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# The valuation of $N$ -phased investment projects under jump-diffusion processes

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## Abstract

In this paper we consider  $N$ -phased investment opportunities where the time evolution of the project value follows a jump-diffusion process. An explicit valuation formula is derived under two different scenarios: in the first case we consider fixed and certain investment costs and in the second case we consider cost uncertainty and assume that investment costs follow a jump-diffusion process.

**Keywords:**  $N$ -fold compound options; sequential investments; jump-diffusion process.

**JEL:** G12; G13; G30; C60.

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# 1 Introduction

As several researchers have noted, R&D investments are essentially real growth options because the value of early projects stems not so much from their expected cash flows as from the follow-up opportunities they may create. At each stage the company may decide to exercise the option or not, that is to continue to invest in the project or to shut it down. This is, for instance, the case of the development of new drugs, which begins with research that leads with some probability to a new compound and which continues with testing and concludes with the construction of a production facility and the marketing of the product. Inventors in this field regularly file applications on a large number of drugs and therapies before knowing whether those drugs will be safe and successful<sup>1</sup>. Given the flexibility and uncertainty involved in such projects, traditional tools fail to capture the value of R&D investments.

In the present paper we consider investment opportunities that are by their nature sequential and where strategically relevant, new information may arrive at each investment stage. This problem can be best modeled as an  $N$ -fold compound option on the commercialization phase where in each of the  $N$  stages the company faces the option of shutting the project down or of continuing its operations, that is, to continue to invest in the project. The arrival of new strategically important information at discrete points in time can be accommodated by modelling the dynamics of the project value as a jump-diffusion process<sup>2</sup>, where the Gaussian diffusion process represents business-as-usual uncertainty and where punctuated jumps at random intervals represent exceptional events such as major project failures or important breakthroughs. Indeed, apart from the obvious market risk, research intensive firms face a number of risks, that, for convenience, we summarize under the heading "rare events". Under the assumption of lognormality of the jump distribution we analytically solve the valuation problem of an  $N$ -staged investment opportunity under two different scenarios. Firstly, we consider the case where investment costs are deterministic and perfectly known at the beginning of the project. Secondly, we consider the case where investment costs are stochastic and unknown at the beginning of the project, but where it is known that they follow a jump-diffusion process.

The paper is organized as follows. Section 2 provides an overview of the related economic litera-

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<sup>1</sup>In the USA, for example, because of FDA regulation, R&D activity for a new drug can be divided into five major phases: (1) discovery and pre-clinical research, (2) phase I clinical trials, (3) phase II clinical trials, (4) phase III clinical trials and (5) regulatory review and approval. Each phase represents an option on a new phase of the process. Therefore, R&D projects can be considered as  $N$ -fold compound options. See, for example, Cassimon et al., (2004) where R&D projects of pharmaceutical companies are valued using 6-fold compound options.

<sup>2</sup>Recent literature argues that jump-diffusion processes better represent the return dynamics of financial and real asset. Such processes may account for fat tails and skewness of probability distributions. See Boyarchenko (2004) for further information.

ture. Section 3 provides a description of the economic model and derives a closed-form solution for a  $N$ -fold compound call option with a mixed jump-diffusion process. An extension to the pricing of a  $N$ -fold compound option where both the underlying project value and the investment cost follow jump-diffusion processes is presented in Section 4. In Section 5, we provide numerical results. The final section concludes the paper.

## 2 Literature review

The literature on the valuation of real investments under jumps is growing quickly. Pennings and Lint (1997) provide a real options model for valuing R&D projects, which assumes a pure jump process for the underlying project value and studies the consequences of this modelling in a real-world investment context. Martzoukos and Trigeorgis (2002) value single stage investment options when the underlying project value follows a log-normal jump-diffusion process involving multiple types of rare events. In this way, they are able to simultaneously represent the discontinuous changes of the project value due to different, unexpected events (i.e., political, technological, competitive etc.). Our model is related to Martzoukos and Trigeorgis's work but we value an  $N$ -staged investment project when the underlying asset undergoes only one class of rare events in each time interval. We also consider the case of investment costs following jump-diffusion dynamics, whereas in Martzoukos and Trigeorgis (2002) investment costs are assumed to be constant. Some papers incorporate different distributions of jump sizes into the valuation problem of real options. Boyarchenko (2004), for example, extends the standard model of irreversible investment under uncertainty to a wide class of jump-diffusion processes, namely Lévy processes. Analytical solutions for (real) option prices with jumps of stochastic size in the underlying asset value are also given by Kou (2002) and Mordecki (2002).

Compound options have been extensively used in the finance literature to evaluate sequential investment opportunities. Geske (1979a) shows that risky securities with sequential payouts can be valued as compound options. Carr (1988) analyzes sequential compound options, of the form of options to acquire subsequent options to exchange an underlying risky asset for another risky asset. Gukhal (2004) derives analytical valuation formulas for 2-fold compound options when the underlying asset follows a log-normal jump-diffusion process. He then applies these results to value extendible options and American call options on stocks that pay continuous and discrete dividends. Using some properties of multivariate normal integrals, the present paper generalizes Gukhal's (2004) result for jump-diffusion compound options to the case of  $N$ -fold compound options and applies the result to the valuation of

sequential investment options in which both the project value and investment cost (i.e., the strike price) follow log-normal jump-diffusion processes. Agliardi and Agliardi (2005) derive a closed-form solution for European-style  $N$ -fold compound call options in the case of time-dependent volatility and interest rate. Their procedure consists of solving  $N$ -nested Black-Scholes partial differential equations<sup>3</sup>. Differently from their approach, we consider a real investment problem and use the risk-neutral argument (Harrison and Kreps, 1979) to calculate the expected present value of the  $N$ -fold compound option and moreover we consider jump-diffusion processes for the underlying values. Lee et. al., (2008) also propose a generalized pricing formula and sensitivity analysis for sequential compound options by using the risk-neutral method but assume that uncertainty is one-dimensional by modeling the underlying value as a geometric Brownian process.

### 3 A valuation formula for sequential investment opportunities

We consider the valuation problem of a risk-neutral venture capitalist who, at time 0, considers to invest in a project whose commercial phase cannot be launched before a pilot phase consisting of  $N$  stages of investment is completed. Let  $T_1$  be the time of the market launch of the product, when, upon bearing the commercialization cost  $I_1$ , the venture capitalist pockets the project value  $V$ . The project payoff at time  $T_1$  is  $\max\{V - I_1, 0\}$  and let  $F_1(V, t)$  denote the value at time  $t$  of this simple investment opportunity. We assume that the commercialization phase is reached upon investing an amount  $I_k$ , at time period  $T_k$ , for  $k = 2, \dots, N$  and with  $T_1 \geq T_2 \geq \dots \geq T_N$ <sup>4</sup>.  $T_N$  is therefore the time period the project starts and  $I_N$  is the start up cost, while  $T_k$  and  $I_k$  are maturities of intermediate phases which lead up to the commercialization phase and their respective investment costs. The  $N$ -staged investment problem may be viewed as a compound option, that is options on options, and its value may be derived in a recursive way. Let us now define a sequence of call options, with value  $F_k$ , on the call option whose value is  $F_{k-1}$ , with exercise price  $I_k$  and expiry date  $T_k$ , for  $k = 2, \dots, N$ . The  $k$ -fold compound option value can be written in a recursive way and its final payoff at the option's maturity date  $T_k$  is given by:

$$F_k(F_{k-1}(V, T_k), T_k) = \max\{F_{k-1}(V, T_k) - I_k, 0\}, \quad (1)$$

<sup>3</sup>In essence, at the first step the underlying option is priced according to the Black-Scholes formula then, compound options are priced as options on the securities whose values have already been found in earlier steps.

<sup>4</sup>Following the capital budgeting literature, we assume that the option's maturities  $T_k$  are deterministic. This is because in many industries (e.g. pharmaceuticals) firm's R&D processes are well-scheduled and follow clear-cut stage-gate process.

for  $k = 2, \dots, N$  and where  $F_{k-1}(V, T_k)$  stands for the value of the underlying compound option at time  $T_k$  and  $I_k$  is the exercise price. According to (1), at time  $T_k$ , the venture capitalist faces the option of investing an amount  $I_k$ , gaining access to stage  $k - 1$  of the project whose value is  $F_{k-1}(V, T_k)$ , or to shut the project down. The option will be exercised if  $F_{k-1}(V, T_k) > I_k$ , that is, if the expected present value of the project at time  $T_k$  exceeds the investment cost.

We assume deterministic investment costs  $I_k$ , for  $k = 1, 2, \dots, N$ , which are perfectly known at time 0; the project value is unknown and uncertain and, denoting by  $V_t$  the time  $t \in [0, T_1]$  evaluation of the project, we assume that  $V_t$  follows a jump-diffusion process:

$$dV_t = \alpha V_t dt + \sigma V_t dz_t + (Y - 1) V_t dq_t, \quad (2)$$

where  $\alpha$  is the drift rate,  $\sigma$  is the volatility of the Brownian part of the process, conditional on no jumps occurring,  $dz$  is a standard Gauss-Wiener process and  $dq$  is a Poisson process with constant intensity  $\lambda (> 0)$ . Therefore,  $dq = 0$  with probability  $1 - \lambda dt$  and  $dq = 1$  with probability  $\lambda dt$ , or, in other words, over a small time period  $dt$ , the probability of a jump in  $V$  is  $\lambda dt$ , where the random variable  $(Y - 1)$  accounts for the relative jump amplitude. The average relative jump size,  $E[Y - 1]$  is denoted by  $K$ , where  $E$  is the expectation operator over the distribution function of  $Y$  under the objective probability measure  $\mathbb{P}$ . We assume that the random variable  $Y$  and the Poisson process  $dq$  are independent of each other and also independent of the Brownian motion  $dz$ .

The project value  $V$  as given by (2) has two sources of uncertainty. The term  $\sigma dz$  corresponds to "business-as-usual" uncertainty, while the term  $dq$  describes rare events. For example, new drugs can turn into mega-selling blockbuster products or alternatively, suffer clinical trial failures and withdrawal from the market. If the Poisson event does not occur ( $dq = 0$ ), then the return dynamics would be identical to those presented by Black and Scholes (1973) and Merton (1973). If, on the other hand the Poisson event occurs, then  $(Y - 1)$  is an impulse function which takes the project value from  $V$  to  $VY$ , where we assume that  $Y$  is drawn from a lognormal distribution with parameter  $(\mu_J, \sigma_J^2)$ <sup>5</sup>. The coefficients  $(\mu_J, \sigma_J^2)$  are constants.

We assume that the venture capitalist achieves risk neutrality by holding a diversified portfolio of activities. In other words, we assume that he invests in activities with negatively correlated risk factors, thereby gaining risk insulation and earning, in expected terms, the exogenously given risk-

<sup>5</sup>That is,  $E[\ln(Y)] = \mu_J$  and  $\text{Var}[\ln(Y)] = \sigma_J^2$ , so  $E[Y] = \exp(\mu_J + \frac{1}{2}\sigma_J^2)$ . We impose this assumption on the distribution of jump sizes since this is the simplest type of model that illustrates the intuition underlying an  $N$ -fold compound options valuation with jumps.

neutral rate of return  $r \geq 0$ . Under this assumption, the risk-neutral project value can be described by the following stochastic differential equation:

$$dV_t = \mu^* V_t dt + \sigma V_t dz_t^* + (Y - 1) V_t dq_t, \quad (3)$$

where  $dz_t^*$  is a standard Wiener process<sup>6</sup>,  $dq$ ,  $Y$  are as above, independently distributed of  $dz_t^*$  and  $\mu^*$  is such that the discounted project value is a martingale under  $\mathbb{Q}$ :

$$\mu^* = r - \lambda K = r - \lambda \left[ \exp \left( \mu_J + \frac{1}{2} \sigma_J^2 \right) - 1 \right].$$

In the sequential investment model, we want to determine the value of the investment opportunity  $F_k(F_{k-1}(V, T_k), T_k)$  at each stage  $T_k$ ,  $k = 2, \dots, N$ , of the project, with  $F_k(F_{k-1}(V, T_k), T_k) = \max\{F_{k-1}(V, T_k) - I_k, 0\}$ , being the boundary condition. Let  $V_k^*$  denote the value of  $V$  such that the underlying option is at the money at time  $T_k$ , i.e.,

$$V_{T_k} = V_k^*$$

where  $V_k^*$  solves:

$$F_{k-1}(V, T_k) - I_k = 0$$

for  $k \geq 2$  and  $V_1^* = I_1$ . Then,

$$F_k[F_{k-1}(V, T_k), T_k] = \begin{cases} F_{k-1}(V, T_k) - I_k & \text{if } V_{T_k} \geq V_k^* \\ 0 & \text{if } V_{T_k} < V_k^* \end{cases}.$$

In other words, if the value of  $V$  at time  $T_k$ , is greater than  $V_k^*$ , the venture capitalist continues to invest in the project, i.e. the compound option will be exercised, while for values less than  $V_k^*$  it will be abandoned. Note that the critical values  $V_k^*$  are determined recursively and their existence and uniqueness are guaranteed in view of the expression of  $F_{k-1}$  (see Remark 2).

Let us define  $n_i$  the number of Poisson arrivals in the time interval  $[T_{i+1}, T_i]$ ,  $i = 1, 2, \dots, N$ , and let

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<sup>6</sup>If through portfolio diversification the venture capitalist replicates the market valuation of the project and if the jump risk is diversifiable, then according to the CAPM  $\alpha = r + \beta\sigma$ , where  $\beta\sigma$  is the risk premium,  $r$  is the risk free interest rate and therefore  $dz^* = dz + \beta dt$ . Note that in such a context the jump risk yields a zero risk premium if it is uncorrelated with the market as a whole, as it is the case, for example, if jumps are due to innovations in technology, actions undertaken by competitors and changes in the firm's strategy. See, for example, Lint and Pennings (1997) and Martzoukos and Trigeorgis (2002).

us set  $T_{N+1} = 0$ . Consequently, let  $s_k = \sum_{i=k}^N n_i$ <sup>7</sup> be the total number of arrivals in the interval  $[0, T_k]$ , for  $k = 1, 2, \dots, N$ . The time interval  $[0, T_1]$  is divided into subintervals of length  $\tau_k = T_k - T_{k+1}$ , for  $k = 1, 2, \dots, N$  with  $\tau_N = T_N$ .

Let  $\sigma_{s_k}^2 = \sigma^2 + \frac{s_k \sigma_J^2}{T_k}$  be the total variance conditional on the occurrence of  $s_k$  jumps in the interval  $[0, T_k]$ , for  $k = 1, 2, \dots, N$ . Moreover, let  $x_t = \ln\left(\frac{V_t}{V_0}\right)$  be the logarithmic return<sup>8</sup>. The correlation between  $x_{T_j}$  and  $x_{T_i}$ , over the overlapping time interval  $T_j < T_i$ , for  $1 \leq i \leq j \leq k$  and  $k = 1, 2, \dots, N$ , conditional on observing  $s_j$  and  $s_i$  jumps, respectively, is:

$$\rho_{s_i s_j} = \frac{\sigma_{s_j} \sqrt{T_j}}{\sigma_{s_i} \sqrt{T_i}}. \quad (4)$$

For any  $k$ ,  $1 \leq k \leq N$ , let  $\Xi_k$  denote a  $k$ -dimensional symmetric correlation matrix with typical element  $\rho_{s_i s_j}$  and unitary elements on the principal diagonal and  $\Xi_1 = 1$ . Let  $\aleph_k(\zeta_k, \dots, \zeta_1; \Theta_k)$  denote the  $k$ -dimensional multinormal cumulative distribution function, with upper limits of integration  $\zeta_k, \dots, \zeta_1$  and correlation matrix  $\Theta_k$ .

Our aim is to obtain a valuation formula for the  $N$ -fold compound option. Let  $V_N^*$  denote the value of  $V$  such that  $F_{N-1}(V, T_N) - I_N = 0$ . Then, if the project value  $V$ , at time  $T_N$ , is greater than the critical value  $V_N^*$ , the venture capitalist should exercise the option, that is to start to invest in the project.

**Proposition 1** *If the project value follows a jump-diffusion process (3), then the expected present value of the  $N$ -staged investment project with final pay-off  $\max\{V - I_1, 0\}$  and with investment costs  $I_k$  at time  $T_k$ ,  $k = 2, \dots, N$  is:*

$$F_N(V, 0) = \prod_{j=1}^N \left[ \sum_{n_j=0}^{\infty} \frac{e^{-\lambda \tau_j} (\lambda \tau_j)^{n_j}}{n_j!} V_0 e^{-\delta_{s_1} T_1} \aleph_N(a_{s_N}, \dots, a_{s_1}; \Xi_N) \right] + \left. - \sum_{j=1}^N \left\{ \prod_{k=j}^N \left[ \sum_{n_k=0}^{\infty} \frac{e^{-\lambda \tau_k} (\lambda \tau_k)^{n_k}}{n_k!} I_j e^{-r T_j} \aleph_{N+1-j}(b_{s_N}, \dots, b_{s_j}; \Xi_{N+1-j}) \right] \right\} \right\}, \quad (5)$$

where:

$$\delta_{s_k} = -\frac{s_k \left( \mu_J + \frac{1}{2} \sigma_J^2 \right)}{T_k} + \lambda \left[ \exp \left( \mu_J + \frac{1}{2} \sigma_J^2 \right) - 1 \right], \text{ for } k = 1, 2, \dots, N,$$

$$b_{s_k} = \frac{\ln \left( \frac{V_0}{V_k^*} \right) + \left( r - \delta_{s_k} - \frac{1}{2} \sigma_{s_k}^2 \right) T_k}{\sigma_{s_k} \sqrt{T_k}},$$

<sup>7</sup>Here the upper limit of the summation refers to the degree of compoundness of the compound option.

<sup>8</sup>The logarithmic return  $x_t$  evolves as:  $dx_t = (r - \lambda K - 0.5\sigma^2) dt + \sigma dz_t^* + \ln(Y) dq_t$ , under the pricing measure  $\mathbb{Q}$ .



$$a_{s_k} = b_{s_k} + \sigma_{s_k} \sqrt{T_k}.$$

**Proof.** See appendix. ■

**Remark 1** *It is easy to see that:*

$$\frac{\partial}{\partial V} F_k = \prod_{j=1}^k \left[ \sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j} (\lambda\tau_j)^{n_j}}{n_j!} e^{-\delta_{s_1} T_1} \mathfrak{N}_k(a_{s_k}, \dots, a_{s_1}; \Xi_k) \right] > 0,$$

and

$$\lim_{V \rightarrow +\infty} F_{k-1}(V, t) = +\infty,$$

and therefore  $V_k^*$  solving  $F_{k-1}(V, T_k) - I_k = 0$  is unique for every  $k$ ,  $1 \leq k \leq N$ .

According to equation (5), the pricing formula has the following interpretation. The price of the jump-diffusion  $N$ -fold compound option can be expressed as the weighted sum of the  $N$ -fold compound option prices where each weight equals the joint probability that a Poisson random variable with constant intensity  $\lambda$  will take on exactly the value  $n_i$  in each time interval  $[T_{i+1}, T_i]$ , for  $i = 1, 2, \dots, N$ .

The expression:

$$\sum_{j=1}^N \left\{ \prod_{k=j}^N \left[ \sum_{n_k=0}^{\infty} \frac{e^{-\lambda\tau_k} (\lambda\tau_k)^{n_k}}{n_k!} \mathfrak{N}_{N+1-j}(b_{s_N}, \dots, b_{s_j}; \Xi_{N+1-j}) \right] \right\},$$

can be interpreted as the joint probability of the multicomponent option expiring in-the-money under the equivalent martingale probability measure, so that the second component in (5) is the present value, computed using risk adjusted probabilities, of the subsequent investment costs<sup>9</sup>. The expression:

$$\prod_{j=1}^N \left[ \sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j} (\lambda\tau_j)^{n_j}}{n_j!} e^{-\delta_{s_1} T_1} \mathfrak{N}_N(a_{s_N}, \dots, a_{s_1}; \Xi_N) \right],$$

can be interpreted as the joint probability that the multicomponent option will be exercised, so that the first component in (5) is the present value of receiving the future cash flows at expiration of the option.

<sup>9</sup>See Lajeri-Chaherli (2002) and Lee et al. (2008, p. 43) for a similar interpretation.

## 4 Sequential investment opportunities and stochastic investment cost

R&D projects often involve considerable cost uncertainty. For example, jumps in the investment cost can be especially important in the development of a new drug by a pharmaceutical company. When a company discovers a new therapeutic target, it has to start a new project, eventually abandoning the current one, with an increase in the investment cost. On the other hand, technological progress can lead to sharp investment cost reductions. In this section we extend the previous model assuming that the investment cost varies over time<sup>10</sup> and that its dynamics are governed by a geometric jump-diffusion process.

Using the same notation as in section 3, the project payoff at time  $T_1$ , the time of market launch, is  $\max\{V - I, 0\}$  where  $V$  and  $I$  are the underlying value and investment cost; let  $W_1(V, I, t)$  be the value at time  $t$  of this simple investment opportunity. The investor observes two random processes  $V$  and  $I$  and must decide at each stage  $T_k$ , for  $k = 1, 2, \dots, N$  and  $T_1 \geq T_2 \geq \dots \geq T_N$ , whether to continue to invest in the project, that is access stage  $k - 1$  of the project, or not. We accordingly define a sequence of call options, with value  $W_k$ , whose underlying value is  $W_{k-1}$ , and with exercise price (i.e. investment cost)  $I_{T_k} p_k$  and expiry date  $T_k$ ,  $p_k > 0$ , for  $k = 2, \dots, N$ , and  $p_1 = 1$ . Note that  $p_k$  is a scaling factor that allows for greater flexibility in the modelling of investment costs. The  $k$ -fold compound option value can be written in a recursive way and its payoff at the option's maturity date  $T_k$  is given by:

$$W_k(W_{k-1}(V, I, T_k), I p_k, T_k) = \max\{W_{k-1}(V, I, T_k) - p_k I_{T_k}, 0\},$$

for  $k = 2, \dots, N$  and where  $W_{k-1}(V, I, T_k)$  stands for the value of the underlying compound option at time  $T_k$ . Thus, at time  $T_k$ , the venture capitalist faces the option of investing an amount  $I_{T_k} p_k$ , and therefore entering stage  $k - 1$  of the project whose value is  $W_{k-1}(V, I, T_k)$ , or to shut the project down. Notice that the option at time  $T_1$  can be viewed as a simple exchange option<sup>11</sup> where the delivery asset is  $I$  and the optioned asset is  $V$  and the phased investment problem can be viewed as a compound exchange problem (see, for example, Carr, 1988).

Under the risk-neutral martingale measure  $\mathbb{Q}$ , the dynamics of the underlying assets are given by

<sup>10</sup>See also Wu and Yen (2007) and Cortellezzi and Villani (2009).

<sup>11</sup>See Lindset (2007) for the pricing of exchange options under jump-diffusion processes.

the following jump-diffusion processes<sup>12</sup>:

$$dV_t = (r - \lambda_1 K_1) V_t dt + \sigma_1 V_t dz_{1,t}^* + (Y_1 - 1) V_t dq_{1,t}, \quad (6)$$

$$dI_t = (r - \lambda_2 K_2) I_t dt + \sigma_2 I_t dz_{2,t}^* + (Y_2 - 1) I_t dq_{2,t}, \quad (7)$$

$\sigma_1$  and  $\sigma_2$  are the respective standard deviations, conditional on no jumps and  $dz_1^*$  and  $dz_2^*$  are standard Brownian motions, under the risk-neutral measure  $\mathbb{Q}$ . The process  $dq_1$  and  $dq_2$  are Poisson random variables with constant rates  $\lambda_1$  and  $\lambda_2$ , respectively, counting the number of jumps. The sizes  $Y_1$  and  $Y_2$  are random variables and it is assumed that  $Y_1$  ( $Y_2$ ) is lognormally distributed with mean  $\mu_{1,J}$  ( $\mu_{2,J}$ ) and variance  $\sigma_{1,J}^2$  ( $\sigma_{2,J}^2$ ).  $K_1$  and  $K_2$  are the average relative jump sizes  $E[Y_1 - 1]$  and  $E[Y_2 - 1]$ , respectively. We assume that the Poisson processes  $dq_1$  and  $dq_2$  and the jump components  $Y_1$  and  $Y_2$  are independent of each other and also independent of the Brownian motions  $dz_1^*$  and  $dz_2^*$ . Finally, we assume that Brownian motion components are correlated, with correlation coefficient  $\varphi_{12}$ , i.e.,  $\text{corr}[dz_1^*, dz_2^*] = \varphi_{12} dt$ .

Let us define by  $V^c$  the price ratio of  $V$  to  $I$ . This allows us to write<sup>13</sup>:

$$\max \{W_{k-1}(V, I, T_k) - p_k I_{T_k}, 0\} = I_{T_k} \cdot \max \{W_{k-1}^c(V^c, T_k) - p_k, 0\},$$

where  $I_{T_k}$  is the numeraire and  $W_{k-1}^c(V^c, T_k) = W_{k-1}(V^c, 1, T_k)$ . Let us denote by  $V_k^{c*}$  the critical price ratio such that the underlying option is at the money at time  $T_k$ , i.e.,  $V_{T_k}^c = V_k^{c*}$ , where  $V_k^{c*}$  solves:

$$W_{k-1}^c(V^c, T_k) - p_k = 0,$$

for  $k \geq 2$  and  $V_1^{c*} = p_1 = 1$ . If the value of  $V^c$  at time  $T_k$ , is greater than the threshold  $V_k^{c*}$ , then the venture capitalist continues to invest in the project, i.e. the compound option will be exercised, while for values less than  $V_k^{c*}$  it will be abandoned.

Let us define  $n_i$  and  $m_i$  the number of event occurrences, respectively, for the project value and investment cost in the time interval  $[T_{i+1}, T_i]$ ,  $i = 1, 2, \dots, N$ . Again,  $T_{N+1}$  is set to zero. Consequently, let  $s_{1,k} = \sum_{i=k}^N n_i$  and  $s_{2,k} = \sum_{i=k}^N m_i$  be the total number of arrivals in the interval  $[0, T_k]$ , for  $k =$

<sup>12</sup>Also in this case we assume that the venture capitalist is diversified, that is that he keeps a portfolio of activities which allows him to value activities in a risk-neutral way.

<sup>13</sup>Given the above mentioned properties of  $V$  and  $I$  it can be shown that the homogeneity theorem holds where  $W_{k-1}(\theta V, \theta I, T_k) = \theta W_{k-1}(V, I, T_k)$  for  $\theta \geq 0$ . See for example Carr (1988) and Geman, El Karoui and Rochet (1995).

1, 2, ..., N. For the following we set  $\tau_k = T_k - T_{k+1}$ , with  $\tau_N = T_N$ .

Let  $\sigma_{s_{1,k}s_{2,k}}^2 = \sigma_c^2 + \frac{s_{1,k}\sigma_{1,J}^2 + s_{2,k}\sigma_{2,J}^2}{T_k}$  be the total variance of a percentage change in the price ratio  $V^c$ , conditional on the occurrence of  $s_{1,k}$  and  $s_{2,k}$  jumps in the time period  $[0, T_k]$  and  $\sigma_c^2 = \sigma_1^2 + 2\sigma_2\sigma_1\varphi_{12} + \sigma_2^2$ . Moreover, let  $x_t^c = \ln\left(\frac{V_t^c}{V_0^c}\right)$  be the logarithmic return<sup>14</sup>. The correlation coefficient between the logarithmic returns  $x_{T_j}^c$  and  $x_{T_i}^c$  over the overlapping time interval  $T_j < T_i$ , for  $1 \leq i \leq j \leq k$  and  $k = 1, 2, \dots, N$ , conditional on the random event occurrences, is:

$$\rho_{s_{1,i}s_{2,j}} = \frac{\sigma_{s_{1,j}s_{2,j}}\sqrt{T_j}}{\sigma_{s_{1,i}s_{2,i}}\sqrt{T_i}},$$

For any  $k$ ,  $1 \leq k \leq N$ , let  $\Phi_k$  denote a  $k$ -dimensional symmetric correlation matrix with typical element  $\rho_{s_{1,i}s_{2,j}}$ .

**Proposition 2** *If the project value  $V$  and the investment cost  $I$  follow jump-diffusion processes (6) and (7), respectively, then the expected present value of a  $N$ -staged investment project with final pay-off  $\max\{V - I, 0\}$  and with investment cost  $p_k I_{T_k}$  at time  $T_k$ ,  $k = 2, \dots, N$ , is:*

$$W_N(V, I, 0) = \prod_{j=1}^N \left[ \sum_{n_j=0}^{\infty} \sum_{m_j=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)\tau_j} (\lambda_1\tau_j)^{n_j} (\lambda_2\tau_j)^{m_j}}{n_j!m_j!} V_0 e^{-\delta_{s_{1,1}}T_1} \aleph_N(c_{s_{1,N}s_{2,N}}, \dots, c_{s_{1,1}s_{2,1}}; \Phi_N) \right] + \left. - \sum_{j=1}^N \left\{ \prod_{k=j}^N \left[ \sum_{n_k=0}^{\infty} \sum_{m_k=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)\tau_k} (\lambda_1\tau_k)^{n_k} (\lambda_2\tau_k)^{m_k}}{n_k!m_k!} I_0 p_j e^{-\delta_{s_{2,j}}T_j} \aleph_{N+1-j}(d_{s_{1,N}s_{2,N}}, \dots, d_{s_{1,j}s_{2,j}}; \Phi_{N+1-j}) \right] \right\} \right\} \quad (8)$$

where:

$$\delta_{s_{i,k}} = -\frac{s_{i,k}(\mu_{i,J} + \frac{1}{2}\sigma_{i,J}^2)}{T_k} + \lambda_i \left[ \exp\left(\mu_{i,J} + \frac{1}{2}\sigma_{i,J}^2\right) - 1 \right], \text{ for } k = 1, 2, \dots, N, \text{ and } i = 1, 2,$$

$$d_{s_{1,k}s_{2,k}} = \frac{\ln\left(\frac{V_k^c}{V_k^{c*}}\right) + \left(\delta_{s_{2,k}} - \delta_{s_{1,k}} - \frac{1}{2}\sigma_{s_{1,k}s_{2,k}}^2\right)T_k}{\sigma_{s_{1,k}s_{2,k}}\sqrt{T_k}},$$

$$c_{s_{1,k}s_{2,k}} = d_{s_{1,k}s_{2,k}} + \sigma_{s_{1,k}s_{2,k}}\sqrt{T_k}.$$

**Proof.** See appendix. ■

(8) can be seen as the weighted sum of the multicomponent exchange option values where each weight equals the joint probability that two Poisson random variables with rates  $\lambda_1$  and  $\lambda_2$  will take on exactly the value  $n_i$  and  $m_i$ , respectively, in each time interval  $[T_{i+1}, T_i]$ , for  $i = 1, 2, \dots, N$ . The

<sup>14</sup>The logarithmic return  $x_t^c$  evolves as:  $dx_t^c = (\lambda_2 K_2 - \lambda_1 K_1 - 0.5\sigma_c^2) dt + \sigma_c dz_t^c + \ln(Y_1) dq_{1,t} - \ln(Y_2) dq_{2,t}$ , under the new risk-neutral measure  $\tilde{\mathbb{Q}}$ . See Appendix for further details.

first component in (8) can be seen as the present value, computed using risk adjusted probabilities, of receiving the future cash flows at expiration of the option. The second component can be seen as the present value of the investment costs.

## 5 Simulation results

In this section we provide some numerical results on multicomound options. In the first part of this section we describe how the model developed in Section 3 can be implemented numerically<sup>15</sup> and afterwards we provide some numerical results. Following Amin (1993) we approximate the continuous time model by a jump-diffusion process in discrete time (see also Martzoukos and Trigeorgis, 2002 and Xu et al., 2003) and we calculate option prices evaluating the expected discounted pay-off of the option using dynamic programming. In particular, we first construct the state space and calculate the corresponding project values, and then value (compound-) options in a recursive way using Markov-transition rates.

Consider first the option on the project value  $V$  with expiry date  $T_1$ . Given the trading period  $[0, T_1]$ , the time interval is divided into  $M$  subintervals of length  $h_M = \frac{T_1}{M}$ , where we define  $M_1 = \frac{T_1}{h_M}$  and thus  $M_1 = M$ . At each time period  $m$ , the state space is a grid, where each consecutive point is spaced  $\sigma\sqrt{h_M}$  apart. Between two consecutive time periods, the grid is shifted upwards by  $\gamma_M = (r - .5\sigma^2) h_M$ . The project's value at time  $m h_M$  if in state  $i$  is then given by:

$$V_i(m) = V(0) e^{m\gamma_M + i\sigma\sqrt{h_M}}$$

The project value can undergo local changes  $i = \pm 1$ , representing the diffusion part, or jumps for  $i \neq \pm 1$ , where the project value can jump possibly to any state in the state space. In order to value options we use Markov transition probabilities within a finite difference scheme. Local changes,  $i = \pm 1$ , have probabilities approximately equal to  $\frac{1}{2}$  (see Amin, 1993), i.e.

$$p_{\pm 1} = \Pr \left\{ \ln [V(t + dt)] - \ln [V(t)] = \gamma_M \pm \sigma\sqrt{h_M} \right\} \simeq 0.5$$

The probability of observing a jump is  $\bar{\lambda}_M = \lambda h_M e^{-\lambda h_M}$ , and, given that a jump occurs, the probability

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<sup>15</sup>Once the change of numeraire has been made, the model described in Section 4 can be implemented in a very similar way.

of observing a jump of size  $l > 1$ , or  $l < -1$  is:

$$p_l = \Pr \{ \ln [V(t+dt)] - \ln [V(t)] = \gamma_M + l\sigma\sqrt{h_M} \} = \\ \aleph [(l + 0.5\sigma)\sqrt{h_M}] - \aleph [(l - 0.5\sigma)\sqrt{h_M}]$$

where  $\aleph(\bullet)$  is the cumulative normal distribution function with mean  $\mu_J$  and variance  $\sigma_J^2$  and where, for simplicity's sake, we set  $\mu_J = -\frac{1}{2}\sigma_J^2$ .

The option value (European style)  $F_1(i, m)$  is calculated in a recursive way, using dynamic programming:

$$F_1(i, m) = e^{-r h_M} \left\{ (1 - \bar{\lambda}_M) \frac{1}{2} [F_1(i+1, m+1) + F_1(i-1, m+1)] + \right. \\ \left. \bar{\lambda}_M [\sum_{l=2}^{\infty} p_l F_1(i+l, m+1) + \sum_{l=2}^{\infty} p_{-l} F_1(i-l, m+1)] \right\} \quad (9)$$

with  $F_1(i, M_1) = f_1[V_i(M_1)] = \max\{V_i(M_1) - I_1, 0\}$  being the pay-off at the expiry date (i.e. boundary condition). Given the properties of the normal distribution, the sums in (9) can be suitably truncated. In the simulations below option values are computed using a 75-nomial scheme.

Compound options can be evaluated in a similar vein. Given  $T_2 < T_1$  the expiration of the compound option, let the pay-off at the expiry date be a function of the underlying option at that date, i.e.  $F_2(i, M_2) = f_2[F_1(i, M_2)] = \max\{F_1(i, M_2) - I_2, 0\}$ , where  $M_2 = \frac{T_2}{h_M}$ . Following (9), the option value can be calculated recursively. This method extends straightforwardly to the case of  $N$ -fold compound options.

In Table 1 we test the accuracy of the procedure by evaluating a 2-fold compound option (call on call option) where a closed form solution is available. Table 1 shows that the algorithm approximates well the closed form solution.

[Insert Table 1]

In Table 2 we provide simulations for a 3-fold compound option (call on call on call option) with strike prices  $I_3 = 5$ ,  $I_2 = 10$  and  $I_1 = 100$  for values of  $\lambda = \{0.4, 0.6, 0.8, 1\}$ . Exercise dates are  $T_3 = 0.2$ ,  $T_2 = 0.35$ , and  $T_1 = 0.5$ . Since by increasing  $\lambda$  we also increase the underlying's total volatility, option prices increase; hence, increasing the average number of jumps (per year) increases the option's value.

[Insert Table 2]

## 6 Conclusion

Phased investments have the property that much of the value of the investment is associated with future cash flows that are contingent on intermediate decisions. Because of this property the analysis of sequential investment projects is one of the most difficult problems. Starting from the difficulty of traditional DCF methods to capture the value of early-stage investments, the real options literature provides advanced models, each focusing on different R&D characteristics. In the present paper we value R&D projects with the following characteristics: 1) two types of uncertainty, i.e., continuous and discontinuous variations in the underlying values, 2) project value and investment cost uncertainties, and 3) compoundness of R&D projects.

## 7 Appendix

### 7.1 Proof of Proposition 1

Under the martingale approach, the value at time 0 of the European  $N$ -fold compound option is given by the following expectation under the risk-neutral measure:

$$F_N(V, 0) = e^{-rT_N} E_0^{\mathbb{Q}} \{ \max [F_{N-1}(V, T_N) - I_N, 0] \}, \quad (10)$$

where  $F_{N-1}$  indicates the value of an  $(N-1)$ -fold compound option and  $V$  is the project value at the maturity date  $T_N$ . The expectation in (10) is in general difficult to solve due to jumps in the project value. We address this problem by conditioning on the random event occurrence, and work with the conditional variable thereafter. Thus:

$$F_N(V, 0) = e^{-rT_N} \sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} E_0^{\mathbb{Q}} \{ \max [F_{N-1}(V, T_N) - I_N, 0] \mid n_N \}. \quad (11)$$

We know that  $F_{N-1}(V, T_N)$  is given by:

$$\begin{aligned} & \prod_{j=1}^{N-1} \left[ \sum_{n_j=0}^{\infty} \frac{e^{-\lambda \tau_j} (\lambda \tau_j)^{n_j}}{n_j!} V_{T_N} e^{-(\delta_{s_1} T_1 - \delta_{s_N} T_N)} \mathfrak{N}_{N-1} \left( a_{s_{N-1}}(V, T_N), \dots, a_{s_1}(V, T_N); \hat{\Xi}_{N-1} \right) \right] + \\ & - \sum_{j=1}^{N-1} \left\{ \prod_{k=j}^{N-1} \left[ \sum_{n_k=0}^{\infty} \frac{e^{-\lambda \tau_k} (\lambda \tau_k)^{n_k}}{n_k!} I_j e^{-r(T_j - T_N)} \mathfrak{N}_{N-j} \left( b_{s_{N-1}}(V, T_N), \dots, b_{s_j}(V, T_N); \hat{\Xi}_{N-j} \right) \right] \right\}, \end{aligned}$$

where:

$$b_{s_k}(V, T_N) = \frac{\ln\left(\frac{V_{T_N}}{V_k^*}\right) + (r - \delta_{s_k} - \frac{1}{2}\sigma_{s_k}^2)T_k - (r - \delta_{s_N} - \frac{1}{2}\sigma_{s_N}^2)T_N}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_N}^2 T_N}},$$

$$a_{s_k}(V, T_N) = b_{s_k}(V, T_N) + \sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_N}^2 T_N},$$

and  $\hat{\Xi}_k$  is a  $k$ -dimensional symmetric correlation matrix with typical element:

$$\hat{\rho}_{s_i s_j} = \sqrt{\frac{\sigma_{s_j}^2 T_j - \sigma_{s_N}^2 T_N}{\sigma_{s_i}^2 T_i - \sigma_{s_N}^2 T_N}}.$$

Moreover, the project value at time  $T_N$  under the risk-neutral probability  $\mathbb{Q}$  and conditioned on  $n_N$  jumps in the interval  $[0, T_N]$ , is:

$$V_{T_N} = V_0 e^{(r - \delta_{s_N} - \frac{1}{2}\sigma_{s_N}^2)T_N + \sigma_{s_N} \sqrt{T_N} \cdot \xi},$$

where  $\xi$  has a standard Gaussian probability law under  $\mathbb{Q}$ . Therefore, the value at time 0 of the European  $N$ -fold compound option is:

$$F_N(V, 0) = e^{-rT_N} \times$$

$$\left\{ \prod_{j=1}^N \left\{ \sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j} (\lambda\tau_j)^{n_j}}{n_j!} \int_{u_N}^{+\infty} n(u) \left[ \phi_{s_N}(u) e^{-(\delta_{s_1} T_1 - \delta_{s_N} T_N)} \mathfrak{N}_{N-1}(\hat{a}_{s_{N-1}}, \dots, \hat{a}_{s_1}; \hat{\Xi}_{N-1}) du \right] \right\} + \right.$$

$$\left. - \sum_{j=1}^{N-1} \left\{ \prod_{k=j}^{N-1} \left\{ \sum_{n_k=0}^{\infty} \frac{e^{-\lambda\tau_k} (\lambda\tau_k)^{n_k}}{n_k!} \int_{u_N}^{+\infty} n(u) \left[ I_j e^{-r(T_j - T_N)} \mathfrak{N}_{N-j}(\hat{b}_{s_{N-1}}, \dots, \hat{b}_{s_j}; \hat{\Xi}_{N-j}) \right] du \right\} \right\} + \right.$$

$$\left. - \sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} \int_{u_N}^{+\infty} n(u) I_N du \right\}, \quad (12)$$

where  $n(\cdot)$  is the normal density function,  $\hat{a}_{s_k} = a_{s_k}(\phi_{s_N}(u), T_N)$ ,  $\hat{b}_{s_k} = b_{s_k}(\phi_{s_N}(u), T_N)$  for  $k = 1, 2, \dots, N-1$ , the function  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  is given by:

$$\phi_{s_N}(u) = V_0 e^{(r - \delta_{s_N} - \frac{1}{2}\sigma_{s_N}^2)T_N + \sigma_{s_N} \sqrt{T_N} \cdot u},$$

and, finally, the constant  $u_N$  is defined implicitly by the equation:

$$u_N = \inf \{u \in \mathbb{R} \mid F_{N-1}[\phi_{s_N}(u), T_N] \geq I_N\}.$$



Straightforward calculations yield:

$$\hat{a}_{s_k} = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} + \frac{1}{2}\sigma_{s_k}^2\right)T_k - \sigma_{s_N}^2 T_N + \sigma_{s_N}\sqrt{T_N} \cdot u}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_N}^2 T_N}},$$

$$\hat{b}_{s_k} = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} - \frac{1}{2}\sigma_{s_k}^2\right)T_k + \sigma_{s_N}\sqrt{T_N} \cdot u}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_N}^2 T_N}},$$

for  $k = 1, \dots, N - 1$ . The last term in (12) can be written in the form:

$$e^{-rT_N} \sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} (\lambda T_N)^{n_N}}{n_N!} I_N \mathfrak{N}_1(b_{s_N}).$$

Using (4) and rearranging terms, it follows that:

$$\hat{\rho}_{s_i s_j} = \frac{(\rho_{s_i s_j} - \rho_{s_i s_N} \rho_{s_j s_N})}{\sqrt{(1 - \rho_{s_i s_N}^2)(1 - \rho_{s_j s_N}^2)}},$$

for  $1 \leq i < j \leq N$ . Therefore, we substitute each term  $\hat{\rho}_{s_i s_j}$  in the matrix  $\hat{\Xi}_{N-j}$  with  $\frac{\rho_{s_i s_j} - \rho_{s_i s_N} \rho_{s_j s_N}}{\sqrt{(1 - \rho_{s_i s_N}^2)(1 - \rho_{s_j s_N}^2)}}$ .

The second term in (12) can be written in terms of the  $N$ -dimensional multinormal cumulative distribution function by applying the following Lemma.

**Lemma 1** *Let  $1 \leq k < N$ , and let  $\tilde{\Xi}_k$  be the matrix obtained from  $\hat{\Xi}_k$  by replacing any element  $\hat{\rho}_{s_i s_j}$  with  $(\rho_{s_i s_j} - \rho_{s_i s_N} \rho_{s_j s_N}) / \sqrt{(1 - \rho_{s_i s_N}^2)(1 - \rho_{s_j s_N}^2)}$ , by setting:*

$$\alpha_{s_k} = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r - \delta_{s_k} - \frac{1}{2}\sigma_{s_k}^2\right)T_k}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_N}^2 T_N}} \text{ and } \beta_{s_k} = \frac{\sigma_{s_N}\sqrt{T_N}}{\sqrt{\sigma_{s_k}^2 T_k - \sigma_{s_N}^2 T_N}},$$

for  $k = 1, \dots, N - 1$ , where  $\alpha_{s_k}$  and  $\beta_{s_k}$  are real numbers, the following identity holds:

$$\int_{-\infty}^{b_{s_N}} n(u) \mathfrak{N}_k\left(\alpha_{s_{N-1}} + u\beta_{s_{N-1}}, \dots, \alpha_{s_{N-k}} + u\beta_{s_{N-k}}; \tilde{\Xi}_k\right) du = \mathfrak{N}_{k+1}(b_{s_N}, \dots, b_{s_{N-k}}; \Xi_{k+1}). \quad (13)$$

**Proof.** It follows by setting  $\frac{b_{s_k}}{\sqrt{1 - \rho_{s_k s_N}^2}} = \alpha_{s_k}$  and  $\frac{-\rho_{s_k s_N}}{\sqrt{1 - \rho_{s_k s_N}^2}} = \beta_{s_k}$ ,  $k = 1, 2, \dots, N - 1$ , and substituting into (13). Then, the second expression of (13) is obtained by using the definition of the standard multivariate normal distribution. ■

Finally, we can write the first term in (12) in terms of the cumulative multivariate normal distribution using Lemma 1, after making the following substitution  $x = u - \sigma_{s_N} \sqrt{T_N}$ .

## 7.2 Proof of Proposition 2

In the proof we apply a change of numeraire<sup>16</sup>. To establish the proposition we need to calculate the dynamics of the process  $V^c = \frac{V}{I}$  under the new risk-neutral measure  $\tilde{\mathbb{Q}}$ . First, we determine  $dV = d\left(\frac{V}{I}\right)$  by applying Itô's Lemma. Computing the derivatives:

$$\begin{aligned} \frac{\partial V^c}{\partial t} &= 0; & \frac{\partial V^c}{\partial V} &= \frac{1}{I}; & \frac{\partial V^c}{\partial I} &= -\frac{V}{I^2}; \\ \frac{\partial^2 V^c}{\partial V^2} &= 0; & \frac{\partial^2 V^c}{\partial I^2} &= \frac{2V}{I^3}; & \frac{\partial^2 V^c}{\partial V \partial I} &= -\frac{1}{I^2}; \end{aligned}$$

and by substituting into Itô's formula:

$$\begin{aligned} dV^c &= \frac{\partial V^c}{\partial t} dt + \frac{\partial V^c}{\partial V} dV + \frac{\partial V^c}{\partial I} dI + \frac{1}{2} \left[ \frac{\partial^2 V^c}{\partial V^2} (dV)^2 + 2 \frac{\partial^2 V^c}{\partial V \partial I} dV dI + \frac{\partial^2 V^c}{\partial I^2} (dI)^2 \right] \\ &= \frac{1}{I} [(r - \lambda_1 K_1) V dt + \sigma_1 V dz_1^* + (Y_1 - 1) V dq_1] - \frac{V}{I^2} [(r - \lambda_2 K_2) I dt + \sigma_2 I dz_2^* + (Y_2 - 1) I dq_2] + \\ &\quad \frac{1}{2} \left\{ -\frac{2}{I^2} [(r - \lambda_1 K_1) V dt + \sigma_1 V dz_1^* + (Y_1 - 1) V dq_1] [(r - \lambda_2 K_2) I dt + \sigma_2 I dz_2^* + (Y_2 - 1) I dq_2] + \right. \\ &\quad \left. \frac{2V}{I^3} [(r - \lambda_2 K_2) I dt + \sigma_2 I dz_2^* + (Y_2 - 1) I dq_2]^2 \right\}. \end{aligned}$$

Neglecting all terms of order  $(dt)^{3/2}$ ,  $(dt)^2$  and above, leads to:

$$dV^c = (r - \hat{\delta}) V^c dt + \sigma_1 V^c dz_1^* - \sigma_2 V^c dz_2^* + (Y_1 - 1) V^c dq_1 - (Y_2 - 1) V^c dq_2, \quad (14)$$

where  $\hat{\delta} = r + \lambda_1 K_1 - \lambda_2 K_2 + \sigma_2^2 - \sigma_1 \sigma_2 \varphi_{12}$ .

Applying the log-transformation for  $I_t$ , under the risk-neutral measure  $\mathbb{Q}$ , it results that:

$$\begin{aligned} I_t &= I_0 \exp \left\{ \left( r - \lambda_2 K_2 - \frac{\sigma_2^2}{2} \right) t + \sigma_2 z_{2,t}^* + \sum_{i=1}^{q_{2,t}} \ln(Y_{2,i}) \right\} \\ &= I_0 \exp \{rt\} \cdot \exp \left\{ -\frac{\sigma_2^2}{2} t + \sigma_2 z_{2,t}^* - \lambda_2 K_2 t + \sum_{i=1}^{q_{2,t}} \ln(Y_{2,i}) \right\}. \end{aligned} \quad (15)$$

<sup>16</sup>See, for example, Geman, El Karoui and Rochet (1995).

In (15), we can interpret the expression:

$$\exp \left\{ -\frac{\sigma_2^2}{2}t + \sigma_2 z_{2,t}^* - \lambda_2 K_2 t + \sum_{i=1}^{q_{2,t}} \ln(Y_{2,i}) \right\},$$

as the Radon-Nikodym derivative of some equivalent measure  $\tilde{\mathbb{Q}}$  with respect to  $\mathbb{Q}$ , since it satisfies the condition:

$$E^{\mathbb{Q}} \exp \left\{ -\frac{\sigma_2^2}{2}t + \sigma_2 z_{2,t}^* - \lambda_2 K_2 t + \sum_{i=1}^{q_{2,t}} \ln(Y_{2,i}) \right\} = 1,$$

for all  $t \geq 0$ . Set:

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp \left\{ -\frac{\sigma_2^2}{2}t + \sigma_2 z_{2,t}^* - \lambda_2 K_2 t + \sum_{i=1}^{q_{2,t}} \ln(Y_{2,i}) \right\},$$

hence, by simple substitution in (15) we can write:

$$I_t = I_0 \exp \{rt\} \cdot \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}.$$

By using the Girsanov's theorem, the process:

$$d\tilde{z}_2 = dz_2^* - \sigma_2 dt, \tag{16}$$

is a Brownian motion under the new risk-neutral measure  $\tilde{\mathbb{Q}}$ . We, therefore, can write  $dz_1^*$  as:

$$dz_1^* = \varphi_{12} dz_2^* + \sqrt{1 - \varphi_{12}^2} dz_3^*, \tag{17}$$

where  $dz_3^*$  is a Brownian motion independent of  $dz_2^*$  under the measure  $\mathbb{Q}$ . By using equations (16) and (17), we can now rewrite the evolution of the asset  $V^c$  under the new risk-neutral measure  $\tilde{\mathbb{Q}}$ :

$$dV^c = (\lambda_2 K_2 - \lambda_1 K_1) V^c dt + \sigma_c V^c dz^c + (Y_1 - 1) V^c dq_1 - (Y_2 - 1) V^c dq_2,$$

with the definitions  $\sigma_c = \sqrt{\sigma_1^2 + 2\sigma_2\sigma_1\varphi_{12} + \sigma_2^2}$  and  $\sigma_c dz^c = (\varphi_{12}\sigma_1 - \sigma_2) d\tilde{z}_2 + \sigma_1\sqrt{1 - \varphi_{12}^2} dz_3^*$  and where  $dz^c$  is a Brownian motion under the risk-neutral measure  $\tilde{\mathbb{Q}}$ .

Given that the jump sizes  $Y_1$  and  $Y_2$  are lognormally distributed with parameters  $(\mu_{1,J}, \sigma_{1,J}^2)$  and  $(\mu_{2,J}, \sigma_{2,J}^2)$ , respectively, by applying the log-transformation for process  $V^c$  allows us to obtain the

explicit value of  $V^c$  under the risk-neutral measure  $\tilde{\mathbb{Q}}$ :

$$V_t^c = V_0^c \exp \left\{ \left( \lambda_2 K_2 - \lambda_1 K_1 - \frac{1}{2} \sigma_c^2 \right) t + \sigma_c z_t^c + \sum_{i=1}^{q_{1,t}} \ln(Y_{1,i}) - \sum_{i=1}^{q_{2,t}} \ln(Y_{2,i}) \right\}.$$

Therefore, the terminal price at time  $t$  under the risk-neutral pricing measure  $\tilde{\mathbb{Q}}$  and conditioned on the number of jumps  $n$  and  $m$  for the project value and investment cost in the time interval  $[0, t]$ , respectively, is:

$$\begin{aligned} V_t^c &= V_0^c \exp \left\{ \left( \lambda_2 K_2 - \lambda_1 K_1 - \frac{1}{2} \sigma_c^2 \right) t + \sigma_c z_t^c + \sum_{i=1}^n \ln(Y_{1,i}) - \sum_{i=1}^m \ln(Y_{2,i}) \right\} \\ &= V_0^c e^{(\delta_{m,2} - \delta_{n,1} - \frac{1}{2} \sigma_{n,m}^2) t + \sigma_{n,m} z_t^c}, \end{aligned}$$

where:

$$\delta_{i,j} = -\frac{i \left( \mu_{j,J} + \frac{1}{2} \sigma_{j,J}^2 \right)}{t} + \lambda_j \left[ \exp \left( \mu_{j,J} + \frac{1}{2} \sigma_{j,J}^2 \right) - 1 \right]$$

and where  $\sigma_{n,m}^2 = \sigma_c^2 + \frac{n\sigma_{1,J}^2 + m\sigma_{2,J}^2}{t}$ .

Let  $I_{T_N} = I_0 e^{rT_N} \cdot \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}$  be the numeraire. Under the martingale approach, the value at time 0 of the European  $N$ -fold compound exchange option is given by the following expectation under the risk-neutral measure:

$$W_N^c(V^c, 0) = e^{-rT_N} E_0^{\tilde{\mathbb{Q}}} \left\{ I_{T_N} \cdot \max [W_{N-1}^c(V^c, T_N) - p_N, 0] \right\},$$

By conditioning on the number of jumps in the interval  $[0, T_N]$ , we obtain:

$$\begin{aligned} W_N^c(V^c, 0) &= \sum_{n_N=0}^{\infty} \sum_{m_N=0}^{\infty} \frac{e^{-(\lambda_1 + \lambda_2)T_N} (\lambda_1 T_N)^{n_N} (\lambda_2 T_N)^{m_N}}{n_N! m_N!} I_0 e^{-\delta_{s_{2,N}} T_N} \times \\ &E_0^{\tilde{\mathbb{Q}}} \left\{ \max [W_{N-1}^c(V^c, T_N) - p_N, 0] \mid n_N, m_N \right\}, \end{aligned} \tag{18}$$

where we know that  $W_{N-1}^c(V^c, T_N)$  is given by:

$$\begin{aligned} & \prod_{j=1}^{N-1} \left[ \sum_{n_j=0}^{\infty} \sum_{m_j=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)\tau_j} (\lambda_1\tau_j)^{n_j} (\lambda_2\tau_j)^{m_j}}{n_j!m_j!} \times \right. \\ & \quad \left. V_{T_N}^c e^{-(\delta_{s_{1,1}}T_1 - \delta_{s_{1,N}}T_N)} \mathfrak{N}_{N-1} \left( c_{s_{1,N-1}s_{2,N-1}}(V^c, T_N), \dots, c_{s_{1,1}s_{2,1}}(V^c, T_N); \hat{\Phi}_{N-1} \right) \right] + \\ & - \sum_{j=1}^{N-1} \left\{ \prod_{k=j}^{N-1} \left[ \sum_{n_k=0}^{\infty} \sum_{m_k=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)\tau_k} (\lambda_1\tau_k)^{n_k} (\lambda_2\tau_k)^{m_k}}{n_k!m_k!} \times \right. \right. \\ & \quad \left. \left. p_j e^{-(\delta_{s_{2,j}}T_j - \delta_{s_{2,N}}T_N)} \mathfrak{N}_{N-j} \left( d_{s_{1,N-1}s_{2,N-1}}(V^c, T_N), \dots, d_{s_{1,j}s_{2,j}}(V^c, T_N); \hat{\Phi}_{N-j} \right) \right] \right\}, \end{aligned}$$

and:

$$d_{s_{1,k}s_{2,k}}(V^c, T_N) = \frac{\ln\left(\frac{V_{T_N}^c}{V_{c^*}^c}\right) + \left(\delta_{s_{2,k}} - \delta_{s_{1,k}} - \frac{1}{2}\sigma_{s_{1,k}s_{2,k}}^2\right)T_k - \left(\delta_{s_{2,N}} - \delta_{s_{1,N}} - \frac{1}{2}\sigma_{s_{1,N}s_{2,N}}^2\right)T_N}{\sqrt{\sigma_{s_{1,k}s_{2,k}}^2 T_k - \sigma_{s_{1,N}s_{2,N}}^2 T_N}},$$

$$c_{s_{1,k}s_{2,k}}(V^c, T_N) = d_{s_{1,k}s_{2,k}}(V^c, T_N) + \sqrt{\sigma_{s_{1,k}s_{2,k}}^2 T_k - \sigma_{s_{1,N}s_{2,N}}^2 T_N}.$$

Finally, equation (18) can be written in integral form as in (12) and solved in a very similar way. The result in Proposition 2 follows.

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TABLE 1						
Option values are obtained using the following values of the parameters: $V=100$ , $I_1=100$ , $\sigma=0.2$ , $\sigma_J=0.2$ , $r=0.02$ , $T_2=0.25$ , $T_1=0.5$ .						
<i>Jump – Diffusion (continuous time)</i>				<i>Jump – Diffusion (discrete time)</i>		
	$M = 200$					
$\lambda$	$I_2= 10;$	$I_2= 12.5;$	$I_2= 15;$	$I_2= 10;$	$I_2= 12.5;$	$I_2= 15;$
1.0	2.25	1.70	1.31	2.20	1.67	1.29
0.8	2.02	1.50	1.14	2.00	1.49	1.13
0.6	1.80	1.31	0.96	1.79	1.30	0.97
0	1.12	0.71	0.45	1.13	0.72	0.45

TABLE 2				
<i>3 – Fold Compound Option (M = 200)</i>				
Option values are obtained using the following values of the parameters: $V=100$ , $I_3=5$ , $I_2=10$ , $I_1=100$ , $\sigma=0.2$ , $\sigma_J=0.2$ , $r=0.02$ , $T_3=0.2$ , $T_2=0.35$ , $T_1=0.5$ .				
$\lambda$	0.4	0.6	0.8	1.0
Option values:	0.66	0.81	0.96	1.10