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ALLOCATING ENVIRONMENTAL COSTS AMONG HETEROGENEOUS SOURCES: THE LINEAR DAMAGE EQUIVALENT MECHANISM*

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Abstract

A group of firms has to divide the costs associated with environmental damages jointly generated as a by-product of their heterogeneous production activities. We propose a specific procedure to assign costs, the *Linear Damage Equivalent Mechanism* (LDE), which satisfies several appealing strategic and axiomatic properties. The LDE induces a strategic game that has an unambiguous noncooperative prediction, a unique Nash equilibrium which is also robust to coalitional deviations; moreover, the equilibrium is efficient. Among its other properties, we find that the LDE is immune to arbitrary changes in the units of account of the outputs.

Keywords: Environmental damages; Cost-sharing; Heterogeneous sources.

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1. Introduction

Environmental problems are often caused by more than one agent. An illustrative example is that of several firms who share a hazardous waste site which may adversely affect neighboring communities. Under the *polluter-pays-principle*, those who pollute are financially responsible for any damages. We ask how polluting agents should divide the costs associated with such damages. The problem is complicated by the fact that, normally, it is not possible to identify the individual contribution of each agent to overall damages. In the *homogeneous case,* firms engage in the same type of damage-causing activity and overall damages depend on the aggregate level of activity. Thus, although we cannot identify the individual contributions to overall damages, we can identify the individual contributions to aggregate activity. We can then use the comparison of levels of activity as a means of determining the respective cost assignments. However, when firms engage in different types of damage-causing activities – the *heterogeneous case* – even such indirect comparisons are no longer feasible.

In this paper, we propose a specific procedure, the *Linear Damage Equivalent Mechanism* (LDE), to solve the cost-sharing problem in heterogeneous cases. We now discuss briefly how it works.

Imagine two firms that, as a result of their productive processes, cause pollution. In the simplest case, damages are a linear function of their output levels y^1 and y^2, that is, $D^L(y^1, y^2) = p^1 y^1 + p^2 y^2$, where $p^i > 0$ represents the marginal contribution of output y^i to damages. In this case, we think all would agree that the obvious apportionment of damages is that in which each agent pays its individual contribution, i.e., $p^i y^i$. This solution is fair: on what basis should one agent pay part of the contribution of the other? Moreover, as the production of one agent does not affect the cost assignment of the other agent, it leads firms to produce the efficient output levels.[1] In consequence, this division of damages leads to an obvious distribution of total surplus (total benefits from production minus resulting damages), namely that in which each agent receives its private benefits from production and pays its individual contribution to damages.

Consider now a more general situation in which it is not possible to separate that part of overall damages for which each agent is responsible. Suppose, however, that the total surplus is equal to that which would have occurred under some, possibly fictitious, linear damage technology. Then, having agreed how to distribute the surplus in the linear context, we might deem such a distribution appropriate here as well. The LDE consists of identifying an appropriate reference linear technology and guarantying each agent the individual surplus it would obtain in the linear case. This permits us to obtain the respective cost assignments of actual damages.

This procedure is based on an original idea suggested by Mas-Colell (1980) in the context of a one input-one output- production model[2], which was later referred to as the *Constant Returns Equivalent Mechanism* (CRE) in Moulin (1987). It consists of selecting an efficient allocation for an arbitrary production function such that the same utility vector is achieved under some, possibly fictitious, constant returns to scale technology.[3] Several papers (Moulin (1987, 1990a, 1990b), Moulin and Roemer (1989) and Fleurbaey and Maniquet (1994)) offer alternative characterizations of this first-best solution on axiomatic grounds.

Unlike the CRE, the LDE is defined not only along the efficient frontier but for the entire feasible set. That is, with each list of feasible outputs, we associate a

[1] Note that this solution to the sharing problem (i.e., each agent paying its own contribution) is valid not only for the linear case but for every damage function separable in both outputs, that is, for every case in which we can perfectly identify the contribution of each injurer to damages.

[2] This model refers to the problem in which agents contribute some homogeneous input (say labor) to jointly produce a single output which is then to be divided among them. The homogeneity is reflected in the fact that input contributions enter additively in the production function.

[3] To be precise, the CRE was originally defined as a *surplus-sharing* mechanism. However, the cost-sharing problem and the surplus sharing problem are reciprocal; the former refers to allocating the costs associated with given outputs and the latter to allocating the benefits associated with given inputs. Consequently, the same mechanism can be easily adapted to both.

reference linear damage technology which will enable us to identify the corresponding assignment of actual damages. Consequently, we can consider the strategic behavior of the agents under the mechanism.

In this paper, we study two types of properties of the LDE: its axiomatic properties (i.e., those which pertain in general) and its strategic properties (i.e., those which pertain in equilibrium). Regarding the former, we establish that the LDE satisfies various equity and/or monotonicity properties which have appeared in the cost-sharing literature. In addition, the LDE is *scale invariant*, that is, the cost assignments are immune to arbitrary rescaling of the units.[4]

Concerning the strategic properties, suppose firms know the LDE will be employed to apportion the damages. Then, the mechanism induces a game where each firm chooses its output level. We establish that the induced game has an unambiguous noncooperative prediction, a unique Nash equilibrium which is also strong (that is, robust to coalitional deviations as well). Moreover, the equilibrium is efficient. This result constitutes a clear advantage with respect to the previous literature in which the CRE allocation was required to be efficient by definition. We instead obtain efficiency as a result of the strategic behavior by firms.

Two novel features of our study are the following. First, we find that under the LDE there may be circumstances under which one agent is subsidized by the remaining agents. This result is in sharp contrast with the existing cost-sharing literature, in which there is no allowance for subsidies. However, as agents share costs, it is conceivable that an otherwise profitable firm might be persuaded to restrict its output level thus generating a positive pecuniary externality for the remaining agents. If it were to do so, it would warrant compensation for its forgone profits.

Second, in heterogeneous cases, the direct comparison of levels of activity has no obvious meaning. Consequently, we propose two measures that capture the relevance

[4]In our opinion, this is an important property for heterogeneous problems because, in principle, the choice of units of account of different outputs may seem arbitrary, as outputs are of a different nature.

of each firm in the problem as a whole: the *relative damage impact* and the *relative profitability*, which evaluate the effect on damages and benefits, respectively, of an increase in one output relative to that of another. We demonstrate that the LDE induces firms to react appropriately: it induces a firm to produce less when it becomes more harmful (that is, when its relative damage impact increases), and to produce more when it becomes more profitable (that is, when its relative profitability increases).

Note that the LDE is defined under a complete information structure, i.e., the regulator is assumed to know both agents' benefit functions and the damage technology. Within this context, we have found a cost-sharing mechanism that is efficient, budget-balanced (i.e., costs are exactly covered) and which induces a game with a unique non-cooperative equilibrium. Alternatively, Green, Kohlberg and Laffont (1976) showed that, in situations in which the regulator knows the damage function but does not have perfect information about the benefit functions of the agents, there is no cost-sharing mechanism which satisfies the three above mentioned properties. Consequently, this delimits the trade-off between performance and information.

To contrast the LDE with other known cost-sharing mechanisms, we distinguish between methods for the homogeneous and for the heterogeneous cases. In the homogeneous case, two prominent cost-sharing methods have been analyzed, both from an axiomatic and a strategic perspective: *Average Cost Pricing* (ACP) and *Serial Cost Sharing* (SCS) (Moulin and Shenker (1992, 1994)). ACP charges the average cost of aggregate production per unit of output, and SCS allocates the incremental costs of production evenly among those firms that produce such units.[5] These methods are characterized by different strategic properties, both of which are satisfied by the LDE. On the one hand, ACP is immune to manipulations by coalitions in the case in which

[5]For example, if production levels are $y^1 \leq y^2 \leq ... \leq y^n$, the cost that would result if the n agents were each to produce y^1 is divided equally among them; the incremental cost when agent 1 produces y^1 and the remaining agents were each to produce y^2 is divided equally among agents 2 through n; and so on.

output is freely transferable among agents, while SCS is not.[6] On the other hand, the strategic game induced by SCS has a unique noncooperative equilibrium (a Nash equilibrium which is also strong) for every profile of convex preferences, provided agents cannot freely transfer output among themselves; ACP does not satisfy this property.[7] Moreover, equilibrium outcomes under either SCS or ACP are generally inefficient.

In the heterogeneous case, there are only interesting strategic results for the discrete case (where output is indivisible). Moulin (1996) proposes the family of *Incremental Cost Sharing Methods* (ICS).[8] Such methods are characterized, under increasing marginal costs and supermodular costs, by the property that they induce a game with a unique noncooperative equilibrium. However, they are not scale invariant.

In the continuous heterogeneous case, there are three main cost-sharing mechanisms: *Shapley-Shubik Cost Sharing* (Shapley (1953), Shubik (1962)), *Aumann-Shapley Pricing* (Billera and Heath (1982), Tauman (1988)) and an extension of *Serial Cost Sharing* for the heterogeneous case (Friedman and Moulin (1995)). The first two do not have nice strategic properties. With respect to the latter, its strategic properties have not yet been explored; however, it does not satisfy scale invariance. Generally, all these cost-sharing mechanisms for the heterogeneous case are inefficient.

To our knowledge, no existing cost-sharing mechanism for the heterogeneous case (either in the complete or in the incomplete information context) satisfies the strategic and efficiency properties of the LDE. Moreover, those methods that have nice strategic properties fail to satisfy scale invariance. Consequently, for the complete information case considered here, the LDE provides an attractive method for allocating environmental costs among heterogeneous sources.

[6] Under SCS, if output were freely transferable, the members of any coalition of agents would find it profitable to "announce" their average production levels.

[7] In general, the game induced by ACP may not have a Nash equilibrium, or it may have multiple Nash equilibria.

[8] If production levels are $(y^1, y^2, ..., y^n)$, ICS consist of constructing a sequence of $\sum_i y^i$ elements, where agent i appears exactly y^i times, and each unit is allocated to the corresponding agent, while it pays the marginal cost.

The remainder paper is organized as follows. In the next section, we present the model and basic definitions. In Section 3, we define the Linear Damage Equivalent Mechanism. In Section 4, we study its axiomatic properties, and in Section 5, we investigate its strategic and efficiency properties. In Section 6, we discuss the possibility of situations in which one agent might be subsidized by the remaining agents. Section 7 contains several comparative statics results . We conclude in Section 8.

2. The Environmental Cost-Sharing Problem

We consider an industrial area composed of n firms, each producing a single output $y^i \in \mathbb{R}_+$, $i = 1, 2, ..., n$. The production process generates benefits for the firms represented by the functions $B^i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i = 1, 2, ..., n$, where B^i is twice continuously differentiable, strictly increasing, strictly concave and such that $B^i(0) = 0$.[9] We denote the set of all such functions by $\mathbb{B}$.

As a by-product of their production activities, the firms jointly generate environmental damages measured by the function $D : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$, where D is twice continuously differentiable, strictly increasing, convex and such that $D(\mathbf{0}) = 0$. $\mathbb{D}$ denotes the set of such functions.

We assume that for any $\mathbf{y} = (y^1, y^2, ..., y^n) \in \mathbb{R}^n_+$, the n agents are financially responsible for the resulting damages $D(\mathbf{y})$. Then, if $\boldsymbol{\phi} = (\phi^1, ..., \phi^n) \in \mathbb{R}^n$ represents the assignment of damages, the *individual surplus* for firm i is defined by $s^i(y^i, \phi^i) \equiv B^i(y^i) - \phi^i$.

Definition 2.1. *A feasible allocation is a vector* $\mathbf{f} = (\mathbf{y}, \boldsymbol{\phi}) \in \mathbb{R}^n_+ \times \mathbb{R}^n$ *such that* $\sum_{i=1}^n \phi^i = D(\mathbf{y})$. *We represent the set of feasible allocations by* $\mathbf{F}$.

[9] We use superscripts to denote agents. For functions of one variable, such as B^i, we use the prime notation to denote derivatives. For functions of several variables, we use subscripts to denote partial derivatives.

Definition 2.2. *A Pareto efficient allocation is a vector* $\mathbf{f} \in \mathbf{F}$ *such that there does not exist* $\hat{\mathbf{f}} \in \mathbf{F}$ *with* $s^i\left(\hat{y}^i, \hat{\phi}^i\right) \geq s^i\left(y^i, \phi^i\right)$ *for every* i*, and with strict inequality for some* i*.* $\mathbf{P}$ *denotes the set of Pareto efficient allocations.*

Let $\mathbb{S}(\mathbf{y})$ denote the total surplus generated by $\mathbf{y} \in \mathbb{R}^n_+$, i.e., $\mathbb{S}(\mathbf{y}) \equiv \sum_{i=1}^n B^i\left(y^i\right) - D(\mathbf{y})$. Under our assumptions, $\mathbb{S}(\mathbf{y})$ has a unique global maximum $\mathbf{y}^* \in \mathbb{R}^n_+$. Therefore, $\mathbf{P}$ can be expressed as follows:

$$\mathbf{P} = \left\{ (\mathbf{y}^*, \boldsymbol{\phi}) \in \mathbb{R}^n_+ \times \mathbb{R}^n \mid \sum_{i=1}^n \phi^i = D(\mathbf{y}^*) \right\} \tag{2.1}$$

Assuming an interior solution, $\mathbf{P}$ is characterized by the following conditions:

$$\begin{aligned} & B^{i\prime}\left(y^{i*}\right) = D_i\left(\mathbf{y}^*\right) \ , i = 1, 2, ..., n \\ & \sum_{i=1}^n \phi^i = D\left(\mathbf{y}^*\right) \\ & y^{i*} \in \mathbb{R}_+, \phi^i \in \mathbb{R}, \ i = 1, 2, ..., n \end{aligned} \tag{2.2}$$

i.e., by unique levels of output $\left(y^{1*}, ..., y^{n*}\right)$ such that marginal benefits equal marginal damages for each i, and by any assignment of the resulting damages.[10]

Until now, we have defined the assignment of damages $\boldsymbol{\phi}$ for a given $\mathbf{y}$. However, we wish to determine a rule to apportion damages resulting from any feasible $\mathbf{y}$. Formally,

Definition 2.3. *A sharing mechanism is a mapping* $\boldsymbol{\phi} : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$*, where* $\boldsymbol{\phi}(\mathbf{y}) = \left(\phi^1(\mathbf{y}), \phi^2(\mathbf{y}) ..., \phi^n(\mathbf{y})\right)$*, such that* $\sum_{i=1}^n \phi^i(\mathbf{y}) = D(\mathbf{y})$*. Let* $\boldsymbol{\Phi}$ *be the set of all sharing mechanisms.*

As in Kranich (1994), we assume that the sharing mechanism is imposed exogeneously and agents take it as given. Each sharing mechanism induces a game among

[10] If we do not allow for bankruptcy, which is the case of the *Linear Damage Equivalent Mechanism* described in Section 3, the cost assignments have both an upper and a lower bound. On the one hand, the individual surplus for agent i is nonnegative, that is, $B^i\left(y^{i*}\right) - \phi^i \geq 0$. On the other hand, the individual surplus for agent i can not be higher than total surplus, i.e., $B^i\left(y^{i*}\right) - \phi^i \leq \sum_{j=1}^n B^j\left(y^{j*}\right) - D\left(\mathbf{y}^*\right)$. Therefore, we have $D\left(\mathbf{y}^*\right) - \sum_{j \neq i} B^j\left(y^{j*}\right) \leq \phi^i \leq B^i\left(y^{i*}\right)$, for all i.

the agents, in the sense that agent i's choice affects agent j's payoff and vice versa. Formally, for every $\phi \in \Phi$, there exists an associated game $\Gamma^{\phi} = \left\{N, \mathbb{R}^n_+, h, \rho\right\}$, where $N = \{1, 2, ..., n\}$ is the set of players; $\mathbb{R}_+$ is the set of strategies for each player; h is the outcome function, i.e., $h : \mathbb{R}^n_+ \longrightarrow \mathbf{F}$, such that $h^i(\mathbf{y}) = (y^i, \phi^i(\mathbf{y}))$ for each i; and ρ is the payoff function, i.e., $\rho : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$, such that $\rho^i(\mathbf{y}) = B^i(y^i) - \phi^i(\mathbf{y})$, for each i.

Before presenting the game-theoretic solution concepts we will concentrate on, we introduce the following notation. Let S denote any coalition of players, that is $S \subseteq N$, $S \neq \emptyset$, and let $\mathbf{y}^S \in \mathbb{R}^S_+$ be a feasible production plan for S, i.e., $\mathbf{y}^S = (y^i)_{i \in S}$. $\left(\hat{\mathbf{y}} \mid \mathbf{y}^S\right) \in \mathbb{R}^n_+$ stands for the vector with ith component y^i, for every $i \in S$, and jth component $\hat{y}^j$, for every $j \notin S$. We abbreviate $\left(\hat{\mathbf{y}} \mid y^{\{i\}}\right)$ by $(\hat{\mathbf{y}} \mid y^i)$.

Definition 2.4. *A Nash equilibrium of Γ^{ϕ} is a vector $\hat{\mathbf{y}} \in \mathbb{R}^n_+$ such that $\rho^i(\hat{\mathbf{y}}) \geq \rho^i(\hat{\mathbf{y}} \mid y^i)$, for all $y^i \in \mathbb{R}_+$, $i = 1, ..., n$. A Nash equilibrium allocation is the outcome resulting from a Nash equilibrium of Γ^{ϕ}, that is, $h(\hat{\mathbf{y}}) = (\hat{\mathbf{y}}, \phi(\hat{\mathbf{y}}))$.*

Definition 2.5. *A strong equilibrium of Γ^{ϕ} is a vector $\hat{\mathbf{y}} \in \mathbb{R}^n_+$ such that there does not exist a coalition $S \subseteq N$, $S \neq \emptyset$, and a deviation $\mathbf{y}^S \in \mathbb{R}^S_+$ for S, such that, for all $i \in S$, $\rho^i\left(\hat{\mathbf{y}} \mid \mathbf{y}^S\right) > \rho^i(\hat{\mathbf{y}})$. A strong equilibrium allocation is the outcome resulting from a strong equilibrium of Γ^{ϕ}, that is, $h(\hat{\mathbf{y}}) = (\hat{\mathbf{y}}, \phi(\hat{\mathbf{y}}))$.*

3. The Linear Damage Equivalent Mechanism

Consider the simple case in which the damage function is linear. Thus, suppose $D^L(\mathbf{y}) = p^1y^1 + p^2y^2 + ... + p^ny^n$, where $p^i > 0$ is the marginal contribution of output y^i to damages. Observe that we can always factor each p^i into a common component $p > 0$ and an idiosyncratic component $\gamma^i > 0$, such that $p^i = p\gamma^i$, for all i. Therefore, we can write $D^L(\mathbf{y}) = p(\gamma^1y^1 + \gamma^2y^2 + ... + \gamma^ny^n)$.

The *standard solution* to the sharing problem in the linear case is for each agent to pay its individual contribution to overall damages. Each agent would then obtain his maximum individual surplus in the following way:

$$\sigma^i\left(p\gamma^i\right) \equiv \max_{y^i \in \mathbb{R}_+}\left\{B^i\left(y^i\right) - p\gamma^i y^i\right\}, \, i = 1, 2, ..., n \tag{3.1}$$

Note that, applying the envelope theorem in (3.1), we obtain $\sigma^{i\prime}\left(p\gamma^i\right) = -\tilde{y}^i\left(p\gamma^i\right) < 0$, where $\tilde{y}^i\left(p\gamma^i\right) \equiv \arg\max_{y^i \in \mathbb{R}_+}\left\{B^i\left(y^i\right) - p\gamma^i y^i\right\}$, and $\sigma^{i\prime\prime}\left(p\gamma^i\right) = -\tilde{y}^{i\prime}\left(p\gamma^i\right) > 0$. That is, maximum individual surplus in the linear case is strictly decreasing and convex in $p\gamma^i$.

Consider now a general nonlinear technology $D \in \mathbb{D}$. Here the solution to the sharing problem is no longer obvious. Suppose $\bar{\mathbf{y}} \in \mathbb{R}^n_+$ is given and imagine that actual surplus $\mathbb{S}(\bar{\mathbf{y}})$ could be achieved by a linear technology D^L, in which agents obtain individual surpluses as in (3.1).

Each γ^i in the linear technology would reflect some idiosyncratic feature of output i with respect to the nonlinear technology. We would suggest that an appropriate measure of these features would be in terms of their marginal contributions to damages evaluated at the efficient output levels. In this way, the linear technology would be such that $\gamma^{i*} = D_i(\mathbf{y}^*)$. for all i. Now, to determine the common component p, we use the assumption that the total surplus is the same under both the linear and the nonlinear problems. That is, we identify $p(\bar{\mathbf{y}})$ implicitly by:

$$\sum_{i=1}^{n} \sigma^i\left(p(\bar{\mathbf{y}})\gamma^{i*}\right) = \max\left\{\sum_{i=1}^{n} B^i\left(\bar{y}^i\right) - D(\bar{\mathbf{y}}), 0\right\} \tag{3.2}$$

Observe that, for a fixed $\bar{\mathbf{y}}$, the right hand side of (3.2) is a nonnegative scalar. Moreover, γ^{i*} is strictly positive for all i, and thus the left hand side is a strictly decreasing function of p, asymptotic to both the vertical and horizontal axes. Therefore, we obtain a mapping $p : \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+$ such that for each $\bar{\mathbf{y}} \in \mathbb{R}^n_+, p(\bar{\mathbf{y}}) \geq 0$ solves (3.2).

(Note that in the homogeneous case, the idiosyncratic components are equal, i.e., $\gamma^{i*} = \gamma^{j*}$, for all i and j. Thus, we can embed these in the common part and simply find $p^H(\mathbf{y}) \equiv p(\mathbf{y})\gamma^{i*}$ rather than $p(\mathbf{y})$).

We thus have two parallel problems that give rise to the same surplus. In the linear case, we agree that the standard distribution of total surplus is the appropriate one. Therefore, in the nonlinear case, we might deem this distribution appropriate as well. Then, under the LDE, the cost assignments are $\boldsymbol{\lambda}(\bar{\mathbf{y}}) = (\lambda^1(\bar{\mathbf{y}}), ..., \lambda^n(\bar{\mathbf{y}}))$ such that:

$$B^i\left(\bar{y}^i\right) - \lambda^i(\bar{\mathbf{y}}) = \sigma^i\left(p(\bar{\mathbf{y}})\gamma^{i*}\right), \text{ for } i = 1, 2, ..., n \tag{3.3}$$

i.e., those that ensure the same individual surpluses as in the reference linear case.

Note that, by (3.2) and (3.3), the resulting cost assignments exactly cover damages, that is, $\sum_{i=1}^{n} \lambda^i(\bar{\mathbf{y}}) = D(\bar{\mathbf{y}})$.

Definition 3.1. *For any* $\mathbf{y} \in \mathbb{R}^n_+$, *the Linear Damage Equivalent Mechanism assigns the vector* $(\mathbf{y}, \boldsymbol{\lambda}(\mathbf{y}))$, *where* $\boldsymbol{\lambda}(\mathbf{y})$ *is given by (3.1), (3.2) and (3.3).*

Remark 1. *Note that, if* B^i *had a fixed component (for instance, a fixed cost of production), that fact would not affect the LDE allocation, since by* (3.1) *and* (3.2), *it would cancel. Therefore, for simplicity, we have considered* $B^i(0) = 0$, *which should not be seen as a restrictive assumption. This means, by* (3.1), *that* $\sigma^i(p\gamma^{i*}) \geq 0$, *for all* $i = 1, 2, ..., n$.

In Figure 1, we represent the LDE allocation for the case of two agents and homogeneous output. On the horizontal axis, we represent individual production y^i as well as aggregate production $(y^1 + y^2)$. On the vertical axis, we represent total damages $D(y^1 + y^2)$ as well as individual cost assignments $\lambda^i(y^1, y^2)$. The figure also includes *iso-individual surplus* curves for each agent. Each iso-individual surplus curve (that is, the set of points (y^i, ϕ^i) for which individual surplus is constant) is strictly increasing and concave in y^i; individual surplus increases to the southeast.[11]

Let $\bar{y}^1$ and $\bar{y}^2$ be the actual levels produced by agents 1 and 2, respectively. These generate total damages $D(\bar{y}^1 + \bar{y}^2)$ to be divided among the agents. The LDE identi-

[11]For a fixed level $\bar{s}^i = B^i(y^i) - \phi^i$, we have $\frac{d\phi^i}{dy^i}|_{\bar{s}^i} = B^{i\prime} > 0$ and $\frac{d^2\phi^i}{dy^{i2}}|_{\bar{s}^i} = B^{i\prime\prime} < 0$.

fies the linear damage technology (or p^H) such that it is possible to obtain the same total surplus as the actual one generated by $\bar{\mathbf{y}} = (\bar{y}^1, \bar{y}^2)$.[12] This is represented by $p^H(\bar{\mathbf{y}})(y^1 + y^2)$. For agent i, $\tilde{y}^i$ is the optimal response with respect to the linear technology, where damages are divided according to the standard solution. This generates maximum individual surplus $\sigma^i\left(p^H(\bar{\mathbf{y}})\right)$. Then, the LDE allocates damages in such a way that, for agent i, $(\bar{y}^i, \lambda^i(\bar{\mathbf{y}}))$ lies on the same iso-individual surplus curve, $\sigma^i\left(p^H(\bar{\mathbf{y}})\right)$. That is, the LDE allocation guarantees each agent the same individual surplus it would obtain if the damage technology were linear and damages were divided according to the standard solution.[13]

4. Axiomatic Properties of the LDE

In this section, we consider various axiomatic properties of cost-sharing mechanisms proposed in the literature, and we compare the performance of the LDE with that of other methods.

Specifically, we consider the following axioms:

Axiom 1. *$\phi \in \Phi$ satisfies weak individual rationality (WIR) if, for all $\mathbf{y} \in \mathbb{R}^n_+$, $y^i = 0$ implies $\phi^i(\mathbf{y}) \leq 0$.*

Before presenting the next axiom, we introduce the following notation. Assume we change the units of account of the outputs according to $\bar{\mathbf{y}} = \boldsymbol{\beta}\cdot\mathbf{y} \equiv (\beta^1 y^1, \beta^2 y^2, ..., \beta^n y^n)$, where $\bar{\mathbf{y}} \in \mathbb{R}^n_+$ are now the relevant quantities. Denote by $\boldsymbol{\phi}_\beta$ the corresponding cost assignments in the transformed problem, where $B^i(y^i) = B^i(\bar{y}^i/\beta^i)$ for all i, and $D(\mathbf{y}) = D(\bar{\mathbf{y}}/\boldsymbol{\beta})$, where $\bar{\mathbf{y}}/\boldsymbol{\beta} \equiv (\bar{y}^1/\beta^1, ..., \bar{y}^n/\beta^n)$.

[12]Recall that the reference linear technology in the homogeneous case is simply $D^L(\mathbf{y}) = p^H(\mathbf{y})(y^1 + y^2)$.

[13]Note that for clarity, we have omitted the aggregate benefit curve. However, since individual surplus is the same at both $\bar{y}^i$ and $\tilde{y}^i$, clearly aggregate surplus at $\bar{y}^1 + \bar{y}^2$ relative to $D(y^1 + y^2)$ is the same as that at $\tilde{y}^1 + \tilde{y}^2$ relative to $p^H(\bar{\mathbf{y}})(y^1 + y^2)$.

Axiom 2. *$\phi \in \Phi$ satisfies scale invariance (SI) if, for every $\boldsymbol{\beta} \in \mathbb{R}^n_{++}$, $\phi(\mathbf{y}) = \phi_{\beta}(\bar{\mathbf{y}}/\boldsymbol{\beta})$, where $\bar{\mathbf{y}} = \boldsymbol{\beta} \cdot \mathbf{y}$.*

Axiom 3. *$\phi \in \Phi$ satisfies demand monotonicity (DM) if $\phi^i_i(\mathbf{y}) > 0$, for every $i = 1, 2, ..., n$.*

In the next axiom, and again in Lemma A.4, we denote by ϕ_D the assignment of costs under the damage technology D.

Axiom 4. *$\phi \in \Phi$ satisfies technological monotonicity (TM) if, for all D, $\hat{D} \in \mathbb{D}$, $\hat{D}(\mathbf{y}) \geq D(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}^n_+$ implies $\phi^i_{\hat{D}}(\mathbf{y}) \geq \phi^i_D(\mathbf{y})$, for all i.*

Next, we define the *stand alone surplus* of agent i as the maximum surplus it would obtain if it were the only agent causing damages, that is, $\pi^i_{S.A.} = \max_{y^i \in \mathbb{R}_+} \mathbb{S}(\mathbf{0} \mid y^i)$.

Axiom 5. *$\phi \in \Phi$ satisfies stand alone upper bound (SAUB) if $s^i(y^i, \phi^i(\mathbf{y})) \leq \pi^i_{S.A.}$, for all $i = 1, 2, ..., n$.*

Intuitively, WIR states that if an agent produces nothing, then it should not be charged a positive amount. This axiom differs from the standard *individual rationality* axiom, in which an agent who produces nothing pays nothing. However, as mentioned in the Introduction, we allow for the possibility of subsidies due to the pecuniary externalities among firms. This is discussed at length in the next section.

SI means that the unit of account in which a given output is measured should not matter, that is, a change in the units should not affect the assignment of costs.

DM states that, as the output level increases, the corresponding cost assignment should also increase.

According to TM, if the damage technology worsens such that actual damages are higher for all $\mathbf{y} \in \mathbb{R}^n_+$ (for example, as a result of improved detection techniques), then all agents should incur an increase in their respective cost assignments. Consequently, no agent should be better-off after this technological change. The opposite result occurs if damages are affected by a positive shock.

Finally, SAUB states that no agent included in the sharing problem should gain while the others lose as a result of the damage technology being convex. That is, all agents should suffer the consequences of the decreasing returns, or the individual surpluses should not exceed those they would obtain on their own.

The performance of the LDE vis-à-vis the above axioms is summarized in the following:

Theorem 4.1. *In general, the LDE satisfies WIR, SI and DM. For the homogeneous case, it also satisfies TM and SAUB.*

The proof is left to the appendix.

SI is an appealing property, especially for heterogeneous problems, where, in principle, it is difficult to compare one output with another. SI says that arbitrary relabeling or rescaling of variables should not affect the allocation of costs. The Shapley-Shubik and Aumann-Shapley mechanisms satisfy this property. However, the Serial Cost Mechanism for the heterogeneous case (Friedman and Moulin(1995)'s proposal for the divisible case) does not. In the discrete case, the Incremental Cost Sharing Mechanisms (Moulin(1996)) also fail to satisfy SI.

DM is important both in terms of equity and incentives. With respect to the latter, it plays a key role: if it were not satisfied, individuals could find it profitable to artificially raise their output levels in order to pay less. Aumann-Shapley Pricing does not satisfy this property.

In the homogeneous case, Average Cost Pricing also satisfies TM. However, this property is not satisfied by either the Serial Cost or the Shapley-Shubik mechanisms.

We now turn our attention to the strategic and efficiency properties of the LDE.

5. Strategic Analysis of the LDE

In this section, we establish that the Nash equilibrium of the game induced by the LDE is unique and it is also strong. We also prove that the equilibrium is efficient.

We begin with the following lemma:

Lemma 5.1. *The mapping* $p : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ *has a unique global minimum at* $\mathbf{y}^*$*; also* $p_{ii} > 0$ *for all* $i = 1, 2, ..., n$.

Proof. Observe that, by (3.1) and (3.2), p and total surplus $\mathbb{S}$ are inversely related, since $\sigma^{i\prime} < 0$ and $\gamma^{i*} > 0$, for all $i = 1, 2, ..., n$. As $\mathbb{S}$ has a unique global maximum at $\mathbf{y}^*$, this will correspond to the unique minimum of p. This proves the first part of the statement.

To prove the second part, we apply the implicit function theorem in (3.2) to obtain:[14]

$$p_i(\mathbf{y}) = -\frac{-B^{i\prime}(y^i) + D_i(\mathbf{y})}{\sum_{k=1}^n \gamma^{k*}\sigma^{k\prime}(p(\mathbf{y})\gamma^{k*})} = \frac{D_i(\mathbf{y}) - B^{i\prime}(y^i)}{\sum_{k=1}^n \gamma^{k*}\tilde{y}^k(p(\mathbf{y})\gamma^{k*})}, \quad i = 1, 2, ..., n. \tag{5.1}$$

Differentiating expression (5.1) with respect to y^i we obtain, for all i:

$$\begin{aligned} p_{ii}(\mathbf{y}) &= \frac{(D_{ii}(\mathbf{y}) - B^{i\prime\prime}(y^i))\left(\sum_{k=1}^n \gamma^{k*}\tilde{y}^k\left(p(\mathbf{y})\gamma^{k*}\right)\right)}{\left(\sum_{k=1}^n \gamma^{k*}\tilde{y}^k(p(\mathbf{y})\gamma^{k*})\right)^2} - \\ &\quad - \frac{\left(\sum_{k=1}^n \left(\gamma^{k*}\right)^2 \tilde{y}^{k\prime}\left(p(\mathbf{y})\gamma^{k*}\right)\right)\frac{\left(D_i(\mathbf{y}) - B^{i\prime}(y^i)\right)^2}{\left(\sum_{k=1}^n \gamma^{k*}\tilde{y}^k(p(\mathbf{y})\gamma^{k*})\right)}}{\left(\sum_{k=1}^n \gamma^{k*}\tilde{y}^k(p(\mathbf{y})\gamma^{k*})\right)^2} > 0 \end{aligned} \tag{5.2}$$

as desired. ■

Therefore, p is strictly convex in each component. In fact, p is U-shaped in y^i, for all i, with a unique minimum characterized by the condition $B^{i\prime}(y^i) = D_i(\mathbf{y} \mid y^i)$. We also have obtained that the unique global minimum of p occurs at $\mathbf{y}^*$.

[14] Recall that $\tilde{y}^i(p\gamma^i) \equiv \arg\max_{y^i \in \mathbb{R}_+}\left\{B^i(y^i) - p\gamma^i y^i\right\}$.

Next, we present the main result of this section.

Theorem 5.2. *The game induced by the LDE has a unique Nash equilibrium that is also strong. Moreover, the corresponding equilibrium allocation is Pareto-efficient.*

Proof. Note that each agent's best response is determined by solving the following problem:

$$\max_{y^i \in \mathbb{R}_+} \left\{ B^i\left(y^i\right) - \lambda^i\left(\mathbf{y}\right) \right\}, \ i = 1, 2, ..., n \tag{5.3}$$

which, by (3.3), is equivalent to:

$$\max_{y^i \in \mathbb{R}_+} \sigma^i\left(p\left(\mathbf{y}\right)\gamma^{i*}\right), \ i = 1, 2, ..., n \tag{5.4}$$

As $\sigma^{i\prime} < 0$ and $\gamma^{i*} > 0$, for all $i = 1, 2, ..., n$, problem (5.4) reduces to

$$\min_{y^i \in \mathbb{R}_+} p\left(\mathbf{y}\right), \ i = 1, 2, ..., n \tag{5.5}$$

Then, by Lemma 5.1, we obtain that the unique Nash equilibrium of the game is $\mathbf{y}^*$, i.e., the unique vector that maximizes total surplus $\mathbb{S}$. Thus, the Nash equilibrium allocation is Pareto-efficient.

To prove that $\mathbf{y}^*$ is robust to coalitional deviations as well, note that, as $\mathbf{y}^* = \arg\max_{\mathbf{y} \in \mathbb{R}^n_+} \mathbb{S}\left(\mathbf{y}\right)$, we have:

$$\mathbb{S}\left(\mathbf{y}^* \mid \mathbf{y}^S\right) \leq \mathbb{S}\left(\mathbf{y}^*\right), \text{ for all } \mathbf{y}^S \in \mathbb{R}^S_+, \ S \subseteq N \tag{5.6}$$

and, as total surplus and p are inversely related, we obtain:

$$p\left(\mathbf{y}^* \mid \mathbf{y}^S\right) \geq p\left(\mathbf{y}^*\right) \tag{5.7}$$

This implies:

$$\sigma^i\left(p\left(\mathbf{y}^* \mid \mathbf{y}^S\right)\gamma^{i*}\right) \leq \sigma^i\left(p\left(\mathbf{y}^*\right)\gamma^{i*}\right), \ i \in S, \ S \subseteq N \tag{5.8}$$

That is, no coalition of agents will find it profitable to deviate from the equilibrium strategy. Therefore, the Nash equilibrium is also strong. ■

As agents' individual surpluses are higher as the associated p is lower, each agent's optimal strategy consists of announcing a level of output that minimizes the associated p, given the other agents' choices. Since all agents have the same incentives, $\mathbf{y}^*$ (the unique minimizer of p) is the unique Nash equilibrium of the game, and it is robust to coalitional deviations.

As described in the Introduction, in the case of homogeneous outputs, Serial Cost Sharing satisfies this strategic property, but with the restriction of nontransferability of output. If transfers were permitted, any coalition of agents would find it profitable, under SCS, to artificially average their output levels. This kind of manipulation can not occur under the LDE, as shown in (5.6), (5.7) and (5.8). In the case of heterogeneous indivisible outputs, the strategic property is satisfied by the family of Incremental Cost Sharing Mechanisms under the assumption of increasing marginal costs and supermodular costs, that is, $D_{ii} > 0$ and $D_{ij} > 0$, respectively. As of yet, there are no such strategic results for the case of heterogeneous real-valued output.

In general, neither SCS nor ICS are efficient.

6. Subsidies under the LDE

In this section, we investigate the possibility of situations in which, under the LDE, one agent might be subsidized by the remaining agents, that is, in which one agent contributes a negative quantity while the remaining agents pay more than total damages. First, we show that this is possible for cases in which the production level of the subsidized agent is sufficiently small. We then characterize, for the two-agent case, the feasible production levels in which one agent is subsidized by the other.[15]

[15] This result is extendable to the n-agent case. We develope the case of n=2 for reasons of simplicity.

Proposition 6.1. *For* $(\mathbf{y} \mid 0^i) \in \mathbb{R}^n_+$ *such that* $\mathbb{S}(\mathbf{y} \mid 0^i) > 0$, *if* $B^{i\prime}(0) > p(\mathbf{y} \mid 0^i)\gamma^{i*}$, *then* $\lambda^i(\mathbf{y} \mid \varepsilon^i) < 0$ *for* $\varepsilon > 0$ *sufficiently small.*

Proof. Consider $(\mathbf{y} \mid 0^i) \in \mathbb{R}^n_+$, such that $\mathbb{S}(\mathbf{y} \mid 0^i) > 0$. By (3.2), this implies $p(\mathbf{y} \mid 0^i) > 0$, as $\gamma^{i*} > 0$. If $B^{i\prime}(0) > p(\mathbf{y} \mid 0^i)\gamma^{i*}$, by (3.1) we have $\sigma^i(p(\mathbf{y} \mid 0^i)\gamma^{i*}) > 0$, which implies $\lambda^i(\mathbf{y} \mid 0^i) < 0$. By the continuity of λ^i, we obtain $\lambda^i(\mathbf{y} \mid \varepsilon^i) < 0$ for ε sufficiently small. ■

This result means that it is possible that agent i is subsidized by the remaining agents if its production level is positive but sufficiently small. This can happen because, as seen in (3.1), the determination of individual surpluses under the LDE depends crucially on p. As such, it is possible that all agents in conjunction (agent i producing ε and agents $j \neq i$ producing y^j) obtain a large surplus and, therefore, generate a small p which implies high individual surpluses. This is in part due to the small amount produced by agent i, who would obtain larger private benefits by producing a larger amount. Therefore, by producing very little, agent i is generating a positive pecuniary externality for the remaining agents, and, in turn, receives compensation.

Note that the LDE is *budget-balanced.* Therefore, as long as $\lambda^i(\mathbf{y} \mid \varepsilon^i) < 0$ for some i, the remaining agents pay more than total damages, that is $\sum_{j \neq i} \lambda^j(\mathbf{y} \mid \varepsilon^i) > D(\mathbf{y} \mid \varepsilon^i)$.

As a consequence of Proposition 6.1, we can now characterize the set of output levels for which a particular agent receives a subsidy from the remaining agents. Consider the following example:

Example 6.2. *Assume there are two agents with benefit functions* $B^i(y^i) = (y^i)^{1/2}$, $i = 1, 2$, *and damages* $D(\mathbf{y}) = (y^1 + y^2)^2$. *Under the LDE, the associated linear economy is* $p^H(\mathbf{y})(y^1 + y^2)$, *and individual surpluses are* $\sigma^i\left(p^H(\mathbf{y})\right) = \frac{1}{4p^H(\mathbf{y})}$, $i = 1, 2$. *Therefore, applying* (3.2), *we have* $p^H(\mathbf{y}) = \frac{1}{2\left[(y^1)^{1/2} + (y^2)^{1/2} - (y^1 + y^2)^2\right]}$. *Let* $Y_{\lambda^i = 0}$ *denote the set of output vectors at which agent* i *pays nothing. By* (3.3), *this set is charac-*

terized by $B^i(y^i) = \sigma^i\left(p^H(\mathbf{y})\right)$. We obtain:

$$Y_{\lambda^1=0} = \left\{\mathbf{y} \in \mathbb{R}^2_+ \;\mid\; \left(y^1\right)^{1/2} - \left(y^2\right)^{1/2} + \left(y^1 + y^2\right)^2 = 0\right\} \tag{6.1}$$

$$Y_{\lambda^2=0} = \left\{\mathbf{y} \in \mathbb{R}^2_+ \;\mid\; \left(y^2\right)^{1/2} - \left(y^1\right)^{1/2} + \left(y^1 + y^2\right)^2 = 0\right\} \tag{6.2}$$

In Figure 2 we represent (6.1) and (6.2). These identify the two regions, I and II, in which $\lambda^1 < 0$ and $\lambda^2 < 0$, respectively. Outside these regions, both agents pay positive amounts. Note that in the example, the efficient output levels are $y^{i*} = 0.25$, $i = 1, 2$. ∎

For the 2-agent case, we can generalize the result obtained in Example 6.2. For a graphic presentation, see Figure 3. Stated formally:[16]

Proposition 6.3. *For n=2, the LDE identifies two disjoint sets of feasible output vectors in which $\lambda^1 < 0$ and $\lambda^2 < 0$, respectively.*

Proof. First, observe that $(0,0)$ belongs to both $Y_{\lambda^1=0}$ and $Y_{\lambda^2=0}$.

Differentiating $\lambda^1(\mathbf{y}) = 0$, we have:

$$\frac{dy^2}{dy^1}\Big|_{Y_{\lambda^1=0}} = -\frac{\lambda^1_1(\mathbf{y})}{\lambda^1_2(\mathbf{y})} \tag{6.3}$$

First, we compute the effect of a variation in y^i on λ^j, $j \neq i$. From (5.1) and appendix equation (A.3), we obtain:

$$\lambda^j_i(\mathbf{y}) = -\sigma^{j\prime}\left(p(\mathbf{y})\gamma^{i*}\right)\gamma^{j*}p_i(\mathbf{y}) = \frac{\gamma^{j*}\tilde{y}^j\left(p(\mathbf{y})\gamma^{j*}\right)}{\sum_{k=1}^n \gamma^{k*}\tilde{y}^k\left(p(\mathbf{y})\gamma^{k*}\right)}\left(D_i(\mathbf{y}) - B^{i\prime}\left(y^i\right)\right) \tag{6.4}$$

[16] For the n-agent case, it is possible that more than one agent is subsidized by the remaining agents. However, by budget-balancedness, it can not be the case that all agents are subsidized at the same time.

Now, from (A.4), (6.3) and (6.4) , we have:

$$\frac{dy^2}{dy^1} \mid_{Y_{\lambda^1=0}} = -\frac{\gamma^{1*}\tilde{y}^1\left(p\left(\mathbf{y}\right)\gamma^{1*}\right)D_1\left(\mathbf{y}\right)+\gamma^{2*}\tilde{y}^2\left(p\left(\mathbf{y}\right)\gamma^{2*}\right)B^{1\prime}\left(y^1\right)}{\gamma^{1*}\tilde{y}^1\left(p\left(\mathbf{y}\right)\gamma^{1*}\right)\left(D_2\left(\mathbf{y}\right)-B^{2\prime}\left(y^2\right)\right)} \tag{6.5}$$

Observe that (6.5) is positive at $(0,0)$ as we have assumed an interior efficient solution. Also, there exists a point $\hat{y} \in Y_{\lambda^1=0}$ such that (6.5) is ∞; for $y \in Y_{\lambda^1=0}$ such that $y^2 \geq \hat{y}^2$, (6.5) is negative; and there exists a unique point $(0,\bar{y}^2)$, $\bar{y}^2 \neq 0$, that belongs to $Y_{\lambda^1=0}$. Observe that, at $(0,\bar{y}^2)$, total surplus is 0.

Doing the same for $\lambda^2(\mathbf{y}) = 0$, we obtain:

$$\frac{dy^2}{dy^1} \mid_{Y_{\lambda^2=0}} = -\frac{\gamma^{2*}\tilde{y}^2\left(p\left(\mathbf{y}\right)\gamma^{2*}\right)\left(D_1\left(\mathbf{y}\right)-B^{1\prime}\left(y^1\right)\right)}{\gamma^{1*}\tilde{y}^1\left(p\left(\mathbf{y}\right)\gamma^{1*}\right)B^{2\prime}\left(y^2\right)+\gamma^{2*}\tilde{y}^2\left(p\left(\mathbf{y}\right)\gamma^{2*}\right)D_2\left(\mathbf{y}\right)} \tag{6.6}$$

Analogously, (6.6) is positive at $(0,0)$; there exists $\check{y} \in Y_{\lambda^2=0}$ such that (6.6) is equal to 0; and for $y \in Y_{\lambda^2=0}$ such that $y^1 \geq \check{y}^1$, (6.6) is negative. Also, there exists a unique point $(\bar{y}^1, 0) \in Y_{\lambda^2=0}$, $\bar{y}^1 \neq 0$, that belongs to $Y_{\lambda^2=0}$, where total surplus is 0.

From (6.5) and (6.6), it is easy to see that, at $(0,0)$, we have:

$$\frac{dy^2}{dy^1} |_{Y_{\lambda^1=0}} > \frac{dy^2}{dy^1} |_{Y_{\lambda^2=0}} \tag{6.7}$$

Moreover, $Y_{\lambda^1=0}$ and $Y_{\lambda^2=0}$ intersect only at $(0,0)$. Otherwise, damages would not be covered.

Considering DM, we obtain the desired result. ■

At this stage, a natural question to ask is whether subsidies can occur in equilibrium? In Example 6.2, the equilibrium is $\mathbf{y}^* = (0.25, 0.25)$, overall damages are 0.25 and the respective cost assignments are $\boldsymbol{\lambda} = (0.125, 0.125)$. Therefore, both agents pay positive amounts.

However, one could imagine a situation in which the production activity of one of the agents were much more harmful than that of the remaining agents. In that case, the efficient solution may be for that agent to produce a small amount, or even to produce nothing. And, in order to provide the necessary inducement for that agent, it might require compensation from the remaining agents. By Theorem 5.2, no one (individual or coalition) would find it profitable to deviate from the equilibrium strategy.

Thus far, we have not found an example which illustrates the presence of subsidies in equilibrium under of the LDE. This remains an open question.

7. Equilibrium Analysis. Some Comparative Statics.

In this section, we introduce two concepts of special importance for the case of heterogeneous outputs: the notions of *relative damage impact* and *relative profitability*. These evaluate the effect on overall damages and benefits, respectively, of an increase in one output relative to that of another, measured in terms of elasticities; they capture the relevance of each firm in the problem as a whole. We use these notions to investigate how the Nash equilibrium of the LDE responds to changes in the impact of a producer on damages or to changes in the profitability of firms.

First, if the relative damage impact of a particular output increases, the LDE unambiguously induces the respective firm to produce less, as the production of this output has become more harmful. Moreover, for a particular class of problems, we find that the corresponding cost assignment decreases in order to induce the firm to produce less.

Alternatively, when the relative profitability of a firm increases, the LDE induces that firm to produce more since, as it is now more profitable, it can generate a larger surplus for society.

To define these formally, let $\mathcal{E}_{D_i}(\mathbf{y})$ be the elasticity of overall damages with respect to y^i, that is, $\mathcal{E}_{D_i}(\mathbf{y}) = D_i(\mathbf{y}) \frac{y^i}{D(\mathbf{y})}$. Analogously, let $\mathcal{E}_{B^i}(\mathbf{y})$ be the elasticity of overall benefits with respect to y^i, that is, $\mathcal{E}_{B^i}(\mathbf{y}) = B^{i\prime}(y^i) \frac{y^i}{\sum_{j=1}^{n} B^j(y^j)}$.

Definition 7.1. *The relative damage impact of firm j with respect to firm i, $j \neq i$, is given by:*

$$RDI_{j,i}(\mathbf{y}) = \frac{\mathcal{E}_{D_j}(\mathbf{y})}{\mathcal{E}_{D_i}(\mathbf{y})} = \frac{D_j(\mathbf{y})\, y^j}{D_i(\mathbf{y})\, y^i} \tag{7.1}$$

For instance, $RDI_{j,i}(\bar{\mathbf{y}}) = k$ means that, at $\bar{\mathbf{y}}$, the impact on D of a one percent increase in y^j is k times greater than that of a one percent increase in y^i.

Definition 7.2. *The relative profitability of firm j with respect to firm i, $j \neq i$, is given by:*

$$RP_{j,i}(\mathbf{y}) = \frac{\mathcal{E}_{B^j}(\mathbf{y})}{\mathcal{E}_{B^i}(\mathbf{y})} = \frac{B^{j\prime}(y^j)\, y^j}{B^{i\prime}(y^i)\, y^i} \tag{7.2}$$

Thus, $RP_{j,i}(\bar{\mathbf{y}}) = m$ means that, at $\bar{\mathbf{y}}$, the impact on overall benefits caused by a one percent increase in y^j is m times greater than that caused by a one percent increase in y^i.

To demonstrate that when the relative damage impact of one output increases, the LDE induces the respective firm to produce less, we first consider the case of $n = 2$ in which $B^i \in \mathbb{B}$ and $D(\mathbf{y}) = D(y^1 + \alpha y^2)$, $\alpha > 1$. Observe that the relative damage impact of firm 2 with respect to firm 1 is $RDI_{2,1}(\mathbf{y}) = \frac{\alpha y^2}{y^1}$, $\mathbf{y} \in \mathbb{R}^2_+$.

Restricting attention to the equilibrium behavior of the firms, the elasticity of output i in equilibrium with respect to α is given by the following:

$$\mathcal{E}_{y^{i*}(\alpha)} = \frac{dy^{i*}(\alpha)}{d\alpha} \frac{\alpha}{y^{i*}(\alpha)} \tag{7.3}$$

We now investigate how the equilibrium of the game induced by the LDE responds to changes in α.

Proposition 7.3. *For $B^i \in \mathbb{B}$, $i = 1, 2$, and $D(\mathbf{y}) = D(y^1 + \alpha y^2)$, $\alpha > 1$, the Nash equilibrium of the game induced by the LDE, namely $(y^{1*}(\alpha), y^{2*}(\alpha))$, is such that*

$\frac{dy^{2*}(\alpha)}{d\alpha} < 0$. *Moreover,* $\frac{dy^{1*}(\alpha)}{d\alpha} = 0$ *if and only if* $\mathcal{E}_{y^{i*}(\alpha)} = -1$, *and* $\frac{dy^{1*}(\alpha)}{d\alpha} > 0$ *(resp.* <0*) if and only if* $\mathcal{E}_{y^{i*}(\alpha)} < -1$ *(resp.* > -1*).*

Proof. By Theorem 5.2, the equilibrium of the game induced by the LDE is characterized by the efficiency conditions:[17]

$$B^{1\prime}\left(y^1\right) = D'\left(y^1 + \alpha y^2\right) \tag{7.4}$$

$$B^{2\prime}\left(y^2\right) = \alpha D'\left(y^1 + \alpha y^2\right) \tag{7.5}$$

From (7.4) and (7.5), we have $B^{2\prime}(y^2) = \alpha B^{1\prime}(y^1)$, from which we can obtain $y^1 = g(\alpha, y^2)$, such that $g_\alpha = -\frac{B^{1\prime}}{\alpha B^{1\prime\prime}} > 0$ and $g_{y^2} = \frac{B^{2\prime\prime}}{\alpha B^{1\prime\prime}} > 0$.

Substituting $y^1 = g(\alpha, y^2)$ in (7.5), we have $B^{2\prime}(y^2) = \alpha D'(g(\alpha, y^2) + \alpha y^2)$. This identifies $y^{2*}(\alpha)$ implicitly. Differentiating this expression with respect to α, we obtain:[18]

$$\frac{dy^{2*}(\alpha)}{d\alpha} = \frac{D'(y^1 + \alpha y^2) + \alpha D''(y^1 + \alpha y^2)(g_\alpha + y^2)}{B^{2\prime\prime}(y^2) - \alpha D''(y^1 + \alpha y^2)(g_{y^2} + \alpha)} < 0 \tag{7.6}$$

Now, substituting $y^{2*}(\alpha)$ in $y^1 = g(\alpha, y^2)$, we have $y^{1*}(\alpha) = g(\alpha, y^{2*}(\alpha))$, and differentiating this with respect to α we obtain:

$$\frac{dy^{1*}(\alpha)}{d\alpha} = \frac{D''(y^1 + \alpha y^2)}{B^{1\prime\prime}(y^1)\left[B^{2\prime\prime}(y^2) - \alpha D''(y^1 + \alpha y^2)(g_{y^2} + \alpha)\right]}\left[B^{2\prime\prime}\left(y^2\right)y^2 + B^{2\prime}\left(y^2\right)\right] \tag{7.7}$$

where $sign\left(\frac{dy^{1*}(\alpha)}{d\alpha}\right) = sign\left(B^{2\prime\prime}(y^2)\,y^2 + B^{2\prime}(y^2)\right)$, evaluated in equilibrium.

By (7.3) and (7.6), and considering also the expressions of g_α and g_{y^2}, we have the

[17] Recall that we assume an interior efficient solution composed of unique efficient output levels.
[18] Expressions (7.6), (7.7), (7.8) and (7.9) are evaluated in equilibrium. We suppress notation of * for reasons of clarity.

following:

$$\mathcal{E}_{y^{2*}(\alpha)} = \frac{\alpha}{y^2}\frac{D'(y^1+\alpha y^2)+\alpha D''(y^1+\alpha y^2)(g_\alpha + y^2)}{B^{2''}(y^2)-\alpha D''(y^1+\alpha y^2)(g_{y^2}+\alpha)} =$$

$$= \frac{1}{y^2}\frac{B^{2'}(y^2)B^{1''}(y^1)+\alpha^2 D''(y^1+\alpha y^2)B^{1''}(y^1)y^2 - D''(y^1+\alpha y^2)B^{2'}(y^2)}{B^{2''}(y^2)B^{1''}(y^1)-D''(y^1+\alpha y^2)B^{2''}(y^2)-\alpha^2 D''(y^1+\alpha y^2)B^{1''}(y^1)}$$

$$< 0 \tag{7.8}$$

and by imposing $\mathcal{E}_{y^{2*}(\alpha)} = -1$, we obtain the condition

$$\left(B^{1''}\left(y^1\right) - D''\left(y^1+\alpha y^2\right)\right)\left(B^{2''}\left(y^2\right)y^2 + B^{2'}\left(y^2\right)\right) = 0\ , \tag{7.9}$$

again evaluated in equilibrium. This condition is satisfied if and only if $B^{2''}(y^2)\,y^2 + B^{2'}(y^2) = 0$, as $B^{1''}(y^1) - D''(y^1+\alpha y^2) < 0$.

Thus, by (7.7). the condition $B^{2''}(y^2)\,y^2 + B^{2'}(y^2) = 0$ is equivalent to $\frac{dy^{1*}(\alpha)}{d\alpha} = 0$, which is the desired result.

From (7.7) and (7.8), it easy to see that, by imposing $\mathcal{E}_{y^{2*}(\alpha)} < -1$ and $\mathcal{E}_{y^{2*}(\alpha)} > -1$, we obtain, respectively, $B^{2''}(y^2)\,y^2 + B^{2'}(y^2) > 0$ and $B^{2''}(y^2)\,y^2 + B^{2'}(y^2) < 0$, which are equivalent to $\frac{dy^{1*}(\alpha)}{d\alpha} > 0$ and $\frac{dy^{1*}(\alpha)}{d\alpha} < 0$, as desired. ■

Intuitively, on the one hand, as α increases both marginal damages increase, leading to a reduction of both y^{1*} and y^{2*} (see (7.4) and (7.5)). On the other hand, comparatively, marginal damages for firm 2 increase more than those for firm 1, which leads to a decrease of y^{2*} and an increase of y^{1*}.

Adding up both effects, we see that y^{2*} decreases. The total effect on y^{1*} depends on which effect is stronger, as they move in opposite directions. For instance, if the total effect on y^{1*} is such that $\frac{dy^{1*}(\alpha)}{d\alpha} < 0$, this means that as α increases, y^{2*} decreases, but the product αy^{2*} increases, that is, a one percent variation in α leads to less than a one percent variation in y^{2*}, which is the case of $\left|\mathcal{E}_{y^{2*}(\alpha)}\right| < 1$. The opposite happens when $\frac{dy^{1*}(\alpha)}{d\alpha} > 0$.

We have established that, as α increases, y^{2*} decreases. But, could it be the case that, although firm 2 is now more harmful, it ends up paying less in the new equilibrium?

Proposition 7.4. *Let $B^i \in \mathbb{B}$, $i = 1,2$, and $D(y) = D(y^1 + \alpha y^2)$, $\alpha > 1$. Then, $\frac{d\lambda^2(\mathbf{y}^*(\alpha))}{d\alpha} < 0$ if and only if $\mathcal{E}_{y^{2*}(\alpha)} < -\left(\frac{\alpha\tilde{y}^2}{\tilde{y}^1+\alpha\tilde{y}^2} + \frac{p^H\tilde{y}^1\tilde{y}^2}{y^{2*}D'(\tilde{y}^1+\alpha\tilde{y}^2)}\right)$.*

Proof. Differentiating $\lambda^2(\mathbf{y}^*(\alpha))$, we obtain:

$$\frac{d\lambda^2}{d\alpha}(\mathbf{y}^*(\alpha)) = \frac{\partial\lambda^2}{\partial\alpha}\big|_{y^*} + \lambda_1^2\frac{dy^{1*}(\alpha)}{d\alpha} + \lambda_2^2\frac{dy^{2*}(\alpha)}{d\alpha} \tag{7.10}$$

where λ_1^2 is 0 in equilibrium (by (6.4)).

From (3.2) and (3.3), we have:[19]

$$\begin{aligned} \frac{\partial\lambda^2}{\partial\alpha}\big|_{y^*} &= -\sigma^{2\prime}\left(p^H\alpha\right)\left[p^H + \alpha\frac{\partial p^H}{\partial\alpha}\big|_{y^*}\right] = \\ &= \frac{\tilde{y}^2}{\tilde{y}^1+\alpha\tilde{y}^2}\left[\alpha y^{2*}D'(\mathbf{y}^*) + p^H\tilde{y}^1\right] \end{aligned} \tag{7.11}$$

From (A.1), evaluated in equilibrium, we obtain:

$$\lambda_2^2\frac{dy^{2*}(\alpha)}{d\alpha} = \alpha D'(\mathbf{y}^*)\frac{dy^{2*}(\alpha)}{d\alpha} \tag{7.12}$$

Thus, adding up (7.11) and (7.12), and considering $\mathcal{E}_{y^{2*}(\alpha)}$, we have:

$$\frac{d\lambda^2}{d\alpha}(\mathbf{y}^*(\alpha)) = y^{2*}D'(\mathbf{y}^*)\left(\frac{\alpha\tilde{y}^2}{\tilde{y}^1+\alpha\tilde{y}^2} + \mathcal{E}_{y^{2*}(\alpha)}\right) + \frac{p^H\tilde{y}^1\tilde{y}^2}{\tilde{y}^1+\alpha\tilde{y}^2} \tag{7.13}$$

[19] The reference linear damage function for $D(y^1+\alpha y^2)$ is simply $p^H(\mathbf{y})(y^1+\alpha y^2)$.

from which we obtain that $\frac{d\lambda^2}{d\alpha}(\mathbf{y}^*(\alpha)) < 0$ if and only if

$$\mathcal{E}_{y^{2*}(\alpha)} < -\left(\frac{\alpha\tilde{y}^2}{\tilde{y}^1+\alpha\tilde{y}^2} + \frac{p^H\tilde{y}^1\tilde{y}^2}{y^{2*}D'(\tilde{y}^1+\alpha\tilde{y}^2)}\right) \tag{7.14}$$

■

When α increases, we observe two effects on the cost assignment of firm 2. On the one hand, an increase in α – a deterioration of the technology – leads to an increase in λ^2, as firm 2 is now more harmful. This is the *punitive effect*, expressed in (7.11). On the other hand, by Proposition 7.3, an increase in α leads to a decrease in y^2 in the new equilibrium, and by DM, a decrease in λ^2. This is the *DM effect* (7.12).

Note that, as α increases, the total change in damages is

$$\frac{d}{d\alpha}D\left(y^{1*}(\alpha)+\alpha y^{2*}(\alpha)\right) = D'y^{2*}(\alpha) + D'\frac{dy^{1*}(\alpha)}{d\alpha} + \alpha D'\frac{dy^{2*}(\alpha)}{d\alpha} \tag{7.15}$$

The first term of the right hand side of (7.15) is negative for the system, in the sense that, as α increases, D also increases and, therefore, total surplus decreases. Due to the nonseparability of damages, both firms are responsible for such an increment in damages in proportions $\frac{\tilde{y}^1}{\tilde{y}^1+\alpha\tilde{y}^2}$ and $\frac{\alpha\tilde{y}^2}{\tilde{y}^1+\alpha\tilde{y}^2}$, respectively. But, as damages increase because firm 2 is now more harmful, there is a compensation from firm 2 to firm 1. Thus, the punitive cost for firm 2 (expressed in (7.11)) has these two components, i.e., the proportion of incremental damages paid by firm 2 and the transfer to firm 1.

The other two components of the right hand side of (7.15) reflect the individual effort in reducing (increasing) D, which is positive (negative) for the system. The last component corresponds to the DM effect of firm 2 (see (7.12)).

Thus, (7.14) means that, if this individual effort in reducing D outweighs the punitive cost imposed on firm 2, then firm 2 imposes a net positive effect on total damages, which is reflected by a reduction in his respective cost assignment.

Now, we consider a general $D \in \mathbb{D}$. Recall the definition of relative damage impact in (7.1). Let $k(\mathbf{y}) = \frac{D_2(\mathbf{y})}{D_1(\mathbf{y})}$. Note that a change in $RDI_{2,1}(\mathbf{y})$ is of the same sign as a change in $k(\mathbf{y})$. Therefore, we can study the sign of the effect of an increase in the relative damage impact of firm 2 as a change in $D \in \mathbb{D}$ to $\tilde{D} \in \mathbb{D}$, such that $\tilde{k}(\mathbf{y}) \geq k(\mathbf{y})$, for all $\mathbf{y} \in \mathbb{R}_+^2$.

Theorem 7.5. *If the relative damage impact of firm 2 with respect to firm 1 increases for all $\mathbf{y} \in \mathbb{R}_+^2$, then y^{2*} will be lower in the equilibrium of the game induced by the LDE.*

Proof. As before, by Theorem 5.2, the equilibrium of the game induced by the LDE is characterized by the efficiency conditions:

$$B^{1\prime}\left(y^1\right) = D_1\left(y^1, y^2\right) \tag{7.16}$$

$$B^{2\prime}\left(y^2\right) = D_2\left(y^1, y^2\right) \tag{7.17}$$

From (7.16), we have a relationship $y^1 = g\left(y^2\right)$ such that

$$g_{y^2} = \frac{dy^1}{dy^2} = \frac{D_{12}}{B^{1\prime\prime} - D_{11}} \tag{7.18}$$

From (7.16) and (7.17), and considering $y^1 = g\left(y^2\right)$, we obtain y^{2*} from:

$$\frac{B^{2\prime}\left(y^{2*}\right)}{B^{1\prime}\left(g\left(y^{2*}\right)\right)} = k\left(g\left(y^{2*}\right), y^{2*}\right) \tag{7.19}$$

and, from (7.19), we obtain $y^{1*} = g\left(y^{2*}\right)$.

The effect of a shift in $k\left(y^1, y^2\right)$ on y^{2*} is going to depend upon the slopes of $\frac{B^{2\prime}(y^2)}{B^{1\prime}(g(y^2))}$ and $k\left(g\left(y^2\right), y^2\right)$.

We first differentiate $\frac{B^{2\prime}(y^2)}{B^{1\prime}(g(y^2))}$ with respect to y^2 to obtain:

$$\frac{d}{dy^2}\left(\frac{B^{2\prime}\left(y^2\right)}{B^{1\prime}\left(g\left(y^2\right)\right)}\right) = \frac{B^{2\prime\prime}B^{1\prime} - B^{2\prime}B^{1\prime\prime}g_{y^2}}{\left(B^{1\prime}\right)^2} =$$

$$= \frac{B^{2''}B^{1'}\left(B^{1''}-D_{11}\right)-B^{2'}B^{1''}D_{12}}{\left(B^{1'}\right)^{2}\left(B^{1''}-D_{11}\right)} \tag{7.20}$$

which is strictly negative if $D_{12} > 0$; otherwise, the sign is ambiguous.

Now differentiate $k\left(g\left(y^2\right), y^2\right)$ with respect to y^2 to obtain:

$$\begin{aligned} \frac{d}{dy^2}\left(k\left(g\left(y^2\right), y^2\right)\right) &= \frac{1}{\left(D_1\right)^2}\left[D_{12}g_{y^2}D_1 + D_{22}D_1 - D_2D_{11}g_{y^2} - D_2D_{12}\right] = \\ &= \frac{D_1\left(B^{1''}D_{22} - D_{11}D_{22} + \left(D_{12}\right)^2\right) - D_2B^{1''}D_{12}}{\left(D_1\right)^2\left(B^{1''}-D_{11}\right)} \end{aligned} \tag{7.21}$$

which is strictly positive when $D_{12} < 0$ and has ambiguous sign otherwise.

To prove that y^2 will be lower in the new equilibrium when $k()$ increases, it is sufficient to ensure that, in equilibrium, the slope of $\frac{B^{2'}}{B^{1'}}$ is always less than the slope of k. Therefore, we claim that:

$$\frac{d}{dy^2}\left(k\left(y^{2*}\right)\right) > \frac{d}{dy^2}\left(\frac{B^{2'}\left(y^{2*}\right)}{B^{1'}\left(g\left(y^{2*}\right)\right)}\right) \tag{7.22}$$

To establish (7.22), suppose, to the contrary, $\frac{d}{dy^2}\left(k\left(y^{2*}\right)\right) \leq \frac{d}{dy^2}\left(\frac{B^{2'}\left(y^{2*}\right)}{B^{1'}\left(g\left(y^{2*}\right)\right)}\right)$. Then, by (7.20) and (7.21), and considering also the efficiency conditions (7.16) and (7.17), we have:

$$\begin{aligned} &\frac{1}{B^{1''}-D_{11}}\left[D_1\left(B^{1''}D_{22} - D_{11}D_{22} + \left(D_{12}\right)^2\right) - D_2B^{1''}D_{12}\right] \\ \leq\; &\frac{B^{2''}B^{1'}\left(B^{1''}-D_{11}\right)-B^{2'}B^{1''}D_{12}}{B^{1''}-D_{11}} \end{aligned} \tag{7.23}$$

evaluated in equilibrium.

Multiplying both sides of (7.23) by $B^{1''} - D_{11} < 0$, dividing both sides by D_1 and considering (7.16), (7.17) and (7.19), we obtain:

$$B^{1\prime\prime}D_{22} - D_{11}D_{22} + (D_{12})^2 \geq B^{2\prime\prime}\left(B^{1\prime\prime} - D_{11}\right) \tag{7.24}$$

However, the left hand side of (7.24) is strictly negative (as B^1 is strictly concave and D is convex) and the right hand side is strictly positive, which is a contradiction.

Therefore, (7.22) holds, and consequently, when k increases, y^2 will be lower in the new equilibrium. ■

Now, we ask what happens in the new equilibrium when the relative profitability of firm 2 increases. Analogously to the relative damage impact, we can study the change of relative profitability directly from a change in $\frac{B^{2\prime}}{B^{1\prime}}$, as they have the same sign.

Theorem 7.6. *If the relative profitability of firm 2 with respect to firm 1 increases for all $y \in \mathbb{R}^n_+$, then y^{2*} will be higher in the equilibrium of the game induced by the LDE.*

We omit the proof as it is similar to that of Theorem 7.5.

To summarize, we have established that, in general, when the relative damage impact of one output versus another increases, the LDE induces the respective firm to produce less. For a particular class of problems, we find that the cost imposed on the now more harmful firm decreases, in order to provide the necessary inducement for that firm to produce less.

Alternatively, when the relative profitability of one output with respect to another increases, the LDE unambiguosly induces the respective firm to produce more.

8. Concluding Comments

In this paper, we have proposed a mechanism to share costs when outputs are heterogeneous, namely the LDE. It consists of finding an appropriate reference linear damage technology in which it is possible to achieve the same total surplus as in the nonlinear problem. Once identified, the total surplus is then divided so as to guarantee each agent the individual surplus it would obtain if the problem were linear. When imposed exogenously, the LDE induces a game among the agents since each agent's choice of production level affects the remaining agents' payoffs. This game has a unique Nash equilibrium that is also robust to coalitional deviations. The equilibrium is, moreover, the unique vector of outputs that maximizes total surplus.

The LDE satisfies several interesting axiomatic properties as well. In general, it satisfies weak individual rationality, scale invariance and demand monotonicity. In the homogeneous case, it also satisfies technological monotonicity and stand alone upper bound. This weaker version of individual rationality has not appeared in the literature. Although, as we mentioned, in light of the externality generated by restricting one's output level, it seems natural in this context.

Next, we introduced the concepts of relative damage impact and relative profitability, two measures of the relevance of each firm in the problem as a whole, and we showed that the LDE induces firms to react appropriately in response to changes in these measures. That is, if the relative damage impact of a firm increases, the LDE induces the firm to produce less; and if the relative profitability of a firm increases, the LDE induces it to produce more.

With respect to the cost- or surplus-sharing literature, we have shown that the LDE performs as well as or better than existing methods. First, both the LDE and the Constant Returns Equivalent Mechanism (CRE) are defined under complete information, but the LDE is applicable to the heterogeneous case, while the CRE pertains only to the homogeneous case. Moreover, the LDE is defined for all feasible allocations, whereas the CRE is defined only along the efficient frontier. As we discussed, we obtain efficiency as a result of the strategic interaction among firms rather than as an assumption.

Second, the LDE shares the nice strategic properties of Serial Cost Sharing (SCS) and Incremental Cost Sharing Mechanisms (ICS), but again it applies to the heterogeneous and continuous case. SCS is defined only for the homogeneous case, and ICS applies only to the discrete case. There are no previous results for the heterogeneous, continuous case. Also, these mechanisms fail to satisfy efficiency and scale invariance.

The main limitation of the LDE is that it requires complete information of the damage technology and the benefit functions of the agents. Concerning the latter, when information is incomplete, we can not identify γ^{i*}, the idiosyncratic components of the reference linear technology.

Several issues remain unresolved. First, in Section 6, we have shown that subsidies can occur under the LDE; however, we have not been able to construct an example in which subsidies occur in equilibrium. This remains an open question. Second, it is unknown whether any of the existing axiomatic characterizations of the CRE can be generalized to the LDE, in the heterogeneous case. In addition, there is the question whether some subset of the properties satisfied by the LDE fully characterize it.

An interesting extension of the model would be to allow the n firms to produce a finite number of heterogeneous outputs. One would consider that each firm produces m goods, where y^{ik} represents the quantity of output k produced by firm i. Damages would be a nonlinear function of the aggregate production amounts, that is $D(y^1, y^2, ..., y^m)$, where $y^k = \sum_{i=1}^{n} y^{ik}$, $k = 1, 2, ..., m$. The problem is interesting because it combines aspects of homogeneity as well as heterogeneity of the outputs.

Concerning the informational requirements of the mechanism, if the damage technology were known but not the benefit functions, we could still use some measure of the relevance (or the impact) of a particular heterogeneous output on damages. The notion of relative damage impact of a firm, introduced in Section 7, captures this relevance at a particular vector of outputs. An advantage of using this concept is that it is immune to the units of account. A first question to address is which vector of output levels should we use as a benchmark to compute the relative damage impacts. Once solved, a second question is whether this measure of the relative damage impact can be used as an indirect tool to determine the respective cost assignments.

But overall, the question of whether it is possible to modify the LDE for use in incomplete information environments remains open.

Appendix

Proof of Theorem 4.1. We separate the parts of the theorem into several lemmas.

Lemma A.1. The LDE satisfies WIR.

Proof. Consider $(\mathbf{y} \mid 0^i) \in \mathbb{R}^n_+$, such that $\mathbb{S}(\mathbf{y} \mid 0^i) \geq 0$.

By (3.1) and the assumption $B^i(0) = 0$, we have $\sigma^i(p(\mathbf{y} \mid 0^i)\gamma^{i*}) \geq 0$. Taking (3.3) into account, we have $\sigma^i(p(\mathbf{y} \mid 0^i)\gamma^{i*}) = B^i(0) - \lambda^i(\mathbf{y} \mid 0^i) \geq 0$, and as $B^i(0) = 0$, $\lambda^i(\mathbf{y} \mid 0^i) \leq 0$ follows. ■

Lemma A.2. The LDE satisfies SI.

Proof. The LDE associates with the nonlinear technology D a linear technology $D^L(\mathbf{y}) = p(\gamma^{1*}y^1 + \gamma^{2*}y^2 + ... + \gamma^{n*}y^n)$, where $\gamma^{i*} = D_i(y^*)$ and y^* is the vector of efficient outputs. Individual surpluses are given by $\max_{y^i}\{B^i(y^i) - p\gamma^{i*}y^i\}$, $i = 1, 2, ..., n$, as expressed in (3.1).

Consider now a rescaling of variables $\boldsymbol{\beta} \in \mathbb{R}^n_{++}$, such that $\bar{\mathbf{y}} = \boldsymbol{\beta} \cdot \mathbf{y}$, and $\bar{\mathbf{y}} \in \mathbb{R}^n_+$ are now the relevant variables. The new efficient levels $\bar{\mathbf{y}}^*$ are obtained by solving the problem $\max_{\bar{\mathbf{y}} \in \mathbb{R}^n_+}\{\sum_{i=1}^n B^i(\bar{y}^i/\beta^i) - D(\bar{\mathbf{y}}/\boldsymbol{\beta})\}$, yielding $\bar{y}^{i*} = \beta^i y^{i*}$, for all i. Observe that, for each $\mathbf{y} \in \mathbb{R}^n_+$, the total surplus is the same under both the original and the transformed problems. We claim that the distribution of surplus is also the same.

The scale factors for the associated linear function in the transformed problem are the following:

$$\bar{\gamma}^{i*} = D_i(\bar{\mathbf{y}}^*/\boldsymbol{\beta})\frac{1}{\beta^i} = D_i(\mathbf{y}^*)\frac{1}{\beta^i} = \frac{\gamma^{i*}}{\beta^i} \tag{A.1}$$

The reference linear damage function for the transformed problem is $D^L(\bar{\mathbf{y}}) = \bar{p}(\bar{\gamma}^{1*}\bar{y}^1 + ... + \bar{\gamma}^{n*}\bar{y}^n)$, which, by (A.1), is equivalent to $D^L(\bar{\mathbf{y}}) = \bar{p}(\gamma^{1*}(\frac{\bar{y}^1}{\beta^1}) + ... +$

$\gamma^{n*}(\frac{\bar{y}^n}{\beta^n}))$.

As total surplus is the same under both the original and the transformed problems, we obtain, by (3.2), that $p(\mathbf{y}) = \bar{p}(\bar{\mathbf{y}})$. Thus, we have:

$$\max_{\bar{y}^i}\left\{B^i\left(\frac{\bar{y}^i}{\beta^i}\right) - \bar{p}\gamma^{i*}\frac{\bar{y}^i}{\beta^i}\right\} = \max_{y^i}\left\{B^i\left(y^i\right) - p\gamma^{i*}y^i\right\} \tag{A.2}$$

Therefore, the individual surplus for each agent in the transformed problem is the same as that of the original problem, as desired. ■

Lemma A.3. The LDE satisfies DM.

Proof. From (3.3), we have, for all $i = 1, 2, ..., n$, and all $y \in \mathbb{R}^n_+$:

$$\lambda^i(\mathbf{y}) = B^i\left(y^i\right) - \sigma^i\left(p(\mathbf{y})\gamma^{i*}\right) \tag{A.3}$$

Differentiating (A.3) with respect to y^i, and considering (5.1), we obtain:

$$\begin{aligned}
\lambda^i_i(\mathbf{y}) &= B^{i\prime}\left(y^i\right) - \sigma^{i\prime}\left(p(\mathbf{y})\gamma^{i*}\right)\gamma^{i*}p_i(\mathbf{y}) = \\
&= \frac{\gamma^{i*}\tilde{y}^i\left(p(\mathbf{y})\gamma^{i*}\right)}{\sum_{k=1}^n \gamma^{k*}\tilde{y}^k\left(p(\mathbf{y})\gamma^{i*}\right)}D_i(\mathbf{y}) + \frac{\sum_{k\neq i}\gamma^{k*}\tilde{y}^k\left(p(\mathbf{y})\gamma^{i*}\right)}{\sum_{k=1}^n \gamma^{k*}\tilde{y}^k\left(p(\mathbf{y})\gamma^{i*}\right)}B^{i\prime}\left(y^i\right) > 0
\end{aligned} \tag{A.4}$$

■

In the following two lemmas, we restrict our attention to the homogeneous case. Let $y = \sum_{i=1}^n y^i$, and denote by $\mathbb{D}^H$ the set of functions $D : \mathbb{R}_+ \to \mathbb{R}_+$, where D is strictly increasing and convex.

Lemma A.4. The LDE satisfies TM on the domain $\mathbb{D}^H$.

Proof. Consider $D, \hat{D} \in \mathbb{D}^H$, such that $\hat{D}(y) \geq D(y)$, where $y = \sum_{i=1}^n y^i$. This implies that $\mathbb{S}_{\hat{D}}(\mathbf{y}) \leq \mathbb{S}_D(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}_+^n$. The reference linear technology in the homogeneous case is $D^L(\mathbf{y}) = p^H\left(\sum y^i\right)$. Therefore, by (3.2), we have $p_{\hat{D}}^H(\mathbf{y}) \geq p_D^H(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}_+^n$; and by (3.1), we then obtain $\sigma_{\hat{D}}^i\left(p_{\hat{D}}^H(\mathbf{y})\right) \leq \sigma_D^i\left(p_D^H(\mathbf{y})\right)$ for $i = 1, 2, ..., n$. Applying (3.3), it follows that $\lambda_{\hat{D}}^i(\mathbf{y}) \geq \lambda_D^i(\mathbf{y})$, for all $\mathbf{y} \in \mathbb{R}_+^n$, $i = 1, 2, ..., n$. ∎

Lemma A.5. The LDE satisfies SAUB on the domain $\mathbb{D}^H$.

Proof. First note that $\pi_{S.A.}^i \geq B^i(y^i) - D(y^i)$, for all i. The reference linear technology in the homogeneous case is $D^L(\mathbf{y}) = p^H\left(\sum y^i\right)$. Therefore, it is sufficient to show that $\sigma^i\left(p^H(\mathbf{y})\right) \leq B^i(y^i) - D(y^i)$. By (3.1), we have $\sigma^i\left(p^H(\mathbf{y})\right) \geq B^i(y^i) - p^H(\mathbf{y})\, y^i$. We claim:

$$p^H(\mathbf{y})\, y^i \geq D\left(y^i\right) \tag{A.5}$$

To prove (A.5), note that, by (3.1) and (3.2), we have:

$$\sum_{i=1}^n \sigma^i\left(p^H(\mathbf{y})\right) = \sum_{i=1}^n B^i\left(y^i\right) - D\left(\sum_{i=1}^n y^i\right) \geq \sum_{i=1}^n B^i\left(y^i\right) - p^H(\mathbf{y})\left(y^1 + ... + y^n\right) \tag{A.6}$$

from which we obtain

$$\frac{D\left(y^1 + ... + y^n\right)}{y^1 + ... + y^n} \leq p^H(y) \tag{A.7}$$

By the convexity of D, and as $y^i \leq \sum_i y^i$,we have $\frac{D(y^i)}{y^i} \leq \frac{D(y^1+...+y^n)}{y^1+...+y^n}$, and, therefore, (A.5) holds. ∎

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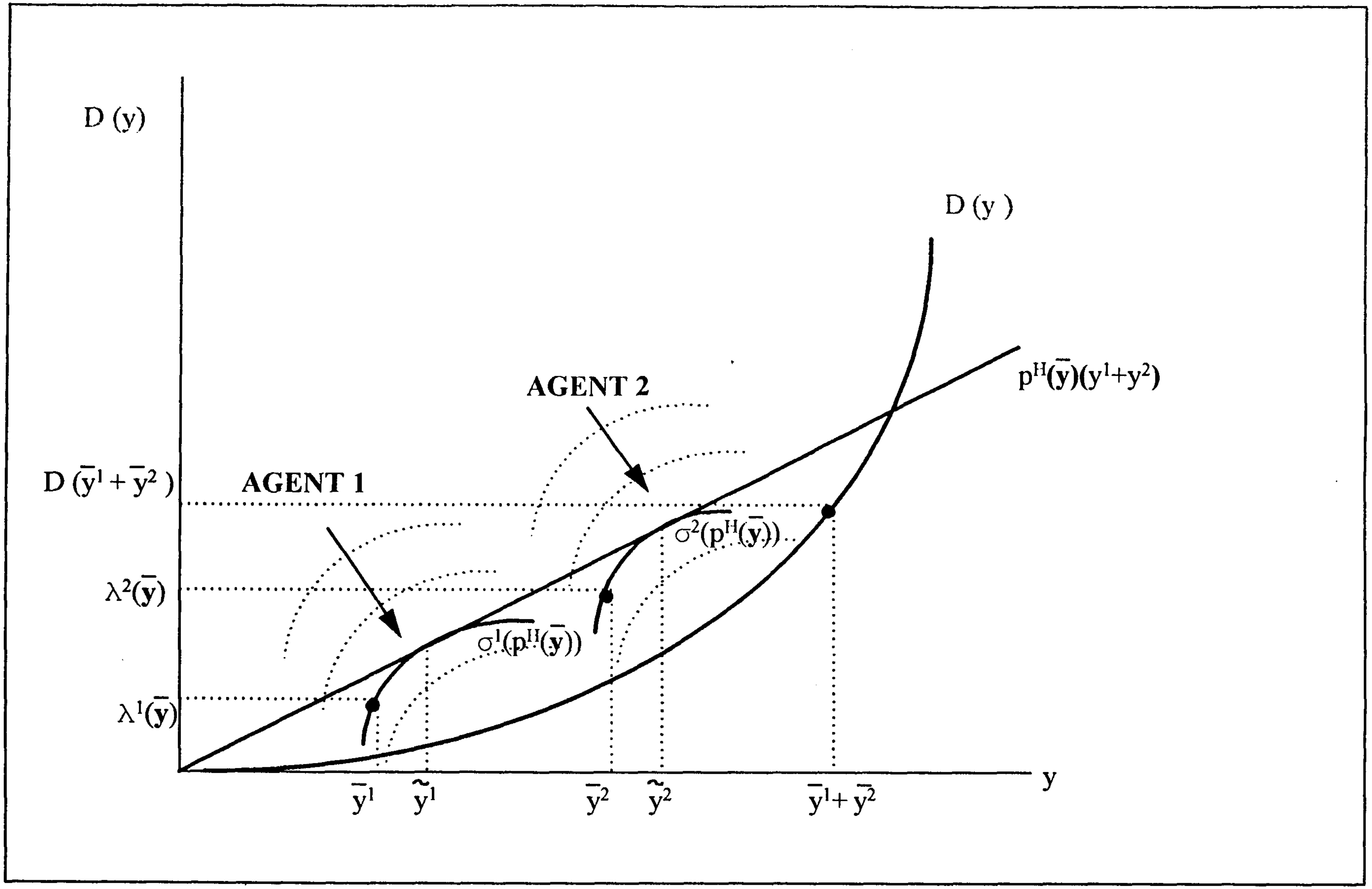

Figure 1. The LDE allocation for the homogeneous case.

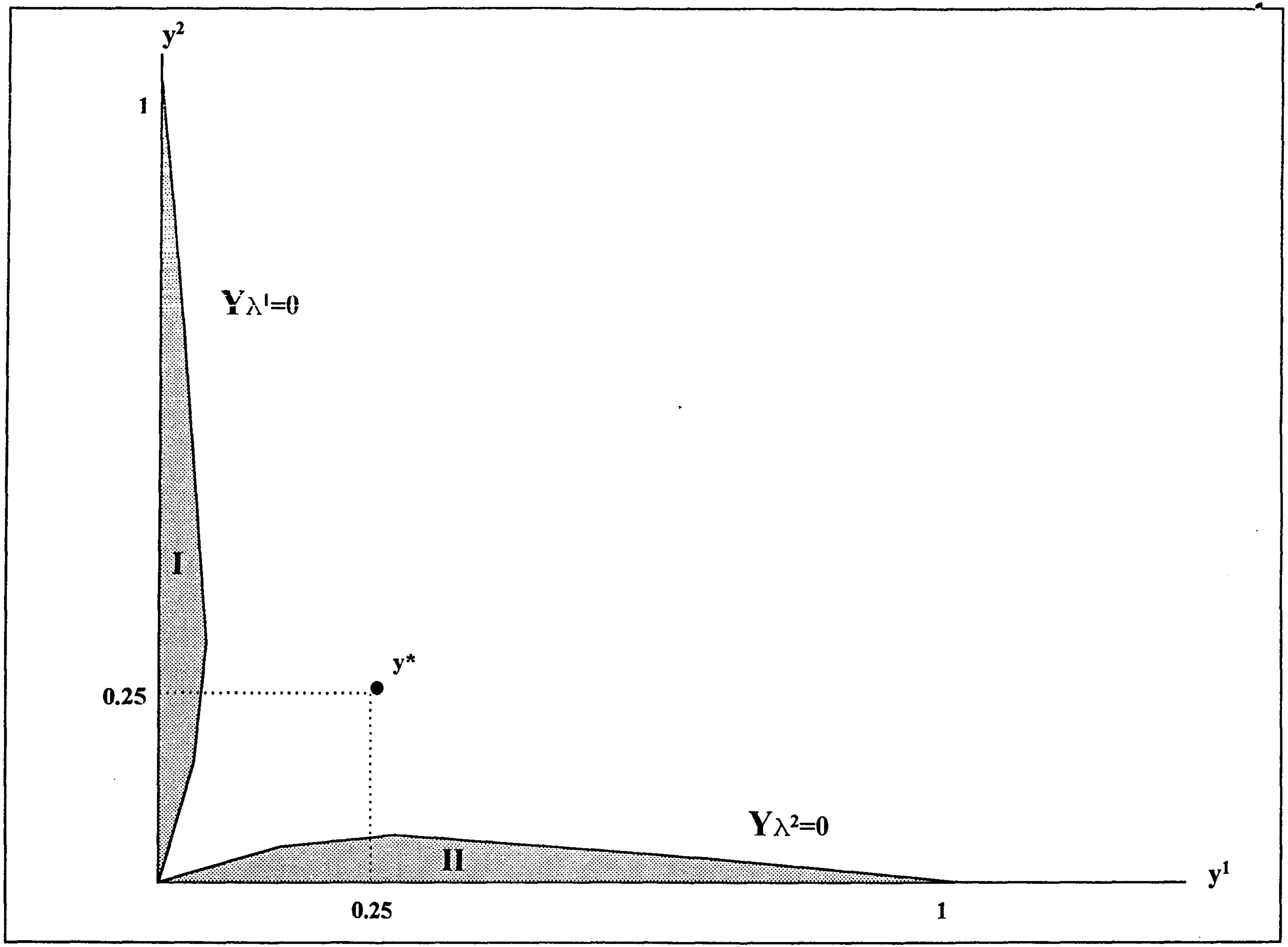

Figure 2. Subsidy regions under the LDE in Example 6.2.

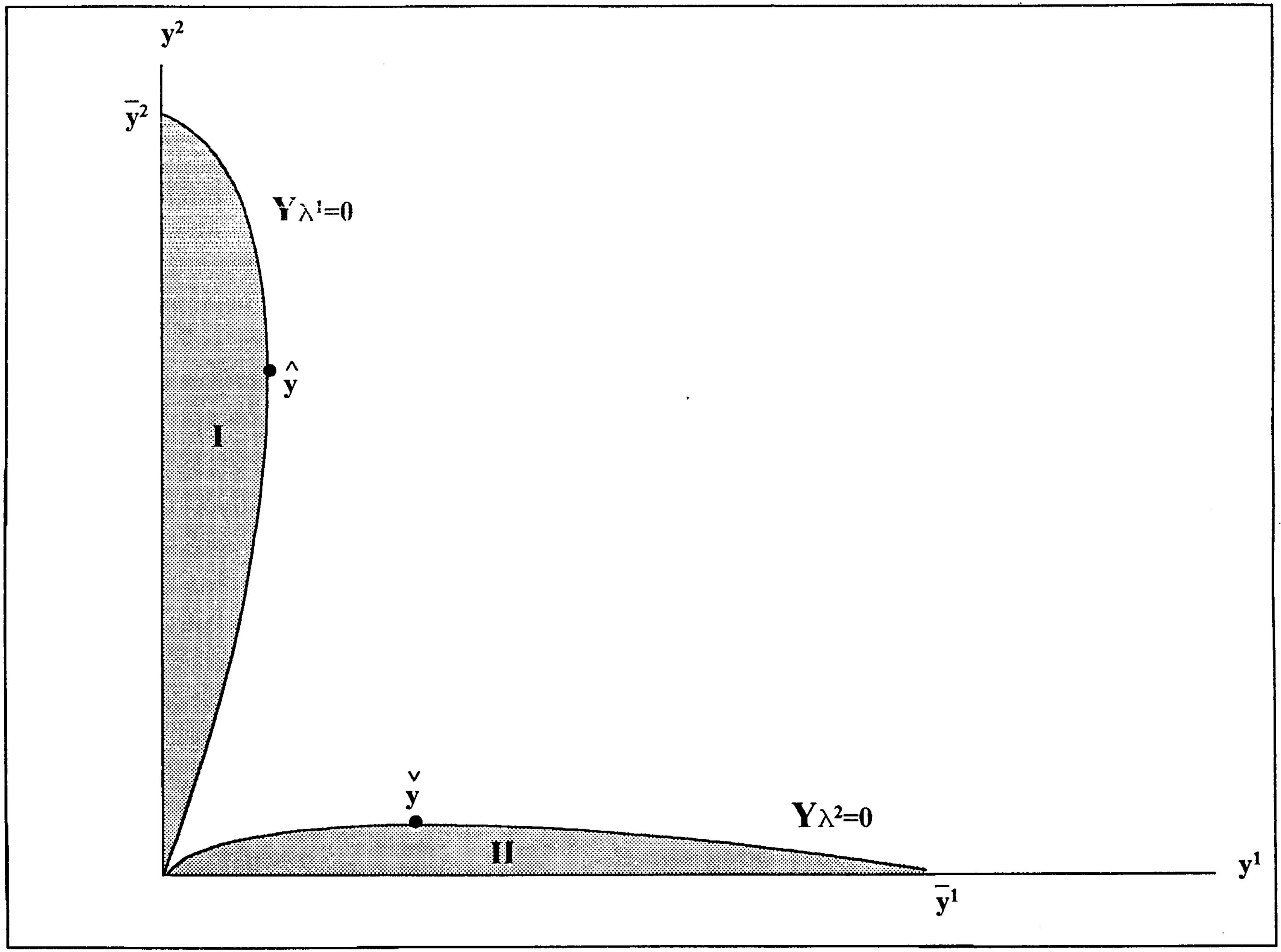

Figure 3. Subsidy regions under the LDE: the general 2-agent case.

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