligned} U_4 &= [\alpha q^G + (1-\alpha)q^B]\widetilde{p} + \delta[\alpha(q^G)^2 + (1-\alpha)(q^B)^2]h \\ &\quad + \delta[\alpha(1-q^G) + (1-\alpha)(1-q^B)]l; \end{aligned}$$

(4) $p_1 = h$: The unique equilibrium strategy is for buyer type (G, h) to randomize, buyer type (B, h) to accept p_1, and buyer types (B, l) and (G, l) to reject p_1, then the seller's payoff is

$$U_5 = [\alpha q^G(1-Y^*) + (1-\alpha)q^B]h + \delta l,$$

where $Y^* = 1 - \frac{(1-\alpha)q^B\gamma^*}{\alpha q^G(1-\gamma^*)}$ is the probability that buyer type (G, h) randomizes to reject p_1 in order to make the seller indifferent in offering $p_2 = l$ or $p_2 = h$ after acceptance of p_1,

$$\gamma(p_1, 1) = \frac{\alpha(1-Y^*)q^G}{\alpha(1-Y^*)q^G + (1-\alpha)q^B} = \gamma^*.$$

Comparing the payoffs above, we find that there always exists a pooling equilibrium with $p_1 = l$, since the highest payoff from a pooling offer U_2 is always greater than any payoff from a semi-separating offer $\max\{U_3, U_4, U_5\}$. Surprisingly, this result implies that the payoffs for a moderately pessimistic seller is no better than those for a pessimistic seller. Intuitively, although the ex ante prior of the h-value buyer types increases for a moderately pessimistic seller, the buyer's strategic behavior makes the seller weakly worse off.

[9] Since the seller is more optimistic after acceptance of p_1 than rejection of p_1, her posterior belief $\gamma(p_1, 1) > \gamma(p_1, 0)$. Therefore, in a mixed strategy equilibrium, $\gamma(p_1, 1) = \gamma^*$ if $\alpha < \gamma^*$ and $\gamma(p_1, 0) = \gamma^*$ if $\alpha > \gamma^*$.

Recall that given $p_1 \in [\underline{p}, l]$ the l-value buyer types may both accept or both reject p_1. Therefore, when the lowest payoff from a pooling offer U_1 is greater than the payoff from offering $p_1 = \widetilde{p}$ or $p_1 = h$, i.e., $\max\{U_4, U_5\}$, any $p_1^* \in [\underline{p}, l]$ can arise in a pooling equilibrium, with both l-value buyer types accepting $p_1 \leq p_1^*$ and rejecting $p_1 > p_1^*$. When U_1 is less than $\max\{U_4, U_5\}$, there exists a pooling offer $p' \in [\underline{p}, l]$ which gives the seller the same payoff as $\max\{U_4, U_5\}$, since $U_2 > \max\{U_4, U_5\}$. So any $p_1^* \in [p', l]$ can arise in a pooling equilibrium, with the l-value buyer types accepting $p_1 \leq p_1^*$ and rejecting $p_1 > p_1^*$.

Proposition 2 (MP Seller: Pooling Equilibria) *When the seller is moderately pessimistic, there always exists a pooling D_1 equilibrium with $p_1 = l$.*

(i) If $U_1 > \max\{U_4, U_5\}$, any $p_1 \in [\underline{p}, l]$ can arise in a pooling equilibrium;

(ii) If $U_1 < \max\{U_4, U_5\}$, any $p_1 \in [p', l]$, with $\underline{p} < p' < l$, can arise in a pooling equilibrium.

Proposition 3 presents the conditions for semi-separating D_1 equilibria. If the lowest payoff from a pooling offer U_1 is greater than the highest payoff from a semi-separating offer, then there is no semi-separating equilibrium. On the contrary, if U_1 is less than the highest payoff from a semi-separating offer, then semi-separating equilibria exist. Furthermore, if $p_1 = \widetilde{p}$ or $p_1 = h$ gives the highest payoff among all semi-separating offers, the equilibrium path of the semi-separating equilibria is unique. If $p_1 = l$ gives the highest semi-separating payoff, then a continuum equilibrium price $p_1 \in [p'', l]$, with $\underline{p} < p'' < l$, arises.

Proposition 3 (MP Seller: Semi-separating Equilibria) *When the seller is moderately pessimistic, the semi-separating D_1 equilibria are characterized as follows.*

(i) If $U_1 > \max\{U_3, U_4, U_5\}$, no semi-separating equilibrium exists;

(ii) If $U_1 < \max\{U_3, U_4, U_5\} = \max\{U_4, U_5\}$, semi-separating equilibria exist and the path is unique, with $p_1 = \widetilde{p}$ or $p_1 = h$;

(iii) If $U_1 < \max\{U_3, U_4, U_5\} = U_3$, any $p_1 \in [p'', l]$, with $\underline{p} < p'' < l$, can arise in a semi-separating equilibrium, so does $p_1 = \widetilde{p}$ or $p_1 = h$ if $\max\{U_4, U_5\} > U_1$.

4.2.2 Moderately Optimistic Seller ($\gamma^* < \alpha < \widehat{\alpha}$)

In this subsection we discuss the pooling equilibria and semi-separating equilibria for a seller with moderately optimistic ex ante beliefs. Similar to last subsection, we start with the seller's payoffs from offering the cutoff prices $p_1 \in \{\underline{p}, l, \widetilde{p}, h\}$.

(1) $p_1 = \underline{p}$: Payoff V_1 is the seller's lowest payoff from a pooling offer with $p_1 = \underline{p}$ and $p_2 = h$, p_1 accepted by all buyer types and p_2 accepted by types with $v_2 = h$.

$$V_1 = \underline{p} + \delta[\alpha q^G + (1-\alpha) q^B] h;$$

(2) $p_1 = l$: Payoff V_2 is the seller's highest payoff from a pooling offer with $p_1 = l$ and $p_2 = h$, p_1 accepted by all buyer types and p_2 accepted by types with $v_2 = h$.

$$V_2 = l + \delta[\alpha q^G + (1-\alpha) q^B] h;$$

Payoff V_3 is the seller's payoff from offering $p_1 = l$, buyer type (G, l) rejects p_1, buyer type (B, l) randomizes, and buyer types (B, h) and (G, h) accept p_1.

$$V_3 = [\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-q^B)(1-X^{**})]l + \delta[\alpha q^G + (1-\alpha)q^B]h,$$

where $X^{**} = \frac{\alpha(1-q^G)(1-\gamma^*)}{(1-\alpha)(1-q^B)\gamma^*}$ is the probability that buyer type (B, l) randomizes to reject p_1 in order to make the seller indifferent in offering $p_2 = l$ or $p_2 = h$ after rejection of p_1,

$$\gamma(p_1, 0) = \frac{\alpha(1-q^G)}{\alpha(1-q^G) + (1-\alpha)X^{**}(1-q^B)} = \gamma^*;$$

(3) $p_1 = \widetilde{p}$: Payoff $V_4 = U_4$ is the seller's payoff when buyer types (B, l) and (G, l) reject p_1, buyer types (B, h) and (G, h) accept p_1, and the seller offers $p_2 = l$ if p_1 is rejected and $p_2 = h$ if p_1 is accepted.

$$\begin{aligned} V_4 &= [\alpha q^G + (1-\alpha)q^B]\widetilde{p} + \delta[\alpha(q^G)^2 + (1-\alpha)(q^B)^2]h \\ &\quad + \delta[\alpha(1-q^G) + (1-\alpha)(1-q^B)]l; \end{aligned}$$

(4) $p_1 = h$: Payoff V_5 is the seller's payoff from offering $p_1 = h$, buyer type (G, h) randomizes, buyer type (B, h) accepts p_1, and buyer types (B, l) and (G, l) reject p_1.

$$V_5 = [\alpha q^G(1-Y^{**}) + (1-\alpha)q^B]h + \delta[\alpha q^G + (1-\alpha)q^B]h,$$

where $Y^{**} = \frac{(1-\alpha)(1-q^B)\gamma^*}{\alpha q^G(1-\gamma^*)} - \frac{1-q^G}{q^G}$ is the probability that buyer type (G, h) randomizes to reject p_1 to make the seller indifferent in offering $p_2 = l$ or $p_2 = h$ after rejection of p_1,

$$\gamma(p_1, 0) = \frac{\alpha Y^{**} q^G + \alpha(1-q^G)}{\alpha Y^{**} q^G + \alpha(1-q^G) + (1-\alpha)(1-q^B)} = \gamma^*.$$

The proof of next two propositions are similar to the proof of Proposition 2 and 3 and are therefore omitted.

Proposition 4 (MO Seller: Pooling Equilibrium) *When the seller is moderately optimistic, the pooling D_1 equilibria are characterized as follows.*

(i) If $V_1 > \max\{V_4, V_5\}$, any $p_1 \in [\underline{p}, l]$ can arise in a pooling equilibrium;

(ii) If $V_1 < \max\{V_4, V_5\} < V_2$, any $p_1 \in [p''', l]$, with $\underline{p} < p''' < l$, can arise in a pooling equilibrium;

(iii) If $V_2 < \max\{V_4, V_5\}$, no pooling equilibrium exists.

Different from the results for a moderately pessimistic seller, case (*iii*) in Proposition 4 implies that it is possible for a semi-separating equilibrium to emerge even if all buyer types accept $p_1 \in (\underline{p}, l]$. That is, when the seller's ex ante belief is sufficiently optimistic, the best pooling offer does not necessarily arise as an equilibrium price. This finding is different from that of Loginova and Taylor (2008). They argue that the seller never offers a first-period price that yields valuable information about the buyer's distribution in a Good equilibrium where all buyer types accept

p_1 less than l. But that conclusion depends on the assumption of no discounting. We find that case (*iii*) of Proposition 4 arises for a non-negligible set of parameters when the discount factor is sufficiently low.

When the seller is moderately optimistic, the conditions for the semi-separating D_1 equilibria are similar to those when the seller is moderately pessimistic.

Proposition 5 (MO Seller: Semi-separating Equilibrium) *When the seller is moderately optimistic, the semi-separating D_1 equilibria are characterized as follows.*

(i) If $V_1 > \max\{V_3, V_4, V_5\}$, no semi-separating equilibrium exists;

(ii) If $V_1 < \max\{V_3, V_4, V_5\} = \max\{V_4, V_5\}$, a semi-separating equilibrium exists and the path is unique, with $p_1 = \widetilde{p}$ or $p_1 = h$;

(iii) If $V_1 < \max\{V_3, V_4, V_5\} = V_3$, any $p_1 \in [p'''', l]$, with $\underline{p} < p'''' < l$, can arise in a semi-separating equilibrium, so does $p_1 = \widetilde{p}$ or $p_1 = h$ if $\max\{V_4, V_5\} > V_1$.

5 Comparison of Expected Revenue

The most important question that this paper is concerned with is whether the seller improves her revenue and gains more monopoly power with the uncertainty about the buyer's value distribution. In this section, we address this issue by comparing the seller's expected revenue in our model with that in the two-period version of Hart and Tirole's (1988) rental model, where the buyer's value distribution is common knowledge.

The two-period version of Hart and Tirole's (1988) rental model is as follows. The buyer has private information about his value, which can be either high or low. The buyer's value is drawn at the beginning of the game and is fixed once realized. In each period $t = 1$ or 2, the seller offers a rental price and the buyer decides whether to accept or reject the offer. Let μ denote the seller's ex ante belief that she is facing a high-value buyer. In order to make a fair comparison, we require that the ex ante probabilities of the high-value buyer in both models be equal, that is, $\mu = \alpha q^G + (1 - \alpha) q^B$. The following proposition compares the expected revenues in the equilibria of the two models for any ex ante belief the seller may have.

Proposition 6 (Revenue Comparison) *If the ex ante probability of high value buyer type in the two-period version of Hart and Tirole's (1988) rental model is the same as in this model, then*

(i) for an optimistic seller, the seller's revenue is higher than in Hart and Tirole;

(ii) for a moderately optimistic seller, if q^B is small enough and q^G is big enough, there exists $\overline{\alpha} \in (\gamma^*, \widehat{\alpha})$ such that, for all $\alpha \in (\overline{\alpha}, \widehat{\alpha})$, the seller's revenue is higher than in Hart and Tirole;

(iii) for a pessimistic and moderately pessimistic seller, there always exists an equilibrium in this model which yields the same revenue as in Hart and Tirole.

When the seller has an optimistic ex ante belief, her revenue in our model is higher than that in Hart and Tirole's (1988). As shown in Proposition 1, the buyer types with $v_1 = h$ accept $p_1 = h$ with probability one since the seller offers $p_2 = h$ independent of whether p_1 is accepted or rejected.

In contrast, in the two-period version of Hart and Tirole's rental model, the high value buyer rejects $p_1 = h$ with a positive probability even if the seller is optimistic enough to offer equilibrium price $p_1 = h$, since otherwise the seller offers $p_2 = l$ after rejection of p_1 and the high-value buyer has an incentive to deviate to reject p_1.

When the seller has a pessimistic or moderately pessimistic ex ante belief, there always exists a pooling equilibrium in our model where the seller offers $p_1 = p_2 = l$ and all buyer types accept the offers. This equilibrium yields the seller the same expected revenue as in Hart and Tirole.

When the seller has a moderately optimistic ex ante belief, she can still be better off than in Hart and Tirole (1988) if the two distributions are sufficiently different, that is, q^B is small enough and q^G is big enough, and the seller's ex ante belief is sufficiently optimistic. However, if the seller's ex ante belief is close to the lower bound of moderately optimistic beliefs, γ^*, then the seller is worse off than in Hart and Tirole (1988).

From Proposition 6 we conclude that, when the seller has sufficiently optimistic ex ante beliefs, the seller is better off compared to the case that the distribution of the buyer's value is common knowledge.

6 Conclusion

In this paper we have considered a two-period repeated bargaining model where the seller offers a price to rent a durable good in each period. The buyer's value of consuming the durable good is drawn from a fixed distribution in each period. The buyer has private information not only about his value in each period, but also about the distribution which his values are drawn from.

We compare the seller's expected revenue in our model with that in the two-period version of Hart and Tirole's (1988) rental model where the distribution of the buyer's value is common knowledge, under the assumption that the ex ante probabilities of high value buyer types are the same in the two models. We find that the seller is better off with the additional layer of uncertainty about the buyer's value distribution when she has sufficiently optimistic ex ante beliefs.

The results we found may cast some light on the longer horizon. In the current two-period model, the seller cannot perfectly learn the buyer's value distribution. It is interesting to examine whether the seller is able to learn the buyer's distribution eventually if she is allowed to employ price experimentation in a finite or an infinite horizon.

On the other hand, this model only allows the seller to rent the durable good. For future research, we are interested in investigating the case where the seller is able to adopt a more general strategy, such as selling the durable good or providing both options of selling and renting the durable good to the buyer.

7 Appendix A: Proofs

Proof of Lemma 3.

Step 1: Suppose $x(p_1, 0) > x(p_1, 1)$. Then $C(G, l) < C(B, l) < l < C(G, h) < C(B, h) < h$. A price p_1 can screen one type from the other types only if $p_1 \in [C(G, l), C(B, l)]$ or $p_1 \in [C(G, h), C(B, h)]$.

If $p_1 \in [C(G, l), C(B, l)]$ and only type (G, l) rejects p_1, then $x(p_1, 0) = 0$, so it contradicts with $x(p_1, 0) > x(p_1, 1)$.

If $p_1 \in [C(G, h), C(B, h)]$ and only type (B, h) accepts p_1, then $x(p_1, 1) = 1$, so it contradicts with $x(p_1, 0) > x(p_1, 1)$.

Step 2: Suppose $x(p_1, 0) < x(p_1, 1)$. Then $l < C(B, l) < C(G, l) < h < C(B, h) < C(G, h)$. A price p_1 can screen one type from the other types only if $p_1 \in [C(B, l), C(G, l)]$ or $p_1 \in [C(B, h), C(G, h)]$.

If $p_1 \in [C(B, l), C(G, l)]$ and only type (B, l) rejects p_1, then $x(p_1, 0) = 1$, so it contradicts with $x(p_1, 0) < x(p_1, 1)$.

If $p_1 \in [C(B, h), C(G, h)]$ and only type (G, h) accepts p_1, then $x(p_1, 1) = 0$, so it contradicts with $x(p_1, 0) < x(p_1, 1)$.

Step 3: Suppose $x(p_1, 0) = x(p_1, 1)$. Then $C(B, l) = C(G, l) = l < C(B, h) = C(G, h) = h$.

If $p_1 \leq l$, all types accept p_1.

If $p_1 > h$, all types reject p_1.

If $l < p_1 \leq h$, both type (B, l) and (G, l) reject p_1 and both type (B, h) and (G, h) accept p_1.

In any case, screening one buyer type cannot happen in equilibrium. ■

Proof of Lemma 4. Suppose $\Psi(p_1, 1) \in (0, 1)$ and $x(p_1, 0) < x(p_1, 1)$ in a PBE. Then $l < C(B, l) < C(G, l) < h < C(B, h) < C(G, h)$, and $\Psi(p_1, 1) \in (0, 1)$ only if $p_1 \in [C(B, l), C(G, h)]$. We will show that it reaches a contradiction for any $p_1 \in [C(B, l), C(G, h)]$.

If $p_1 \in [C(B, l), C(G, l))$, then only type (B, l) rejects p_1 and $x(p_1, 0) = 1 \geq x(p_1, 1)$.

If $p_1 \in (C(B, h), C(G, h)]$, then only type (G, h) accepts p_1 and $x(p_1, 1) = 0 \leq x(p_1, 0)$.

If $p_1 \in (C(G, l), C(B, h))$, then $\gamma(p_1, 0) = \underline{\gamma}(\alpha) < \alpha < \overline{\gamma}(\alpha) = \gamma(p_1, 1)$ and $x(p_1, 0) \geq x(p_1, 1)$.

Denote X' as the probability for type (G, l) to reject $p_1 = C(G, l)$ and Y' as the probability for type (B, h) to reject $p_1 = C(B, h)$.

If $p_1 = C(G, l)$,

$$\gamma(p_1, 0) = \frac{\alpha X'(1 - q^G)}{\alpha X'(1 - q^G) + (1 - \alpha)(1 - q^B)} < \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1, 1) = \frac{\alpha q^G + \alpha(1 - X')(1 - q^G)}{\alpha q^G + (1 - \alpha)q^B + \alpha(1 - X')(1 - q^G)} > \overline{\gamma}(\alpha),$$

so $x(p_1, 0) \geq x(p_1, 1)$.

If $p_1 = C(B, h)$,

$$\gamma(p_1, 0) = \frac{\alpha(1 - q^G)}{\alpha(1 - q^G) + (1 - \alpha)(1 - q^B) + (1 - \alpha)Y'q^B} < \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha q^G}{\alpha q^G + (1-\alpha)(1-Y')q^B} > \overline{\gamma}(\alpha),$$

so $x(p_1,0) \geq x(p_1,1)$.

Since every case leads to a contradiction with $x(p_1,0) < x(p_1,1)$, the seller offers $x(p_1,0) \geq x(p_1,1)$ in a PBE if $\Psi(p_1,1) \in (0,1)$. ■

Proof of Lemma 5. (i) $\Psi(p_1,1) \in (0,1)$ and $x(p_1,0) > x(p_1,1) \Rightarrow \alpha \in [\widetilde{\alpha}, \widehat{\alpha}]$.

Suppose $\Psi(p_1,1) \in (0,1)$, $x(p_1,0) > x(p_1,1)$ and $\alpha \in (0,\widetilde{\alpha}) \cup (\widehat{\alpha},1)$. Then $C(G,l) < C(B,l) < l < C(G,h) < C(B,h) < h$. $\Psi(p_1,1) \in (0,1)$ only if $p_1 \in [C(G,l), C(B,h)]$. We will show that it reaches a contradiction for any $p_1 \in [C(G,l), C(B,h)]$.

If $p_1 \in [C(G,l), C(B,l))$, then only type (G,l) rejects p_1 and $x(p_1,0) = 0 \leq x(p_1,1)$.

If $p_1 \in (C(G,h), C(B,h)]$, then only type (B,h) accepts p_1 and $x(p_1,1) = 1 \geq x(p_1,0)$.

If $p_1 \in (C(B,l), C(G,h))$ and $\alpha < \widetilde{\alpha}$, then $\gamma(p_1,1) = \overline{\gamma}(\alpha) < \gamma^*$ and $x(p_1,1) = 1 \geq x(p_1,0)$.

If $p_1 \in (C(B,l), C(G,h))$ and $\alpha > \widehat{\alpha}$, then $\gamma(p_1,0) = \underline{\gamma}(\alpha) > \gamma^*$ and $x(p_1,0) = 0 \leq x(p_1,1)$.

Denote X as the probability for type (B,l) to reject $p_1 = C(B,l)$ and Y as the probability for type (G,h) to reject $p_1 = C(G,h)$.

If $p_1 = C(B,l)$,

$$\gamma(p_1,0) = \frac{\alpha(1-q^G)}{\alpha(1-q^G) + (1-\alpha)X(1-q^B)} > \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha q^G}{\alpha q^G + (1-\alpha)q^B + (1-\alpha)(1-X)(1-q^B)} < \overline{\gamma}(\alpha).$$

When $\alpha < \widetilde{\alpha}$, $\gamma(p_1,1) < \overline{\gamma}(\alpha) < \gamma^*$ and $x(p_1,1) = 1 \geq x(p_1,0)$. When $\alpha > \widehat{\alpha}$, $\gamma(p_1,0) > \underline{\gamma}(\alpha) > \gamma^*$ and $x(p_1,0) = 0 \leq x(p_1,1)$.

If $p_1 = C(G,h)$,

$$\gamma(p_1,0) = \frac{\alpha Y q^G + \alpha(1-q^G)}{\alpha Y q^G + \alpha(1-q^G) + (1-\alpha)(1-q^B)} > \underline{\gamma}(\alpha)$$

and

$$\gamma(p_1,1) = \frac{\alpha(1-Y)q^G}{\alpha(1-Y)q^G + (1-\alpha)q^B} < \overline{\gamma}(\alpha).$$

When $\alpha < \widetilde{\alpha}$, $\gamma(p_1,1) < \overline{\gamma}(\alpha) < \gamma^*$ and $x(p_1,1) = 1 \geq x(p_1,0)$. When $\alpha > \widehat{\alpha}$, $\gamma(p_1,0) > \underline{\gamma}(\alpha) > \gamma^*$ and $x(p_1,0) = 0 \leq x(p_1,1)$.

Therefore, every case is contradictory to $x(p_1,0) > x(p_1,1)$.

(ii) It is directly derived from Lemma 4 and (i) of Lemma 5. ■

Proof of Proposition 1.

Part 1: Pessimistic Seller

Step 1: We first show that it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

(1) Suppose $x(p_1, 0) > x(p_1, 1)$. Lemma 5 implies that either all buyer types accept p_1 or all buyer types reject p_1. Given $\alpha < \widetilde{\alpha}$ and $x(p_1, 0) > x(p_1, 1)$, it must be the case that all buyer types reject p_1, otherwise $x(p_1, 1) = 1$. Since $x(p_1, 0) > x(p_1, 1)$ and all buyer types reject p_1, $p_1 > C(B, h)$, which is less than h. But Lemma 7 shows that a PBE cannot pass criterion D_1 if all buyer types reject $p_1 < h$. Therefore, $p_1 \geq h$ and all buyer types reject p_1.

(2) Suppose $x(p_1, 0) < x(p_1, 1)$. Given $\alpha < \widetilde{\alpha}$, Lemma 5 implies that $p_1 \leq C(B, l) = \min_{\theta_1}\{C(\theta_1)\}$, which is greater than l, and all buyer types accept p_1. But Lemma 7 shows that a PBE cannot pass criterion D_1 if all buyer types accept $p_1 > l$. Therefore, $p_1 \leq l$ and all buyer types accept p_1.

(3) Suppose $x(p_1, 0) = x(p_1, 1)$. All buyer types accept p_1 if and only if $p_1 \leq v_1$ for any p_1.

Combining three cases above, it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

Step 2: Given the buyer's strategy, the seller offers $p_1 = l$ or $p_1 = h$, and always offers $p_2 = l$ on the equilibrium path. The respective payoffs for the seller is:

$$\begin{cases} \pi(l) = l + \delta l; \\ \pi(h) = \alpha h + \delta l. \end{cases}$$

Given $\alpha < \widetilde{\alpha} < \gamma^*$, it is optimal to offer $p_1 = l$.

Part 2: Optimistic Seller

Step 1: Similar to the pessimistic seller, we first show that it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

(1) Suppose $x(p_1, 0) > x(p_1, 1)$. Given $\alpha > \widehat{\alpha}$, Lemma 5 implies that $p_1 \leq C(G, l) = \min_{\theta_1}\{C(\theta_1)\} < l$, and all buyer types accept p_1. Lemma 7 shows it passes criterion D_1.

(2) Suppose $x(p_1, 0) < x(p_1, 1)$. Given $\alpha > \widehat{\alpha}$, Lemma 5 implies that $p_1 > C(G, h) = \max_{\theta_1}\{C(\theta_1)\} > h$, and all buyer types reject p_1. Lemma 7 shows it passes criterion D_1.

(3) Suppose $x(p_1, 0) = x(p_1, 1)$. All buyer types accept p_1 if and only if $p_1 \leq v_1$ for any p_1.

Combining three cases above, it is the unique D_1 equilibrium strategy for the buyer to accept p_1 if and only if $p_1 \leq v_1$.

Step 2: Given the buyer's strategy, the seller offers $p_1 = l$ or $p_1 = h$, and always offers $p_2 = h$ on the equilibrium path. The respective payoffs for the seller is:

$$\begin{cases} \pi(l) = l + \delta \alpha h; \\ \pi(h) = \alpha h + \delta \alpha h. \end{cases}$$

Since $\alpha > \widehat{\alpha} > \gamma^*$, it is optimal to offer $p_1 = h$. ■

Proof of Lemma 6. We try to derive all the possible buyer's strategies following different second-period strategy of the seller.

(1) Suppose $x(p_1, 0) < x(p_1, 1)$.

We have $l < C(B, l) < C(G, l) < h < C(B, h) < C(G, h)$. Lemma 4 implies that p_1 must be accepted or rejected with probability one given $x(p_1, 0) < x(p_1, 1)$. Given $x(p_1, 0) < x(p_1, 1)$, when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 \le C(B,l)$ and all buyer types accept p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 > C(G,h)$ and all buyer types reject p_1. Lemma 7 shows that the equilibrium cannot pass criterion D_1 if all types accept $p_1 > l$. Thus, when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 \le l$ and all types accept p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 > C(G,h) > h$ and all buyer types reject p_1.

(2) Suppose $x(p_1, 0) = x(p_1, 1)$.

We have $C(G,l) = C(B,l) = l < C(G,h) = C(B,h) = h$. Therefore, for $p_1 \le l$ all buyer types accept p_1, and for $p_1 > h$ all types reject p_1. For $p_1 \in (l,h]$, type (B,l) and (G,l) reject p_1 and type (B,h) and (G,h) accept p_1, so $x(p_1,0) = 1$ and $x(p_1,1) = 0$ given $\widetilde{\alpha} < \alpha < \widehat{\alpha}$, which leads to a contradiction.

(3) Suppose $x(p_1, 0) > x(p_1, 1)$.

We have $C(G,l) < C(B,l) < l < C(G,h) < C(B,h) < h$. Next we divide all the possibilities into three cases.

Case 1: p_1 is accepted or rejected with probability one, i.e., $\Psi(p_1, 1) \in \{0, 1\}$. So when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 > C(B,h)$ and all buyer types reject p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 \le C(G,l)$ and all buyer types accept p_1. Lemma 7 shows that the equilibrium cannot pass criterion D_1 if all buyer types reject $p_1 < h$. Thus, when $\widetilde{\alpha} < \alpha < \gamma^*$, $p_1 > h$ and all buyer types reject p_1. When $\gamma^* < \alpha < \widehat{\alpha}$, $p_1 \le C(G,l) < l$ and all buyer types accept p_1.

Case 2: p_1 is accepted and rejected with a positive probability, i.e., $\Psi(p_1, 1) \in (0, 1)$, and the seller plays pure strategy in the second period, i.e., $x(p_1, 0) = 1$ and $x(p_1, 1) = 0$. Lemma 3 shows that no p_1 separates a single type from other types. So $p_1 \in (C(B,l), C(G,h)] = (\underline{p}, \widetilde{p}]$. Type (B,l) and (G,l) reject p_1 and type (B,h) and (G,h) accept p_1.

Case 3: p_1 is accepted and rejected with a positive probability, i.e., $\Psi(p_1, 1) \in (0, 1)$, and the seller plays mixed strategy in the second period, i.e., $0 < x(p_1, 0) - x(p_1, 1) < 1$. Then either $x(p_1, 0) = 1$ and $x(p_1, 1) \in (0, 1)$ or $x(p_1, 0) \in (0, 1)$ and $x(p_1, 1) = 0$, since the knife-edge condition $\alpha = \gamma^*$ is omitted. The former implies $\gamma(p_1, 0) < \gamma^*$ and $\gamma(p_1, 1) = \gamma^*$, and the latter implies $\gamma(p_1, 0) = \gamma^*$ and $\gamma(p_1, 1) > \gamma^*$. Therefore, $\gamma(p_1, 1) = \gamma^*$ when $\widetilde{\alpha} < \alpha < \gamma^*$, and $\gamma(p_1, 0) = \gamma^*$ when $\gamma^* < \alpha < \widehat{\alpha}$. From Lemma 3, it is not possible for type (G,l) or (B,h) to randomize, otherwise the seller can at least sometimes separate type (G,l) or (B,h) from other types. So only type (B,l) and (G,h) may play mixed strategy.

When $\widetilde{\alpha} < \alpha < \gamma^*$, type (B,l) randomizes to reject p_1 with probability X^*, (G,l) rejects p_1, and (G,h) and (B,h) accept p_1. Then

$$\gamma(p_1, 1) = \frac{\alpha q^G}{\alpha q^G + (1-\alpha) q^B + (1-\alpha)(1-X^*)(1-q^B)} = \gamma^*.$$

Type (B,l) is indifferent from accepting and rejecting p_1, then

$$l - p_1 + \delta q^B x(p_1, 1)(h - l) = \delta q^B (h - l).$$

So type (B,l) rejects $p_1 \in (\underline{p}, l]$ with probability $X^* = 1 + \frac{q^B}{1-q^B} - \frac{\alpha q^G (1-\gamma^*)}{(1-\alpha)(1-q^B)\gamma^*}$, and the seller offers $x(p_1, 1) = 1 - \frac{l - p_1}{\delta q^B (h-l)}$, $x(p_1, 0) = 1$.

When $\gamma^* < \alpha < \widehat{\alpha}$, type (B,l) randomizes to reject p_1 with probability X^{**}, (G,l) rejects p_1, and (G,h) and (B,h) accept p_1. Then

$$\gamma(p_1,0) = \frac{\alpha(1-q^G)}{\alpha(1-q^G)+(1-\alpha)X^{**}(1-q^B)} = \gamma^*.$$

Type (B,l) is indifferent from accepting and rejecting p_1, then

$$l - p_1 = \delta q^B x(p_1,0)(h-l).$$

So type (B,l) rejects $p_1 \in (\underline{p}, l]$ with probability $X^{**} = \frac{\alpha(1-q^G)(1-\gamma^*)}{(1-\alpha)(1-q^B)\gamma^*}$, and the seller offers $x(p_1,0) = \frac{l-p_1}{\delta q^B(h-l)}$ and $x(p_1,1) = 0$.

When $\widetilde{\alpha} < \alpha < \gamma^*$, type (G,h) randomizes to reject p_1 with probability Y^*, (B,l) and (G,l) reject p_1, and (B,h) accepts p_1. Then

$$\gamma(p_1,1) = \frac{\alpha(1-Y^*)q^G}{\alpha(1-Y^*)q^G+(1-\alpha)q^B} = \gamma^*.$$

Type (G,h) is indifferent from accepting and rejecting p_1, then

$$h - p_1 + \delta q^G x(p_1,1)(h-l) = \delta q^G(h-l).$$

So type (G,h) rejects $p_1 \in (\widetilde{p}, h]$ with probability $Y^* = 1 - \frac{(1-\alpha)q^B\gamma^*}{\alpha q^G(1-\gamma^*)}$, and the seller offers $x(p_1,1) = 1 - \frac{h-p_1}{\delta q^G(h-l)}$ and $x(p_1,0) = 1$.

When $\gamma^* < \alpha < \widehat{\alpha}$, type (G,h) randomizes to reject p_1 with probability Y^{**}, (B,l) and (G,l) reject p_1, and (B,h) accepts p_1. Then

$$\gamma(p_1,0) = \frac{\alpha Y^{**}q^G + \alpha(1-q^G)}{\alpha Y^{**}q^G + \alpha(1-q^G) + (1-\alpha)(1-q^B)} = \gamma^*.$$

Type (G,h) is indifferent from accepting and rejecting p_1, then

$$h - p_1 = \delta q^G x(p_1,0)(h-l).$$

So type (G,h) rejects $p_1 \in (\widetilde{p}, h]$ with probability $Y^{**} = \frac{(1-\alpha)(1-q^B)\gamma^*}{\alpha q^G(1-\gamma^*)} - \frac{1-q^G}{q^G}$ and the seller offers $x(p_1,0) = \frac{h-p_1}{\delta q^G(h-l)}$ and $x(p_1,1) = 0$.

Lemma 6 comes from the combination of three steps. ■

Proof of Proposition 2. *Step 1:* First we show that $U_2 > \max\{U_4, U_5\}$ for $\widetilde{\alpha} < \alpha < \gamma^*$. Given this, there always exists a pooling equilibrium with $p_1 = l$ on the equilibrium path and all buyer types accepting $p_1 \in [\underline{p}, l]$.

$$\begin{aligned} & U_4 - U_2 \\ = \ & \delta(1-\alpha)q^B l(q^G-1) + \delta\alpha q^G l(q^G-1) \\ & +\delta(1-\alpha)q^B h(q^B-q^G) + [\alpha q^G h + (1-\alpha)q^B h - l] \\ < \ & 0 \end{aligned}$$

Each item on the right hand side of the equation is negative for $\widetilde{\alpha} < \alpha < \gamma^*$.

By plugging Y^* into the definition of U_5, $U_5 = \frac{1-\alpha}{1-\gamma^*} q^B h + \delta l$, which is decreasing in α. So

$$\begin{aligned} & U_5 - U_2 \\ < \; & \frac{1-\widetilde{\alpha}}{1-\gamma^*} q^B h - l \\ = \; & \frac{h}{q^G + q^B - l/h}(l/h - q^G)(l/h - q^B) < 0. \end{aligned}$$

Step 2: (i)If $U_1 > \max\{U_4, U_5\}$, for an arbitrary $p_1^* \in [\underline{p}, l]$, assume all buyer types accept $p_1 \in [\underline{p}, p_1^*]$, type (B, l) and (G, l) reject $p_1 \in (p_1^*, l]$, and type (B, h) and (G, h) accept $p_1 \in (p_1^*, l]$, then p_1^* is the optimal p_1.

(ii) Since $U_1 = \underline{p} + \delta l < \max\{U_4, U_5\} < U_2 = l + \delta l$, there exists $p' \in (\underline{p}, l)$ such that $u(p') = p' + \delta l = \max\{U_4, U_5\}$. For an arbitrary $p_1^* \in [p', l]$, assume all buyer types accept $p_1 \in [\underline{p}, p_1^*]$, type (B, l) and (G, l) reject $p_1 \in (p_1^*, l]$, and type (B, h) and (G, h) accept $p_1 \in (p_1^*, l]$. Then p_1^* is the optimal p_1 given $u(p') = \max\{U_4, U_5\}$. ■

Proof of Proposition 3. (i) By definition U_3, U_4 and U_5 are the potential highest payoffs in a semi-separating equilibrium. If the lowest payoff from a pooling offer, U_1, is greater than $\max\{U_3, U_4, U_5\}$, there is no semi-separating equilibrium.

(ii) For all $p_1 \in (\underline{p}, l]$, the buyer can adopt two semi-separating equilibrium strategies: 1) types with $v_1 = l$ reject p_1 and types with $v_1 = h$ accept p_1, or 2) types with $v_1 = h$ accept p_1, type (G, l) rejects p_1 and type (B, l) randomizes. If the first strategy is adopted at $p_1 \in (\underline{p}, l]$, the seller's payoff by offering p_1 is less than U_4. If the second strategy is adopted, the payoff is weakly less than U_3, which is less than $\max\{U_4, U_5\}$. Therefore, when the buyer adopts either of these two strategies, given $U_1 < \max\{U_4, U_5\}$, a semi-separating equilibrium exists and the path is unique, with $p_1 = \widetilde{p}$ or $p_1 = h$, depending on whether U_4 or U_5 is larger.

(iii) Define $U(p_1, X^*) = [\alpha q^G + (1-\alpha) q^B + (1-\alpha)(1-q^B)(1-X^*)]p_1 + \delta l$, which is increasing in $p_1 \in (\underline{p}, l]$. First suppose $\max\{U_4, U_5\} < U_1 < U_3$. By definition $U(\underline{p}, X^*) < U_1 < U_3 = U(l, X^*)$. Therefore, there exists $p'' \in (\underline{p}, l)$ such that $U(p'', X^*) = U_1$. For any arbitrary $p_1^* \in [p'', l]$, assume the buyer uses the second strategy for $p_1^* \leq p''$ and uses the first strategy described in part (ii) for $p_1^* > p''$, then $p_1^* \in [p'', l]$ is the optimal p_1.

Then suppose $U_1 < \max\{U_4, U_5\} < U_3$. Since $U(\underline{p}, X^*) < U_1 < U_3 = U(l, X^*)$, $U(\underline{p}, X^*) < \max\{U_4, U_5\} < U(l, X^*)$. Define $p'' \in (\underline{p}, l)$ such that $U(p'', X^*) = \max\{U_4, U_5\}$. If for any arbitrary $p_1^* \in [p'', l]$, the buyer uses the second strategy described in part (ii) for $p_1^* \leq p''$ and uses the first strategy for $p_1^* > p''$, then $p_1^* \in [p'', l]$ is the optimal p_1. If for any $p_1 \in (\underline{p}, l]$, the buyer uses the first strategy, then $p_1 = \widetilde{p}$ or $p_1 = h$ is optimal, depending on whether U_4 or U_5 is larger. ■

Proof of Proposition 6.

Step 1: We first describe the equilibrium in the two-period version of Hart and Tirole's rental model.

In period 2, both types accept p_2 if and only if $p_2 \leq v_2$ and reject p_2 otherwise. In the first period, the l-type buyer accepts p_1 if and only if $p_1 \leq l$ and reject p_1 otherwise. If $\mu < l/h$, the

h-type buyer accepts $p_1 \leq h - \delta(h-l)$ and reject $p_1 > h - \delta(h-l)$. If $\mu > l/h$, the h-type buyer accepts $p_1 \leq h - \delta(h-l)$, randomizes to accept $p_1 \in (h-\delta(h-l), h]$ with probability $y^* = \frac{\mu h - l}{\mu(h-l)}$, and reject $p_1 > h$. Therefore, if $\mu < l/h$, the seller offers $p_1 = p_2 = l$; if $l/h < \mu < \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}$, the seller offers $p_1 = h - \delta(h-l)$, $p_2 = h$ if p_1 is accepted and $p_2 = l$ if p_1 is rejected; if $\mu > \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}$, the seller offers $p_1 = p_2 = h$. The seller's revenue in the equilibrium is as follows.

$$\pi = \begin{cases} l + \delta l, & \text{if } \mu < l/h; \\ \mu[h - \delta(h-l)] + \delta\mu h + \delta(1-\mu)l = \mu h + \delta l, & \text{if } l/h < \mu < \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}; \\ \mu y^* h + \delta \mu h = \frac{\mu h^2 - hl + \delta\mu h^2 - \delta\mu h l}{h-l} & \text{if } \mu > \frac{hl+\delta l(h-l)}{hl+\delta h(h-l)}. \end{cases}$$

Step 2: Next we compare the revenue in our model with that in Hart and Tirole, assuming that $\mu = \alpha q^G + (1-\alpha)q^B$. Notice that $\alpha > \gamma^*$ is equivalent to $\mu > l/h$. For convenience, denote $W_1 = \mu h + \delta l$ and $W_2 = \mu y^* h + \delta \mu h$.

(i) For an optimistic seller ($\alpha > \widehat{\alpha}$), there is a unique equilibrium outcome as shown in Proposition 2, and the seller's revenue in our model is

$$\begin{aligned} & (\alpha q^G + (1-\alpha)q^B)h + \delta(\alpha q^G + (1-\alpha)q^B)h \\ = \; & \mu h + \delta \mu h \\ > \; & \max\{W_1, W_2\}. \end{aligned}$$

So the seller's revenue in our model is higher than in Hart and Tirole's.

(ii) For a moderately optimistic seller ($\gamma^* < \alpha < \widehat{\alpha}$), it suffices to compare the potential optimal revenues W_1 and W_2 in Hart and Tirole with the potential optimal revenues V_2, V_4 and V_5 in our model. Our proof consists of the following results.

Result 1:

$$W_1 - V_2 = (1-\delta)(\mu h - l) > 0$$

Result 2:

$$W_1 - V_4 = \delta(1-\alpha)q^B(q^G - q^B)h + \delta\mu(1-q^G)l > 0.$$

Result 3:

$$V_5 > W_2 \;\text{ if }\; q^G > 1 - q^B + q^B(l/h).$$

$$\begin{aligned} & V_5 - W_2 \\ = \; & [\alpha q^G(1-Y^{**}) + (1-\alpha)q^B]h - \mu y^* h \\ = \; & \alpha(q^G - q^B)\left(\frac{1-q^B}{q^G - l/h} - \frac{1}{1-l/h}\right)h \\ & + \left[q^B - \frac{(1-q^B)(l/h - q^B)}{q^G - l/h} + \frac{l/h - q^B}{1-l/h}\right]h \end{aligned}$$

If $(1-q^B)(1-l/h) < q^G - l/h$, then $V_5 > W_2$ when

$$\alpha < \frac{l/h - q^B}{q^G - q^B} - \frac{q^B}{q^G - q^B}\frac{1}{\frac{1-q^B}{q^G - l/h} - \frac{1}{1-l/h}}.$$

The RHS of the inequality is decreasing in q^G and converges to 1 when $q^G \to 1$, therefore the RHS of the inequality is greater than 1, so the inequality always holds when $(1-q^B)(1-l/h) < q^G - l/h$.

Result 4: There exists $\overline{\alpha} \in (\gamma^*, \widehat{\alpha})$ such that, for $\alpha \in (\overline{\alpha}, \widehat{\alpha})$, $W_2 > W_1$ if $q^G > 1 - q^B + q^B(l/h)$ and $q^B < \frac{\delta(l/h)(1-l/h)}{l/h+\delta(1-l/h)}$.

$$\begin{aligned} & W_2 > W_1 \\ \Rightarrow \quad & \mu > \frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} \\ \Rightarrow \quad & \alpha > \frac{1}{q^G - q^B}[\frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B] \equiv \overline{\alpha} \end{aligned}$$

It is easy to show that $\overline{\alpha} > \gamma^*$. Next we need to show the conditions under which $\overline{\alpha} < \widehat{\alpha}$.

$$\begin{aligned} & \overline{\alpha} < \widehat{\alpha} \\ \Leftrightarrow \quad & \frac{1}{q^G - q^B}[\frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B] < \frac{1-q^B}{(1-q^B)-(q^G - l/h)} \cdot \frac{l/h - q^B}{q^G - q^B} \\ \Leftrightarrow \quad & \frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B < \frac{(1-q^B)(l/h - q^B)}{(1-q^B)-(q^G - l/h)} \end{aligned}$$

At the same time,

$$\begin{aligned} & q^G - l/h > (1-q^B)(1-l/h) \\ \Leftrightarrow \quad & (1-q^B) - (q^G - l/h) < (1-q^B) - (1-q^B)(1-l/h) \\ \Leftrightarrow \quad & (1-q^B)(l/h) > (1-q^B) - (q^G - l/h) \\ \Leftrightarrow \quad & \frac{l/h - q^B}{l/h} < \frac{(1-q^B)(l/h - q^B)}{(1-q^B)-(q^G - l/h)} \end{aligned}$$

To show $\overline{\alpha} < \widehat{\alpha}$, it is sufficient to show that

$$\frac{l/h + \delta(l/h)(1-l/h)}{l/h + \delta(1-l/h)} - q^B < \frac{l/h - q^B}{l/h},$$

which is satisfied when $q^B < \frac{\delta(l/h)(1-l/h)}{l/h+\delta(1-l/h)}$.

Combining Result 1, 2, 3, and 4, we have shown that, if $q^G > (1-q^B)(1-l/h) + l/h$ and $q^B < \frac{\delta(l/h)(1-l/h)}{l/h+\delta(1-l/h)}$, there exists $\overline{\alpha} \in (\gamma^*, \widehat{\alpha})$ such that, for $\alpha \in (\overline{\alpha}, \widehat{\alpha})$, $V_5 > W_2 > W_1 > \max\{V_2, V_4\}$. Therefore, V_5 is the optimal revenue in our model and it is higher than the optimal revenue in the two-period version of Hart and Tirole (1988).

(iii) For a pessimistic seller or moderately pessimistic seller ($\alpha < \gamma^*$), there always exists a pooling equilibrium in which the seller offers $p_1 = p_2 = l$ and all buyer types accept the offer as shown in Proposition 1 and 3. This equilibrium yields revenue $l + \delta l$, which is the same as in Hart and Tirole's (1988). ■

8 Appendix B: Criterion D_1

The following definition of Criterion D_1 is modified from Cho and Kreps (1987).

Consider a fixed equilibrium on the continuation of p_1, with action $a_1 \in \{0, 1\}$ reached with zero probability. Suppose $x(p_1, 1)$ and $x(p_1, 0)$ is the seller's equilibrium strategy.

Step 1: Find the sets of all (mixed) responses ϕ by the seller that would cause type $\theta_1 = (d, v_1)$ to defect from the equilibrium and to be indifferent. If $a_1 = 0$ is the out-of-equilibrium action, form the sets

$$D_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d x(p_1, 1)(h - l) < \delta q^d \phi(h - l), \phi \in [0, 1]\},$$

$$D^0_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d x(p_1, 1)(h - l) = \delta q^d \phi(h - l), \phi \in [0, 1]\}.$$

If $a_1 = 1$ is the out-of-equilibrium action, form the sets

$$D_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d \phi(h - l) > \delta q^d x(p_1, 0)(h - l), \phi \in [0, 1]\},$$

$$D^0_{\theta_1} \equiv \{\phi : (v_1 - p_1) + \delta q^d \phi(h - l) = \delta q^d x(p_1, 0)(h - l), \phi \in [0, 1]\}.$$

Step 2: For a given out-of-equilibrium action a_1, if for some type θ_1 there exists a second type $\widetilde{\theta}_1$ with $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{\widetilde{\theta}_1}$, then the combination (θ_1, a_1) may be pruned from the continuation game following p_1.

Step 3: Check whether the fixed equilibrium is still sequentially rational given that the seller's belief is restricted to the buyer types who survive from Step 2. If not, then the equilibrium does not survive from D_1.

Given a PBE, if the corresponding equilibrium in all the continuation games following $p_1 \in \mathbb{R}$ survives from D_1, then we say that the PBE survives from D_1.

The effect of applying criterion D_1 in our model is summarized in the following lemma.

Lemma 7 *The equilibrium in the continuation game can not pass criterion D_1 if all buyer types accept $p_1 > l$ or all buyer types reject $p_1 < h$; The equilibrium in the continuation game passes criterion D_1 if all buyer types accept $p_1 \leq l$ or all buyer types reject $p_1 \geq h$.*

Proof of Lemma 7. Part 1: Suppose all buyer types accept $p_1 > l$. Then $x(p_{1,}1) > x(p_1, 0)$ and $x(p_{1,}1) = 1$ without considering the knife-edge case that $\alpha = \gamma^*$. Since $\max\{x(p_{1,}1) - x(p_1, 0)\} = 1$ and all types accept p_1, $p_1 \leq \min_{(d,v_1)}\{v_1 + \delta q^d(h - l)\} = l + \delta q^B(h - l)$ by the definition of cutoff value.

Apply the definition of D_1 in the case that $a_1 = 0$ is the out-of-equilibrium message and form the sets D_{θ_1} and $D^0_{\theta_1}$ for each buyer type θ_1. So $D_{\theta_1} = \{\phi : \phi > x(p_1, 1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0, 1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1, 1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0, 1]\}$. Therefore, for $x(p_{1,}1) = 1$ and $p_1 \in (l, l + \delta q^B(h - l)]$, $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(B,l)}$ for all $\theta_1 \neq (B, l)$. All the combinations $(\theta_1, a_1 = 0)$ with $\theta_1 \neq (B, l)$ are pruned from the game. Given the seller's belief is restricted on type (B, l) after rejection, $x(p_1, 0) = 1$ and it is contradictory to $x(p_{1,}1) > x(p_1, 0)$. So the equilibrium fails to pass criterion D_1.

Part 2: Suppose all buyer types accept $p_1 \leq l$. From Part 1, $D_{\theta_1} = \{\phi : \phi > x(p_1, 1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0, 1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1, 1) + \frac{v_1 - p_1}{\delta q^d(h-l)}, \phi \in [0, 1]\}$.

If $p_1 = l$ and $\alpha < \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B, h), (G, h)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = \{1\}$ for $\theta_1 \in \{(B, l), (G, l)\}$.

If $p_1 = l$ and $\alpha > \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,h),(G,h)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = [0,1]$ for $\theta_1 \in \{(B,l),(G,l)\}$.

If $p_1 < l$ and $\alpha < \gamma^*$, then $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1.

If $p_1 < l$ and $\alpha > \gamma^*$, then either $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1 or $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(G,l)}$. If the latter happens, the seller's belief is restricted on type (G,l) after rejection and she offers $x(p_1,0) = 0$. It is still sequential rational for all buyer types θ_1 to accept $p_1 < l$ given $x(p_1,1) = x(p_1,0) = 0$.

In all the cases above, the equilibrium passes criterion D_1.

Part 3: Suppose all buyer types reject $p_1 < h$. Then $x(p_{1,}0) > x(p_1,1)$ and $x(p_{1,}0) = 1$ without considering the knife-edge case that $\alpha = \gamma^*$. Since $\max\{x(p_{1,}0) - x(p_1,1)\} = 1$ and all types reject p_1, $p_1 \geq \max\limits_{(d,v_1)}\{v_1 - \delta q^d(h-l)\} = h - \delta q^B(h-l)$ by the definition of cutoff value.

Apply the definition of criterion D_1 in the case that $a_1 = 1$ is the out-of-equilibrium message. So $D_{\theta_1} = \{\phi : \phi > x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$. Then for $x(p_{1,}0) = 1$ and $p_1 \in [h - \delta q^B(h-l), h)$, $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(B,h)}$ for all $\theta_1 \neq (B,h)$. All the combinations $(\theta_1, a_1 = 1)$ with $\theta_1 \neq (B,h)$ are pruned from the game. Given the seller's belief is restricted on type (B,h) after acceptance, $x(p_{1,}1) = 1$ and it is contradictory to $x(p_{1,}0) > x(p_1,1)$. So the equilibrium fails to pass Criterion D_1.

Part 4: Suppose all buyer types reject $p_1 \geq h$. From Part 3, $D_{\theta_1} = \{\phi : \phi > x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$ and $D^0_{\theta_1} = \{\phi : \phi = x(p_1,0) + \frac{p_1 - v_1}{\delta q^d(h-l)}, \phi \in [0,1]\}$.

If $p_1 = h$ and $\alpha < \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,l),(G,l)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = \{1\}$ for $\theta_1 \in \{(B,h),(G,h)\}$.

If $p_1 = h$ and $\alpha > \gamma^*$, $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for $\theta_1 \in \{(B,l),(G,l)\}$ and $D_{\theta_1} \cup D^0_{\theta_1} = [0,1]$ for $\theta_1 \in \{(B,h),(G,h)\}$.

If $p_1 > h$ and $\alpha < \gamma^*$, then $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1.

If $p_1 > h$ and $\alpha > \gamma^*$, then either $D_{\theta_1} \cup D^0_{\theta_1} = \emptyset$ for all buyer types θ_1 or $D_{\theta_1} \cup D^0_{\theta_1} \subsetneq D_{(G,h)}$. If the latter case happens, the seller's belief is restricted on type (G,h) after acceptance and $x(p_1,1) = 0$. Then it is still sequential rational for all buyer types θ_1 to reject $p_1 > h$ given $x(p_1,1) = x(p_1,0) = 0$.

In all the cases above, the equilibrium passes criterion D_1. ■

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