Efficient Trading Strategies in the Presence of Market Frictions*

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First draft: March 1993. This version: September 1999

Abstract

In this paper we provide a price characterization of efficient consumption bundles in multiperiod economies with market frictions. Efficient consumption bundles are those that are chosen by at least one rational agent with monotonic state-independent and risk-averse preferences and a given future endowment. Frictions include dynamic market incompleteness, proportional transaction costs, short selling costs,

*We have strongly benefitted from conversations with Jean-Marc Bonnisseau, Bernard Dumas, Philip Dybvig, Lars Hansen, Chi-Fu Huang, Jean-Michel Lasry, José Scheinkman, and Michael Woodford. We would like to thank seminar participants at the Massachusetts Institute of Technology, New York University (joint seminar with Columbia University), the Université de Toulouse, the 1994 meetings of the Western Finance Association in Santa Fe, the 1995 Institute for Advanced Study meeting on the mathematical problems in finance, the Courant Institute of Mathematical Sciences, and the second Columbia-Jafee 1998 conference on the mathematics of finance, for their numerous comments. We would also like to thank the editor, Kerry Back, and an anonymous referee for their insightful suggestions, and in particular for greatly simplifying the proof of a key result of this paper. The supports of New York University and of the Laboratoire d'Econométrie de l'Ecole Polytechnique are gratefully acknowledged. Shlomith Zuta provided excellent research assistance. This research was developed while Kallal was affiliated with New York University. Salomon is not responsible for any statement or conclusions herein, and no opinions, theories, or techniques presented herein in any way represent the position of Salomon Brothers Inc.

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borrowing costs, taxes, and others. We characterize the inefficiency cost of a trading strategy - the difference between the investment it requires and the largest amount required by any rational agent to obtain the same utility level - and we propose a measure of portfolio performance based on it. We also show that the arbitrage bounds on a contingent claim to consumption cannot be tightened based on efficiency arguments without restricting preferences or endowments. We examine the efficiency of common investment strategies in economies with borrowing costs due to asymmetric information, short selling costs, or bid-ask spreads. We find that market frictions generally change and typically shrink the set of efficient investment strategies, shifting investors away from well-diversified strategies into low cost ones, and for large frictions into no trading at all. Hence we observe strategies that become inefficient with market frictions, as well as strategies that are rationalized by market frictions.

1. Introduction

In this paper we characterize efficient contingent claims to future consumption (consumption bundles) in multiperiod economies with uncertainty, taking a wide range of market frictions into account,[1] such as dynamic market incompleteness, proportional transaction costs, short sales costs and restrictions, borrowing costs and constraints, taxes, and potentially other imperfections. An efficient consumption bundle is defined as an optimal choice for at least one agent with Von Neumann-Morgenstern preferences and a concave, strictly increasing utility function. If a consumption bundle is inefficient, we compute the lower bound on its efficiency loss across agents with different preferences but given future endowment: this gives a measure of inefficiency that does not rely on a specific utility function. It also allows us to define a measure of portfolio performance. We also show that the arbitrage bounds on a contingent claim cannot be tightened based on efficiency arguments without restricting preferences or endowments. We apply these results to commonly used trading and hedging strategies, and we provide examples of efficient strategies that become inefficient in the presence of market frictions, as well as examples of inefficient strategies that are rationalized by market frictions. Indeed, market frictions generally change and tend to shrink the set of efficient strategies, shifting investors away from diversified investment strategies into low cost strategies, and for large frictions into no trading at all.

In economies without any market imperfections, Dybvig (1988a) provides a useful characterization of efficient consumption bundles, based on the unique positive linear pricing rule (i.e. Arrow-Debreu price vector) that prices traded securities in the absence of arbitrage.[2] A consumption bundle is efficient if and only if it provides at least as much consumption in cheaper states of the world, i.e. in states of the world with lower Arrow-Debreu prices. A new model is then developed, the payoff distribution pricing model

[1] There is already a substantial related literature that studies optimal portfolio and consumption problems with market frictions. Among others, Constantinides (1986), Davis and Norman (1990), Duffie and Sun (1990) and Dumas and Luciano (1991) for the bid-ask spread case, and Cvitanic and Karatzas (1993), He and Pearson (1991), and Tuckman and Vila (1992) for the short sales constraints case. However these studies take preferences as given and derive optimal solutions: they do not provide a characterization of the set of optimal solutions when preferences belong to a general class. Pelsser and Vorst (1996) performs some simulations to study the efficiency of portfolio strategies under transaction costs.

[2] Dybvig and Ross (1982) and Peleg and Yaari (1975) also treat the incomplete market case.

(PDPM), and the size of the inefficiency of a consumption bundle is measured by the difference between its market price (or the investment required to replicate it) and the price of the cheapest consumption bundle with the same distribution - called its "distributional price". Therefore a consumption bundle is efficient if and only if its market price is equal to its " distributional price". This leads to a measure of portfolio performance based on the PDPM which, unlike previous performance measures based on mean-variance analysis, avoids making unrealistic assumptions about preferences and/or the distribution of returns. In Dybvig (1988b) the PDPM is used to analyze trading strategies that are commonly used by practitioners, such as stop-loss or lock-in strategies and rolled-over portfolio insurance. It is found, under a reasonable parametrization of securities returns, that these strategies have an inefficiency cost of the order of 0.5% per year, a substantial amount.

In order to obtain a price characterization of efficient consumption bundles, we first characterize the opportunity set of available returns in arbitrage-free economies with market frictions in terms of linear pricing rules. It is well-known that in arbitrage-free frictionless economies with complete markets there exists a unique positive linear pricing rule that prices any contingent claim and the opportunity set of available consumption is a hyperplane (see for instance Cox and Ross [1976], Harrison and Kreps [1979], Harrison and Pliska [1979], Duffie and Huang [1986], and Back and Pliska [1990]). In this case, the shadow prices at the optimum - the intertemporal marginal rates of substitution - are the same for all agents. On the other hand, in economies with market frictions the pricing rule is generally not linear. However, for a wide range of market imperfections including dynamic market incompleteness, short selling costs and constraints, borrowing costs and constraints, and proportional transaction costs it can be shown that the pricing rule is sublinear[3] (i.e. positively homogeneous and subadditive) and that the opportunity set of available consumption is a convex cone (see Jouini and Kallal [1995a and 1995b and 1999]). This means that in such economies the pricing rule is the maximum of a family of underlying linear pricing rules, which can be interpreted as the different implicit shadow prices - the intertemporal marginal rates of substitution - for different potential agents. For instance, in incomplete markets each underlying linear pricing rule corresponds to a

[3] A pricing rule π is sublinear if $\pi(\lambda x)=\lambda\pi(x)$ and $\pi(x+y)\leq\pi(x)+\pi(y)$, for all comsumption vectors x and y and all positive real number λ.

martingale[4] measure of the price processes of traded securities normalized by a numeraire, and is associated to a possible "fictitious" completion of the initial market (as defined by Karatzas et al. [1991]). In economies with bid-ask spreads, the set of underlying linear pricing rules is the set of martingale measures of all the processes that lie between the normalized bid and ask price processes of traded securities and that can be transformed into a martingale (see Jouini and Kallal [1995a]). In economies with short sales constraints, it is the set of probability measures that transform the normalized price processes of traded securities into supermartingales. Economies where short selling and borrowing are possible but costly, can be analyzed in similar terms and are consistent with our approach (see Jouini and Kallal [1995b]). However, economies where there are higher charges for odd lots, or other fixed transaction costs, do not fall in this framework (indeed, the positive homogeneity of the pricing rule is violated).

This description of the opportunity set of available returns in economies with market frictions enables us to characterize efficient consumption bundles. If we denote by π the sublinear pricing rule and by K the set of underlying pricing rules E, we have $\pi(c) = max\{E(c) : E \in K\}$, and a contingent claim c is efficient if and only if it provides at least as much net consumption in "cheaper" states of the world. However "cheaper" is not defined with respect to the sublinear pricing rule π but with respect to one of the positive underlying linear pricing rules E in K that "prices" c, i.e. that satisfies $\pi(c) = E(c)$. It also allows us to compute the size of the inefficiency of a contingent claim, i.e. the difference between the investment it requires and the largest amount needed by any rational agent, with a given future endowment, to get the same utility level. We show that it is equal to the difference between the investment it requires and the minimum investment necessary to obtain a claim with the same distribution *or* a convex combination of such claims (the "utility price"). Even though the "utility price" coincides with the "distributional price" in the frictionless case, in general it is strictly smaller: this is because, due to the frictions, some distributions of consumption are inefficient. However, we show that the "utility price" of a consumption bundle is the largest of its "distributional prices" in the underlying frictionless economies. We also show that the largest amount needed by

[4] A martingale is a process that is constant on average. A supermartingale is a process that is nonincreasing on average. A submartingale is a process that is nondecreasing on average.

any rational agent with *any* future endowment to get the same utility level as with c is equal to the arbitrage bound $\pi(c)$. Hence arbitrage bounds cannot be tightened based on efficiency arguments without restricting preferences or endowments. Also, in frictionless complete markets, hedging and investment decisions can be separated into two distinct stages: duplicate the liability to be hedged and invest optimally the remaining funds. In the presence of market frictions, however, hedging and investment decisions are intimately related and cannot be separated. Although perfect hedging (duplication) is not always optimal, we find that strategies that minimize the cost of obtaining a payoff at least equal to a given liability have a zero inefficiency cost. These results allow us to define a measure of portfolio performance that does not rely on mean-variance analysis (and avoids the problems associated with it: see Dybvig [1988 a] and Dybvig and Ross [1985]), taking market frictions into account. A correct measure of portfolio performance must trade off the additional frictional costs of alternative investment strategies against their incremental benefit from diversification. Market frictions generally change and typically shrink the set of efficient strategies, shifting investors away from well-diversified investment strategies into low cost strategies, and when frictions are large enough into no trading at all. We also apply these results to economies with market frictions such as different borrowing and lending rates due to asymmetries of information, short selling costs and bid-ask spreads, and we evaluate the inefficiency of investment strategies commonly followed by practitioners. We observe trading strategies that become inefficient as bid-ask spreads are introduced. We also show that high borrowing costs, especially if they increase with leverage, can rationalize strategies such as stop-loss that are inefficient in frictionless markets.

The remainder of the paper is organized as follows. In section 2 we provide a price characterization of efficient consumption bundles and a preference-free characterization of their inefficiency cost (which leads to a measure of portfolio performance), and we investigate tightening the arbitrage bounds. In section 3 we apply the results of section 2 to evaluate numerically the impact of some market imperfections on the efficiency of commonly used trading strategies. All proofs are in the appendix.

2. Efficient trading strategies

2.1 The economy

We consider a multiperiod economy with a finite number of dates and n equiprobable states of the world where investors have some initial wealth w_0 and some uncertain future endowment x in R^n, and maximize their expected utility of future consumption that occurs at the final date T. We denote by π the pricing rule, i.e. agents have to pay $\pi(c)$ units of initial wealth in order to obtain a consumption bundle $c = (c_1, \ldots, c_n)$ that gives the right to c_i units of consumption at the final date T in each state of the world $i = 1, \ldots, n$. We shall make the following:

Assumption 2.1 :

(i) The pricing rule π is sublinear, i.e. $\pi(\lambda x) = \lambda\pi(x)$ and $\pi(x + y) \leq \pi(x) + \pi(y)$, for all $x, y \in R^n$ and all nonnegative real number λ.

(ii) The pricing rule π is arbitrage free, i.e. $\pi(c) > 0$ for any nonzero consumption bundle $c = (c_1, \ldots, c_n)$ such that $c_i \geq 0$ for all states of the world $i = 1, \ldots, n$.

(iii) The pricing rule π satisfies $\pi(1) = -\pi(-1) = 1$.

(iv) The pricing rule π is nondecreasing, i.e. $\pi(x) \leq \pi(y)$, for all $(x, y) \in R^n$ such that $x \leq y$.

Part (i) means that the price of a consumption bundle is proportional to the quantity purchased and that it is less expensive to purchase a portfolio of consumption bundles than to purchase each consumption bundle separately. Note that this implies that $\pi(c) \geq -\pi(-c)$ for any consumption bundle c (indeed $\pi(0) = \pi(c - c) \leq \pi(c) + \pi(-c)$ and $\pi(0) = 0$), i.e. the price at which c can be bought is larger than or equal to the price at which it can be sold. Part (ii) means that there are no arbitrage opportunities, i.e. no free consumption bundles that are nonnegative in every state of the world and strictly positive in at least one. This is a minimum requirement for any model. Part (iii) means that the riskless asset can be bought and sold without any frictions and that the riskless rate is equal to zero. This assumption can be made with little loss of generality: indeed, it amounts to the normalization of all consumption bundles and their prices by a numeraire, e.g. a consumption bundle that is strictly positive in every state of the world and that can be bought and sold without any frictions. Note that in an economy with riskless and risky assets, it is not necessary to take the riskless asset as a numeraire and we can normalize by a risky asset as far as it is strictly positive in all the states of the world. If that risky asset can be bought and sold without any frictions, assumption (*iii*) is then satisfied and different borrowing and lending rates can be taken into account in this framework (see Jouini and Kallal (1995b)). Part (*iv*) is, for instance, satisfied by an equilibrium pricing rule. Indeed, no rational agent will accept to pay more for less.

In multiperiod economies where in order to transfer wealth from the initial date to the future agents can trade a finite number of securities, then $\pi(c)$ is the minimum cost of obtaining a payoff equal to (or larger than) the contingent claim c in all states of the world. Note that in this case $-\pi(-c)$ and $\pi(c)$ can also be interpreted as arbitrage bounds on the price of c : indeed, investors would not pay more than $\pi(c)$ for c, and would not sell it for less than $-\pi(-c)$, since in both cases a better outcome can be reached through securities trading. If markets are complete and frictionless then π is linear. On the other hand, if markets are dynamically incomplete, if borrowing is restricted or the borrowing rate is larger than the lending rate, if short selling securities is restricted or costly, or if there are bid-ask spreads (i.e. proportional transaction costs) then π is sublinear as in Assumption 2.1 (see Jouini and Kallal [1995a and 1995b]). This sublinear representation of the pricing rule is in fact the reduced form of multiperiod models encompassing a larger class of market frictions that includes taxes (see Chen [1992]) and various portfolio constraints, but excludes higher charges for odd lots or other fixed transaction costs. We also have

Proposition 2.1 : *For any pricing rule satisfying Assumption 2.1 there exists a closed convex set K of "underlying" linear pricing rules or "risk-neutral" probability measures $E = (e_1, \ldots, e_n)$, with $e_1 + \ldots + e_n = 1$, that are nonnegative, i.e. $e_i \geq 0$ for $i = 1, \ldots, n$, and where at least one element E^* of K is strictly positive,*[5] *i.e. $e_i^* > 0$ for $i = 1, \ldots, n$, such that $\pi(c) = max\{E(c) : E \in K\}$ for all $c \in R^n$. A linear pricing rule $E \in K$ "prices" c if it satisfies $\pi(c) = E(c)$.*

If markets are complete and frictionless then the set of underlying linear pricing rule K contains a unique element, and hence the set of feasible consumption $\{c \in R^n : \pi(c) \leq w_0\}$ is a hyperplane and the shadow price vector at the optimum - the vector of intertemporal marginal rates of substitution

[5] It also means that K is the closure of the set of its strictly positive elements.

- is the same for all potential agents. On the other hand, if the markets are incomplete and/or there are frictions such that the pricing rule π is only sublinear as in Assumption 2.1, then the set of underlying linear pricing rule K contains more than one element, and the set of feasible consumption $\{c \in R^n : \pi(c) \leq w_0\}$ is only a convex cone.[6] In this case, there are different implicit shadow prices - the underlying linear pricing rules - for different potential agents. We now review four cases of multiperiod economies with frictions such that the pricing rule is sublinear and satisfies Assumption 2.1.

Case 1 (Incomplete Markets) : If markets are dynamically incomplete then the set of underlying linear pricing rules K is the set of martingale measures of the traded securities normalized price processes (see Jouini and Kallal [1995a], as well as Karatzas et al. [1991] and He and Pearson [1991] for the concept of "least favorable fictitious completion"). ∎

Case 2 (Bid-Ask spreads, i.e Proportional Transaction Costs) : If the traded securities can be bought at a price (the ask) that is potentially higher than the price (the bid) at which they can be sold, then the set of underlying linear pricing rules K is the set of martingale measures of any price process between the normalized bid and ask price processes (see Jouini and Kallal [1995a]). ∎

Case 3 (Short Sales Constraints and Short Selling Costs) : If agents are subject to short sales constraints, i.e. if securities cannot be held in negative quantities, then the set of underlying linear pricing rules K is the set of supermartingale measures of the traded securities normalized price processes. The case where short sales are not completely restricted but costly can be treated analogously by introducing shadow securities that cannot be held in positive quantities and have a higher expected return. (see Jouini and Kallal [1995b], as well as Dybvig and Ross [1986] for the two-period case). ∎

Case 4 (Different Borrowing and Lending Rates) : If agents can borrow and lend at possibly different rates, net of default risk (e.g. if asymmetries of information prevent good borrowers from differentiating themselves from bad ones), then the set of underlying linear pricing rules K is equal to the set of martingale measures of the traded securities normalized price processes, where the normalizing numeraire is any instantaneously riskless asset with a rate of return between the borrowing and the lending rate (see

[6] A set C is a convex cone if $x+y \in C$ and $\lambda x \in C$ for all $x, y \in C$ and all nonnegative number λ.

Jouini and Kallal [1995b]). ■

Note that in cases 2, 3, and 4 markets might be dynamically complete and we still have more than one underlying linear pricing rule, i.e. different implicit shadow prices for different potential agents. Moreover, we shall make

Assumption 2.2 : *All the states of the world are equiprobable*[7] *and agents have preferences of the Von Neumann-Morgenstern type: they maximize expected utility, and have a concave strictly increasing utility function. This means that agents prefer more to less, are risk-averse, and only care about the distribution of consumption.*[8]

2.2 Efficient consumption bundles

A contingent claim (and hence the minimum cost trading strategy that leads to it) is efficient if there exists a rational agent for which it is an optimal choice, given his uncertain future endowment. This future endowment - which can be the result of the investor having written a contract contingent on the state of the world, or the result of earnings derived from human capital - is taken as given in this analysis. More formally, we propose

Definition 2.1 : *A contingent claim $c^* \in R^n$ is (resp. strictly) efficient, given an uncertain future endowment $x \in R^n$, if there exists an initial wealth w_0 and a utility function $u \in \mathcal{U}$ (resp. $\mathcal{U}_{sc}$), where $\mathcal{U}$ (resp. $\mathcal{U}_{sc}$) denotes the set of weakly (resp. strictly) concave and strictly increasing Von Neumann-Morgenstern preferences, such that c^* solves $max\{u(c+x) : \pi(c) \leq w_0\}$.*

This is the same definition as in the frictionless case except that the budget constraint is in terms of the sublinear pricing rule π, which is the maximum of a family of nonnegative linear pricing rules. Hence, the budget constraint $\pi(c) \leq w_0$ is a collection of linear budget constraints $E(c) \leq w_0$, for all E in K. Also, since agents have strictly increasing preferences, an efficient contingent claim c^* makes the budget constraint binding and the initial wealth w_0 for which it is an optimal choice must be equal to $\pi(c^*)$.

Also, in frictionless complete markets the optimal consumption problem $\max\{u(c+x) : \pi(c) \leq w_0\}$ can be separated in two steps[9]: first hedge the

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[7] Most results go through unchanged if states of the world are not assumed to be equiprobable, except that state prices must be normalized by actual probabilities in all statements. Also, if we view the finite model as an approximation for continuous distributions this assumption can be made without loss of generality.

[8] We exclude state-dependent preferences (although we allow uncertain future endowments).

[9] Indeed the problem $\max\{u(c+x):\pi(c)\leq w_0\}$ can be written $\max\{u(\tilde{c}):\pi(\tilde{c}-x)\leq w_0\}$, where $\tilde{c}$ denotes net

uncertain future endowment x, which provides an amount $\pi(x)$, and then solve for the optimal net consumption bundle $\tilde{c}$ subject to the budget constraint $\pi(\tilde{c}) \leq w_0 + \pi(x)$. Hence, changing the uncertain future endowment is equivalent to changing the initial wealth. This means that the set of optimal *net* consumption bundles is unaffected by the presence of an uncertain future endowment, and that any trading strategy can be rationalized by assuming a particular uncertain future endowment. This is no longer true in the presence of market frictions, where we have

Theorem 2.1 : *Given an uncertain future endowment $x = (x_1, \ldots, x_n) \in R^n$, a contingent claim $c^* = (c_1^*, \ldots, c_n^*) \in R^n$ is (resp. strictly) efficient if and only if there exists $E^* = (e_1^*, \ldots, e_n^*) \in K$, the set of underlying linear pricing rules, such that*

(i) E^* prices c^*, i.e. $E^*(c^*) = \pi(c^*)$,

(ii) $c^* + x$ is in (resp. strict) reverse order of E^*, i.e. $c_i^* + x_i > c_j^* + x_j$ implies $e_i^* \leq e_j^*$ for all $i, j = 1, \ldots, n$ (resp. $c_i^* + x_i > c_j^* + x_j$ implies $e_i^* < e_j^*$ for all $i, j = 1, \ldots, n$).

This says that a contingent claim is efficient if and only if it gives the right to at least as much *net* consumption in "cheaper" - according to a positive linear pricing rule that prices it - states of the world. If there are no market frictions there exists a unique linear pricing rule and we find that a consumption bundle is efficient if and only if it entitles to at least as much net consumption in "cheaper" - according to the unique linear pricing rule - states of the world, i.e. if it is the cheapest consumption bundle with its distribution (Dybvig [1988a]).

Roughly speaking, this characterization follows from the first-order conditions: marginal utilities of consumption must be proportional to the linear pricing rule corresponding to one of the binding linear budget constraints. From the assumption that agents are risk-averse, marginal utilities are decreasing, which implies that net consumption must be higher in cheaper (according to the binding linear pricing rule) states of the world. The difficulty, however, is that we are dealing with a continuum of constraints. This Theorem generalizes the price characterizations obtained by Peleg and Yaari (1975) and by Dybvig and Ross (1982) in the incomplete markets case.[10]

consumption, and since π is linear in this case it is equivalent to $\max\{u(\tilde{c}) : \pi(\tilde{c}) \leq w_0 + \pi(x)\}$.

[10] Even though the formulation of Peleg and Yaari (1975) is more general than the incomplete markets case, it does not explicitly or implicitly include our framework with market frictions.

At this point we are able to appreciate the impact of market frictions on the efficiency of a contingent claim c^*. Market frictions enlarge the set of underlying linear pricing rules from a single one to a continuum. This makes it easier to satisfy part (ii), i.e. find a linear pricing rule that is in reverse order of the net consumption bundle $c^* + x$. However, it makes it harder to satisfy part (i), i.e. this linear pricing rule must price c^*. Therefore, market frictions do not always make inefficient strategies become efficient or efficient strategies become inefficient: both situations can happen. Indeed, as market frictions increase investors move away from trading strategies that are optimally diversified over time - but have higher costs - towards strategies that have lower costs but are less than optimally diversified, and if costs are higher than any potential diversification benefits the set of efficient trading strategies shrinks to not trading at all (see section 3).

For instance, from Theorem 2.1 we see that the strategy that consists in hedging the uncertain future endowment x by duplicating the contingent claim $-x$ is not necessarily efficient. Indeed, there might not even be any strictly positive measure in K that prices it.[11] However, we have

Remark 2.1 : *Given an uncertain future endowment $x \in R^n$, duplicating the contingent claim $-x$ is an efficient strategy if and only if there exists $E \in K$, the set of underlying linear pricing rules, that is strictly positive and prices $-x$, i.e. is such that $\pi(-x) = E(-x)$. In particular, this will be the case when frictions are sufficiently small so that all the underlying linear pricing rules in K are strictly positive.*

2.3 Inefficiency size

If a contingent claim is found to be inefficient we would like to evaluate the size of its inefficiency, i.e. have a measure of how far it is from being optimal. We propose the following

Definition 2.2 : *The "inefficiency cost" of a contingent claim $c^* \in R^n$, given an uncertain future endowment $x \in R^n$, is the difference $\pi(c^*) - V_x(c^*)$,*

[11] This is the case, for instance, whenever there exists a contingent claim $-\tilde{x}$ that strictly dominates $-x$ but is cheaper to duplicate than $-x$ (see Bensaid et al. [1992] for examples of this situation with options). In this case, it is clear that duplicating the claim $-x$ is not optimal, and that if a pricing rule E prices $-x$, and if $-\tilde{x}$ minimizes the cost of dominating $-x$, we have $E(-x)=\pi(-x)=\pi(-\tilde{x})$ and hence $E(-\tilde{x})=E(-x)$, which implies that E cannot be strictly positive since $-\tilde{x}$ strictly dominates $-x$.

where

$$V_x(c^*) = \sup_{u\in\mathcal{U}}\{\min_c\{\pi(c) : u(c+x) \geq u(c^*+x)\}\}$$

denotes the "utility price" of c^*.

Indeed, $V_x(c^*)$ is the largest amount that is required by rational agents with an uncertain future endowment x in order to get the same utility level as with the consumption bundle c^*. Hence, $\pi(c^*) - V_x(c^*)$, which is equal to $\inf_{u\in\mathcal{U}}\{\pi(c^*) - \min_c\{\pi(c) : u(c+x) \geq u(c^*+x)\}\}$, is the smallest discrepancy, across all rational agents with future endowment x, between the cost of obtaining c^* and the price at which it would be an optimal choice. Also, we have $\pi(c^*) \geq V_x(c^*)$, i.e. our measure of inefficiency is always nonnegative. Moreover, if c^* is efficient then $V_x(c^*) = \pi(c^*)$ and our measure of inefficiency is equal to zero. Also note that this measure of inefficiency does not depend on the choice of a specific utility function.

In dynamically complete frictionless markets the utility price of a contingent claim coincides with the minimum cost of achieving the same distribution of consumption (see Dybvig [1988a]), and efficiency is equivalent to cost minimization of achieving a distribution of consumption. Even though efficiency always implies cost minimization, the converse in not true in imperfect markets (see the Example in the appendix). Hence, in looking for a cost characterization of our measure of inefficiency we shall consider the set of consumption bundles that are equal to (or larger than) a convex combination of consumption bundles with a given distribution. We then have

Lemma : *For every consumption bundle* $\tilde{c} \in R^n$ *we have*

$$\{c : u(c) \geq u(\tilde{c}), \forall u \in \mathcal{U} \text{ (resp.}\mathcal{U}_{sc})\}$$
$$= \{ \text{ convex hull of the bundles distributed as } \tilde{c}\} + R^n_+.$$

Note that this is a new alternative characterization of second-order stochastic dominance (see Rothschild and Stiglitz [1970] for other characterizations). This allows us to prove

Theorem 2.2 : *Given an uncertain future endowment* $x \in R^n$, *for every contingent claim* $c^* \in R^n$ *the utility price of* c^*, *satisfies* [12]

[12] Note that it implies that our utility price coincides with the utility price defined with respect to the smaller class of strictly increasing and strictly concave Von Neumann-Morgenstern preferences $\mathcal{U}_{sc}$. This means that the utility price is somewhat robust to changes in the class of preferences that is considered.

$$\begin{aligned}(i) V_x(c^*) &= \min\{\pi(\tilde{c}-x) : \tilde{c} \text{ is a conv. comb. of bundles distributed as } c^*+x\} \\ &= \min\{\pi(c) : u(c+x) \geq u(c^*+x) \text{ for all } u \in U \text{ (resp. } \mathcal{U}_{sc})\}\end{aligned}$$

(ii) $V_x(c^*) = max\{P_x(c^*, E) : E \in K\}$,
where

$$\begin{aligned}P_x(c^*, E) &= \min\{E(\tilde{c}-x) : \tilde{c} \text{ is distributed as } c^*+x\} \\ &= P_0(c^*+x, E) - E(x)\end{aligned}$$

is the utility price of c^* *in the frictionless economy defined by the linear pricing rule* E*, and* K *is the set of underlying linear pricing rules associated to* π*. Hence,*

$$\max_{E\in K} P_0(c^*+x, E) - \pi(x) \leq V_x(c^*) \leq \max_{E\in K} P_0(c^*+x, E) + \pi(-x)$$

(iii) $V_x(c^*) = max\{\int_0^1 F^{-1}_{c^*+x}(y) F^{-1}_E(1-y)dy - E(x) : E \in K\}$,
where K *is the set of underlying linear pricing rules associated to* π, $F_{\tilde{c}}(z)$ *is the probability that the random variable* $\tilde{c}$ *is less than or equal to* z *(and similarly for* F_E*), and* $F^{-1}(y) = \min\{z : F(z) \geq y\}$ *for all* $y \in (0, 1)$ *is the inverse of the cumulative distribution function* F.

Part (i) says that the utility price $V_x(c^*)$ of the contingent claim c^* is equal to the cost of the cheapest claim that leads to a net consumption bundle distributed as c^*+x *or* that is a convex combination of consumption bundles distributed as c^*+x. Note that this implies that given a future endowment x, $V_x(c^*)$ only depends on the distribution of net consumption c^*+x. It also says that $V_x(c^*)$ is equal to the cost of the cheapest contingent claim that makes every rational agent at least as happy as with the net consumption bundle c^*+x. Note that in the frictionless case, because the pricing rule π is linear, the minimum $\min\{\pi(\tilde{c}-x) : \tilde{c}$ *is a convex combination of bundles distributed as* $c^*+x\}$ is attained for a consumption bundle that has the same distribution as c^*+x. Hence the utility price coincides with the minimum cost of achieving a given distribution of net consumption. In imperfect markets though, this minimum is only attained for convex combinations of consumption bundles that have the same distribution as c^*+x.

Part (ii) is analogous to Proposition 2.1 which says that $\pi(c^*)$ is the largest of the prices of c^* for the underlying linear pricing rules in $K : V_x(c^*)$

is the largest of the utility prices of c^* in the underlying frictionless economies. Moreover, if there is no uncertain future endowment (i.e. if $x = 0$), $V_0(c^*)$ is the largest of the distributional prices of c^* in the underlying frictionless economies. This implies[13] that if we can find a consumption bundle $\tilde{c}$ with the same distribution as $c^* + x$ and that is in reverse order of a linear pricing rule $\tilde{E}$ in K that prices $\tilde{c} - x$, then the utility price of c^* is equal to $V_x(c^*) = \pi(\tilde{c} - x) = \tilde{E}(\tilde{c} - x)$.

Note that unlike in frictionless markets, in the presence of market frictions it can happen that even though a claim is not efficient, its inefficiency cost is nonetheless equal to zero. However, as can readily be seen from the definition, the fact that a contingent claim has a zero inefficiency cost implies that it is arbitrarily close from being an optimal choice for some rational agents. Moreover, we have the following price characterization:

Theorem 2.3 : *Given an uncertain future endowment $x \in R^n$, the inefficiency cost $\pi(c^*) - V_x(c^*)$ of a contingent claim $c^* = (c_1^*, \ldots, c_n^*) \in R^n$, is equal to zero if and only if there exists $E^* = (e_1^*, \ldots, e_n^*) \in K$, such that*

(i) E^* prices c^*, i.e. $E^*(c^*) = \pi(c^*)$,

(ii) $c^* + x$ is in reverse order of E^*, i.e. $c_i^* + x_i > c_j^* + x_j$ implies $e_i^* \leq e_j^*$ for all $i, j = 1, \ldots, n$.

Note that this is almost the characterization of efficient contingent claims obtained in Theorem 2.1, except that the linear pricing rule E^* does not need to be strictly positive.[14] For instance, we have

Remark 2.2 : *Given an uncertain future endowment $x \in R^n$, the minimum cost strategies that dominate $-x$ have a zero inefficient cost.*

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[13] Indeed, by Theorem 2.2 (*ii*) we have $V_x(c^*) \geq P_x(c^*, \tilde{E}) = P_0(c^* + x, \tilde{E}) - \tilde{E}(x) = \tilde{E}(\tilde{c}) - \tilde{E}(x) = \tilde{E}(\tilde{c} - x) = \pi(\tilde{c} - x)$ and by Theorem 2.2 (*i*) we have $V_x(c^*) \leq \pi(\tilde{c} - x)$.

[14] If the set of underlying linear pricing rules K has a finite number of extreme points (which is the case in most models with a finite number of periods and states of the world, and all the examples in this paper), we also have that a contingent claim $c^* \in R^n$ is strictly efficient if and only if it is the unique solution of $\min\{\pi(c) : c + x$ is at least equal to a convex combination of claims distributed as $c^* + x\}$. The assumption is needed to avoid situations where marginal rates of substitution are required to be unbounded at the optimum. One could expand the set of utility functions to lexicographic or hyperreal-valued utility functions, as in Blume et al. (1991a and 1991b), which allow infinite marginal rates of substitution. We are grateful to an anonymous referee for this point.

This means that as extreme as it may seem in some specific cases[15] dominating the uncertain future endowment, while minimizing the cost of doing so, cannot be ruled out on efficiency grounds only: it is a well-diversified strategy (albeit one may rule out the preferences that "rationalize" it).

2.4 Arbitage bounds and utility bounds

As a consequence of Theorem 2.3 we also have

Corollary 2.1 : *For every contingent claim $c^* \in R^n$, we have*

$$max\{V_x(c^*) : x \in R^n\} = \pi(c^*).$$

This means that even though for a given uncertain future endowment x the "utility upper bound" $V_x(c^*)$ might be strictly lower than the arbitrage upper bound $\pi(c^*)$ and the "utility lower bound" $-V_x(-c^*)$ might be strictly higher than the arbitrage lower bound $-\pi(-c^*)$, the widest range of "utility bounds" across all possible uncertain future endowments coincides with the interval of arbitrage bounds. Hence, if neither preferences nor endowments are observable, efficiency arguments do not lead to tighter bounds on the price of a contingent claim c^* than the simple arbitrage bounds $\pi(c^*)$ and $-\pi(-c^*)$. In order to achieve tighter bounds, further restrictions on preferences and/or endowments are necessary.

2.5 Portfolio performance

In this section, we apply the results of the previous sections to the measure of portfolio performance. As in Dybvig (1988a), in measuring performance we follow the tradition of comparing some investment strategy - and its distribution of payoffs - to the alternative of trading in a given securities market: the benchmark market. However, we do not assume that it is frictionless, and because of this we have to take the uncertain future endowment into account since investment and hedging decisions can no longer be separated. Ignoring these frictions would make the benchmark market available to investors more attractive than it actually is. This effect is mitigated by the fact that the investment strategy itself is subject to the same frictions. The previous results will allow us to evaluate the net effect.

[15] For instance if a risky asset follows a geometric brownian motion with a positive bid-ask spread, a minimum cost strategy that dominates a call option consists in buying and holding the risky asset (see Soner et al. [1995]).

Given an uncertain future endowment x, an investment strategy is evaluated on the basis of the distribution F_{c+x} of its net payoff $c+x$, where c might depend on information not available to the agents (but only to the portfolio manager), allowing for information-trading and private investments outside the benchmark market. The benchmark market is described by the set K of underlying linear pricing rules that summarize the investment opportunities that are available. For utility pricing, by Theorem 2.2 (iii) the relevant characteristic of the benchmark market is the set of cumulative distribution functions of the underlying linear pricing rules $\{F_E : E \in K\}$. The following Theorem is the counterpart of Theorem 4 in Dybvig (1988a) for the frictionless case, and is a consequence of our Theorem 2.2.

Corollary 2.2 : *Suppose that an investment strategy leads from an initial wealth w_0 to a cumulative net distribution of payoffs F_{c+x}, where x is the uncertain future endowment. Let $V_x(c) = max\{\int_0^1 F_{c+x}^{-1}(y) F_E^{-1}(1-y)dy - E(x) : E \in K\}$. Then,*

(i) If $w_0 < V_x(c)$, we have superior performance, i.e. there exists a rational agent with concave and strictly increasing Von Neumann-Morgenstern preferences who prefers receiving the net distribution of payoffs F_{c+x} to trading in the benchmark market. Moreover, the largest amount such a rational agent would pay to switch is $V_x(c) - w_0 > 0$.

(ii) If $w_0 = V_x(c)$, we have ordinary performance, i.e. every rational agent with concave and strictly increasing Von Neumann-Morgenstern preferences weakly prefers trading in the benchmark market to receiving the distribution of payoffs F_{c+x}. However, the lowest amount such a rational agent would pay to switch is equal to zero.

(iii) If $w_0 > V_x(c)$, we have inferior performance, i.e. every rational agent with concave and strictly increasing Von Neumann-Morgenstern preferences strictly prefers trading in the benchmark market to receiving the distribution of payoffs F_{c+x}. Moreover, the lowest amount such a rational agent would pay to switch is $w_0 - V_x(c) > 0$.[16]

[16] These are in fact the infimum or the supremum over all the concave strictly increasing Von Neumann-Morgenstern preferences, which are not necessarily attained for a specific utility function.

Hence, by comparing the initial investment required by an investment strategy to the utility price of the distribution of its payoff we can evaluate its performance. If the utility price is lower than the initial investment, the portfolio is not well-diversified and is underperforming. If the utility price is equal to the initial investment, the portfolio is well-diversified and it is performing as it should. If the utility price is larger than the initial investment, the manager has superior ability and/or information, and/or is subject to lower transaction costs, and the portfolio is overperforming.

As argued in Dybvig (1988a) this provides an alternative to the Security Market Line (SML) in measuring portfolio performance. As opposed to the SML analysis, this alternative gives a correct evaluation even when superior performance is due to private information. Indeed, the SML is based on mean-variance analysis,[17] and even if securities returns are assumed to be jointly normally distributed, they will typically not be normal once conditioned on information (see Dybvig and Ross [1985]).

3. Examples and numerical results

In this section we examine examples of a multiperiod economy (the binomial economy) where agents can trade a riskless asset, paying a continuously compounded interest rate r, and a risky asset that follows a multiplicative binomial process with an initial value $S(0)$ and an actual probability of $\frac{1}{2}$ of going "up" by $u = exp(\mu\frac{T}{n} + \sigma\sqrt{\frac{T}{n}})$ or "down" by $d = exp(\mu\frac{T}{n} - \sigma\sqrt{\frac{T}{n}})$ each period and at each node, where T denotes the length of the investment horizon, and n the number of periods. We shall assume that $\sigma > |\mu - r|\sqrt{\frac{T}{n}}$ to ensure the absence of arbitrage. In this example all states of the world are equiprobable and the results of section 2 apply. It is well-known that this binomial process converges (as the number of periods n goes to infinity) to a geometric Brownian Motion process with drift (instantaneous expected return) $\mu + \frac{1}{2}\sigma^2$ and volatility σ (see Cox, Ross and Rubinstein [1979]).

3.1 Bid-ask spreads: proportional transaction costs

In this example transacting in the risky asset is costly, and the transaction cost is proportional to the quantity transacted. If one share of the risky asset

[17] Mean-variance analysis can be justified either by assuming normally distributed returns or by assuming quadratic utility. However, the latter assumption implies undesirable properties such as nonmonotonic preferences and increasing absolute risk aversion.

is worth S at a given time and state of the world, we assume that it can be bought for $(1+\kappa)S$ and can be sold for $(1-\kappa)S$: the bid-ask spread (per share) is equal to $2\kappa S$, where κ is a nonnegative constant. We assume that the risky asset has a nonnegative expected excess return (i.e. $\frac{u+d}{2} \geq exp(r\frac{T}{n})$), i.e. the risk-neutral probability p_u of going "up" from each node is less than or equal to 0.5.

Even for very small (but strictly positive) κ, the strategy that consists in dynamically duplicating a call option on the risky asset (for any strike price P) is inefficient when n is large. In the frictionless economy (where $\kappa = 0$) by Theorem 2.1 the efficient consumption bundles are those that are in the same order as the price of the risky asset (since $p_u \leq 0.5$). This means that if the price of the risky asset is higher in a state of the world than in another, so is the payoff of any efficient consumption bundle. Since the payoff of a call option with strike price P (and physical delivery) is equal to $c^* = \max\{(1-\kappa)S(T) - P, 0\}$ at its expiration date, it satisfies this requirement. However, the trading strategy that duplicates this payoff requires frequent portfolio rebalancing: if $\kappa > 0$ it can be shown (see Soner et al. [1995]) that as the number of periods n grows to infinity the total cost incurred is at least equal to the cost $(1+\kappa)S(0)$ of purchasing one share of the risky asset at the initial date. Since the payoff of this investment strategy is $(1-\kappa)S(T)$ at the final date T, it strictly dominates the payoff of the call option. This shows that duplicating the call option is inefficient, regardless of its strike price, as long as $\kappa > 0$. Note that by Theorem 2.2 the "utility price" $V_0(c^*)$ of any consumption bundle c^* is at most equal to $exp(-rT)E_{\frac{1}{2}}(c^*)$, the present value of its expected value with respect to the actual probability measure.[18] Hence, the inefficiency cost of c^* satisfies $\pi(c^*) - V_0(c^*) \geq \pi(c^*) - exp(-rT)E_{\frac{1}{2}}(c^*)$, where $\pi(c^*)$ denotes the minimum cost of achieving or dominating c^*. In our example (when n goes to infinity) this means $\pi(c^*) - V_0(c^*) \geq (1+\kappa)S(0) - exp(-rT)E_{\frac{1}{2}}(\max\{S(T) - P, 0\})$. For example, if $\kappa = 0.1\%$, $r = 6\%$, $\sigma = 20\%$, $\mu + \frac{1}{2}\sigma^2 = 12\%$, $P = S(0)$, and $T = 1$ year, then the inefficiency cost of hedging an at-the-money call option is at least equal to 83.99% of the value of the underlying risky asset, an enormous amount. Nevertheless, the cheapest hedging strategy for that call

[18] Indeed, since after normalizing by $exp(rT)$ we have $V_0(c^*) = \min\{\pi(c) : c \in \Sigma(c^*)\}$, where $\Sigma(c^*)$ is the set of convex combination of contingent claims distributed as c^*, and since $(E_{\frac{1}{2}}(c^*), \ldots, E_{\frac{1}{2}}(c^*))$ belongs to $\Sigma(c^*)$ this implies $V_0(c^*) \leq \max\{E(E_{\frac{1}{2}}(c^*), \ldots, E_{\frac{1}{2}}(c^*)) : E \in K\} = E_{\frac{1}{2}}(c^*)$.

option costs $(1+\kappa)S(0)$ (see Soner et al. [1995]) but has a zero inefficiency cost by Theorem 2.3 and Remark 2.2. By Theorem 2.1 and Theorem 2.2 we also have

Remark 3.1 : *If* $Log(\frac{1+\kappa}{1-\kappa}) > max\{|(\mu-r)T+nLog(\frac{exp(\sigma\sqrt{\frac{T}{n}})+exp(-\sigma\sqrt{\frac{T}{n}})}{2})|, |\mu-r|T\}$, *e.g. if* $Log(\frac{1+\kappa}{1-\kappa}) > |\mu+\frac{1}{2}\sigma^2-r|T$ *when* n *is large enough, then the only efficient trading strategies consist in investing in the riskless asset. The "utility price"* $V_0(c^*)$ *of any consumption bundle* c^* *is then equal to* $V_0(c^*) = exp(-rT)E_{\frac{1}{2}}(c^*)$, *the present value of its expectation with respect to the actual probability measure with equiprobable states of the world.*

For instance, consider the parameters above where $(\mu+\frac{1}{2}\sigma^2-r)T = 6\%$. In this case, if κ is equal to 3% or more this condition is satisfied and the only strategy that is efficient consists in investing in the riskless asset. To put this number into prospective, Amihud and Mendelson (1991) report that the typical bid-ask spread on Treasury Notes and Bonds is roughly equal to 0.03% of face value, i.e. $\kappa = 0.015\%$. On the other hand, Sharpe (1987) reports an average bid-ask spread of 0.52% for large capitalization stocks (larger than 1.5 billion dollars), up to 6.55% for small capitalization stocks (smaller than 10 million dollars). When the typical commission rate of 1% charged by retail brokers is taken into account this means that κ averages from 1.25% to 4.25% for stocks, depending on their size.

3.2 Short selling costs/Different borrowing and lending rates

In this example the risky asset has no bid-ask spread and it can be sold short, but it costs an annualized percentage c of the asset value to do so over any period of time (see Tuckman and Vila [1992]). To model this situation we assume that the risky asset cannot be held in negative quantities and we introduce a shadow risky asset $\tilde{S}$ that cannot be held in positive quantities and that has a higher expected return: $\tilde{S}(0) = S(0)$, $\tilde{u} = exp((\mu+c)\frac{T}{n}+\sigma\sqrt{\frac{T}{n}})$ and $\tilde{d} = exp((\mu+c)\frac{T}{n}-\sigma\sqrt{\frac{T}{n}})$. We shall analyze the efficiency of a stop-loss trading strategy by an investor who expects the risky asset to have a negative excess return (i.e. $\mu-r+\frac{1}{2}\sigma^2 \leq 0$) and sells it short, but liquidates the position if unexpected[19] losses exceed a given threshold percentage ϵ of the initial investment. This is plausible if investors disagree on the actual probability distribution of returns of the risky asset. Note that the short interest on the NYSE averaged about 3.5 billion shares in 1997, almost 2%

[19] This is a slight difference with the strategy in Dybvig (1988b) where the threshold applies to actual losses.

of the total number of shares listed, more than 6 time the average daily volume,[20] and according to Engel and Boyd (1983, chap. 22) short selling normally accounts for 6% to 8% of transactions.[21] We have by Theorem 2.2

Remark 3.2 : *The utility price of any contingent claim c^* is equal to $V_0(c^*) = E_\beta(\tilde{c})$ where $\tilde{c}$ is distributed as c^* and is in reverse order of E_β, the probability measure such that the conditional probability of going "up" at each node is $\beta = max\{([\alpha_1, \alpha_2] \cup [1-\alpha_2, 1-\alpha_1]) \cap [0, \frac{1}{2}]\}$, with $\alpha_1 = \frac{exp((r-c)\frac{T}{n})-d}{u-d}$ and $\alpha_2 = \frac{exp(r\frac{T}{n})-d}{u-d}$.*

This provides us with a simple algorithm (together with Feller [1950, vol. 1, chap. 14] for the distribution of payoffs of the trading strategies) for computing the utility price of any contingent claim c^* and evaluating the inefficiency cost of any trading strategy (the computation of $\pi(c^*)$ can be carried out by backward induction as shown in Jouini and Kallal [1995 b]). If $T = 1$ year, $r = 8\%$, $\mu + \frac{1}{2}\sigma^2 - r = -4\%$, $\sigma = 20\%$, and $\epsilon = 10\%$, when short selling is costless we find that the inefficiency cost of the stop-loss strategy is equal to 0.28% (see Dybvig [1988b]), but if the short selling cost is equal to $c = 1\%$ it is reduced to 0.2%, and it is totally eliminated if the short selling cost is as high as $c = 4\%$. More generally we have by Theorem 2.1

Remark 3.3 : *The stop-loss strategy is efficient if and only if the cost c of short selling the risky asset is equal to $-(\mu - r + \frac{1}{2}\sigma^2)$, i.e. to the negative of its expected excess return over the riskless rate. This means that if the short selling costs are high enough they rationalize strategies such as stop-loss that are inefficient in frictionless markets.*[22]

To put these costs into prospective note that short selling a specific stock requires posting a 50% initial margin, and that the proceeds from the short sale are typically not available to the investor (although large institutional investors are generally able to negotiate a much lower rental fee, it tends to increase sharply with the desirability of the short sale).[23] In this case, $c = r$, which is equal to 8% in our example. In the bond market, short

[20] See The Wall Street Journal, June 22, 1998, page C12.

[21] Arguably, a good portion of the short positions have hedging motives in situations such as mergers and acquisitions, the purchase of options or convertible securities, and tax management.

[22] We would obtain exactly the same result for other strategies studied in Dybvig (1988b) such as lock-in strategies and rolled-over portfolio insurance.

[23] See Cox and Rubinstein (1985, p. 50), and Sharpe (1987, p. 34).

sales are performed through repurchase agreements in which the short seller lends money at the repo rate and takes the bond as collateral. If the bond is "special", meaning that it is particularly hard to borrow - which is typically the case for the most liquid benchmark bonds - its repo rate can be sharply lower than the repo rate on general collateral. The short selling cost c is then the sum of the bid-ask spread on the repo rate, and the difference between the repo rate and the repo on general collateral. Stigum (1983) reports typical values of c between 0.25% and 0.65%, but we can have $c = r$ if the bond is impossible to borrow and the short seller is forced to fail on its delivery. Since this is likely to happen when the bond has outperformed, stop-loss strategies are plausibly rationalized by such shortselling costs.

We can similarly analyze the case where the borrowing rate is higher than the riskless lending rate (net of the ex-ante probability of default). This can occur because of asymmetric information between borrowers and lenders, and the inability of good borrowers to differentiate themselves from bad ones. We examine a stop-loss strategy that consists in borrowing some amount at the initial date and investing it in the risky asset, liquidating the position whenever unexpected losses exceed a given threshold fraction ϵ of the initial investment. If $T = 1$ year, $r = 8\%$, $\mu + \frac{1}{2}\sigma^2 - r = 10\%$, $\sigma = 20\%$, and $\epsilon = 10\%$, when there are no borrowing costs we find that the inefficiency cost of the stop-loss strategy is equal to 0.79%, but if the borrowing cost is equal to 3% it is reduced to 0.49%, and it is totally eliminated if the cost is as high as 10%. Again, we find that this strategy is rationalized by borrowing costs equal to the expected excess return of the risky asset $\mu + \frac{1}{2}\sigma^2 - r$.

To put these borrowing costs into prospective, note that individual investors can borrow with their home as collateral at a spread of roughly 1%, that they typically pay a spread of 2.5% to borrow against their stock holdings, and a spread of the order of 10% on their credit card balance (uncollateralized borrowing). As far as corporations are concerned, the spread at which they can borrow typically depends on their leverage. For instance, the average spread at which AAA companies can borrow is roughly 0.4% whereas it is roughly 5.5% for B companies[24]. According to *Standard & Poor's Credit Week* (November 8, 1993, p. 41-2) the median total debt as a percentage of capitalization was 21.9% for AAA companies and 65.9% for B companies

[24] Although it is difficult to estimate ex-ante probabilities of default, the available evidence suggests that these spreads go well beyond this component.

during the three-year period 1990-1992. In order to take these stylized facts into account we assume that leverage is observable and that it enables borrowers to partially differentiate themselves. Suppose that the borrowing cost is equal to $a + bR$ with $a = 0.5\%$ and $b = 3\%$, where R is the debt-to-equity ratio. In this case, the stop-loss strategy is rationalized by such borrowing costs if the liquidation threshold is equal to 50% or higher.

5. Conclusion

In this paper, we have provided a general price characterization of efficient (i.e. optimal for at least one rational agent with concave and strictly increasing Von Neumann-Morgenstern preferences) consumption bundles in arbitrage-free multiperiod economies with market frictions. The opportunity set in such economies can be characterized in terms of a sublinear pricing rule that is the maximum of a convex set of underlying frictionless nonnegative linear pricing rules. We have shown that a contingent claim is efficient if and only if it gives the right to at least as much net consumption in "cheaper" states of the world, where "cheaper" is meant with respect to an underlying linear pricing rule that "prices" the contingent claim. We have then defined a conservative measure of the potential inefficiency of a contingent claim as the difference between its minimum cost to achieve and the maximum amount it would cost any rational agent to get at least the same utility level (the "utility price", which does not depend on any specific utility function). We have shown that the utility price coincides with the "distributional price" (i.e. the minimum amount it costs to obtain the same distribution of consumption) in frictionless economies, but that it is in general smaller in economies with market frictions, and that it is equal to the minimum amount it costs to obtain the same distribution of consumption or a convex combination of consumption bundles with the same distribution. Furthermore, we have proved that it is not possible to tighten the arbitrage bounds on a contingent claim to consumption based on efficiency arguments without restricting preferences or endowments. Also, we have exploited these results to propose a measure of portfolio performance in imperfect markets without relying on strong assumptions on preferences such as the Security Market Line analysis. We have then applied these results to commonly used trading and hedging strategies in the presence of different borrowing and lending rates due to asymmetries of information, short selling costs, and bid-ask spreads. We have given examples of efficient trading strategies that become inefficient with market frictions, as well as examples of inefficient strategies that are rationalized by market frictions. Indeed, the presence of market frictions genarally changes and tends to shrink the set of efficient strategies, shifting investors away from diversified investment strategies into low cost strategies, and for large costs into no trading at all.

Appendix

First, recall that for a convex function $F : \Omega \to R$, where Ω is an open subset of R^n, the subgradient of F at $x \in \Omega$ is defined by $\partial F(x) = \{p \in R^n : p\cdot(y-x) \leq F(y)-F(x) \text{ for all } y \in \Omega\}$. Furthermore, following Clarke (1983, Theorem 2.5.1) we have that $\partial F(x)$ is the convex hull of $\{\lim_{n\to\infty} F'(x_n) : (x_n) \text{ converges to } x \text{ and } F \text{ is differentiable at } x_n\}$. For a concave function G we define ∂G as $-\partial(-G)$.

Example : Inefficient Distributions of Returns

Consider a two-period economy with two equiprobable states of the world, "up" and "down". We shall assume that the riskless rate is equal to zero (this is merely a normalization) and that investors can buy and sell a risky asset that pays off S_u in state "up" and S_d in state "down", with $S_u > S_d$, at an ask price $S^b = bS_u + (1-b)S_d$ and a bid price $S^a = aS_u + (1-a)S_d$, with $1 > b > a > 0$. We have already found that the set of risk-neutral measures (or underlying linear pricing rules) is equal to $K_{a,b} = \{(p^*, 1-p^*) : p^* \in [a, b]\}$, and hence the minimum cost to obtain a consumption bundle (c_u, c_d) is equal to $ac_u + (1-a)c_d$ if $c_u \leq c_d$, and is equal to $bc_u + (1-b)c_d$ otherwise. Suppose without loss of generality that $c_u < c_d$, which means that (c_u, c_d) would be efficient in a frictionless world. The distributional price of (c_u, c_d), i.e. the minimum cost to get a consumption claim distributed as (c_u, c_d), is then equal to $\min\{ac_u + (1-a)c_d, bc_d + (1-b)c_u\}$. It is then easy to check that if $a < \frac{1}{2} < b$ then $\min\{ac_u + (1-a)c_d, bc_d + (1-b)c_u\} > \frac{c_u+c_d}{2}$. Since any rational agent with preferences satisfying Assumption 2.2 weakly prefers the consumption bundle $(\frac{c_u+c_d}{2}, \frac{c_u+c_d}{2})$ to (c_u, c_d) and to (c_d, c_u), and since the consumption bundle $(\frac{c_u+c_d}{2}, \frac{c_u+c_d}{2})$ only costs $\frac{c_u+c_d}{2}$ to obtain, this shows that the distribution of payoffs of (c_u, c_d) as a whole is inefficient: neither (c_u, c_d) or (c_d, c_u) will ever be chosen by a rational agent, no matter what his utility function is.

Note that this example is not a degenerate one. Both consumption bundles (c_u, c_d) and (c_d, c_u) are in the opportunity set and neither of them is dominated by a consumption bundle that costs the same amount to obtain.[25] Moreover, for any given set of payoffs S_u and S_d for the risky asset we can

[25] In the presence of market frictions this would not violate the absence of arbitrage. See Bensaid et al. (1992) for an example and a discussion of this point.

find a transaction cost large enough to make any distribution of consumption other than the riskless one (i.e. $c_u = c_d$) inefficient. And if there is *any* positive bid-ask spread around an initial price of $\frac{S_u+S_d}{2}$ for the risky asset then the only efficient distribution of consumption is the riskless one. ∎

Proof of Proposition 2.1 : See Jouini (1999). ∎

Proof of Theorem 2.1 : We shall treat here the case where the uncertain future endowment x is equal to zero. The case where $x \neq 0$ is an immediate extension. First, note that by Proposition 2.1 and following Clarke (1983, Theorem 2.8.6) $\partial\pi(x)$ is defined for all $x \in R^n$ and $\partial\pi(x) \subset \{E \in K : E(x) = \pi(x)\}$. Moreover, if c^* is efficient (resp. strictly efficient), there exists $u \in \mathcal{U}$ (resp. $u \in \mathcal{U}_{sc}$) such that c^* solves $\max\{u(c) : \pi(c) \leq \pi(c^*)\}$, and by Rockafellar (1970, Theorems 28.2 and 28.3) there exists a nonnegative real number λ such that $0 \in -\partial u(c^*) + \lambda\partial\pi(c^*)$. Also, by definition of $\mathcal{U}$ (resp. $\mathcal{U}_{sc}$) there exists a concave (resp. strictly concave) and strictly increasing function $U : R \to R$ such that for all $c = (c_1, \ldots, c_n) \in R^n, u(c) = \frac{1}{n}[U(c_1) + \ldots + U(c_n)]$ and we have that $\partial u(c^*) = \prod_{i=1}^{n}[U'_+(c_i), U'_-(c_i)]$ where $U'_-(x)$ and $U'_+(x)$ are the left and the right derivatives of U at x, respectively. Consequently, there exists $E^* \in K$ and $\lambda \geq 0$ such that $\lambda E^* \in \prod_{i=1}^{n}[U'_+(c_i^*), U'_-(c_i^*)]$. Since U is concave (resp. strictly concave) and strictly increasing, if $c_i^* > c_j^*$ we have $0 < U'_+(c_i^*) \leq U'_-(c_i^*) \leq U'_+(c_j^*) \leq U'_-(c_j^*)$ (resp. $0 < U'_+(c_i^*) \leq U'_-(c_i^*) < U'_+(c_j^*) \leq U'_-(c_j^*)$). This implies that $\lambda > 0$ and that E^* is strictly positive and in reverse order (resp. strict reverse order) of c^*.

Conversely, let $E^* = (e_1^*, \ldots, e_n^*) \in K$ be strictly positive and such that $E^*(c^*) = \pi(c^*)$ and c^* is in reverse order of E^*. Consider the function $g : R \to R$, which is right-continuous, piecewise linear with potential discontinuity and change of slope at $c_1^*, \ldots, c_n^*$. The function g is then entirely defined by its values and left limits at each point $c_i^* : g(c_i^*) = \inf\{e_k^* : c_k^* = c_i^*\}$ and $g(c_i^*)^- = \sup\{e_k^* : c_k^* = c_i^*\}$, and by the equations : $g(c) = g(c^*_{min}) + (c^*_{min} - c)$ for $c < c^*_{min}$ and $g(c) = g(c^*_{max})exp(c^*_{max} - c)$ for $c > c^*_{max}$ (where c^*_{min} and c^*_{max} are respectively the smallest and the largest values taken by the c_i^*'s). It is clear that g is positive nonincreasing (resp. decreasing) and if we define $U^* : R \to R$ by $U^*(x) = \int_0^x g(t)dt$, then U^* is concave (resp. strictly concave) and strictly increasing. Furthermore, for every $i = 1, \ldots, n$ we have $e_i^* \in [g(c_i^*), g(c_i^*)^-] = [{U^*}'_+(c_i^*), {U^*}'_-(c_i^*)] = \partial U^*(c_i^*)$ which implies (for a convex program) that c^* solves $\max\{u(c) : E^*(c) \leq E^*(c^*)\}$, where u is defined on R^n by $u(c_1, \ldots, c_n) = \frac{1}{n}[U^*(c_1) + \ldots + U^*(c_n)]$. Consequently c^*

solves $\max\{u(c) : \pi(c) \le \pi(c^*)\}$. ∎

Proof of Remark 2.1 : If there exists $E \in K$ that is strictly positive and prices $-x$ then, by Theorem 2.1, duplicating $-x$ (with a zero net payoff) is efficient since the null vector is in reverse order of any linear pricing rule and hence of E. Conversely, if there is no such $E \in K$ then duplicating $-x$ is not efficient since part (i) of Theorem 2.1 is not satisfied. ∎

Proof of the Lemma : Let $\mathcal{P}(c^*) = \{c : u(c) \ge u(c^*),\ \text{for all } u \in \mathcal{U}\}$, and $\Sigma(c^*)$ be the convex hull of the permutations of the vector c^* (i.e. the convex hull of the consumption bundles that are distributed as c^*). It is clear that if a consumption bundle is distributed as c^*, it is giving the same utility as c^* to every agent with preferences in $\mathcal{U}$. Hence, by concavity of the preferences in $\mathcal{U}$, any convex combination of consumption bundles distributed as c^* provides at least the same utility as c^* to every agent with preferences in $\mathcal{U}$. And by monotonicity of the preferences in $\mathcal{U}$, any bundle that dominates such a convex combination provides at least the same utility as c^* to every agent with preferences in $\mathcal{U}$. Hence we must have $\Sigma(c^*) + R^n_+ \subset \mathcal{P}(c^*)$. Note that since $\{c : u(c) \ge u(c^*),\ \text{for all } u \in \mathcal{U}\} \subset \{c : u(c) \ge u(c^*),\ \text{for all } u \in \mathcal{U}_{sc}\}$, we would reach the same conclusion if we defined $\mathcal{P}(c^*)$ as $\mathcal{P}(c^*) = \{c : u(c) \ge u(c^*),\ \text{for all } u \in \mathcal{U}_{sc}\}$.

Conversely, let $c \in \mathcal{P}(c^*)$ and suppose that $c \notin \Sigma(c^*) + R^n_+$. Consider $\tilde{c}^*$ (resp. $\tilde{c}$), the permutation of c^* (resp. c) that satisfies $\tilde{c}^*_1 \le \tilde{c}^*_2 \le \ldots \le \tilde{c}^*_n$ (resp. $\tilde{c}_1 \le \tilde{c}_2 \le \ldots \le \tilde{c}_n$). We have that $\tilde{c} \in \mathcal{P}(c^*)$ and $\tilde{c} \notin \Sigma(c^*) + R^n_+$ (indeed, $\mathcal{P}(c^*)$ and $\Sigma(c^*) + R^n_+$ are stable by permutation of coordinates). Since $\Sigma(c^*)+R^n_+$ is closed and $\{\tilde{c}\}$ is compact, by a standard strict separation Theorem (see Luenberger [1969]), there exists a nonzero vector $p \in R^n$ such that $p \cdot \tilde{c} < \inf\{p \cdot x : x \in \Sigma(c^*) + R^n_+\}$. It is easy to see that p must be nonnegative and that we have $p \cdot \tilde{c} < p \cdot x$ for all permutations x of c^*. Consider $\bar{p}$, the permutation of p satisfying $\bar{p}_1 \ge \bar{p}_2 \ge \ldots \ge \bar{p}_n$. We then have $\bar{p} \cdot \tilde{c} \le p \cdot \tilde{c}$ and since $p \cdot \tilde{c} < p \cdot x$ for all permutations x of c^*, we also have that $p \cdot \tilde{c} < \bar{p} \cdot x$ for all permutations x of c^*. In particular, we have that $\bar{p} \cdot \tilde{c} < \bar{p} \cdot \tilde{c}^*$. Let us now consider a concave, strictly increasing real function $U : R \to R$ such that $U'(\tilde{c}^*_i) = \bar{p}_i$ for all $i = 1, \ldots, n$ and let us define the utility function $u \in \mathcal{U}$ by $u(c) = \frac{1}{n}(U(c_1) + \ldots + U(c_n))$ for all $c \in R^n$. We have that $u'(\tilde{c}^*) \cdot (\tilde{c} - \tilde{c}^*) < 0$ and consequently, by concavity of u, $u(\tilde{c}) < u(\tilde{c}^*)$ or equivalently $u(c) < u(c^*)$. This contradicts the fact that $c \in \mathcal{P}(c^*)$ and shows that $\mathcal{P}(c^*) \subset \Sigma(c^*) + R^n_+$, which concludes the proof.

If instead we define $\mathcal{P}(c^*)$ by $\mathcal{P}(c^*) = \{c : u(c) \geq u(c^*), \text{ for all } u \in \mathcal{U}_{sc}\}$, let $U_q : R \to R$ be defined by $U_q(x) = U(x) - \frac{1}{q}exp(-x)$, for every positive integer q. Since U is concave and strictly increasing U_q is strictly concave and strictly increasing for every positive integer q. Hence, the utility function $u_q(c) = \frac{1}{n}(U_q(c_1) + \ldots + U_q(c_n))$, for all $c \in R^n$, belongs to $\mathcal{U}_{sc}$ for every positive integer q. Moreover since $u(\tilde{c}) < u(c^*)$ we have $u_q(\tilde{c}) < u_q(c^*)$ for q sufficiently large, which contradicts the fact that $c \in \mathcal{P}(\rfloor^*)$ and concludes the proof. ∎

Proof of Theorem 2.2: We shall treat here the case where the uncertain future endowment x is equal to zero. The case where $x \neq 0$ is an immediate extension. Let us show that for every $c^* \in R^n$ we have $\sup_{u \in \mathcal{U}}\{\min\{\pi(c) : u(c) \geq u(c^*)\}\} = \min\{\pi(c) : c \in \mathcal{P}(c^*)\}$, where $\mathcal{P}(c^*) = \{c : u(c) \geq u(c^*), \text{ for all } u \in \mathcal{U}\}$. It is obvious that we have $\min\{\pi(c) : c \in \mathcal{P}(c^*)\} \geq \sup_{u \in \mathcal{U}}\{\min\{\pi(c) : u(c) \geq u(c^*)\}\}$. Recall that we have assumed that there exists a probability measure $\tilde{E} \in K$ that is strictly positive (i.e. $\tilde{e}_i > 0$ for all $i = 1, \ldots, n$), and let $m = \frac{1}{2}\inf_i \tilde{e}_i$, and $M = \sup_i |c_i^*| + 1 - log(m)$. Consider $\mathcal{U}_0$, the class of utility functions u that belong to $\mathcal{U}$ and that satisfy $u(c) = \frac{1}{n}(U(c_1) + \ldots + U(c_n))$ with $U(x) \leq M + m(x - M)$ for $x \geq M$ and $U(x) \leq x + M$ for $x \leq -M$ (by monotonicity of U this implies that $U(x) \leq M$ for all $x \in [-M, M]$). Clearly, we have $\sup_{u \in \mathcal{U}_0} \min\{\pi(c) : u(c) \geq u(c^*)\} \leq \sup_{u \in \mathcal{U}} \min\{\pi(c) : u(c) \geq u(c^*)\} \leq \min\{\pi(c) : \text{ for all } u \in \mathcal{U}, u(c) \geq u(c^*)\}$.

Suppose for now that K is a singleton $\{\bar{E}\}$ (the frictionless market case) and consider an efficient permutation $\bar{c}$ of c^* (i.e. a permutation of c^* that is in reverse order of $\bar{E}$). Then derive the function $\bar{U} : R \to R$ from $\bar{E}$ and $\bar{c}$ as U^* was derived from E^* and c^* in the proof of Theorem 2.1. Let the utility function $\bar{u}$ be defined by $\bar{u}(c) = \frac{1}{n}(\bar{U}(c_1) + \ldots + \bar{U}(c_n))$. By construction we have $\bar{u}(\bar{c}) = \max\{\bar{u}(c) : \pi(c) \leq \pi(\bar{c})\}$ and hence it is easy to see that $\pi(\bar{c}) = \min\{\pi(c) : \bar{u}(c) \geq \bar{u}(\bar{c})\}$. This implies that $\pi(\bar{c}) = \min\{\pi(c) : \text{ for all } u \in \mathcal{U}, u(c) \geq u(\bar{c})\}$, and since $u(\bar{c}) = u(c^*)$ for all $u \in \mathcal{U}$, combinining the previous two equalities we obtain $\min\{\pi(c) : \bar{u}(c) \geq \bar{u}(c^*)\} = \min\{\pi(c) : \text{ for all } u \in \mathcal{U}, u(c) \geq u(c^*)\}$. Moreover it is easy to verify[26] that $\bar{u}$ belongs to $\mathcal{U}_0$ (where $\mathcal{U}_0$ is defined above in relation to

[26]Indeed, recall that $\bar{U}(x) = \int_0^x g(t)dt$ where $g(M) = g(c^*_{max})exp(c^*_{max} - \sup_i |c_i^*| - 1 + log(m)) \leq exp(-1+$

$\tilde{E}$ which is fixed in the whole proof) and this implies that $\sup_{u \in \mathcal{U}_0} \min\{\pi(c) : u(c) \geq u(c^*)\} \geq \min\{\pi(c) : \text{ for all } u \in \mathcal{U}, u(c) \geq u(c^*)\}$ and combining this inequality with the inequalities already obtained we have $\sup_{u \in \mathcal{U}_0} \min\{\pi(c) : u(c) \geq u(c^*)\} = \min\{\pi(c) : \text{ for all } u \in \mathcal{U}, u(c) \geq u(c^*)\}$.

Let us now turn to the more general case where K is not reduced to a singleton (the case with market frictions). Let $W(c^*) = \sup_{u \in \mathcal{U}_0} \min\{\pi(c) : u(c) \geq u(c^*)\}$, which is also equal to $W(c^*) = \sup_{u \in \mathcal{U}_0} \min\{\pi(c) : u(c) \geq u(c^*), \text{ and } \pi(c) \leq \pi(c^*)\}$. Consider some $u \in \mathcal{U}_0$ and let $c \in R^n$ such that $u(c) \geq u(c^*)$. We then have that $\sum_{c_i \geq M}[M + m(c_i - M)] + \sum_{c_i \leq -M}[c_i + M)] + \sum_{-M \leq c_i \leq M} M \geq nu(c^*)$, which implies $m \sum_{c_i \geq M} c_i + \sum_{c_i \leq -M} c_i + nM \geq nu(c^*)$, and hence $\sum_{c_i \geq M} c_i + \sum_{c_i \leq -M} c_i \geq \frac{nu(c^*) - nM}{m}$. Moreover, if $\pi(c) \leq \pi(c^*)$ we have $\tilde{E}(c) \leq \pi(c^*)$ which means $\sum_{i=1}^n \tilde{e}_i c_i \leq \pi(c^*)$ and this implies $\alpha \sum_{c_i \geq M} c_i + \beta \sum_{c_i \leq -M} c_i \leq \pi(c^*) + nM$ where $\alpha = \inf_i \tilde{e}_i = 2m > 0$ and $0 < \beta = \sup_i \tilde{e}_i < 1 - m$. Subtracting the previous inequalities we obtain $(\alpha - m) \sum_{c_i \geq M} c_i + (\beta - 1) \sum_{c_i \leq -M} c_i \leq \pi(c^*) + 2nM - nu(c^*)$ and hence $m \sum_{c_i \geq M} c_i + m \sum_{c_i \leq -M}(-c_i) \leq \pi(c^*) + 2nM - nu(c^*)$ which implies that $c_i \leq \sup\{M, \frac{\pi(c^*) + 2nM - nu(c^*)}{m}\}$. Let $B_u(c^*) = \{c : u(c) \geq u(c^*) \text{ and } \pi(c) \leq \pi(c^*)\}$, which is then bounded (by the previous inequalities) and hence compact. This gives us $\min\{\pi(c) : u(c) \geq u(c^*)\} = \min_{c \in B_u(c^*)} \max\{E(c) : E \in K\} = \max\{\min_{c \in B_u(c^*)} E(c) : E \in K\}$ by the min-max Theorem (see Luenberger [1969, Theorem 1, p. 208]). Hence, we have $W(c^*) = \sup_{u \in \mathcal{U}_0} \max\{\min_{c \in B_u(c^*)} E(c) : E \in K\} = \max\{\sup_{u \in \mathcal{U}_0} \min_{c \in B_u(c^*)} E(c) : E \in K\}$, and this implies that $W(c^*) \geq \max\{\sup_{u \in \mathcal{U}_0} \min_{c : u(c) \geq u(c^*)} E(c) : E \in K\}$. We have already seen that if $E \in K$ is strictly positive then $\sup_{u \in \mathcal{U}_0} \min\{E(c) : u(c) \geq u(c^*)\} = \min\{E(c) : c \in \mathcal{P}(c^*)\}$ and hence[27] we obtain $W(c^*) \geq \max\{\min_{c \in \mathcal{P}(c^*)} E(c) : E \in K\}$. Combining this inequality with the proof of part (ii), where we show that $\max\{\min_{c \in \mathcal{P}(c^*)} E(c) :$

$log(m)) \leq m$ which implies that $\bar{U}(x) \leq M + m(x - M)$ whenever $x \geq M \geq c^*_{max}$. Moreover, if $x \leq -M$ then $x \leq c^*_{min}$ and $g(x) = g(c^*_{min}) + c^*_{min} - x$. Moreover, in this case we have $c^*_{min} - x \geq c^*_{min} + \sup_i |c^*_i| + 1 - log(m) \geq 1$ which implies $0 \geq U(-M) \geq U(x) + (-M - x)$.

[27] Recall that K is the closure of its strictly positive elements.

$E \in K\} = \min\{\pi(c) : c \in \mathcal{P}(c^*)\}$, we obtain that $W(c^*) \geq \min\{\pi(c) : c \in \mathcal{P}(c^*)\}$ and hence $W(c^*) = \min\{\pi(c) : c \in \mathcal{P}(c^*)\}$. Since we have that $\mathcal{U}_0 \subset \mathcal{U}_{sc} \subset \mathcal{U}$ this gives us the result : $V(c^*) = \min\{\pi(c) : c \in \mathcal{P}(c^*)\}$ and $V(c^*) = \sup_{u \in \mathcal{U}_{sc}} \min\{\pi(c) : u(c) \geq u(c^*)\}$. This concludes the proof of part (i).

We are now going to show that for all $c^* \in R^n$, $\min\{\pi(c) : c \in \mathcal{P}(c^*)\} = \max\{\min_{c \in \mathcal{P}(c^*)} E(c) : E \in K\}$ where $\mathcal{P}(c^*) = \{c : u(c) \geq u(c^*),$ for all $u \in \mathcal{U}\}$. By the Lemma we have $\mathcal{P}(c^*) = \Sigma(c^*) + R^n_+$, where $\Sigma(c^*)$ is the convex hull of the permutations of c^* (i.e. of the bundles distributed as c^*). Since π is nondecreasing we have $\min\{\pi(c) : c \in \mathcal{P}(c^*)\} = \min\{\pi(c) : c \in \Sigma(c^*)\} = \min_{c \in \Sigma(c^*)} \max\{E(c) : E \in K\}$ and since $\Sigma(c^*)$ and K are convex and compact we have $\min\{\pi(c) : c \in \mathcal{P}(c^*)\} = \max\{\min_{c \in \Sigma(c^*)} E(c) : E \in K\}$ by the min-max Theorem (see Luenberger [1969, Theorem 1, p. 208]). Moreover, since each $E \in K$ is a linear functional, we have $\min\{E(c) : c \in \Sigma(c^*)\} = \min\{E(c) : c$ is a permutation of $c^*\}$ and hence $\min\{\pi(c) : c \in \mathcal{P}(c^*)\} = \max\{P(c^*, E) : E \in K\}$. This concludes the proof of part (ii).

Part (iii) is a direct consequence of part (ii) and of Theorem 3 in Dybvig (1988a). ∎

Proofs of Theorem 2.3 (and footnote 17): We shall treat here the case where the uncertain future endowment x is equal to zero. The case where $x \neq 0$ is an immediate extension. If c^* is strictly efficient, by Theorem 2.1 it is in strict reverse order of a strictly positive linear pricing rule E^* that prices it, which implies that: $E(c^*) \leq E^*(c^*) \leq E^*(c)$, for all $E \in K$ and for all $c \in \Sigma(c^*) + R^n_+$, where $\Sigma(c^*)$ is the convex hull of the permutations of c^* (i.e. of the consumption bundles that are distributed as c^*). Hence, the pair (E^*, c^*) is a saddle-point and it solves $\min_{c \in \Sigma(c^*) + R^n_+} \{\max\{E(c) : E \in K\}\}$ and $\max\{\{\min_{c \in \Sigma(c^*) + R^n_+} E(c)\} : E \in K\}$. Moreover, suppose that there exists $c' \neq c^*$ that also solves $\min_{c \in \Sigma(c^*) + R^n_+} \{\max\{E(c) : E \in K\}\}$. Then $\frac{c' + c^*}{2}$ is strictly preferred to c^* by every strictly risk averse agent (since c' is weakly preferred to c^*) and costs at most $\min_{c \in \Sigma(c^*)} \{\max\{E(c) : E \in K\}\} = \pi(c^*) = \pi(c')$, which contradicts the strict efficiency of c^*.

Conversely, suppose that c^* is the unique solution of $\min_{c \in \Sigma(c^*) + R^n_+} \{\max\{E(c) :$

$E \in K\}\}$, then it also solves $\min_{c \in \Sigma(c^*)} \{\max\{E(c) : E \in K\}\}$. This is equivalent, by the min-max Theorem (see Rockafellar [1970]), to the existence of $E^* \in K$ such that (E^*, c^*) is a saddle point, i.e. such that $E(c^*) \leq E^*(c^*) \leq E^*(c)$, for all $E \in K$ and for all $c \in \Sigma(c^*)$. This means that there exists $E^* \in K$ that prices c^* and is in reverse order of c^*. This proves Theorem 2.3.

In order to conclude the proof of footnote 17 we also need E^* to be positive and in strict reverse order of c^*. Suppose first that E^* is in fact in strict reverse order of c^*. We are going to show that in this case we can construct $\tilde{E}$ that is also in strict reverse order of c^* and that is strictly positive. Indeed, suppose that (assuming, without loss of generality, that E^* is in nondecreasing order) we have $e_1^* = \ldots = e_k^* = 0$ and $0 < e_{k+1}^* \leq \ldots \leq e_n^*$ for some $n > k \geq 1$. Since c^* is in strict reverse order of E^* we then have $c_1^* = \ldots = c_k^*$ and $c_k^* \geq c_{k+1}^* \geq \ldots \geq c_n^*$. Let $\epsilon > 0$ and consider the consumption bundle $c' = (c_1^* + \epsilon, \ldots, c_k^* + \epsilon, c_{k+1}^*, \ldots, c_n^*)$. If $c_k^* > c_{k+1}^*$ we then have that $\pi(c^*) = E^*(c^*) = E^*(c')$ and $E^*(c') = \pi(c')$ since if there were a measure $E \in K$ such that $E(c') > E^*(c^*)$ it would also satisfy $E(c^*) > E^*(c^*)$ which is impossible since $E^*(c^*) = \pi(c^*)$. This contradicts the uniqueness of the solution of $\min\{\pi(c) : c \in \mathcal{P}(c^*)\}$ since $c' \neq c^*$. If $c_k^* = c_{k+1}^*$ then we either have that $\pi(c^*) = E^*(c^*) = E^*(c') = \pi(c')$ (and again a contradiction) or we obtain a strictly positive measure $\tilde{E} \in K$ that is in strict reverse order of c^* and prices it. Indeed, if $E^*(c') < \pi(c')$ then there is a measure in K that is identical to E^* but puts more weight on $c_1^* = \ldots = c_k^*$ and less weight on the c_j^* 's such that $c_j^* = c_{k+1}^*$ and $j \geq k+1$. We can then construct (by convexity of K) $\tilde{E} \in K$ that prices c^*, is still in strict reverse order of c^* and is such that $\tilde{e}_1, \tilde{e}_2, \ldots,$ or e_k is strictly positive and $\tilde{e}_j > 0$ for $j > k$. Repeating this reasoning (k times at most) we obtain $\tilde{E}$ that is strictly positive, in strict reverse order of c^* and prices it. To conclude the proof, we only need to show that there exists a measure $E^* \in K$ that prices c^* and is in strict reverse order of c^*. Let $A = \{x : \pi(c^* + x) \leq \pi(c^*)\}$ and $B = \{x : c^* + x \in \Sigma(c^*)\}$ (note that 0 is an extreme point of B). Since K has a finite number of extreme points, π is polyhedral and A is polyhedral. Since 0 belongs to A, the convex cone A' generated by A is then closed (see Rockafellar [1970, Theorem 19-7]). Let B' be the convex cone generated by B, since 0 belongs to B which is polyhedral B' is closed. It is also easy to show that $B' \cap (-B') = \{0\}$, and this implies that there exists an affine hyperplane H such that $0 \notin H$, $\tilde{B} = H \cap B'$ is compact and B' is the convex cone generated by $\tilde{B}$ (see Bourbaki [1981, chapter II-7-3]). Moreover, it is easy to show that $A' \cap B' = \emptyset$ and hence that $A' \cap \tilde{B} = \emptyset$. Moreover, since A' is closed and $\tilde{B}$ is compact there exists $\epsilon > 0$ such that $(\tilde{B} + B(0,\epsilon)) \cap A' = \emptyset$ (where $B(0,\epsilon)$ is the closed ball of center 0 and radius ϵ). Note $\tilde{B}_\epsilon = \tilde{B} + B(0,\epsilon)$ and let B'_ϵ be the cone generated by $\tilde{B}_\epsilon$. Since $\tilde{B}_\epsilon$ is convex and compact and does not contain 0, then B'_ϵ is a closed convex cone, and it is easy to show that $A' \cap B'_\epsilon = \{0\}$. Moreover, by construction B'_ϵ has a nonempty interior and we have $0 \notin int(B'_\epsilon)$ and hence $int(B'_\epsilon) \cap A' = \emptyset$. Hence, by Eidelheit separation Theorem, there exists a nonzero linear map f such that for all $(a,b) \in A' \times B'_\epsilon, f(a) \leq 0 \leq f(b)$ (see Luenberger [1969]). This means that if $E_i(c^* + x) \leq \pi(c^*)$ for every extreme point E_i of K then $f(x) \leq 0$. In other words, if $E_i(c^* + x - \pi(c^*)e) \leq 0$ for every extreme point E_i of K then $f(c^* + x - \pi(c^*)e) \leq f(c^*) - \pi(c^*)f(e)$ (where e is the vector with all components equal to one). Since f is bounded above on a cone it is necessarily bounded by 0 and by Farkas' Lemma we have that f is a nonnegative linear combination of the (finite) extreme points of K and $f(c^*) - \pi(c^*)f(e) \geq 0$. Renormalizing f if necessary, f then belongs to K and $f(c^*) \geq \pi(c^*)$, which implies $f(c^*) = \pi(c^*)$ (this means that f prices c^*). Now let b a nonzero vector of B; since B' is the cone generated by $\tilde{B}$, there exists $\tilde{b} \in \tilde{B}$ and a real number $\lambda > 0$ such that $b = \lambda\tilde{b}$ and $B(\tilde{b},\epsilon) \subset B'_\epsilon$. Hence f is nonnegative on $B(\tilde{b},\epsilon)$ and since $f \neq 0$ we must have $f(\tilde{b}) > 0$ and therefore $f(b) > 0$. Since $f(0) = 0$, f attains its minimum on B' at 0 only. This shows that f attains its minimum on $\Sigma(c^*)$ at the point c^* only, which implies that f is in strict reverse order of c^* and concludes the proof of footnote 17. ∎

Proof of Remark 2.2 : Let the payoff $-\tilde{x} \geq -x$ of a minimum cost strategy that dominates $-x$. This means $\pi(-\tilde{x}) = \pi(-x)$. If $E^* \in K$ prices $-x$ we have $\pi(-x) = E^*(-x) \leq E^*(-\tilde{x}) \leq \pi(-\tilde{x})$ which implies $\pi(-x) = \pi(-\tilde{x}) = E^*(-\tilde{x}) = E^*(-x)$, i.e. E^* prices $-\tilde{x}$ as well. Hence E^* assigns a zero price to the states of the world where $-\tilde{x}$ strictly dominates $-x$. Since $-\tilde{x} + x$ equals zero in the other states, E^* is in reverse order of $-\tilde{x} + x$. By Theorem 2.3 this shows that the minimum cost dominating strategy has no inefficiency cost. ∎

Proof of Corollary 2.1 : We have $V_x(c^*) \leq \pi(c^*)$ by Definition 2.2. Moreover, $V_{-c^*}(c^*) = \pi(c^*)$. Indeed, any pricing rule in K that prices c^* is in reverse order of the net contingent claim equal to zero and hence, by Theorem 2.3, c^* has a zero inefficient cost given an uncertain future endowment $-c^*$. ∎

Proof of Corollary 2.2 : This is a direct consequence of Theorem 2.2. ∎

Proof of Remark 3.1 : We shall assume that all prices and payoffs at time t have been normalized by $exp(rt)$. Denote by $E_{\frac{1}{2}}$ the expectation with respect to the actual probability measure with conditional probability 0.5 of going "up" from each node in the tree. Using the inequalities satisfied by κ we have, for every date t and every node $\omega(t)$, we have $(1-\kappa)S(t) \leq E_{\frac{1}{2}}((1-\kappa)S(T) \mid \omega(t)) \leq (1+\kappa)S(t)$ or $(1-\kappa)S(t) \leq E_{\frac{1}{2}}((1+\kappa)S(T) \mid \omega(t)) \leq (1+\kappa)S(t)$. This proves that $E_{\frac{1}{2}}$ belongs to K.

We shall now prove $V_0(c^*) = E_{\frac{1}{2}}(c^*)$ for all c^*. By Theorem 2.2 (ii) we have $V_0(c^*) = \max\{P_0(c^*, E) : E \in K\}$, hence $V_0(c^*) \geq P_0(c^*, E_{\frac{1}{2}})$. Moreover $P_0(c^*, E_{\frac{1}{2}}) = E_{\frac{1}{2}}(c^*)$ since all states of the world are equiprobable under $E_{\frac{1}{2}}$. This implies $V_0(c^*) \geq E_{\frac{1}{2}}(c^*)$. By Theorem 2.2 (i) we have $V_0(c^*) = \min\{\pi(c) : c \in \Sigma(c^*)\}$. Since $(E_{\frac{1}{2}}(c^*), \ldots, E_{\frac{1}{2}}(c^*)) \in \Sigma(c^*)$ this implies $V_0(c^*) \leq \max\{E(E_{\frac{1}{2}}(c^*), \ldots, E_{\frac{1}{2}}(c^*)) : E \in K\}$ and hence $V_0(c^*) \leq E_{\frac{1}{2}}(c^*)$.

Suppose that c is not riskless, then there exist i and j such that $c_i > c_j$. Because the inequalities satisfied by κ are strict, $E_{\frac{1}{2}}$ belongs to the relative interior of K. Define the linear pricing rule $E = E_{\frac{1}{2}} + (0, \ldots, 0, \epsilon, 0, \ldots, 0, -\epsilon, 0, \ldots, 0)$, where ϵ is the i-th element and $-\epsilon$ is the j-th element (without loss of generality). If $\epsilon > 0$ is sufficiently small E belongs to K. Since $E(c) > E_{\frac{1}{2}}(c)$ this implies $\pi(c) > E_{\frac{1}{2}}(c)$ and hence $\pi(c) > V_0(c)$. This shows that the only efficient consumption bundles are the riskless ones. ∎

Proof of Remark 3.2 : Define $K = [\alpha_1, \alpha_2]^N$ and $K^* = [K \cup (1-K)] \cap [0, \frac{1}{2}]^N$, with $\alpha_1 = \frac{exp((r-c)\frac{T}{n})-d}{u-d}$ and $\alpha_2 = \frac{exp(r\frac{T}{n})-d}{u-d}$, where N is the total number of nodes in the tree (except the terminal ones). K^* is the set of probability measures defined by conditional probabilities (of the "up" state) that are not larger than $\frac{1}{2}$ and belong to $[\alpha_1, \alpha_2] \cup [1-\alpha_2, 1-\alpha_1]$. By Theorem 2.2 (ii) we have $V(c^*) = \max\{P(c^*, E) : E \in K\}$, where $P(c^*, E) = \min\{E(c) : c$ is distributed as $c^*\}$. We then have $V(c^*) = \max\{P(c^*, E) : E \in K^*\}$, since we can reorder c^* to match the switch in conditional probabilities from the "up" state to the "down" state, without changing its distribution. We are now going to prove that $(E_\beta, \tilde{c})$ is a saddle point, where $\tilde{c}$ is a permutation of c^* which (i) is in reverse order of E_β, and (ii) whenever two states of the world have the same number of "ups" (and hence the same weights for the probability measure E_β) $\tilde{c}$ has a (weakly) higher payoff in the state that is "higher up" in the tree. This means that we shall prove that $(E_\beta, \tilde{c})$ satisfies $E(\tilde{c}) \leq E_\beta(\tilde{c}) \leq E_\beta(c)$ for every c distributed as c^* and every $E \in K^*$, and this will prove $V(c^*) = E_\beta(\tilde{c})$. Note that as far as computing $V(c^*)$ is concerned we can use any $\tilde{c}$ that is in reverse order of E_β since they all give the same value for $E_\beta(\tilde{c})$. Also, $E_\beta(\tilde{c}) \leq E_\beta(c)$ follows immediately from (i). In order to prove $E(\tilde{c}) \leq E_\beta(\tilde{c})$ for all $E \in K^*$, we shall proceed by backward induction and prove it for the expectations conditioned on each node. Let $E \in K^*$, then the inequality on the conditional expectations obviously holds at the final date. Assuming that it holds for an arbitrary date t, let us prove that it holds for date $t-1$ as well, i.e. $E(\tilde{c} \mid \omega(t-1)) \leq E_\beta(\tilde{c} \mid \omega(t-1))$, for every date $t-1$, every node $\omega(t-1)$ and every $E \in K^*$. Let the successors of a $t-1$ node $\omega(t-1)$ be $\omega(t-1, up)$ and $\omega(t-1, down)$, and let α and $1-\alpha$ be their conditional probabilities under E. We have $E(\tilde{c} \mid \omega(t-1)) = \alpha E(\tilde{c} \mid \omega(t-1, up)) + (1-\alpha)E(\tilde{c} \mid \omega(t-1, down))$. We shall now distinguish two cases: either $E(\tilde{c} \mid \omega(t-1, up)) \geq E(\tilde{c} \mid \omega(t-1, down))$ or $E(\tilde{c} \mid \omega(t-1, up)) \leq E(\tilde{c} \mid \omega(t-1, down))$. In the first case, since E belongs to K^* we have $\alpha \leq \beta$, we obtain $E(\tilde{c} \mid \omega(t-1)) \leq \beta E(\tilde{c} \mid \omega(t-1, up)) + (1-\beta)E(\tilde{c} \mid \omega(t-1, down))$, and by our induction hypothesis this leads to $E(\tilde{c} \mid \omega(t-1)) \leq \beta E_\beta(\tilde{c} \mid \omega(t-1, up)) + (1-\beta)E_\beta(\tilde{c} \mid \omega(t-1, down))$, i.e. $E(\tilde{c} \mid \omega(t-1)) \leq E_\beta(\tilde{c} \mid \omega(t-1))$. In the second case we have $E(\tilde{c} \mid \omega(t-1, up)) \leq E(\tilde{c} \mid \omega(t-1, down)) \leq E_\beta(\tilde{c} \mid \omega(t-1, down)) \leq E_\beta(\tilde{c} \mid \omega(t-1, up))$. The first inequality is by assumption, the second by induction hypothesis, and the third by the properties (i) and (ii) satisfied by $\tilde{c}$. This implies $E(\tilde{c} \mid \omega(t-1)) \leq E_\beta(\tilde{c} \mid \omega(t-1))$. ∎

Proof of Remark 3.3 : It is easy to see that when $c = -(\mu - r + \frac{1}{2}\sigma^2)$ the linear pricing rule with equal prices for all states of the world is in K, and is equal to E_β of Remark 3.2. Since the stop-loss strategy only involves short selling the risky asset (and investing in the riskless asset), it is easy to show that this linear pricing rule prices its payoff. It then follows from Remark 3.2 and Theorem 2.1 that the stop-loss strategy is efficient. ∎

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