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Discussion Paper 2011-18

Institut de Recherches Économiques et Sociales
de l'Université catholique de Louvain

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April 4, 2011

*We thank Hippolyte d'Albis, Bruno Decreuse, David de la Croix, Nicolas Gravel, Alan Kirman, Patrick Pintus, Vladimir Veliov and participants in conferences and seminars in Lisbon, Marseille and La Rochelle for stimulating comments. Financial support from the Belgian ARC contract 09/14-018 on sustainability is gratefully acknowledged.

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Abstract

We reconsider the optimal population size problem in a continuous time economy populated by homogenous cohorts with a fixed life span. Linear production functions in the labor input and standard rearing costs are also considered. First, we study under which conditions the successive cohorts will be given the same consumption per capita. We show that this egalitarian rule is optimal whatever the degree of altruism when life spans are infinite. However, when life spans are finite, this rule can only be optimal in the Benthamite case, *i.e.* when the degree of altruism is maximal. Second, we prove that under finite life spans the Millian welfare function leads to optimal extinction at finite time whatever the lifetime. In contrast, the Benthamite case is much more involved: for isoelastic utility functions, it gives rise to two threshold lifetime values, say $T_0 < T_1$: below T_0, finite time extinction is optimal; above T_1, balanced growth paths are optimal. In between, asymptotic extinction is optimal. Third, optimal consumption and population dynamics are given in closed-form.

Key words: Optimal population size, Benthamite Vs Millian criterion, finite lives, optimal extinction

JEL numbers: D63, D64, C61, 0 40

1 Introduction

As recently outlined by Dasgupta (2005), the question of optimal population size traces back to antiquity. For example, Plato concluded that the number of citizens in the ideal city-state is 5,040, arguing that it is divisible by every number up to ten and have as many as 59 divisors, which would allow for the population to "... suffice for purposes of war and every peacetime activity, all contracts for dealings, and for taxes and grants" (cited in Dasgupta, 2005). A considerable progress has been made since then! A fundamental contribution to this normative debate is due to Edgeworth (1925) who considered the implications of total utilitarianism, originating from the classical Benthamite welfare function, for population and standard of living, in comparison with the alternative average utilitarianism associated with the Millian welfare function (see also Dasgupta, 1969). Edgeworth was the first to claim that total utilitarianism leads to a bigger population size and lower standard of living. Subsequent literature has aimed to study the latter claim in different frameworks. A first important inspection is due to Nerlove, Razin and Sadka (1985), who examined the robustness of Edgeworth's claim to parental altruism. They show that the claim still holds when the utility function of adults is increasing in the number of children and/or the utility of children.[1]

The analysis uses simple arguments within a static model. Dynamic extensions were considered later. A question arises as to the robustness of Edgeworth's claim when societies experience long periods (say infinite time periods) of economic growth. Two endogenous growth papers with apparently opposite conclusions are worth mentioning here.[2] Razin and Yuen (1995) confirm Edgeworth's claim in an endogenous growth model driven by human capital accumulation with an explicit trade-off between economic growth and demographic growth deriving from an underlying time allocation between education and children rearing. In contrast, Palivos and Yip (1993) showed that Edgeworth's claim cannot hold for the realistic parameterizations of their model. Palivos and Yip used the framework of endogenous growth driven by an AK production function. The determination of optimal population size relies on the following trade-off: on one hand, the utility function depends explicitly on population growth rate; on the other, population growth has the standard linear dilution effect on physical capital accumu-

[1] A connected philosophical literature is population ethics, as illustrated by the writings of Parfit for example (see Parfit, 1984).

[2] A more recent contribution to the optimal population size literature within the Ramsey framework can be found in Arrow et al. (2010).

lation. Palivos and Yip proved that in such a framework the Benthamite criterion leads to a smaller population size and a higher growth rate of the economy provided the intertemporal elasticity of substitution is lower than one (consistently with empirical evidence), that is when the utility function is negative. Indeed, a similar result could be generated in the setting of Razin and Yuen (1995) when allowing for negative utility functions.[3]

This paper is a contribution to the literature of optimal population size under endogenous growth in line with Palivos and Yip, and Razin and Yuen. It has the following four distinctive features:

1. First of all, it departs from the current literature by bringing into the analysis the life span of individuals. We shall assume that all individuals of all cohorts live a fixed amount of time, say T. The value of T will be shown to be crucial in the optimal dynamics and asymptotics of the model. Early exogenous increases in life expectancy have been invoked to be at the dawn of modern growth in several economic theory and historical demography papers (see for example Boucekkine et al., 2002), explaining a substantial part of the move from demographic and economic stagnation to the contemporaneous growth regime. We shall examine the normative side of the story. Our study can be also understood as a normative appraisal of natural selection. Admittedly, a large part of the life spans of all species is the result of a complex evolutionary process (see the provocative paper of Galor and Moav, 2007). Also it has been clearly established that for many species life span correlates with mass, genome size, and growth rate, and that these correlations occur at differing taxonomic levels (see for example Goldwasser, 2001).[4] Our objective here is to show that the lifetime value is a dramatic determinant of optimal population size, which could be naturally connected to more appealing issues like for example the determinants of species' extinction. This point is made clear hereafter.

2. Second, in comparison with the AK models surveyed above which do not display transitional dynamics, our AK-like model does display transitional dynamics because of the finite lifetime assumption (just like in the AK vintage capital model studied in Boucekkine et al., 2005, and Fabbri and Gozzi, 2008). The deep reason of these different behaviors

[3]See also Boucekkine and Fabbri (2010).

[4]Of course, part of the contemporaneous increase of humans' life span is, in contrast, driven by health spending and medical progress. We shall abstract from the latter aspect, and take a fully exogenous view of life spans. See Arrow et al. (2010) for a model with health expenditures allowing to endogenize life spans.

is that the finite life span setting we use allows to take into account the whole age-distribution structure of the population that does have a key role in the evolution of the system. Indeed the engine of the transitional dynamics of detrended variables is the rearrangement of the shares among the cohorts. This clearly distinguishes the approach we use from the models with "radioactive" decay of the population (and from our benchmark $T = +\infty$ case) where all the individual are identical.

3. Among the many new research lines allowed by our setting, optimal population extinction at finite time or asymptotically can be studied. Extinction is an appealing topic that has been much more explored in natural sciences than in economics. A few authors have already tried to investigate it both positively or normatively. On the positive side, one can mention the literature of the Easter Island collapse, and in particular the work of de la Croix and Dottori, 2008).[5] On the normative side, one can cite the early work of Baranzini and Bourguignon (1995) or more recently Boucekkine and de la Croix (2009). Interestingly enough, the former considers a stochastic environment inducing an uncertain lifetime but the modelling leads to the standard deterministic framework once the time discounting is augmented with the (constant) survival probability.

4. Fourth, in order to address analytically the dynamic issues mentioned above, we shall consider a minimal model in the sense that we do not consider neither capital accumulation (as in Palivos and Yip) nor natural resources (as in Makdissi, 2001, and more recently Boucekkine and de la Croix, 2009). We consider one productive input, population (that's labor), and, in contrast to Palivos and Yip, the instantaneous utility function does not depend on population growth rate, that is we remove intratemporal (or instantaneous since time is continuous) altruism. Nonetheless, we share with the latter constant returns to scale: we therefore have an AN model with N the population size.[6] By taking this avenue, population growth and economic growth will coincide in contrast to the previous related AK literature (and in particular to Razin and Yuen, 1995). However, we shall show clearly that the difference between the outcomes of the Millian and Benthamite cases is much

[5] Again the literature on the Easter Island collapse is much more abundant in natural sciences and applied mathematics, see for example Basener and Ross (2004).

[6] Boucekkine and de la Croix (2009) have decreasing returns to labor, infinite lifetime and natural resources which depletion depends on population size.

sharper regarding optimal dynamics than long-term growth (which is the focus of the related existing AK literature).

Resorting to AN production functions and removing intratemporal altruism and capital accumulation has the invaluable advantage to allow for (nontrivial) analytical solutions to the optimal dynamics in certain parametric conditions. In particular, we shall provide optimal dynamics in closed-form in the two polar cases where the welfare function is Millian Vs Benthamite, and to an intermediate welfare function, namely a case of impure altruism as referred to in the related literature. It is important to notice here that considering finite lifetimes changes substantially the mathematical nature of the optimization problem under study. Because the induced state equations are no longer ordinary differential equations but delay differential equations, the problem is infinitely dimensioned. Problems with these characteristics are tackled in Boucekkine et al. (2005), Fabbri and Gozzi (2008) and recently by Boucekkine, Fabbri and Gozzi (2010). We shall follow the dynamic programming approach used in the two latter papers. Because some of the optimization problems studied in this paper present additional peculiarities, a nontrivial methodological progress has been made along the way. The main technical details on the dynamic programming approach followed are however reported in the appendix given the complexity of the material.

Main findings

Several findings will be highlighted along the way. At the minute, we enhance three of them.

1. A major contribution of the paper is the striking implication of finite life spans for the optimal consumption pattern across cohorts. Indeed, we study under which conditions the successive cohorts will be given the same consumption per capita, which in our case amounts to fixing a constant fertility rate over time. We show that provided growth is optimal (which rules out the Millian case, see below), this egalitarian rule is optimal whatever the degree of altruism when life spans are infinite. However, when life spans are finite, this rule can only be optimal in the Benthamite case, suggesting that when individuals' lifetime is finite, the degree of altruism should be maximal to support an egalitarian consumption rule across generations.[7]

[7] There exists a huge theoretical literature on optimal intergenerational consumption rules. A large part of it takes an axiomatic approach, see Epstein (1986) or Asheim (1991). In contrast to the two latter papers in which population is constant, ours is more about optimal population size, which explains our focus on this problem.

2. Moreover, because the longer individuals live the larger their contribution to resources of the economy, a natural outcome of our model is that population and therefore economic growth (given the AN production structure) are optimal only if the individuals' lifetime is large enough. In this sense, our model is in accordance with unified growth theory (Galor and Moav, 2002, and Boucekkine, de la Croix and Licandro, 2002). Beside this outcome, the analysis brings out nontrivial results on when finite time Vs asymptotic extinction cases arise, and on the demographic and technological shocks needed to escape from extinction and to move to an optimal growth regime.

3. Last but not least, the analysis illustrates the crucial role of the degree of altruism in the shape of the optimal allocation rules for given finite life span, and the framework allows for striking clear-cut analytical results. Effectiveley we highlight dramatic differences between the Millian and Benthamite cases in terms of optimal dynamics, which is to our knowledge a first contribution to this topic (the vast majority of the papers in the topic only are working on balanced growth paths). While the Millian welfare function leads to optimal population extinction at finite time whatever individuals' lifetime, the Benthamite case does deliver a much more complex picture. We identify the existence of two threshold values for individuals' lifetime, say $T_0 < T_1$: below T_0, finite time extinction is optimal; above T_1, balanced growth paths (at positive rates) are optimal. In between, asymptotic extinction is optimal.

The paper is organized as follows. Section 2 describes the optimal population model, gives some technical details on the maximal admissible growth and the boundedness of the associated value function, and displays some preliminary results on extinction. Section 3 studies the infinite lifetime case as a benchmark. Section 4 derives the optimal dynamics corresponding to the Millian Vs Benthamite cases. Section 5 studies the case of impure (or imperfect) altruism. Section 6 concludes. The Appendices A and B are devoted to collect most of the proofs.

2 The optimal population size problem

2.1 The model

Let us consider a population in which every cohort has a fixed finite life span equal to T. Assume for simplicity that all the individuals remain perfectly active (i.e. they have the same productivity and the same procreation ability) along their life time. Moreover assume that, at every moment t, if $N(t)$ denotes the size of population at t, the size $n(t)$ of the cohort born at time t is bounded by $M \cdot N(t)$, where $M > 0$ measures the maximal (time-independent) biological reproduction capacity of an individual.

The dynamic of the population size $N(t)$ is then driven by the following delay differential equation (in integral form):

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s, \qquad t \geq 0, \tag{1}$$

and

$$n(t) \in [0, MN(t)], \qquad t \geq 0. \tag{2}$$

The past history of $n(r) = n_0(r) \geq 0$ for $r \in [-T, 0)$ is known at time 0: it is in fact the initial datum of the problem. This features the main mathematical implication of assuming finite lives. Pointwise initial conditions, say $n(0)$ or $N(0)$, are no longer sufficient to determine a path for the state variable, $N(t)$. Instead, an initial function is needed. The problem becomes infinitely dimensioned, and the standard techniques do not immediately apply. Summarizing, (1) becomes:

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s, \; n(r) = n_0(r) \geq 0 \; \text{ for } r \in [-T, 0), \; N(0) = \int_{-T}^{0} n(r)\,\mathrm{d}r. \tag{3}$$

Note that the constraint (2) together with the positivity of n_0 ensure the positivity of $N(t)$ for all $t \geq 0$.

We consider a closed economy, with a unique consumption good, characterized by a labor-intensive aggregate production function exhibiting constant returns to scale, that is

$$Y(t) = aN(t). \tag{4}$$

Note that by equation (1) we are assuming that individuals born at any date t start working immediately after birth. Delaying participation into the labor market is not an issue but adding another time delay into the model will only complicate unnecessarily the computations without altering substantially our findings. Note also that there is no capital accumulation in our model. The linearity of the production technology is necessary to generate long-term growth, it is also adopted in the related bulk of papers surveyed in the introduction. If decreasing returns were introduced, that is $Y(t) = aN^{\alpha}$ with $\alpha < 1$, growth will vanish, and we cannot in such a case connect life span with economic and demographic growth. This said, we shall comment along the way on how the results of the paper could be altered if one switches from constant to decreasing returns, namely from endogenous to exogenous growth.

Output is partly consumed, and partly devoted to raising the newly born cohort, say rearing costs. In this benchmark we assume that the latter costs are linear in the size of the cohort, which leads to the following resource constraint:

$$Y(t) = N(t)c(t) + bn(t) \tag{5}$$

where $b > 0$. Again we could have assumed that rearing costs are distributed over time but consistently with our assumption of immediate participation in the labor market, we assume that these costs are paid once for all at time of birth. On the other hand, the linearity of the costs is needed for the optimal control problem considered above to admit closed-form solutions. As it will be clear along the way, this assumption is much more innocuous than the AN production function considered. This seems rather natural: if extinction is optimal for linear costs, extinction is likely to hold *a fortiori* for stronger strictly convex costs.

Let us describe now accurately the optimal control problem handled. The controls of the problem are $n(\cdot)$ and $c(\cdot)$ but, using (4) and (5), one obtains

$$n(t) = \frac{(a - c(t))N(t)}{b}. \tag{6}$$

so we have only to choose $c(t)$. Since we want both per-capita consumption and the size of new cohorts to remain positive, we need to ensure:

$$0 \leq c(t) \leq a. \tag{7}$$

In other words we consider the controls $c(\cdot)$ in the set[8]

$$\mathcal{U}_{n_0} := \left\{ c(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) \ : \ \text{eq. (7) holds for all } t \geq 0 \right\}.$$

We shall consider the following social welfare function to be maximized within the latter set of controls:

$$\int_0^{+\infty} e^{-\rho t} u(c(t)) N^{\gamma}(t) \, \mathrm{d}t, \tag{8}$$

[8]The space $L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+)$ in the definition of $\mathcal{U}_{n_0}$ is defined as $L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) := \left\{ f \colon [0, +\infty) \to \mathbb{R}_+ \ : \ f \text{ measurable and } \int_0^T |f(x)| \, \mathrm{d}x < +\infty, \ \forall T > 0 \right\}$.

where $\rho > 0$ is the time discount factor, $u\colon (0, +\infty) \to (0, +\infty)$ is a continuous, strictly increasing and concave function, and $\gamma \in [0, 1]$ is a parameter allowing to capture the altruism of the social planner. More precisely, γ measures the degree of *intertemporal altruism* of the planner in that the term $N^\gamma(t)$ is a determinant of the discount rate at which the welfare of future generations is discounted. While intratemporal welfare is not considered here (as mentioned in the introduction section) in order to extract closed-form solution to optimal dynamics, intertemporal altruism is kept to study the two polar cases outlined above: indeed, $\gamma = 0$ covers the case of average utilitarianism, that's the Millian social welfare function, and $\gamma = 1$ is the Benthamite social welfare function featuring total utilitarianism. We shall also solve an intermediate case, $0 < \gamma < 1$, in Section 5. Our modeling of intertemporal altruism is nowadays quite common. Recently, Strulik (2005) and Bucci (2008) have studied the impact of population growth on economic growth within endogenous growth settings, keeping population growth exogenous and introducing intertemporal altruism as above. In particular, Strulik (2005) shows that exogenous population growth rate has a positive impact on economic growth through intertemporal altruism. In our framework, population growth is endogenous in line with Palivos and Yip (1993).

Remark 2.1 Notice also that we only consider positive utility functions. Indeed, our modeling implicitly implies that the value of not living is zero. A (strictly) negative utility function therefore implies that not living is superior to living, implying that the the optimal cohort is zero. As a result, for any initial conditions and any lifetime, T, the planner will choose extinction at finite time. This argument is formalized in the discussion paper version of the paper.[9]

2.2 Maximal admissible growth

We begin our analysis by giving a sufficient condition ensuring the boundedness of the value function of the problem. The arguments used are quite intuitive so we mostly sketch the proofs.[10].

Consider the state equation (3) with the constraint (7). Given an initial datum $n_0(\cdot) \geq 0$ (and then $N(0) = \int_{-T}^{0} n_0(r)\,\mathrm{d}r$), we consider the admissible control defined as $c_{MAX} \equiv 0$. This control obviously gives the maximal population size allowed, associated with $n_{MAX}(t) = \frac{a}{b}N(t)$ by equation (6):

[9] see Proposition 2.3 of the discussion paper version at: `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`. Also, utility functions with no sign restriction are explcitly handled in this paper, see Section 3.2.2.

[10] The reader interested in technical details in the proofs of Lemma 2.1 and Proposition 2.1 is reported to Propositions 2.1.6, 2.1.10 and 2.1.11 in Fabbri and Gozzi (2008).

it is the control/trajectory in which all the resources are assigned to raising the newly born cohorts with nothing left to consumption. Call the trajectory related to such a control $N_{MAX}(\cdot)$. By definition $N_{MAX}(\cdot)$ is a solution to the following delay differential equation (written in integral form):

$$N_{MAX}(t) = \int_{(t-T)\wedge 0}^{0} n_0(s)\,\mathrm{d}s + \frac{a}{b}\int_{(t-T)\vee 0}^{t} N_{MAX}(s)\,\mathrm{d}s. \tag{9}$$

The characteristic equation of such a delay differential equation is[11]

$$z = \frac{a}{b}\left(1 - e^{-zT}\right). \tag{10}$$

It can be readily shown (see e.g. Fabbri and Gozzi, 2008, Proposition 2.1.8) that if $\frac{a}{b}T > 1$, the characteristic equation has a unique strictly positive root ξ. This root belongs to $(0, \frac{a}{b})$ and it is also the root with maximal real part. If $\frac{a}{b}T \leq 1$, then all the roots of the characteristic equation have non-positive real part and the root with maximal real part is 0. In that case, we define $\xi = 0$. We have that (see for example Diekmann et al., 1995, page 34), for all $\epsilon > 0$,

$$\lim_{t\to\infty} \frac{N_{MAX}(t)}{e^{(\xi+\epsilon)t}} = 0, \tag{11}$$

and that the dynamics of $N_{MAX}(t)$ are asymptotically driven by the exponential term corresponding to the root of the characteristic equation with the largest real part. As it will be shown later, this result drives the optimal economy to extinction when individuals' lifetime is low enough. At the minute, notice that since $N_{MAX}(\cdot)$ is the trajectory obtained when all the resources are diverted from consumption, it is the trajectory with the largest population size. More formally, one can write:

Lemma 2.1 *Consider a control $\hat{c}(\cdot) \in \mathcal{U}_{n_0}$ and the related trajectory $\hat{N}(\cdot)$ given by (1). We have that*

$$\hat{N}(t) \leq N_{MAX}(t), \qquad \text{for all } t \geq 0.$$

The previous lemma, coupled with property (11), straightforwardly implies the following sufficient condition for the value function of the problem to be bounded:

Proposition 2.1 *The following hypothesis*

$$\rho > \gamma\xi \tag{12}$$

[11] As for any linear dynamic equation (in integral or differential form), the characteristic equation is obtained by looking at exponential solutions, say e^{zt}, of the equation.

is sufficient to ensure that the value function

$$V(n_0) := \sup_{\hat{c}(\cdot)\in\mathcal{U}_{n_0}} \int_0^{+\infty} e^{-\rho t} u(\hat{c}(t))\hat{N}^{\gamma}(t)\,\mathrm{d}t$$

is finite (again we denoted with $\hat{N}(\cdot)$ the trajectory related to the control $\hat{c}(\cdot)$).

The proofs of the two results above follow the line of Propositions 2.1.10 and 2.1.11 in Fabbri and Gozzi (2008), proving that $0 \leq V(n_0) < +\infty$ using an upper bound for every admissible strategy.

We are now ready to provide the first important result of the paper highlighting the case of asymptotic extinction.

2.3 A preliminary extinction result

We provide now a general extinction property inherent to our model. Recall that when $\frac{a}{b}T \leq 1$, all the roots of the characteristic equation of the dynamic equation describing maximal population, that is equation (9), have non-positive real part, which may imply that maximal population goes to zero asymptotically (asymptotic extinction). The next proposition shows that this is actually the case for any admissible control in the case where $\frac{a}{b}T < 1$.

Proposition 2.2 *If $\frac{a}{b}T < 1$ then every admissible control drives the system to extinction.*

The proof is in the Appendix B. The value of individuals' lifetime is therefore crucial for the optimal (and non-optimal) population dynamics. This is not fully surprising: if people do not live long enough to bring in more resources than it costs to raise them, then one might think that eventually the population falls to zero. The proposition identifies indeed a threshold value independent of the welfare function (and so independent in particular of the strength of intertemporal altruism given by the parameter γ) such that, if individuals' lifetime is below this threshold, the population will vanish asymptotically. While partly mechanical, the result has some interesting and nontrivial aspects. First of all, one would claim that in a situation where an individual costs more than what she brings to the economy, the optimal population size could well be zero at finite time. Our result is only about asymptotic extinction. As we shall show later, whether finite time extinction could be optimal requires an additional conditions, notably on the degree of altruism. Precisely, we will show that finite time extinction is always optimal in the Millian case while in the Benthamite case, and even under $\frac{a}{b}T < 1$, we can have finite time or asymptotic extinction depending on other parameters of the model.

Second, the result is interesting in that it identifies an explicit and interpretable threshold value, equal to $\frac{b}{a}$, for individuals' lifetime: the larger the productivity of these individuals, the lower this threshold is, and the larger the rearing costs, the larger the threshold is.[12] An originally non-sustainable economy can be made sustainable by two types of exogenous impulses: technological shocks (via a or b) or demographic shocks (via T).[13]

Remark 2.2 Before getting to the analysis of the Millian Vs Benthamite social welfare function, let us discuss briefly the robustness of our results in this section to departures from the linearity assumptions made on the cost and production functions. Introducing a strictly convex rearing function, say replacing bn by bn^{β} with $\beta > 1$, will obviously not alter the message of the extinction Proposition 2.2 and 2.3. Things are apparently more involved if we move from the linear production function $Y = aN$ to $Y = aN^{\alpha}$, with $\alpha < 1$. First note that in such a case the resource constraint (5) becomes

$$aN(t)^{\alpha} = Y(t) = N(t)c(t) + bn(t)$$

that is

$$c(t) = aN(t)^{\alpha-1} - b\frac{n(t)}{N(t)}. \tag{13}$$

The trajectory of maximum population growth (found taking $c(t) \equiv 0$) is now the solution of

$$\dot{N}_{MAX}(t) = \frac{a}{b}\left(N_{MAX}^{\alpha}(t) - N_{MAX}^{\alpha}(t-T)\right)$$

This equation has two equilibrium points: $\overline{N}_0 = 0$ which is unstable, and $\overline{N}_1 > 0$ which is asymptotically stable and attracts all positive data. This implies that the existence result of Proposition 2.1 holds for all $\rho > 0$ and the result of Proposition 2.2 does not hold.

3 The infinite lifetime case as a benchmark

In order to disentangle accurately the implications of finite lives, we start with the standard case where agents live forever. Some preliminary manipulations are needed. First we need to rewrite the optimal control problem using $n(\cdot)$ as a control instead of $c(\cdot)$: using (4) and (5) we obtain

[12] If $T = \frac{b}{a}$, not all the admissible trajectories drive the system to extinction: indeed if we have for example the constant initial datum $N(t) = 1$ for all $t < 0$ or $n(t) = a/b$ for all $t < 0$, the (admissible) maximal control $N_{MAX}(t)$ allows to maintain the population constant equal to 1 for every t.

[13] This is largely consistent with unified growth theory- see Galor and Weil (1999), Galor and Moav (2002), and Boucekkine, de la Croix and Licandro (2002).

$$c(t) = \frac{aN(t) - bn(t)}{N(t)}. \tag{14}$$

Since we want per-capita consumption to remain positive, we need $n(t) \leq \frac{a}{b}N(t)$, so that:

$$0 \leq n(t) \leq \frac{a}{b}N(t) \qquad t \geq 0. \tag{15}$$

The previous constraint can be rewritten by requiring $n(t)$ to be in the set

$$\mathcal{V}_{n_0} := \left\{ n(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) \ : \ \text{conditions (15) hold for all } t \geq 0 \right\}. \tag{16}$$

More importantly, we shall use from now on explicit utility functions in order to get closed-form solutions. More precisely, we choose the isoelastic function $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma \in (0, 1)$.[14]. With the latter utility function choice, the functional (8) can be rewritten as

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^{\gamma-(1-\sigma)}(t)\, \mathrm{d}t. \tag{17}$$

Last but not least, it should be noted that our problem is not likely to be concave for every value of γ because of endogenous fertility, and notably the altruism term, N^γ, in the objective function. Indeed, one can straightforwardly show that the problem is concave if and only if $\gamma \in [1 - \sigma, 1]$. Throughout this paper, we shall use dynamic programming, which which always give sufficient optimality condition even in the absence of concavity: recall that Euler equations are not sufficient for optimality if concavity is not guaranteed. This said, we will use the latter for purpose of clarification and interpretation (of course provided concavity is met).

Let us see what happens in the limit case $T = +\infty$. In this situation, the problem reduces to maximizing the functional

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(a - \frac{bn(t)}{N(t)}\right)^{1-\sigma}}{1-\sigma} N^\gamma(t)\, \mathrm{d}t \tag{18}$$

[14]The values considered for σ guarantee the positivity of the utility function. It might be argued following Palivos and Yip (1993) that such values imply unrealistic figures for the intertemporal elasticity of substitution, which require $\sigma > 1$. We show in the discussion paper version of the paper that our main results still hold qualitatively on the utility function: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, whose positivity is compatible, under certain scale conditions, with the more realistic $\sigma > 1$. See Section 3.2.2 at `http://halshs.archives-ouvertes.fr/docs/00/53/60/73/PDF/DTGREQAM2010_40.pdf`.

for the system driven by the state equation:

$$\dot{N}(t) = n(t), \qquad N(0) = N_0 \tag{19}$$

and constraints $n \in \left[0, \frac{a}{b}N\right]$. We have the following result:

Proposition 3.1 *In the described limit case $T = +\infty$, when the functional is given by (18) the condition*

$$\rho > \frac{a}{b}\gamma \tag{20}$$

is necessary and sufficient to ensure the boundedness of the functional.

Moreover we have the following.

(i) If $\frac{a}{b}\gamma \geq \rho(1-\sigma)$ (which implies $\gamma > 0$) then the optimal trajectory in feedback form is given by $n = \theta N$ where $\theta := \left(\frac{\rho}{\gamma}(1 - 1/\sigma) + a/(b\sigma)\right)$, the optimal trajectory and control can be written explicitly as $N^*(t) = N_0 e^{\theta t}$ and $n^*(t) = \theta N_0 e^{\theta t}$. The related trajectory of the per-capita consumption is constant over time and is given by $c^*(t) = a - b\theta$.

(ii) If $\frac{a}{b}\gamma < \rho(1-\sigma)$ then the optimal control is $n^* \equiv 0$, the optimal trajectory is $N^*(t) \equiv N_0$ and the optimal per-capita consumption is given by $c^*(t) = a$.

The proof is in Apendix B. This result allows to distinguish quite sharply between the Millian and Benthamite cases. Actually, Proposition 3.1 goes much beyond the two latter cases and highlights the role of the altruism parameter, γ, in the optimal dynamics. In particular, the proposition identifies a threshold value for the latter parameter, $\bar{\gamma} = \frac{b\rho(1-\sigma)}{a}$, under which the optimal outcome is to never give birth to any additional individual, so that population size is always equal to its initial level (permanent zero fertility case). This already means that for fixed cost, technological and preference parameters, a Millian planner will always choose this permanent zero fertility rule. In contrast, a Benthamite planner can choose to implement a nonzero fertility rule, leading to growing population (and production) over time provided: (i) the productivity parameter a is large enough, and/or (ii) the cost parameter b is small enough, and/or (iii) the time discount rate ρ (resp. σ) is small (resp. large) enough, which are straightforward economic conditions. The same happens for impure altruism cases ($0 < \gamma < 1$) although in such cases the "compensation", in terms of productivity or cost for example, should be higher with respect to the Benthamite case for the planner to launch a growing population regime. An interesting special case is $\gamma = 1 - \sigma$, which will be shown to have some peculiar analytical implications under finite lives in Section 5, ultimately allowing to get a closed-form solution to the optimal dynamics. For comparison with Section 5, let us isolate this case.

Corollary 3.1 *Under $\gamma = 1 - \sigma$, and provided $\rho \leq \frac{a}{b} < \frac{\rho}{\gamma}$, case (i) of Propostion 3.1 applies.*

A much more intriguing property is that consumption per capita is constant over time when growth is optimal whatever the value of γ ($\gamma \neq 0$). Note that this property derives entirely from the fact that the optimal size of new cohorts relative to the size of total population, that is $\frac{n(t)}{N(t)}$, is constant, equal to parameter θ. We shall interpret this ratio as a reproduction or fertility rate. It should be noted that the proposition implies that this ratio is increasing in γ since $\sigma < 1$, which is consistent: the larger the altruism parameter, the larger the fertility rate chosen by the planner. The fact that the optimal fertility ratio is constant whatever the altruism parameter (provided growth is optimal) is indeed intriguing. One way to understand how intriguing it is is to search for some formal equivalence with the standard AK model. One can readily show that the case $\gamma = 1 - \sigma$ studied in the Corollary just above is formally identical to the standard AK model with zero capital depreciation where the investment to capital ratio plays the role of the fertility rate in our model.[15] Therefore, the constancy of the optimal fertility rate is a mere consequence of the AK (or AN) structure in the case of the Corollary.

So why optimal fertility is constant for every γ compatible with growth in our setting? One way to visualize this case better is to compute the first-order conditions of our optimal control problem when concavity is ensured. As mentioned above, our optimal control problem is concave if and only if $1 - \sigma \leq \gamma \leq 1$, thus including the AK-equivalent value $\gamma = 1 - \sigma$ and the Benthamite configuration. In particular, the problem is strictly concave when $1 - \sigma < \gamma \leq 1$ ensuring uniqueness in the cases which are not AK-equivalent. Let us rewrite our problem slightly to put forward the fertility rate $m = \frac{n}{N}$ as the control variable. The objective function becomes

$$\int_0^{+\infty} e^{-\rho t} \frac{(a - bm(t))^{1-\sigma}}{1-\sigma} N^{\gamma}(t) \, \mathrm{d}t$$

and the state equation: $\dot{N}(t) = m(t)N(t)$, with $N(0) = N_0$ given. If $\lambda(t)$ is the adjoint (or co-state) variable, the first-order necessary (and sufficient by

[15] Indeed, one can see that in this case the AK model amounts to maximizing the functional $\int_0^{+\infty} e^{-\rho t} \frac{\left(A - \frac{i(t)}{K(t)}\right)^{1-\sigma}}{1-\sigma} K^{1-\sigma}(t) \, \mathrm{d}t$ under the state equation $\dot{K}(t) = i(t)$. Therefore, our problem is formally identical to this AK model if and only if $\gamma = 1 - \sigma$.

concavity) conditions are:

$$\lambda = b(a - bm)^{-\sigma}\ N^{\gamma - 1}\ e^{-\rho t},$$

$$-\dot{\lambda} = \lambda m + \gamma \frac{(a - bm)^{1-\sigma}}{1-\sigma}\ N^{\gamma-1}\ e^{-\rho t},$$

with the transversality condition $\lim_{t \to +\infty} \lambda N = 0$. The first equation is the optimality condition for the current value Hamiltonian with respect to m: it leads to equalizing the adjoint variable and the marginal (dis)utility of the fertility rate divided by N.[16] The second equation is the adjoint or Euler equation: as usual it stipulates that the decrease in the (social) value of an individual should reflects its future and present contributions to social welfare, the second term of the sum in the right-hand side representing obviously the contemporaneous impact on welfare of an additional individual. It is trivial to eliminate the co-state variable by differentiating the first condition with respect to time and substituting it in the adjoint equation. Using the state equation $\dot{N}(t) = m(t)N(t)$, one can show after trivial but tedious computations that the dynamics of the fertility rate are independent of the actual size of population, which is the crucial property of the model under infinite lives. Indeed, one can easily show that these dynamics are driven by:

$$\dot{m} = \kappa\ (a - bm)\ (\theta - m),$$

where κ is a constant depending on the parameters of the model, and θ is given in the Proposition 3.1. Under strict concavity of the problem, which occurs for example in the Benthamite case as explained above, the solution $m(t) = \theta, \forall t$ is therefore the unique solution to the problem as proved in the latter proposition. As one can see, this property comes from the fact that the dynamics of the fertility rate m are independent of the population size, N, and this property is true whatever the strength of altruism measured by γ. Such an outcome is totally non-trivial: an increase N for given λ does increase the fertility rate by the first-order condition with respect to m shown above, but as fertility goes up, the marginal value of population, that is λ, drops when $\gamma > 1 - \sigma$,[17]which covers the case where the optimal control problem is concave, inducing a second round effect on m. Our main result therefore implies that this second round effect exactly cancels the former first round effect given the specifications of our model.

[16]That is because we use the auxiliary control m instead of the original n: an increase in n by 1 increases N by 1 but an increase in m by 1 increases N by actual N.

[17]Trivial computations lead to :$-\frac{\dot{\lambda}}{\lambda} = (1 - \frac{\gamma}{1-\sigma})\ m + \frac{\gamma a}{b(1-\sigma)}$.

The analysis above of the infinite life case is a useful benchmark. We shall examine hereafter the finite life case, and show how the latter changes the results. We can already anticipate one interesting conceptual difference: while extinction cannot be even feasible in the infinite life case (since the size of cohorts are not allowed to be negative), the latter is a potential outcome when individuals don't leave for ever. In particular, if the zero fertility (or zero procreation) regime uncovered for example in the Millian case in the benchmark case turns out to be also optimal under finite lives, it would lead to extinction. More importantly, it goes without saying that the main property outlined in the benchmark case, that is the independence of fertility rate dynamics of the actual size of population, is not guaranteed to hold under finite lives: while there is no population destruction or "depreciation" when individuals' lifetime is infinite, we do have such a phenomenon under finite span. Indeed, one could write the law of motion of population size as:

$$\dot{N} = n(t) - \delta(t)\, n(t),$$

where $\delta(t) = \frac{n(t-T)}{n(t)}$ is the endogenous population "destruction" rate implied by our model. In the benchmark case, this rate was nil, it is endogenous in the finite life case.

4 Optimal population dynamics under finite lives

In this section, we perform the traditional comparison between the outcomes of the polar Benthamite Vs Millian cases. Nonetheless, our comparison sharply departs from the existing work (like in Nerlove et al., 1985, or Palivos and Yip, 1993) in that we are able to extract a closed-form solution to optimal dynamics, and therefore we compare the latter. Traditional comparison work only considers steady states.[18] This focus together with the finite lifetime specification allows to derive several new results.

4.1 The Millian case, $\gamma = 0$, under finite lives

This case can be treated straightforwardly. Indeed, in the absence of intertemporal altruism, the functional (8) reduces to

$$\int_0^{+\infty} e^{-\rho t} u(c(t))\, \mathrm{d}t. \tag{21}$$

[18]As mentioned above, Palivos and Yip have an AK model, so their model does not display transition dynamics.

and, since we can freely choose $c(t) \in [0, a]$ for all $t \geq 0$, the following claim is straightforward:

Proposition 4.1 *Consider the problem of maximizing (8) with $\gamma = 0$ subject to the state equation (3) and the constraint (7). Then the optimal control is given by $c^*(t) \equiv a$, so that, from (6), $n^*(t) \equiv 0$. So the value function is constant in N_0 and its value and equal to $\frac{u(a)}{\rho}$.*

Remark 4.1 One can ask what is the meaning of the functional along the optimal path for $t \geq T$ i.e. once the population is extinct. One could overcome the problem changing the optimal control problem and restricting the set of the admissible controls to those satisfying the stronger constraint $N(t) > 0$ for all t. In this case the problem would not have an optimal control but it would be easy to prove that every maximizing sequence weakly converges to $c^*(\cdot)$ (and the sequence of the utilities of the sequence would converge to $\frac{u(a)}{\rho}$). Otherwise, at least for positive utility functions $u(\cdot)$ one could consider the functional $\int_0^\tau e^{-\rho t} u(c(t))\,dt$ where τ is the extinction time. Even in this case every maximizing sequence would converge (in a suitable sense) to $c^*(\cdot)$ (and the sequence of the utilities of the sequence would converge to $\frac{u(a)}{\rho}$). So in both cases, even if the control $c^*(\cdot)$ would not be, strictly speaking, optimal (because it is not admissible), it is a good reference to capture the system's optimal behavior.

Since the objective function depends only on consumption, and since it is increasing in the latter, the optimal control $c^*(t) \equiv a$, or equivalently $n^*(t) \equiv 0$, is obvious: in the Millian case, it is always optimal to not procreate. A direct implication of this property is finite-time extinction:

Corollary 4.1 *For the solution of the optimal control problem described in Proposition 4.1, population extinction occurs at a certain time $\bar{t} \leq T$.*

Some comments are in order here. In our set-up, the absence of intertemporal altruism makes procreation sub-optimal at any date. And this property is independent of the deep parameters of the problem: it is independent of the value of individuals' lifetime, T, of the value of intertemporal elasticity of substitution (determined by σ), and of the technological parameters, a and b. One would think that a higher enough labor productivity, a, and/or a lower enough marginal cost, b, would make procreation optimal at least along a transition period. This does not occur at all. Much more than in the AK model built up by Palivos and Yip, our benchmark enhances the implications of intertemporal altruism, which will imply a much sharper distinction between the outcomes of the Millian Vs Benthamite cases. This will be clarified in the next section. Before, it is worth pointing out that Proposition 4.1 is robust to departures from linearity. Indeed, the finite-time extinction result does not at all depend on the linear cost function, $bn(t)$, adopted. Even if we consider a more general cost $C(n(t))$, the behavior of the system does not change in the Millian case: in this case the production would be again equal to $Y(t) = aN(t)$, resulting in $c(t) = a - C(n(t))/N(t)$, so, any admissible function $C(\cdot)$ would work (for example $C(0) = 0$ and $C(\cdot)$ increasing and strictly convex): again optimal $c(t)$ should be picked in the interval $[0, a]$, and as before, one would have to choose $c(t) = a$ or $n(t) = 0$, leading to finite-time extinction.

Last but not least, it is worth pointing out that the optimal finite time extinction property identified here holds also under decreasing returns: the result described in Proposition 4.1 can be replicated without changes. Again only per-capita consumption enters the utility function and again the highest per-capita consumption is obtained taking $n \equiv 0$. Note that, differently from the linear case, here the per-capita consumption is not bounded by a but, when the population approaches to extinction, thanks to (13), tends to infinity, so in a sense the incentive to choose $n = 0$ is even greater.

4.2 The Benthamite case, $\gamma = 1$, under finite lives

We now come to the Benthamite case. This case is much more complicated than the first one. In particular, the mathematics needed to characterize the optimal dynamics is complex, relying on advanced dynamic programming techniques in infinite-dimensioned Hilbert spaces. Technical details are given in Appendix A. To get a quick idea of the method, we summarize here the steps taken.

1. First of all, we have to define a convenient functional Hilbert space. We denote by $L^2(-T, 0)$ the space of all functions f from $[-T, 0]$ to $\mathbb{R}$ that are Lebesgue measurable and such that $\int_{-T}^{0} |f(x)|^2 \, dx < +\infty$. It is an Hilbert space when endowed with the scalar product $\langle f, g \rangle_{L^2} = \int_{-T}^{0} f(x)g(x) \, dx$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-T, 0)$ (with the scalar product $\langle (x_0, x_1), (z_0, z_1) \rangle_{M^2} := x_0 z_0 + \langle x_1, z_1 \rangle_{L^2}$) that will contain the states of the studied system.

2. Then we translate our initial optimal control problem of a delay differential equation as an optimal control problem of an ordinary differential equation in this infinite dimensioned Hilbert space.

3. Finally we write the corresponding Hamilton-Jacobi-Bellman equation in the Hilbert space, and we seek for explicit expressions for the value function, which in turn gives the optimal feedback in closed form.

The same technique is used to handle the impure altruism case studied in the next section. Here, since $\gamma = 1$, the functional (17) simplifies into

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^\sigma(t)\, \mathrm{d}t. \tag{22}$$

For the value-function to be bounded, we can use the general sufficient condition (12): when $\gamma = 1$, it amounts to

$$\rho > \xi. \tag{23}$$

Recall that we have $\xi = 0$ when (10) does not have any strictly positive roots, i.e. when $\frac{a}{b}T \leq 1$. Moreover if we define

$$\beta := \frac{a}{b}(1 - e^{-\rho T}). \tag{24}$$

then equation (23) implies[19]

$$\rho > \beta \quad \Longleftrightarrow \quad \frac{\rho}{1 - e^{-\rho T}} > \frac{a}{b}. \tag{25}$$

The following theorem states a sufficient parametric condition ensuring the existence of an optimal control and characterizes it.

Theorem 4.1 *Consider the functional (22) with $\sigma \in (0, 1)$. Assume that (23) holds and let β given by (24). Then there exists a unique optimal control $n^*(\cdot)$.*

- *If*

$$\beta \leq \rho(1-\sigma) \quad \Longleftrightarrow \quad \frac{\rho}{1 - e^{-\rho T}} \leq \frac{a}{b} \cdot \frac{1}{1-\sigma}, \tag{26}$$

  then the optimal control is $n^*(\cdot) \equiv 0$ and we have extinction at time T.

- *If*

$$\beta > \rho(1-\sigma) \quad \Longleftrightarrow \quad \frac{\rho}{1 - e^{-\rho T}} > \frac{a}{b} \cdot \frac{1}{1-\sigma}, \tag{27}$$

  then we call

$$\theta := \frac{a}{b} \cdot \frac{\beta - \rho(1-\sigma)}{\beta\sigma} = \frac{a}{b}\left[\frac{1}{\sigma} - \frac{\rho(1-\sigma)}{\beta\sigma}\right] = \frac{1}{\sigma}\frac{a}{b} + \frac{\rho}{1 - e^{-\rho T}}\left(1 - \frac{1}{\sigma}\right) \tag{28}$$

  and we have $\theta \in (0\ , \frac{a}{b})$. The optimal control $n^*(\cdot)$ and the related trajectory $N^*(\cdot)$ satisfy

$$n^*(t) = \theta N^*(t). \tag{29}$$

[19] see e.g. Fabbri and Gozzi 2008, equation (15).

Along the optimal trajectory the per-capita consumption is constant and its value is

$$c^*(t) = \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - b\theta \in (0, a) \qquad \textit{for all } t \geq 0. \tag{30}$$

Moreover the optimal control $n^*(\cdot)$ is the unique solution of the following delay differential equation

$$\begin{cases} \dot{n}(t) = \theta\left(n(t) - n(t-T)\right), & \textit{for } t \geq 0 \\ n(0) = \theta N_0 & \\ n(s) = n_0(s), & \textit{for all } s \in [-T, 0). \end{cases} \tag{31}$$

The proof is in Appendix A. In contrast to the Millian case, there is now room for optimal procreation, and therefore for both demographic and economic growth. When $\gamma = 1$, intertemporal altruism is maximal, and such an ingredient may be strong enough in certain circumstances (to be specified) to offset the anti-procreation forces isolated in the analysis of the Millian case. Some comments on the optimal control identified are in order here specially in comparison with the benchmark infinite lifetime case.

1. First of all, one has to notice that the condition for growth in Theorem 4.1, $\beta > \rho(1-\sigma)$ leads exactly to the growth condition uncovered in the benchmark infinite lives case by making T going to infinity and putting $\gamma = 1$. The same can be claimed on the optimal constant fertility rate, θ, which now depends on individuals' lifetime: it is an increasing function of life spans and it converges to the constant fertility rate identified in the benchmark case when T goes to infinity and $\gamma = 1$. Notice that the longer individuals' lives, the larger the fertility rate since individuals' are active for a longer time in our model. This anti-demographic transition mechanism can be counter-balanced if one introduces fixed labor time and costly pensions. This extension goes beyond the objectives of this paper.

2. Second, and related to the previous comparison point, the Benthamite case with finite lives displays qualitatively the same growth regime as in the benchmark infinite lifetime case: both consumption per capita and the fertility rate are constant over time. As mentioned in Section 3, such an outcome is far from obvious: finite lives introduce a depreciation term in the law of motion of population size, which does not exist when people live forever, and it is unclear that the state independence property outlined in the benchmark for fertility optimal

dynamics can survive to this depreciation term. Theorem 4.1 shows that it does. However, we shall show in Section 5 that in contrast to the infinite lifetime case, the constancy of the optimal fertility rate and the corresponding intergenerational egalitarian consumption rule do not necessarily hold under impure altruism, and seems specific to the Benthamite social welfare function.

3. Third, one has to notice that finite time extinction is still possible in the Benthamite case. This occurs when parameter β is low enough. By definition, this parameter measures a kind of adjusted productivity of the individual: productivity, a, is adjusted for the fact that individuals live a finite life (through the term $1 - e^{-\rho T}$), and also for the rearing costs they have to pay along their lifetime. If this adjusted productivity parameter is too small, the economy goes to extinction at finite time. And this possibility is favored by larger time discount rates and intertemporal elasticities of substitution (under $\sigma < 1$). Longer lives, better productivity and lower rearing costs can allow to escape from this scenario, although even in such cases, the economy is not sure to avoid extinction asymptotically (see Proposition 4.2 below). In particular, it is readily shown that condition (27), ruling out finite time extinction, is fulfilled if and only if $T > T_0$, where $T_0 = -(1/\rho)\ln(1 - \rho(1-\sigma)b/a)$ is the threshold value induced by (27), which depends straightforwardly on the parameters of the model.

4. Finally it is worth pointing out that there is a major difference between the finite life Benthamite case and the benchmark infinite life case: while the latter does not exihibit any transitional dynamics, the former does. Equation (31) gives the optimal dynamics of cohort's size $n(t)$. This linear delay differential equation is similar to the one analyzed by Boucekkine et al. (2005) and Fabbri and Gozzi (2008). The dynamics depend on the initial function, $n_0(t)$, and on the parameters θ and T in a way that will be described below. They are generally oscillatory reflecting replacement echoes as in the traditional vintage capital theory (see Boucekkine et al., 1997). In our model, the mechanism of generation replacement induced by finite life spans is the engine of these oscillatory transitions.

We now dig deeper in the dynamics properties and asymptotics of optimal trajectories. The following proposition summarizes the key points.

Proposition 4.2 *Consider the functional (22) with $\sigma \in (0,1)$. Assume that (23) and (27) hold, so $\theta \in (0\,,\frac{a}{b})$. Then*

- *If $\theta T < 1$ then $n^*(t)$ (and then $N^*(\cdot)$) goes to 0 exponentially.*
- *If $\theta T > 1$ then the characteristic equation of (31)*

$$z = \theta\left(1 - e^{-zT}\right), \tag{32}$$

has a unique strictly positive solution h belonging to $(0,\theta)$ while all the other roots have negative real part; h is an increasing function of T. Moreover the population and cohort sizes both converge to an exponential solution at rate h[20]*:*

$$\lim_{t\to\infty} \frac{n^*(t)}{e^{ht}} = \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\,\mathrm{d}s > 0$$

and

$$\lim_{t\to\infty} \frac{N^*(t)}{e^{ht}} = \frac{1 - e^{-hT}}{h} \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\,\mathrm{d}s > 0$$

Finally, convergence is generally oscillatory.

The proof is in Appendix B. The proposition above highlights the dynamic and asymptotic properties of the optimal control when finite time extinction is ruled out, that it is when $T > T_0$.

Indeed the proposition adds another threshold value $T_1 > T_0$ on individuals' lifetime: we have extinction in finite time when $T < T_0$, asymptotic extinction when individuals' lifetime is between T_0 and T_1, population and economic growth when $T > T_1$. Notice that the emergence of asymptotic extinction is consistent with Proposition 2.2[21] and that it is new with respect to the Millian case where optimal extinction takes place at finite time whatever the individuals' lifetime.[22]

The existence of such second threshold T_1 would be trivial if θ be independent of T. Since θ do depend on T the argument can be made precise observing that the function $T \mapsto T\theta(T)$ is strictly increasing in T, at least as long as $\theta(T)$ remains in $\left[0, \frac{a}{b}\right)$, which is the interval in which our main theorem works. This allows to formulate the following important result.

Corollary 4.2 *Under the conditions of Theorem 3.1, there exist two threshold values for individuals' lifetime, T_0 and T_1, $0 < T_0 < T_1$ such that:*

[20] Observe that $\left(1 - e^{(-s-T)h}\right)$ is always positive for $s \in [-T, 0]$ and the constant $\frac{1}{1-T(\theta-h)}$ can be easily proved to be positive too.

[21] Here the threshold is indeed larger than the one identified in Proposition 2.2, see Corollary 4.2

[22] Of course, in this case, the longer the lifetime, the later extinction will take place.

1. for $T < T_0$, finite-time extinction is optimal,

2. for $T_0 < T < T_1$, asymptotic extinction is optimal,

3. for $T > T_1$, economic and demographic growth (at positive rate) is optimal.

Proposition 4.2 brings indeed further important results. If individuals' lifetime is large enough (i.e. above the threshold T_1), then both the cohort size and population size will grow asymptotically at a strictly positive rate. In other words, these two variables will go to traditional balanced growth paths (BGPs). Proposition 4.2 shows that the longer the lifetime, the higher the BGP growth rate, which is a quite natural outcome of our setting. Moreover, consistently with standard endogenous growth theory, the levels of the BGPs depend notably on the initial conditions, here the initial function $n_0(t)$. Proposition 4.2 derives explicitly these long-run levels and their dependence on the initial datum is explicitly given. More aspects should be singled out.

A first one has to do with the shape of the optimal paths. One would like to know how they look like once growth is taken out, that is after detrending. Proposition 4.2 shows that in contrast to consumption per capita which is constant over time, both population size and cohort size (detrended) generally show oscillatory convergence. In Section 5, we show that such optimal demographic dynamics are likely to emerge even under imperfect altruism. The mechanism behind is the so-called replacement echoes, which is induced by the finite life characteristic (see Boucekkine et al., 1997, for details on this mechanism). The degree of altruism does not matter in the emergence of these replacement dynamics. However, as it will be clear in Section 5, the fertility rate is constant when altruism is perfect, not under imperfect altruism.

A second observation concerns the precise role of initial conditions. Proposition 4.2 shows that once finite-time extinction is ruled out and since the growth rate h is such that $1 - T(\theta - h) > 0$, the long-run level of cohort and population sizes are positively correlated with the "historical" values of the cohort size (that is $n_0(t)$'s values for $t < 0$). So once finite-time extinction is ruled out, the process of optimal economic and demographic development designed here will not alter the historical ranking in terms of population levels. Since output only depends on labor input, the same conclusion can be made for output levels. This said, and given that both optimal per capita consumption and fertility rates are constant and independant of the initial data, the Benthamite case does also generate convergence outcomes: even if two countries differ in their historical demography, they will be assigned the same amount of consumption and children per capita by the social planner. This is a quite peculiar property for an AK-type model, it is driven by its endogenous fertility component, and crucially by the hypothesis of perfect altruism ($\gamma = 1$) as it is outlined in the next section. Finally, one can visualize the role of the age distribution of populations in the long-run. Indeed, Proposition 4.2 allows to see how different age-distributions of population affect the long run levels: if we consider two initial populations $n_0^1(\cdot)$ and $n_0^2(\cdot)$ having the same number of individuals (i.e. $\int_{-T}^0 n_0^1(r)\,\mathrm{d}r = \int_{-T}^0 n_0^2(r)\,\mathrm{d}r$) with n_0^2 younger than n_0^1 (formally this means that $\int_{-T}^s n_0^1(r)\,\mathrm{d}r \geq \int_{-T}^s n_0^2(r)\,\mathrm{d}r$ for all $s \in [-T, 0]$) one has that the level of optimal population is higher for n_0^2 (the younger one).

Before getting to the next section, it is worth commenting a bit on what would deliver the Benthamite case in the absence of growth, that is when the production function has the form $Y(t) = aN^\alpha(t)$ with $\alpha < 1$. Needless to say, in such a case, long-term growth being ruled out, the picture cannot be replicated by construction. Recall that, in the decreasing returns case, the trajectory of maximum population growth (found taking $c(t) \equiv 0$) is given by

$$\dot{N}_{MAX}(t) = \frac{a}{b}\left(N_{MAX}^\alpha(t) - N_{MAX}^\alpha(t-T)\right),$$

which has two equilibrium points, $N_0 = 0$ which is unstable, and $N_1 > 0$ which is asymptotically stable. In such a case, one expects finite time extinction to never occur, and in the absence of growth, convergence to an equilibrium point N_2 smaller than the maximal one N_1 to set in.

5 The case of impure altruism

In this section we study the intermediate case $\gamma = 1-\sigma$, with $0 < \sigma < 1$. We consider here the intermediate case $\gamma = 1 - \sigma$ since it is a good and "cheap" wary to address such crucial question. Indeed from the mathematical point of view, and in contrast to the case $\gamma = 1$ handled above (and to the case $\gamma \neq 1-\sigma$), the case $\gamma = (1-\sigma)$ leads to the same infinitely dimensioned optimal control problem solved out explicitly by Fabbri and Gozzi (2008) using dynamic programming.[23] Moreover, by varying σ in $(0\ ,1)$, one can extract

[23]Indeed, these authors identified a closed-form solution to the Hamilton-Jacob-Bellman equation induced by the optimal growth model with AK technology and "one-hoss-shay" depreciation, i.e. all machines of any vintage are operated during a fixed time T. The objective function (with obvious notations) is $\int_0^{+\infty} e^{-\rho t}\frac{(ak(t)-i(t))^{1-\sigma}}{1-\sigma}\,\mathrm{d}t$ under the state equation $k(t) = \int_{t-T}^t i(\tau)\,\mathrm{d}\tau$, which is formally identically to our problem if and only if $\gamma = 1-\sigma$.

some insightful lessons on the outcomes of our optimal control problem for any γ in $(0\,,1)$.[24]

A crucial question arising from the findings of the previous section is how the huge gap between the outcomes of the Millian and the Benthamite cases is altered when the intertemporal altruism parameter γ varies in $(0\,,1)$. The answer is that, for $\gamma \in (0\,,1)$, the optimal dynamics show some similarities with the Benthamite case concerning notably the optimal extinction properties and the oscillatory dynamics exhibited by population and cohort's sizes but they are also quite different in some aspects like the optimal consumption and fertility rate dynamics. [25]

As in the previous section, we reformulate the optimal control problem using $n(\cdot)$ as a control in the set

$$\mathcal{V}_{n_0} := \left\{ n(\cdot) \in L^1_{\mathrm{loc}}(0, +\infty; \mathbb{R}_+) \ : \ \text{conditions (15) hold for all } t \geq 0 \right\},$$

while the objective function becomes

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(aN(t) - bn(t)\right)^{1-\sigma}}{1-\sigma} \, \mathrm{d}t. \tag{33}$$

Also, as discussed in Subsection 2.2, we call ξ the unique strictly positive root of equation

$$z = \frac{a}{b}\left(1 - e^{-zT}\right),$$

if it exists, otherwise we pose $\xi = 0$. From Subsection 2.2, we know that $\xi > 0$ if individuals' lifetime is large enough: $T > \frac{b}{a}$. The condition (12) needed for the boundedness of the value function becomes:

$$\rho > \xi(1-\sigma). \tag{34}$$

It is then possible to characterize the optimal control of our problem as follows:

Theorem 5.1 *Consider the optimal control problem driven by (3), with constraint (15) and functional (33). If (34) and the following condition (needed to rule out corner solutions)*

$$\frac{\rho - \xi(1-\sigma)}{\sigma} \leq \frac{a}{b} \tag{35}$$

[24]In contrast to the infinite horizon case, we have found no way to identify an explicit solution to the Hamilton-Jacobi-Bellman equation under finite lives for any value of the altruism parameter, γ.

[25]This fact can be also assessed (with some hard mathematical work) in the case $\gamma \neq 1-\sigma$ studying the qualitative properties of the optimal dynamics through the dynamic programming approach.

are satisfied, then, along the unique optimal trajectory $n^*(\cdot)$ *and the related optimal trajectory* $N^*(\cdot)$*, we have*

$$n^*(t) = \frac{a}{b} N^*(t) - \Lambda e^{gt} \tag{36}$$

where

$$g := \frac{\xi - \rho}{\sigma} \tag{37}$$

and

$$\Lambda := \left(\frac{\rho - \xi(1-\sigma)}{\sigma} \cdot \frac{a}{b\xi} \right) \left(\int_{-T}^{0} \left(1 - e^{\xi r}\right) n_0(r)\, \mathrm{d}r \right).$$

Moreover $n^*(\cdot)$ *is characterized as the unique solution of the following delay differential equation:*

$$\begin{cases} \dot{n}(t) = \dfrac{a}{b}\left(n(t) - n(t-T)\right) - g\Lambda e^{gt}, & t \geq 0 \\ n(0) = \frac{a}{b}\left(N_0 - \Lambda\right) \\ n(r) = n_0(r), \qquad r \in [-T, 0). \end{cases}$$

The proof is in Appendix B, it is a simple adaptation of previous work of Fabbri and Gozzi (2008). Two important comments should be already made. First of all, one can see that the properties extracted in the theorem above are not applicable to the limit case $\gamma = 1$ because this amounts to study the limit case $\sigma = 0$: in the latter case, magnitudes, like the growth rate g given in equation (37), are not defined. In contrast, the theorem can be used to study possible dynamics of optimal controls when γ is close to zero, or when σ is close to one (but not equal to 1 of course). When $\gamma = 0$, we know from Section 4.1 that we have optimal extinction at finite-time whatever the value of $\sigma > 0$. Theorem 5.1 shows that when γ is close to zero (but not equal to zero), finite-time extinction is not the unique optimal outcome: population may even grow at a rate close to $g = \xi - \rho$ which might well be positive if the lifetime T is large enough (see a finer characterization below). In this sense, the impure altruism cases considered mimic to a large extent the properties identified for the Benthamite configuration. Nonetheless, Theorem 5.1 highlights important specificities of the latter case. A major difference comes from the fact that the fertility rate can be hardly constant when altruism is imperfect. As a consequence, per capita consumption can neither be constant. [26] Recall that in the Benthamite case, optimal

[26]Formally, the main difference between the Benthamite and imperfect altruism cases is the term $g\Lambda e^{gt}$. The implications of this term for the asymptotic properties of the model are not all immediate, we give them in a proposition hereafter.

consumption per capita and the fertility rate are constant and independent of the initial procreation profile. This does reflect the specificity of the latter case: when intertemporal altruism is maximal, the social planner abstracts from the initial conditions when fixing optimal consumption level and the fertility rate. Under intermediate altruism, the planner takes into account the initial data, and the optimal dynamics of the latter variables do adjust to this data. Notice finally that the fact that optimal fertility rate and per capita consumption are non-constant in the impure altruism case goes at odds with the findings of Corollary 3.1: when $\gamma = 1 - \sigma$, the latter variables are constant (no transition dynamics) in the benchmark infinite time case. That's to say, finite life spans do significantly matter! Another and more direct way to get this crucial aspect is to visualize the role of population's age distribution, which is irrelevant when lifetimes are infinite. Here observe that the younger the population, the higher the value of Λ. Therefore, the optimal decision $n(t) = aN(t)/b - \Lambda e^{gt}$ means that for a younger population an higher per-capita consumption and a lower fertility are optimal in the short-run. Still, as it will be shown below, the latter quantities are independent in the long run of the initial age-distribution of the population and the age-share profile converges to the "exponential" one.

The closed-form solution identified allows for a much finer characterization of the impure altruism case. For example, one can show in detail how close the impure altruism case is to the Benthamite configuration studied in Section 4.1. Indeed, condition (35) rules out finite time extinction as an optimal outcome: if it is not verified, we get, as in Section 4.2, a case of optimal finite time extinction. Since the root ξ is an increasing function of the life span T (this will be proved in next proposition) one can also interpret condition (35) as putting a first threshold value for T below which finite extinction is optimal. Above this first threshold, either sustainable positively growing or asymptotically vanishing populations (and economies) are optimal. In particular, note that when $T < \frac{b}{a}$, $\xi = 0$ and therefore $g < 0$: in this case we necessarily have asymptotic extinction. Sustainable growth is not guaranteed even if $T > \frac{b}{a}$ because even if in this case the root $\xi > 0$, it is not necessarily bigger than ρ for g to be necessarily positive. Just like in the Benthamite case, there should exist a second threshold value of life span above which positive growth is optimal. This is formalized in the next proposition.

Proposition 5.1 *Under the hypotheses of Theorem 5.1, fixed a and b, the constant ξ and then the growth rate g are strictly increasing in $T \in \left(\frac{b}{a}, +\infty\right)$. Moreover, once the constants a and b are chosen, writing g as function of T,*

$g(T)$, we have that

$$\lim_{T \to \frac{b}{a}^+} g(T) = \frac{-\rho}{\sigma}$$

and

$$\lim_{T \to +\infty} g(T) = \frac{\frac{a}{b} - \rho}{\sigma}.$$

So, if $\frac{a}{b} > \rho$, there exists $T_1 \in \left(\frac{b}{a}, +\infty\right)$ such that, for every $T > T_1$ the growth rate of the population g is positive while, for every $T < T_1$, g is negative.

Finally the transition dynamics in the impure altruism case can be described in detail.

Proposition 5.2 *Under the hypotheses of Theorem 5.1 the following limits exist*

$$\lim_{t \to \infty} \frac{n^*(t)}{e^{gt}} =: n_L$$

and

$$\lim_{t \to \infty} \frac{N^*(t)}{e^{gt}} =: N_L.$$

Moreover, if $g \neq 0$ we have:

$$n_L = \frac{\Lambda}{\frac{a}{bg}(1 - e^{-gT}) - 1}$$

and

$$N_L = \frac{b}{a}(n_L + \Lambda) = \frac{\Lambda(1 - e^{-gT})}{\frac{a}{b}(1 - e^{-gT}) - g} = n_L \cdot \frac{1 - e^{-gT}}{g}.$$

In particular, if $\rho > \xi$ (i.e. if $T < T_1$) in the long run $N(t)$ and $n(t)$ go to zero exponentially, if $\rho < \xi$ ((i.e. if $T > T_1$) they grow exponentially with rate g defined in (37), if $\rho = \xi$ they stabilize respectively to n_L and N_L. Moreover

$$\lim_{t \to \infty} c^*(t) = \lim_{t \to \infty} \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - \frac{bg}{1 - e^{-gT}}.$$

Finally $c(t)$, detrended $n(t)$ and detrended $N(t)$ exhibit oscillatory convergence to their respective asymptotic values.

The proposition shows that, as in the Benthamite case and despite the extra non-autonomous term, the economy will converge to a balanced growth path at rate g given in equation (37). As before, the long-run levels corresponding to total population and cohort sizes depend on the initial procreation profile via the parameter Λ. It should be noted here that despite the latter feature, both per capita consumption and the fertility rate are independent of the parameter Λ in the long-run. Therefore, and though the two latter variables do show up transition dynamics, they converge to magnitudes which are independent of the initial conditions, contrary to the traditional AK model. So the intermediate cases studied here give rise to a weaker form of convergence in the standards of living compared to the Benthamite case. This said, in all cases where growth is optimal in the long run, we have the same picture: differences in historical demography yield different long term optimal population sizes (the younger is the population the higher is the level N_L) but identical optimal per capita consumption and fertility rates in the long run.

6 Conclusion

In this paper, we have introduced the realistic assumption of finite lives into an otherwise standard optimal population size problem. By taking advantage of some recent developments in the optimization of infinite-dimensioned problems, we have been able to fully characterize the optimal dynamics of the resulting problems. Within a very simple AN setting, we have highlighted the role of the value of individuals' lifetime in optimal dynamics, and the highly differentiated optimal outcomes of the Millian Vs Benthamite cases. We have also characterized finely the implications of some intermediate welfare functions.

Of course, our analytical approach cannot be trivially adapted to handle natural extensions of our model (through the introduction of capital accumulation or natural resources for example, or the incorporation of nonlinear production functions). We believe however that this first step into the analysis of optimal dynamics in optimal population size problems is an important enrichment of the ongoing debate. It is especially interesting because it follows from a very natural assumption: individuals have finite lives, and this feature can only be crucial for the outcomes of the optimal population size problem.

References

K. Arrow, A Bensoussan, Q. Feng, and S. P. Sethi. The genuine savings criterion and the value of population in an economy with endogenous fertility rate, in R. Boucekkine, N. Hritonenko and Y. Yatsenko (eds.), *Optimal Control of Age-structured Populations in Economy, Demography, and the Environment*, Taylor and Francis, Boca Raton, 2010.

G. Asheim. Unjust intergenerational allocations. Journal of Economic Theory 54, 350-371 (1991)

A. Baranzini and F. Bourguignon. Is sustainable growth optimal? . International Tax and Public Finance 2, 341-356 (1995).

W. F. Basener and D. Ross. Booming and Crashing Populations and Easter Island. SIAM Journal of Applied Mathematics 65, 684-701 (2004).

A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter. *Representation and control of infinite dimensional system. Second edition.* Birkhäuser, Boston, 2007.

R. Boucekkine, G. Fabbri and F. Gozzi. Maintenance and investment: complements or substitutes? A reappraisal. Journal of Economic Dynamics and Control 34, 2420-2439 (2010).

R. Boucekkine and G. Fabbri. Assessing the Parfit's Repugnant Conclusion within a canonical endogenous growth set-up. DP 2010-20, department of economics, University of Glasgow (2010).

R. Boucekkine and D. de la Croix. Optimal apocalypse. Manuscript, IRES, Université catholique de Louvain (2009).

R. Boucekkine, O. Licandro, L. Puch and F. del Río. Vintage capital and the dynamics of the AK model. Journal of Economic Theory 120, 39-76 (2005).

R. Boucekkine, M. Germain and O. Licandro. Replacement Echoes in the Vintage Capital Growth Model. Journal of Economic Theory 74, 333-348 (1997).

R. Boucekkine, D. de la Croix and O. Licandro. Vintage human capital, demographic trends and growth. Journal of Economic Theory 104, 340-375 (2002).

A. Bucci. Population growth in a model of economic growth with human capital accumulation and horizontal R&D. Journal of Macroeconomics 30, 1124-1147 (2008).

P. Dasgupta. Regarding optimum population. Journal of Political Philosophy 13, 414-442 (2005).

P. Dasgupta. On the concept of optimum population. Review of Economic Studies 36, 295-318 (1969).

D. de la Croix and D. Dottori. Easter Islands collapse: a tale of a population race. Journal of Economic Growth 13, 27-55 (2008).

O. Diekmann, S. A. Van Gils, S. M. Verduyn Lunel and H. O. Walther. *Delay equations*. Springer, Berlin, 1995.

F. Edgeworth. *Papers relating to political economy*. Vol. III, Macmillan, London, 1925.

L. Epstein. Intergenerational consumption rules: an axiomatization of utilitarianism and egalitarianism. Journal of Economic Theory 38, 280-297 (1986).

G. Fabbri and F. Gozzi. Solving optimal growth models with vintage capital: The dynamic programming approach. Journal of Economic Theory 143, 331-373 (2008).

O. Galor and O. Moav. The Neolithic revolution and contemporary variations in life expectancy . DP 2007-14, Brown University (2007).

O. Galor and O. Moav. Natural selection and the origin of economic growth. Quarterly Journal of Economics 117, 1133-1191 (2002).

O. Galor and D. N. Weil. From Malthusian stagnation to modern growth. The American Economic Review 89, 150–154 (1999).

L. Goldwasser. The biodemography of life span: resources, allocation and metabolism. Trends in Ecology and Evolution 16, 536-538 (2001).

J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*. Springer-Verlag, Berlin, 1993.

P. Makdissi. Population, ressources naturelles et droits de propriété. Annales d'Economie et Statistique 61, 93-104 (2001).

M. Nerlove, A. Razin and E. Sadka. Population size: individual choice and social optima. The Quarterly Journal of Economics 100(2), 321-334 (1985).

T. Palivos and C.K. Yip. Optimal population size and endogenous growth. Economics letters 41(1), 107-110 (1993).

D. Parfit. *Reasons and persons*. Oxford University Press, Oxford, 1984.

A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.

A. Razin and C. Yuen. Utilitarian tradeoff between population growth and income growth. Journal of Population Economics 8, 81-87 (1995).

H. Strulik. The Role of human capital and population growth in R&D-based models of economic growth. Review of International Economics 13, 129-145 (2005).

J. Yong and X. Y. Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*. Springer Verlag, Berlin, 1999.

A The case $\gamma = 1$: the infinite-dimensional setting and the proof of Theorem 4.1

We denote by $L^2(-T,0)$ the space of all functions f from $[-T,0]$ to $\mathbb{R}$ that are Lebesgue measurable and such that $\int_{-T}^0 |f(x)|^2 \, \mathrm{d}x < +\infty$. It is an Hilbert space when endowed with the scalar product $\langle f, g\rangle_{L^2} = \int_{-T}^0 f(x)g(x)\,\mathrm{d}x$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-T,0)$ (with the scalar product $\langle (x_0,x_1),(z_0,z_1)\rangle_{M^2} := x_0 z_0 + \langle x_1, z_1\rangle_{L^2}$). Following Bensoussan et al. (2007) (see Chapter II-4 and in particular Theorem 5.1), given an admissible control $n(\cdot)$ and the related trajectory $N(\cdot)$, if we define $x(t) = (x_0(t), x_1(t)) \in M^2$ for all $t \geq 0$ as

$$\begin{cases} x_0(t) := N(t) \\ x_1(t)[r] := -n(t-T-r), \qquad \text{for all } r \in [-T,0), \end{cases} \tag{38}$$

we have that $x(t)$ satisfies the following evolution equation in M^2:

$$\dot{x}(t) = A^* x(t) + B^* n(t). \tag{39}$$

where A^* is the adjoint of the generator of a C_0-semigroup[27] A defined as[28]

$$\begin{cases} D(A) \overset{def}{=} \{(\psi_0,\psi_1) \in M^2 \; : \; \psi_1 \in W^{1,2}(-T,0), \; \psi_0 = \psi_1(0)\} \\ A \colon D(A) \to M^2, \qquad A(\psi_0,\psi_1) \overset{def}{=} (0, \tfrac{\mathrm{d}}{\mathrm{d}s}\psi_1) \end{cases} \tag{40}$$

and B^* is the adjoint of $B \colon D(A) \to \mathbb{R}$ defined as $B(\psi_0,\psi_1) := (\psi_1[0] - \psi_1[-T])$. Moreover, using the new variable $x \in M^2$ defined in (38) we can rewrite the welfare functional as

$$\int_0^{+\infty} e^{-\rho t} \frac{(a x_0(t) - b n(t))^{1-\sigma}}{1-\sigma} x_0^\sigma(t)\,\mathrm{d}t.$$

Our optimal control problem of maximizing the welfare functional (22) over the set $\mathcal{V}_{n_0}$ defined in (16) with the state equation (3) can be equivalently rewritten as the problem of maximizing the functional above with the state equation (39) over the same set $\mathcal{V}_{n_0}$ (if we read x_0 instead of N in the definition (16)). The value function V depends now on the new variable x that can be expressed in term of the datum n_0 using (38) for $t = 0$. The associated Hamilton-Jacobi-Bellman equation for the unknown v is[29]:

$$\rho v(x) = \langle x, ADv\rangle_{M^2} + \sup_{n \in \left[0, \frac{a}{b} x_0\right]} \left(nBDv(x) + \frac{(a x_0 - bn)^{1-\sigma}}{1-\sigma} x_0^\sigma \right). \tag{41}$$

As far as

$$BDv > a^{-\sigma} b \tag{42}$$

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[27] See e.g. Pazy (1983) for a standard reference to the argument.

[28] $W^{1,2}(-T,0)$ is the set $\left\{ f \in L^2(-T,0) \; : \; \partial_\omega f \in L^2(-T,0) \right\}$ where $\partial_\omega f$ is the distributional derivative of f.

[29] Dv is the Gateaux derivative.

the supremum appearing in (41) is a maximum and the unique maximum point is strictly positive (since $x_0 > 0$) and is

$$n_{max} := \frac{a}{b}\left(1 - \left(\frac{BDv(x)}{a^{-\sigma}b}\right)^{-1/\sigma}\right)x_0 \tag{43}$$

so (41) can be rewritten as

$$\rho v(x) = \langle x, ADv\rangle_{M^2} + \frac{a}{b}x_0 BDv(x) + \frac{\sigma}{1-\sigma}x_0\left(\frac{1}{b}BDv(x)\right)^{1-\frac{1}{\sigma}}. \tag{44}$$

When

$$BDv \leq a^{-\sigma}b \tag{45}$$

then the supremum appearing in (41) is a maximum and the unique maximum point is $n_{max} := 0$. In this case (41) can be rewritten as

$$\rho v(x) = \langle x, ADv\rangle_{M^2} + \frac{a^{1-\sigma}x_0}{1-\sigma} \tag{46}$$

We expect that the value function of the problem is a (the) solution of the HJB equation. Since it is not hard to see that the value function is 1-homogeneous, we look for a linear solution of the HJB equation. We have the following result:

Proposition A.1 *Suppose that (23) (and then (25)) holds and* $\sigma \in (0,1)$*. If*

$$\beta > \rho(1-\sigma) \tag{47}$$

then the function

$$v(x) := \alpha_1\left(x_0 + \int_{-T}^{0} x_1(r)e^{\rho r}\,\mathrm{d}r\right) \tag{48}$$

where

$$\alpha_1 = a^{1-\sigma}\frac{1}{\beta}\left(\frac{1-\sigma}{\sigma}\cdot\frac{\rho-\beta}{\beta}\right)^{-\sigma}$$

is a solution of (44) in all the points s.t. $x_0 > 0$*.*

On the other side, if

$$\beta \leq \rho(1-\sigma) \tag{49}$$

then the function

$$v(x) := \alpha_2\left(x_0 + \int_{-T}^{0} x_1(r)e^{\rho r}\,\mathrm{d}r\right) \tag{50}$$

where

$$\alpha_2 = \frac{a^{1-\sigma}}{\rho(1-\sigma)}$$

is a solution of (46) in all the points s.t. $x_0 > 0$*.*

Proof. Let $i = 1, 2$. We first observe that the function v is C^1 (since it is linear). Setting $\phi(r) = e^{\rho r}$, $r \in [-T, 0]$ we see that its first derivative is constant and is

$$Dv(x) = \alpha_i (1, \phi) \qquad \text{for all } x \in M^2$$

Looking at (40) we also see that such derivative belongs to $D(A)$ so that all the terms in (41) make sense. We have $ADv(x) = (0, \alpha_i \rho \phi)$ and $BDv(x) = \alpha_i (1 - e^{-\rho T})$. Then, thanks to (47) (resp. (49)) we have that (42) (resp. (45)) is satisfied and (41) can be written in the form (44) (resp. (46)). To verify the statement we have only to check it directly: the left hand side of (44) (resp. (46)) is equal to $\rho \alpha_i \left(x_0 + \langle x_1, \phi \rangle_{L^2} \right)$. The right hand side is, for $i = 1$,

$$\langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + \frac{a}{b} x_0 \alpha_1 (1 - e^{-\rho T}) + \frac{\sigma}{1 - \sigma} x_0 \left(\frac{1}{b} \alpha_1 (1 - e^{-\rho T}) \right)^{1 - \frac{1}{\sigma}}$$

$$= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \alpha_1 \beta + \frac{\sigma}{1 - \sigma} x_0 \left(\frac{\alpha_1 \beta}{a} \right)^{1 - \frac{1}{\sigma}}$$

$$= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \frac{\alpha_1 \beta}{a} \left[1 + \frac{\sigma}{1 - \sigma} \left(\frac{\alpha_1 \beta}{a} \right)^{-\frac{1}{\sigma}} \right].$$

Since the expression in square brackets is equal to $a\rho/\beta$ thanks to the definition of α_1, we have the claim for $i = 1$. For $i = 2$ the right hand side of (46) is (using the expression of α_2 above)

$$\langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \frac{a^{1-\sigma}}{1 - \sigma} x_0 = \langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \alpha_2 \rho x_0$$

and this proves the claim for $i = 2$. □

Once we have a solution of the Hamilton-Jacobi-Bellman equation we can prove that it is the value function and so use it to find a solution of our optimal control problem in feedback form.

Theorem A.1 *Suppose that (23) (and then (25)) holds and $\sigma \in (0, 1)$. If (47) holds then the function v defined in (48) is the value function V and there exists a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ and the related trajectory $x^*(\cdot)$ satisfy the following equation:*

$$n^*(t) = \frac{a}{b} \left(1 - (\alpha_1 \beta)^{-\frac{1}{\sigma}} \right) x_0^*(t) = \theta x_0^*(t) \tag{51}$$

where θ is given by (28). If (49) is satisfied then the function v defined in (50) is the value function V and there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ is identically zero.

Proof. The proof follows the arguments of the one of Proposition 2.3.2. in Fabbri and Gozzi (2008) with various modifications due to peculiarity of our problem. We do not write the details for brevity. □

Proof of Theorem 4.1. Theorem 4.1 is nothing but Theorem A.1 once we write again $N^*(\cdot)$ instead of $x_0^*(\cdot)$. In particular (51) becomes (29). Finally, if we write $N^*(t)$ as $\int_{t-T}^{t} n(s)\,\mathrm{d}s$ and we take the derivative in (51) we obtain (31). □

B Other proofs

Proof of Proposition 2.2. Thanks to Lemma 2.1 it is enough to prove the statement for $N_{MAX}(t)$. Let us take $\bar{t} \in \arg\max_{s\in[T,2T]} N_{MAX}(s)$ (the argmax is non-void because N_{MAX} is continuous on $[0,+\infty)$). We have that $N_{MAX}(\bar{t}) = \frac{a}{b}\int_{\bar{t}-T}^{\bar{t}} N_{MAX}(s)\,\mathrm{d}s \leq a/b(2T-\bar{t})\max_{s\in[0,T]} N_{MAX}(s) + a/b(\bar{t}-T)N_{MAX}(\bar{t})$ so $N_{MAX}(\bar{t}) \leq \frac{a/b(2T-\bar{t})}{1-a/b(\bar{t}-T)}\max_{s\in[0,T]} N_{MAX}(s)$. Observe that, for all $\bar{t} \in [T,2T]$ we have that $\frac{a/b(2T-\bar{t})}{1-a/b(\bar{t}-T)} \in \left[0, \frac{a}{b}T\right]$, so $\max_{s\in[T,2T]} N_{MAX}(s) \leq \frac{a}{b}T\max_{s\in[0,T]} N_{MAX}(s)$. In the same way we can prove that, for all positive integer n, $\max_{s\in[nT,(n+1)T]} N_{MAX}(s) \leq \left(\frac{a}{b}T\right)^n \max_{s\in[0,T]} N_{MAX}(s)$. Since, by hypothesis, $\left(\frac{a}{b}T\right) < 1$ we have that $\lim_{t\to+\infty} N_{MAX}(t) = 0$ and then the claim. □

Proof of Proposition3.1. We give the proof in the case $\gamma > 0$. The case $\gamma = 0$ is simpler.

Part (i): Since the control problem is now one dimensional the value function v in this case depends only on the variable N. The associated Hamilton-Jacobi-Bellman equation is given by

$$\rho v(N) = \sup_{n\in[0,aN/b]}\left(nv'(N) + \frac{\left(a-\frac{bn}{N}\right)^{1-\sigma}}{1-\sigma}N^{\gamma}\right) = 0.$$

One can directly verify that the function $v(N) = \alpha N^{\gamma}$, where $\alpha := \frac{b}{\gamma}\left(\frac{\rho b - a\gamma}{\gamma}\frac{1-\sigma}{\sigma}\right)^{-\sigma}$, is a solution of the above Hamilton-Jacobi-Bellman equation. So, using a standard verification argument (see for example Yong and Zhou (1999) Section 5.3), one proves that such v is indeed the value function of the problem and that the induced feedback map, given by

$$n = \phi(N) := \arg\max_{n\in[0,aN/b]}\left(nv'(N) + \frac{\left(a-\frac{bn}{N}\right)^{1-\sigma}}{1-\sigma}N^{\gamma}\right) = \theta N,$$

is the (unique) optimal feedback map of the problem. It turns out that the related trajectory, i.e. the unique solution of $\dot{N}^*(t) = \phi(N^*(t)) = \theta N^*(t)$, $N^*(0) = N_0$, i.e. $N^*(t) = N_0 e^{\theta t}$ is the (unique) optimal trajectory of the problem and so that the control $n^*(t) = \theta N^*(t) = \theta N_0 e^{\theta t}$ is the (unique) optimal control. The expression for $c^*(t)$ follows using (6). Evaluating the utility along the trajectory $N^*(t)$ one can verify that the condition (20) is indeed necessary and sufficient for the boundedness of the functional. Part (ii) can be proved using the same kind of arguments. □

Proof of Proposition 4.2. Since $n^*(\cdot)$ solves (31) it can be written (see Diekmann et al., 1995, page 34) as a series

$$n^*(t) = \sum_{j=1}^{\infty} p_j(t) e^{\lambda_j t}$$

where $\{\lambda_j\}_{j=1}^{+\infty}$ are the roots of of the characteristic equation (32) (studied in Fabbri and Gozzi, 2008, Proposition 2.1.8) and $\{p_j\}_{j=1}^{N}$ are $\mathbb{C}$-valued polynomial. If $\theta T > 1$, as already observed in Subsection 2.2 there exists a unique strictly positive root $\lambda_1 = h$. Moreover $h \in (0, \theta)$ and it is also the root with biggest real part (and it is simple). The polynomial p_1 associated to h is a constant (since h is simple) and can be computed explicitly (see for example Hale and Lunel (1993) Chapter 1, in particular equations (5.10) that gives the expansion of the fundamental solution and Theorem 6.1) obtaining that p_1 is constant and

$$p_1(t) \equiv \frac{\theta}{1 - T(\theta - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s$$

this gives the limit for $n(t)^*/e^{ht}$. The limit for $N(t)^*/e^{ht}$ follows from the relation $N^*(t) = \int_{t-T}^{t} n^*(s)\, \mathrm{d}s$.

If $\theta T < 1$ each λ_j, for $j \geq 2$, has negative real part while $\lambda_1 = 0$ is the only real root. But again if we compute explicitly the polynomial p_1 (again a constant value) related to the root 0 we have

$$p_1(t) \equiv \frac{\theta N_0 + (-\theta) \int_{-T}^{0} n_0(r)\, \mathrm{d}r}{1 + \theta T} = \frac{\theta N_0 - \theta N_0}{1 + \theta T} = 0.$$

so only the contributions of the roots with negative real parts remain. This concludes the proof of the claims related to asymptotic behavior of detrended variables.

Let us prove now that h is an increasing function of T (recall that θ depends on T too). We use the implicit function theorem. Define

$$F(\lambda, T) = \theta(T)(1 - e^{-T\lambda}) - \lambda.$$

Given T such that $\theta(T)T > 1$ one has that $F(\lambda, T)$ is concave in λ, $F(0, T) = 0$ and $F(h, T) = 0$ (recall that $h \in (0, \theta(T))$). So it must be

$$\left.\frac{\partial}{\partial \lambda} F(\lambda, T)\right|_{\lambda = h} = \theta(T) T e^{-Th} - 1 < 0.$$

Moreover, since by the definition of θ in (28) we easily get $\theta'(T) > 0$, we have:

$$\frac{\partial F(h,T)}{\partial T} = \theta'(T)(1 - e^{-Th}) + \theta(T)he^{-Th} > 0$$

Now, by the implicit function theorem we have

$$\frac{\mathrm{d}h}{\mathrm{d}T} = -\frac{\partial F}{\partial T}\left(\frac{\partial F}{\partial \lambda}\Big|_{\lambda=h}\right)^{-1} > 0$$

and this concludes the proof. □

Proof of Theorem 5.1. The statements follows from Lemma 2.3.3 and Theorem 2.3.4 of Fabbri and Gozzi (2008): here we have the control variable n instead of i and the state variable N instead of k. The state equation is the same. To rewrite the objective functional exactly in the form of the problem treated in Fabbri and Gozzi (2008) we only need to write

$$aN(t) - bn(t) = b\left(\frac{a}{b}N(t) - n(t)\right)$$

so the functional becomes

$$b^{1-\sigma}\int_0^{+\infty} e^{-\rho t}\frac{\left(\frac{a}{b}N(t) - n(t)\right)^{1-\sigma}}{1-\sigma}\,\mathrm{d}t.$$

The constant $b^{1-\sigma}$ as it does not changes the optimal trajectories. Dropping it the functional is the same as the one of Fabbri and Gozzi (2008) where the constant a is substituted here by $\frac{a}{b}$. □

Proof of Proposition 5.1. Define $F(z,T) = \left(\frac{a}{b} - \frac{a}{b}e^{-zT}\right) - z$. For a fixed T, the function $F(z,\cdot)\colon \mathbb{R} \to \mathbb{R}$, $z \mapsto F(z,T)$, is concave and it can be seen with elementary arguments that, when $\frac{a}{b}T > 1$, it has has exactly two zeros, the first in 0 and the second in $\xi(T) > 0$ and so $\frac{\partial F}{\partial z}(\xi(T),T) < 0$ i.e. $\left(T\frac{a}{b}e^{-\xi T} - 1\right) < 0$. To show that $\xi(T)$ (and then $g(T)$) is increasing in T in the interval $T \in (b/a, +\infty)$ we can apply the implicit function theorem:

$$\frac{\mathrm{d}\xi}{\mathrm{d}T} = -\left(\frac{\partial F}{\partial z}|_{z=\xi}\right)^{-1}\left(\frac{\partial F}{\partial T}\right) = -\left(T\frac{a}{b}e^{-\xi T} - 1\right)^{-1}\left(\xi\frac{a}{b}e^{-\xi T}\right) > 0. \tag{52}$$

So, since $\xi(T)$ (and then $g(T)$) is continuous in T and strictly increasing, there exist the two limits (that can be non-finite) $\underline{\xi} := \lim_{T\to\frac{b}{a}^+}\xi(T)$ and $\overline{\xi} := \lim_{T\to+\infty}\xi(T)$. Moreover, since $\xi(T)$ is strictly positive and increasing in T one has that $\lim_{T\to\infty} e^{-\xi(T)T} = 0$, so, since, for all $T > b/a$, $F(\xi(T),T) = 0$, we have

$$0 = \lim_{T\to+\infty} F(\xi(T),T) = \lim_{T\to+\infty}\left(\frac{a}{b} - \frac{a}{b}e^{-\xi(T)T}\right) - \xi(T) = \frac{a}{b} - \overline{\xi}.$$

This implies $\overline{\xi} = \frac{b}{a}$ and so $\lim_{T\to+\infty} g(T) = \frac{\frac{a}{b}-\rho}{\sigma}$. The same argument allows to get the statement when $T \to \frac{b}{a}^+$. □

Proof of Proposition 5.2. Arguing as in the proof of Theorem 5.1 the statement is equivalent to that of Proposition 2.3.5 in Fabbri and Gozzi (2008). □

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ISSN 1379-244X D/2011/3082/018