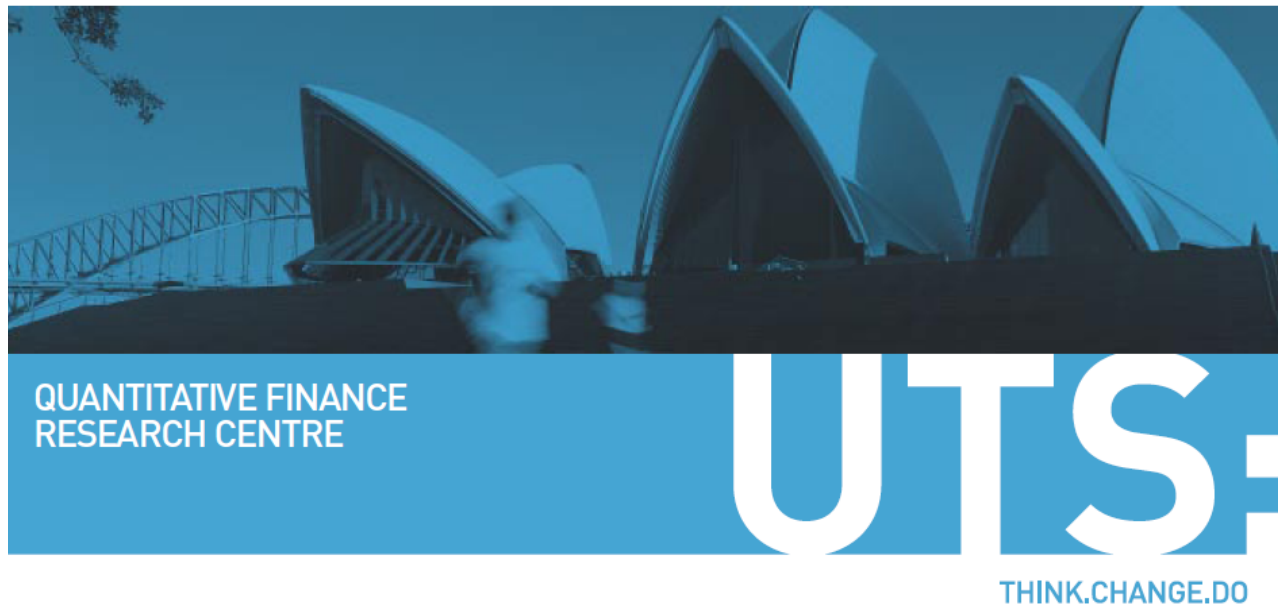


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Calibration of the Multi-Currency LIBOR Market Model

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Abstract

This paper presents a method for calibrating a multicurrency lognormal LIBOR Market Model to market data of at-the-money caps, swaptions and FX options. By exploiting the fact that multivariate normal distributions are invariant under orthonormal transformations, the calibration problem is decomposed into manageable stages, while maintaining the ability to achieve realistic correlation structures between all modelled market variables.

Keywords: Currency options, LIBOR Market Model, exchange rate risk, interest rate risk

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1 Introduction

Ever since the seminal paper of Miltersen, Sandmann and Sondermann (1997) introducing a term structure model of directly market-observable interest rates such as LIBOR,¹ developed further by Brace, Gatarek and Musiela (1997) and Jamshidian (1997), what has become known as the LIBOR Market Model (LMM) is the industry standard for pricing fixed income derivatives. Subsequently, this has given rise to a rich literature of model extensions, numerical methods and diverse applications of the LMM. One strand of this literature was initiated by Schlögl (2002b), a work which extended the LMM to multiple currencies linked by forward exchange rates. Through appropriate reinterpretation, the framework of the multicurrency LMM can be applied to the joint modelling of various market risks, including interest rates and inflation (Mercurio (2005)) or interest rates and commodities (Pilz and Schlögl (2009)).

For the multicurrency LMM itself, a key open question remains how to best calibrate the model to the market. In this paper we focus on calibration of the lognormal multicurrency LMM to volatilities implied by at-the-money caps, swaptions and FX options,² while also matching historically estimated correlations between market variables as closely as possible.

The fit to market-implied at-the-money volatilities is achieved by applying the single-currency LMM calibration method of Pedersen (1998) in multiple stages, modifying it where necessary for the calibration of forward FX volatilities and taking into account the no-arbitrage relationships identified in Schlögl (2002b). In order to fit target correlations, we exploit the fact that a system of volatility vectors is invariant under orthonormal rotation. In this manner, one can change the correlation between, say, interest rates in two different currencies without affecting each of the single-currency calibrations.

A similar approach was taken in Pilz and Schlögl (2009) to achieve simultaneous calibration of interest rate and commodity volatilities in a single currency. However, calibration in the FX/interest rate case introduces interesting additional problems. Not only are there more market-implied volatilities which enter into the calibration (i.e. FX options and fixed income derivatives in *each* currency), but also *implied correlations* when one considers all FX options in a “currency triangle,” say USD/EUR, USD/JPY and EUR/JPY.

The paper is organised as follows: Section 2 introduces notation and reviews the construction of the multicurrency LMM. The calibration method is laid out in Section 3, followed by examples in Section 4 demonstrating its effectiveness on market data³ before and during the credit crisis. For the reader’s convenience, the Pedersen (1998) calibration method is reviewed in the appendix, as is the relationship between volatilities of market rates given in terms of fixed maturities (as in the construction of the

¹Or even earlier, if one takes into account that practitioners were using the Black (1976) formula to price caplets well before the aforementioned papers provided a theoretically consistent interest rate term structure model justifying this practice.

²The basic LMM has been extended to allow for calibration to implied volatility skews and smiles (see e.g. Brace (2007)). Incorporating these extensions into the calibration method described in the present paper is the topic of further research.

³The source of the market data is SuperDerivatives.

LMM) and in terms of fixed times to maturity (as used in the Pedersen calibration).

2 The multicurrency LIBOR Market Model (LMM)

The model which will be calibrated to market data below is the extension (as in Schlögl (2002b)) of the lognormal LIBOR Market Model (Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), Jamshidian (1997)). Given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0, T^*]}, \mathbf{P}_{T^*})$ satisfying the usual conditions, let $\{W_{T^*}(t)\}_{t \in [0, T^*]}$ denote a d -dimensional standard Wiener process and assume that the filtration $\{\mathcal{F}_t\}_{t \in [0, T^*]}$ is the usual \mathbf{P}_{T^*} -augmentation of the filtration generated by $\{W_{T^*}(t)\}_{t \in [0, T^*]}$.

The model is set up on the basis of assumptions **(BP.1)** and **(BP.2)** of Musiela and Rutkowski (1997):

(BP.1) For any date $T \in [0, T^*]$, the price process of a zero coupon bond $B(t, T)$, $t \in [0, T]$ is a strictly positive special martingale⁴ under \mathbf{P}_{T^*} .

(BP.2) For any fixed $T \in [0, T^*]$, the *forward process*

$$F_B(t, T, T^*) = \frac{B(t, T)}{B(t, T^*)}, \quad \forall t \in [0, T]$$

follows a martingale under \mathbf{P}_{T^*} .

These assumptions are assumed to hold for zero coupon bond prices in each currency. In the present paper we consider three currencies modelled simultaneously.⁵ For ease of exposition (and to link to the actual market data used in the calibration examples below), we label these currencies USD, EUR and JPY, where USD is the primary reference (i.e. “domestic”) currency, and EUR and JPY are the secondary (i.e. “foreign”) currencies. Thus **(BP.1)** and **(BP.2)** apply for $\{B^{\text{USD}}(t, T), B^{\text{USD}}(t, T^*), \mathbf{P}_{T^*}^{\text{USD}}\}$, $\{B^{\text{EUR}}(t, T), B^{\text{EUR}}(t, T^*), \mathbf{P}_{T^*}^{\text{EUR}}\}$ and $\{B^{\text{JPY}}(t, T), B^{\text{JPY}}(t, T^*), \mathbf{P}_{T^*}^{\text{JPY}}\}$, respectively.

Following Schlögl (2002b), let $X(t)$ denote the spot exchange rate in terms of units of domestic currency per unit of foreign currency (i.e. we have $X^{\text{USD/EUR}}$ and $X^{\text{USD/JPY}}$) and assume:

(X.1) The spot exchange rate processes $X(t)$, $t \in [0, T^*]$, are strictly positive special martingales under the measures \mathbf{P}_{T^*} .

Time T forward LIBORs $L(t, T)$ with accrual periods of length δ are defined in each currency in terms the corresponding zero coupon bond prices as

$$L(t, T) = \delta^{-1} \left(\frac{B(t, T)}{B(t, T + \delta)} - 1 \right) \quad (1)$$

⁴Musiela and Rutkowski (1997) define a *special martingale* as a process χ which admits a decomposition $\chi = \chi_0 + M + A$, where $\chi_0 \in \mathbb{R}$, M is a real-valued local martingale and A is a real-valued predictable process of finite variation.

⁵It is straightforward, though notationally tedious, to extend our approach to more than three currencies.

For calibration purposes, the multi-currency LMM is defined on a maturity time grid $T_j = j\delta$, for $j = 0, \dots, n_f$, where δ is the greatest common divisor of the *market* LIBOR forward times over all currencies. For instance, USD LIBORs refer to period of 3 months, whereas EUR and JPY LIBORs to a period of 6 months, which gives a δ of 3 months. The *model* forward LIBORs thus all have an accrual period of length δ , even though some market forward LIBORs have accrual periods of integer multiples of δ . The n_f forward LIBORs in the discrete-tenor LIBOR Market Model for each currency are assumed to have deterministic volatility $\lambda(t, T_j)$.⁶ The stochastic dynamics are assumed to be driven by a d_I -dimensional standard Brownian motion W , i.e. the $\lambda(t, T_j)$ are d_I -dimensional vector-valued functions of t , and assumed to be piecewise constant on the calendar time grid $t = (t_0 = 0, t_1, \dots, t_{n_c})$. Thus we have for each currency a volatility matrix $\Lambda = \{\lambda_{i,j,k}\} \in \mathbb{R}^{n_c, n_f, d_I}$, where $\lambda_{i,j,k}$ is the volatility of the k -th stochastic factor of $L(t, T)$, i.e. the Libor rate at calendar time $t_{i-1} \leq t < t_i$ with time to maturity $T_{j-1} \leq T < T_j$. Those matrices are denoted by Λ^{USD} , Λ^{EUR} and Λ^{JPY} for each currency. The number of stochastic factors d_I refers to the interest rate markets only.

It follows from (1) that the forward LIBORs are martingales under their “native” forward measures, i.e.

$$dL(t, T_j) = L(t, T_j)\lambda(t, T_j)dW_{T_{j+1}}(t) \quad (2)$$

where $W_{T_{j+1}}$ is a d_I -dimensional standard Brownian motion under the time T_{j+1} forward measure \mathbf{P}_{T_j} , and the volatility specification chosen for the forward LIBORs fixes the relationships between the forward measures \mathbf{P}_{T_j} and \mathbf{P}_{T^*} .⁷

Time T forward exchange rates $X(t, T)$ are defined as

$$X(t, T) := \frac{\tilde{B}(t, T)X(t)}{B(t, T)} \quad (3)$$

where $B(t, T)$ and $\tilde{B}(t, T)$ are zero coupon bonds in domestic and foreign currency, respectively. For example,

$$X^{\text{USD/EUR}}(t, T) := \frac{B^{\text{EUR}}(t, T)X^{\text{USD/EUR}}(t)}{B^{\text{USD}}(t, T)} \quad (4)$$

which is a martingale under $\mathbf{P}_T^{\text{USD}}$. The stochastic dynamics are assumed to be driven by a d_C -dimensional standard Brownian motion W , i.e.

$$dX^{\text{USD/EUR}}(t, T_j) = X^{\text{USD/EUR}}(t, T_j)\lambda^{\text{USD/EUR}}(t, T_j)dW_{T_{j+1}}^{\text{USD}}(t) \quad (5)$$

⁶This means that approximate formulas for market caplet prices must be used in currencies where the market forward LIBORs have accrual periods of multiples of δ — we use the “frozen drift approximation,” as in the market-standard approximate swaption formulas in the LMM (see e.g. Brigo and Mercurio (2001)). Furthermore, the discrete-tenor LIBOR Market Model requires the use of interpolation to price instruments which do not match the tenor grid (see Schlögl (2002a)) — however, this is not required for the calibration instruments that we consider here, where slight mismatches due to differing daycount conventions and holiday calendars can be safely ignored.

⁷See e.g. Musiela and Rutkowski (1997).

where $W_{T_{j+1}}^{\text{USD}}$ is a d_C -dimensional standard Brownian motion under the time T_{j+1} forward measure $\mathbf{P}_{T_j}^{\text{USD}}$.

The volatilities are parameterized in the same way as for the interest forward rates, i.e. volatilities are given by $\Lambda = \{\lambda_{i,j,k}\} \in \mathbb{R}^{n_c, m_f, d_C}$. The number of calendar times n_c is the same as before, since we have the same calendar time discretization for all assets. The number of forward times m_f could differ from n_f , and the same is true for d_C , the number of stochastic factors.

From the definition (1) of forward LIBOR and Ito's Lemma it follows that the volatility of the zero coupon bond price quotients $B(t, T_j)/B(t, T_{j+1})$, $\gamma(t, T_j, T_{j+1})$, is given by

$$\gamma(t, T_j, T_{j+1}) = \frac{\delta L(t, T_j)}{1 + \delta L(t, T_j)} \lambda(t, T_j) \quad \forall t \in [0, T_j] \quad (6)$$

Consequently, it follows from (3) that forward LIBOR and forward exchange rate volatilities must satisfy no-arbitrage conditions of the type

$$\lambda^{\text{USD/EUR}}(t, T_{j-1}) = \gamma^{\text{EUR}}(t, T_{j-1}, T_j) - \gamma^{\text{USD}}(t, T_{j-1}, T_j) + \lambda^{\text{USD/EUR}}(t, T_j) \quad (7)$$

for all maturities and for each currency pair.

Note that the $\gamma(t, T_j, T_{j+1})$ are not deterministic, but rather depend on the values $L(t, T_j)$ of forward LIBOR. In order to implement (7), we therefore employ the commonly used "frozen coefficient" approximation by setting

$$\bar{\gamma}(t, T_j, T_{j+1}) = \frac{\delta L(0, T_j)}{1 + \delta L(0, T_j)} \lambda(t, T_j) \quad \forall t \in [0, T_j] \quad (8)$$

and replacing (7) by

$$\lambda^{\text{USD/EUR}}(t, T_{j-1}) = \bar{\gamma}^{\text{EUR}}(t, T_{j-1}, T_j) - \bar{\gamma}^{\text{USD}}(t, T_{j-1}, T_j) + \lambda^{\text{USD/EUR}}(t, T_j) \quad (9)$$

So far we allowed different numbers of stochastic factors d_I and d_C for interest and foreign exchange markets. To include these no-arbitrage conditions in the calibration objective function, all volatility matrices need to have the same number of factors, $d = \max\{d_I, d_C\}$, which is realized by extending the smaller matrices in the factor dimension and filling the new entries with zeros (initially).

Furthermore, given the exchange rates between (at least) three currencies, we have the *currency triangle relation*, i.e. in our example

$$X^{\text{JPY/EUR}}(t, T_j) = \frac{X^{\text{USD/EUR}}(t, T_j)}{X^{\text{USD/JPY}}(t, T_j)} \quad (10)$$

By Ito's Lemma this implies the triangular relation for the volatility matrices

$$\lambda^{\text{JPY/EUR}}(t, T) = \lambda^{\text{USD/EUR}}(t, T) - \lambda^{\text{USD/JPY}}(t, T). \quad (11)$$

This condition on FX volatilities will also form part of the calibration objective.

3 Calibration Approach

The calibration method is divided into three steps which will be discussed in detail in the following sections. The first step concerns only the calibration of the three interest rate markets. Volatilities and correlations of forward interest rates within each currency and cross-correlations of forward interest rates for different currencies are calibrated.

In the second step each FX market is calibrated separately, which determines all exchange rate volatilities and correlations between forwards within each exchange rate. Finally, in the third step the results obtained from the second step will be modified in order to match the triangle relation (11). This triangle relation does not only determine the volatilities of one of the exchange rates, but also the cross-correlations between all forwards of different exchange rates.

The overall calibration of the multi-currency LMM is illustrated by applying the procedures proposed in each step to a real world example, which consists of the currencies US Dollar (USD), Euro (EUR) and Japanese Yen (JPY) and is calibrated for the 8th May 2008. As noted above, we will refer to the USD as the primary (“domestic”) currency and to EUR and JPY as secondary currencies, since the exchange rates involving USD are historically more liquidly traded than JPY/EUR. This also suggests to calibrate volatilities for those exchange rates involving USD to market instruments and to derive the JPY/EUR rate volatilities by the triangle relation.

For clarity of exposition, we restrict ourselves in the following technical description of the multi-currency LMM calibration to the case of three currencies and maintain the notation USD for the major currency and EUR, JPY for the secondary ones.

3.1 1. Step: Calibration of the Interest Rate Markets

For the calibration of a standard LMM, we use the method of Pedersen (1998), which has been found to be effective and robust.⁸ For the reader’s convenience, Pedersen’s method is briefly reviewed in Appendix A. However, since our calibration proceeds in stages, most other methods for the calibration of the single-currency LMM also would be compatible with our approach.

Corresponding to the notation in Section 2, we assume to have for each currency a volatility matrix $\Lambda = \{\lambda_{i,j,k}\} \in \mathbb{R}^{n_c, n_f, d_I}$, where $\lambda_{i,j,k}$ is the volatility of the k -th stochastic factor of $L(t, T)$, i.e. the Libor at calendar time $t_{i-1} \leq t < t_i$ with time to maturity $T_{j-1} \leq T < T_j$. Recall that those matrices are denoted by Λ^{USD} , Λ^{EUR} and Λ^{JPY} for each currency. The number of stochastic factors d_I refers to the interest rate markets only.

Aggregated volatilities are denoted by $V = \{v_{i,j}\} \in \mathbb{R}^{n_c, n_f}$ and relate to the (decomposed) volatilities above through

$$v_{i,j}^2 = \sum_{k=1}^{d_I} \lambda_{i,j,k}^2.$$

⁸See e.g. Choy, Dun and Schlögl (2004).

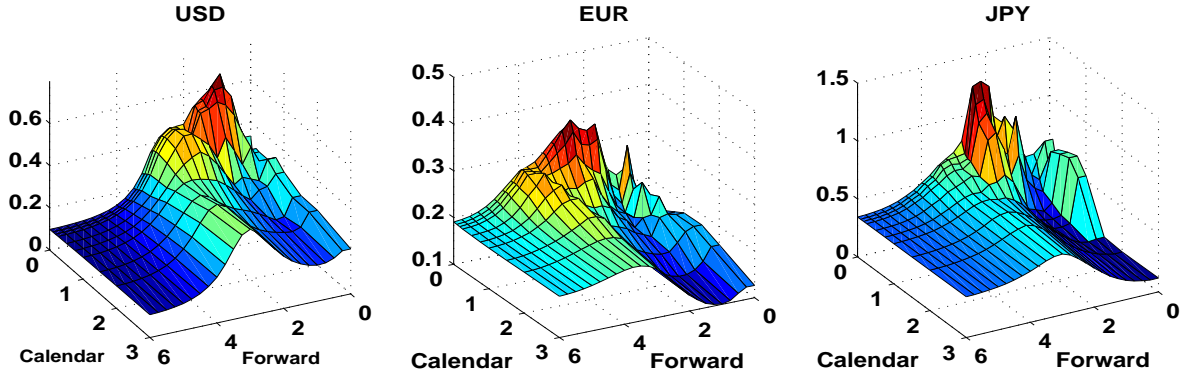


Figure 1: The calibrated forward rate volatilities V^{USD} , V^{EUR} and V^{JPY} for the calibration date 8th May 2008.

In our example, we calibrate the single Libor Market Models by the method proposed in Pedersen (1998). The correlation matrices for forward rates of the same currency but with different forward times are estimated historically based on a 3-month period preceding the calibration date. The calendar time vector is set to the year fraction equivalent of

$$t_c = [0M, 1M, 2M, 3M, 6M, 9M, 1Y, 1Y6M, 2Y, 3Y].$$

The market instruments used are caplets, caps and swaptions with times to maturity up to 3 years and swaption tenors of 1, 2 and 3 years. Hence, a 3M forward rate curve up to 6 years forward time is required. A relatively large number of 11 stochastic factors is allowed for each LMM, but this is mainly in view of the cross-correlation adjustment discussed in the rest of this sub-section, i.e. the number of factors has to be large enough in order to jointly model all three IR markets. The scale parameters of the objective function as required in Pedersen's approach are set to $\text{scale}_{\text{fwd}} = 1e - 04$ and $\text{scale}_{\text{cal}} = 1e - 04$.

Figure 1 shows the calibrated (aggregated) volatilities for the USD, EUR and JPY interest forward rates, and Figure 2 shows the model and market prices of the caps, caplets and swaptions used for calibration.

The cross-correlations between the forward Libors of different currencies have not yet been taken into account when calibrating Λ^{USD} , Λ^{EUR} and Λ^{JPY} separately. In a second fit these volatility matrices are linked according to exogenously given cross-correlations by utilizing the fact that multivariate normal distributions are invariant under orthonormal transformations.

Consider a set of orthonormal transformation matrices $Q_i^{\text{EUR}} = \{q_{k_1, k_2}^{\text{EUR}}\} \in \mathbb{R}^{d_I, d_I}$ and $Q_i^{\text{JPY}} = \{q_{k_1, k_2}^{\text{JPY}}\} \in \mathbb{R}^{d_I, d_I}$ for each calendar time $t_i > 0$. Multiplying the decomposed volatilities for EUR and JPY with the corresponding transformation, e.g. for EUR $\Lambda_i^{\text{EUR}} Q_i^{\text{EUR}}$, where $\Lambda_i^{\text{EUR}} \in \mathbb{R}^{n_f, d_I}$ is the

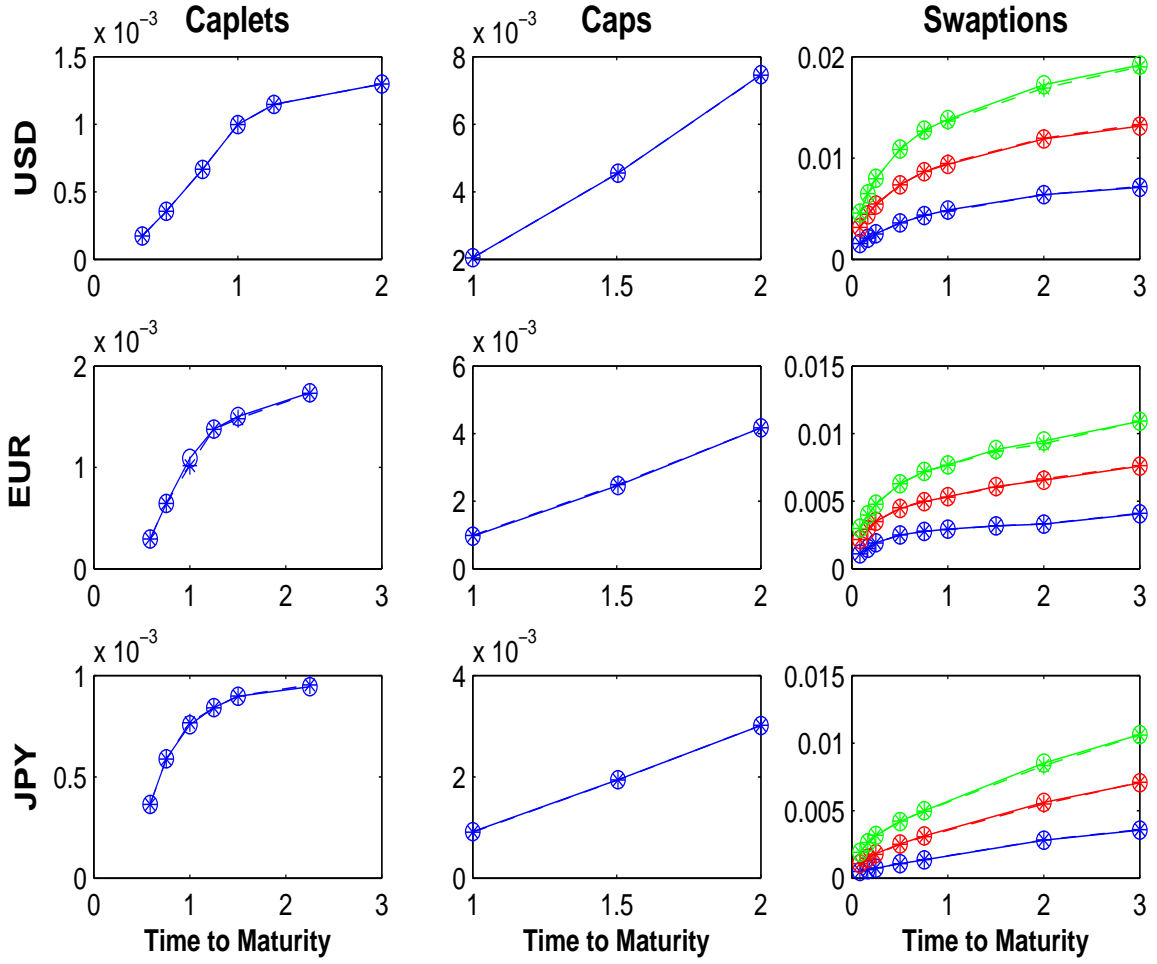


Figure 2: Fit of model prices (stars) to market prices (circles). The swaptions tenors are 1Y (blue), 2Y (red) and 3Y (green).

sub-matrix of Λ^{EUR} for a fixed t_i , results in modified decomposed volatilities, but unchanged aggregated volatilities V^{EUR} and V^{JPY} . This is due to the rotational invariance of the multivariate normal distribution. As a consequence this allows to fit cross-correlations by finding suitable transformations, without changing the fit to market prices or the smoothness of the previously determined volatilities.

The fitting of the transformation matrix has to be carried out for each calendar time $t_i > 0$ separately. The model intrinsic cross-covariance for, say, USD versus EUR is given by

$$\tilde{\Sigma}_i^{\text{USDEUR}} = V_i^{\text{USD}}(V_i^{\text{EUR}}Q_i^{\text{EUR}})^\top,$$

where V_i denotes the vector of volatilities for different forward times and calendar time t_i of V^{USD} and V^{EUR} , respectively. Accordingly, the model cross-correlation is given by

$$\tilde{C}_i^{\text{USDEUR}} = (V_i^{\text{USD}}(V_i^{\text{EUR}}Q_i^{\text{EUR}})^\top) \odot \{\text{diag}\{(V_i^{\text{USD}}(V_i^{\text{USD}})^\top)\} \text{diag}\{(V_i^{\text{EUR}}(V_i^{\text{EUR}})^\top)\}^{-\frac{1}{2}}\},$$

where diag (applied to a matrix) returns the diagonal vector, and the square root and the multiplication (\odot) have to be applied component-wise. Due to the orthonormality of Q_i^{EUR} its multiplication with V_i^{EUR} can be omitted in the right part of the multiplication above.

The target cross-correlation is the exogenously specified matrix C^{USDEUR} , which yields the target cross-covariance matrix

$$\Sigma_i^{\text{USDEUR}} = \sqrt{\text{diag}\{(V_i^{\text{USD}}(V_i^{\text{USD}})^\top)\text{diag}\{(V_i^{\text{EUR}}(V_i^{\text{EUR}})^\top)\}^\top} \odot C^{\text{USDEUR}},$$

again with component-wise multiplication and square root.

In order to match target and model cross-correlations, transformation matrices Q_i^{EUR} and Q_i^{JPY} have to be found that minimize the loss value $r_1 + r_2$ composed by:

- A matrix norm, e.g. Frobenius norm, of the difference between target and model matrix

$$\begin{aligned} r_1 &= \zeta_1 \sum_{\mathcal{X} \in \mathcal{M}} \|\tilde{\Sigma}_i^{\mathcal{X}} - \Sigma_i^{\mathcal{X}}\| \\ \text{or} \\ r_1 &= \zeta_1 \sum_{\mathcal{X} \in \mathcal{M}} \|\tilde{C}_i^{\mathcal{X}} - C_i^{\mathcal{X}}\|. \end{aligned}$$

The set $\mathcal{M} = \{\text{USDEUR}, \text{USDJPY}, \text{JPYEUR}\}$ contains all cross-relations.

- The orthonormality condition

$$r_2 = \zeta_2 \sum_{\mathcal{X} \in \mathcal{N}} \|Q_i^{\mathcal{X}}(Q_i^{\mathcal{X}})^\top - I_{d_I}\|,$$

where I_{d_I} is the d_I -dimensional identity matrix and $\mathcal{N} = \{\text{EUR}, \text{JPY}\}$.

The parameters ζ_1 and ζ_2 are nicety parameters that allow to put different weights on the loss criteria.

Remark 1 *The cross-correlation fitting is successively done for each calendar time t_i , which is crucial for the feasibility of the procedure, since otherwise the optimization problem becomes to large.*

As for the intra-currency correlations historically estimated cross-correlations from the preceding 3 month period are used in the example. The scale parameters are set to $\zeta_1 = 1$ and $\zeta_2 = 100$. The target and the fitted cross-correlations are plotted in Figure 3 for the 1M calendar time. The differences between target and fitted cross-correlations for the worst fit (with respect to the Frobenius norm) are shown in Figure 4.

3.2 2. Step: Calibration of the Foreign Exchange Markets

In this step the FX volatilities for the exchange rates USD/EUR, USD/JPY and JPY/EUR are calibrated separately to market call option prices. The volatilities are parameterized in the same way as for the interest forward rates, i.e. volatilities are given by $\Lambda = \{\lambda_{i,j,k}\} \in \mathbb{R}^{n_c, m_f, d_C}$. The number of calendar times n_c is the same as before, since we have the same calendar time discretization for all assets. The number of forward times m_f could differ from n_f , and the same is true for d_C ,

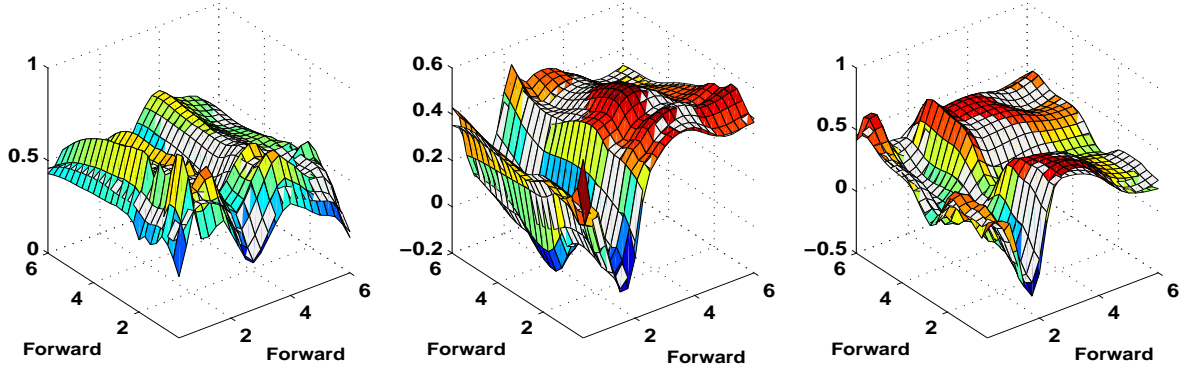


Figure 3: Target (gray) and fitted (colored) cross-correlation matrices for calendar time $t_1 = 1M$.

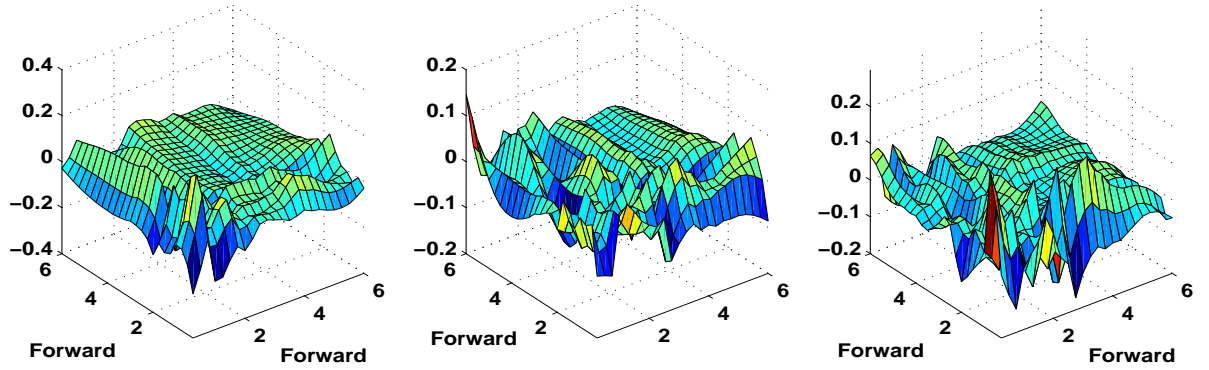


Figure 4: The differences between fitted and target cross-correlations for the worst fit (w.r.t. Frobenius norm) over all calendar times.

the number of stochastic factors. Notation introduced in the previous section, like $V^{\text{USD/EUR}}$ for aggregated volatilities and $Q^{\text{USD/EUR}}$ for transformation matrices, will be used accordingly here and in the sequel.

The calibration is subject to the following loss criteria, which will be discussed in detail subsequently:

1. Quality of fit of the model prices compared to market prices
2. Smoothness of the volatility surface
3. No-arbitrage condition of the multi-currency LMM

Since all FX markets will be calibrated separately in this step, the description below is for a generic exchange rate FX out of the set USD/EUR, USD/JPY and JPY/EUR.

The major intention of the calibration procedure is to fit the model parameters such that market prices are matched. The corresponding loss value is given by

$$q = \eta_q \sum_{j=1}^N (C_j - \tilde{C}_j)^2,$$

where N is the number of observable option prices and C_j and \tilde{C}_j denote the market price and the model option price, respectively.

In most cases less options will be available for calibration than the volatility matrix has parameters (entries). Therefore, additional conditions applied to the aggregated volatility matrices are introduced in order to push the volatility surface to a smooth shape. Let $V = \{v_{i,j}\} \in \mathbb{R}^{n_c, m_f}$ denote the aggregated volatility matrix for any of the exchange rates. To our experience it is appropriate to allow for the following three different smoothing criteria:

- Smoothness in calendar time

Volatilities for the same forward time and neighbouring calendar times should not differ too much,

$$s_1 = \eta_1 \sum_{j=1}^{m_f} \sum_{i=1}^{n_c-1} (v_{i+1,j} - v_{i,j})^2.$$

- Decreasing volatilities in forward time

This criterion implements the Samuelson effect, which states that forward volatilities tend to decrease with increasing time to maturity,

$$s_2 = \eta_2 \sum_{i=1}^{n_c} \sum_{j=1}^{m_f-1} (\max\{v_{i,j+1} - v_{i,j}, 0\})^2.$$

- Smoothness in forward time

This criterion imposes the volatility term structure to be smooth in forward time direction for each fixed calendar time. Assigning a large weight would force the volatility to be flat in the forward time, which is usually not desirable. But to our experience a small weight assigned to this criterion contributes to a smoother volatility surface.

$$s_3 = \eta_3 \sum_{i=1}^{n_c} \sum_{j=1}^{m_f-1} (v_{i,j+1} - v_{i,j})^2.$$

Finally, we seek to enforce the (approximate) no-arbitrage condition (9) for each forward exchange rate in combination with the two corresponding interest rates. The no-arbitrage condition is checked

separately for every calendar time and forward maturity using decomposed volatilities. So far we allowed different numbers of stochastic factors d_I and d_C for interest and foreign exchange markets. For the no-arbitrage condition all volatility matrices need to have the same number of factors, $d = \max\{d_I, d_X\}$, which is realized by extending the matrices in the factor dimension and filling the new entries with zeros.

Next, for each calendar time t_i and maturity time T_j volatilities in terms of absolute maturities are calculated, say $\lambda_{i,j,k}^{\text{USD/EUR}}$ and $\lambda_{i-1,j,k}^{\text{USD/EUR}}$. It is shown in Appendix B how to obtain volatilities for absolute maturities from volatility matrices for times to maturity (see also Remark 2 below). Then the Libor volatilities $\lambda_{i,j,k}^{\text{USD}}$, $\lambda_{i,j,k}^{\text{EUR}}$ and from those the ‘‘frozen coefficient’’ forward process volatilities $\bar{\gamma}_{i,j,k}^{\text{USD}}$, $\bar{\gamma}_{i,j,k}^{\text{EUR}}$ are constructed. Note, that all these volatilities are now vectors with d entries. With these pre-calculations the loss value for the multi-currency LMM (here for the case USD/EUR) is defined by

$$a = \eta_\alpha \sum_{i=1}^{n_c} \sum_{j=1}^{m_f} \sum_{k=1}^d (\lambda_{i-1,j,k}^{\text{USD/EUR}} - \bar{\gamma}_{i,j,k}^{\text{EUR}} + \bar{\gamma}_{i,j,k}^{\text{USD}} - \lambda_{i,j,k}^{\text{USD/EUR}})^2.$$

Finally, the loss function of the second step is given by $q + s_1 + s_2 + s_3 + a$.

Remark 2 *Volatilities like $\lambda_k(t, T)$ and $v(t, T)$ for the forward interest rates as well as for the FX rates could be either denoted in terms of absolute maturity times or relative times to maturity. For practical purposes times to maturity appear more appropriate to us. In particular, for given calendar and forward time grids they allow for a more flexible representation of volatilities (see also Appendix B), and also Pedersen’s single LMM calibration approach is delineated for times to maturity. However, in the literature typically volatilities for absolute maturities are used, and both notations are equivalent in the sense that for any of both representations there is a volatility matrix in the other representation generating the same model prices (albeit both matrices will not necessarily have the same calendar time grid). For application purposes it is necessary to implement a conversion method like proposed in Appendix B.*

Remark 3 *The no-arbitrage condition is verified with respect to the FX forward time grid, which is the reason why m_f is the upper limit in the second sum above. We further assume that the Libor forward grid given by $T_j = j\delta$ is always of finer granularity than the FX forward grid. In other words, no neighbouring FX forward times have a difference less than δ . In practice, FX rate forwards are commonly traded at the very short end of the curve that differ only a few days or weeks. We will address in Section <Reference here> below the problem of how to include those forwards in the calibration.*

Unfortunately, the FX markets trade forwards at the front end of the curve with maturities that differ only days or weeks. We will address in Section <Reference here> below the problem of how to calibrate these cases.

3.3 3. Step: Calibration of the Triangle Relation

The final step is to include the the triangle relation (10), which is a no-arbitrage condition and becomes relevant when three exchange rate combinations for three currencies are jointly modeled.

In order to fulfill this equality we need to modify the volatilities for $\Lambda^{\text{JPY/EUR}}$, but do not want to destroy the market fit obtained in step 2 for volatilities involving the reference currency USD. Therefore, the idea is to re-calibrate the secondary market JPY/EUR by fitting for each calendar time t_i a transformation matrix $Q_i^{\text{USD/EUR}}$, such that JPY/EUR market option prices are matched by the model using the volatility matrix received from the triangular relation, i.e. by

$$\Lambda_i^{\text{JPY/EUR}} = \Lambda_i^{\text{USD/EUR}} Q_i^{\text{USD/EUR}} - \Lambda_i^{\text{USD/JPY}}.$$

Again, we mean by $\Lambda_i^{\text{JPY/EUR}} \in \mathbb{R}^{n_f \times d}$ and $Q_i^{\text{JPY/EUR}} \in \mathbb{R}^{d \times d}$ the appropriate sub-matrices for calendar time t_i .

It is important to note that a fit of the $Q_i^{\text{USD/EUR}}$ also determines the cross-correlations between all of the FX rates, and here is no need to fall back on historical correlation estimates.

For the definition of the loss value we use similar criteria as in the steps 1 and 2 above.

1. Quality of fit to market prices

$$q = \xi_q \sum_{j=1}^N (C_j^{\text{JPY/EUR}} - \tilde{C}_j^{\text{JPY/EUR}})^2,$$

where N is the number of observable option prices and by $\tilde{C}_j^{\text{JPY/EUR}}$ the model option price is meant.

2. Orthonormality of the transformation matrix

$$r_2 = \xi_2 \| Q_i^{\text{USD/EUR}} (Q_i^{\text{USD/EUR}})^\top - I_d \|,$$

where $\| \cdot \|$ is a matrix norm (e.g. Frobenius norm) and I_d the d -dimensional identity matrix.

As before, ξ_q and ξ_2 are parameters controlling the optimization.

Remark 4 *Unfortunately, the separation into step 2 and 3 for calibrating the FX markets exhibits a particular weakness. By applying the fitted transformations $Q_i^{\text{USD/EUR}}$ the multi-currency LMM no-arbitrage conditions calibrated in step 2 are destroyed. The reason why step 3 has been separated from step 2 is that the joint calibration of all three FX volatility matrices would result in fitting a total of $3(n_c m_f d)$ parameters. In our approach we fit in step 2 3 problems with $(n_c m_f)$ parameters each first, and in step 3 (n_c) -many problems with d^2 parameters each. In Section <Reference here> an alternative approach will be addressed that overcomes this problem.*

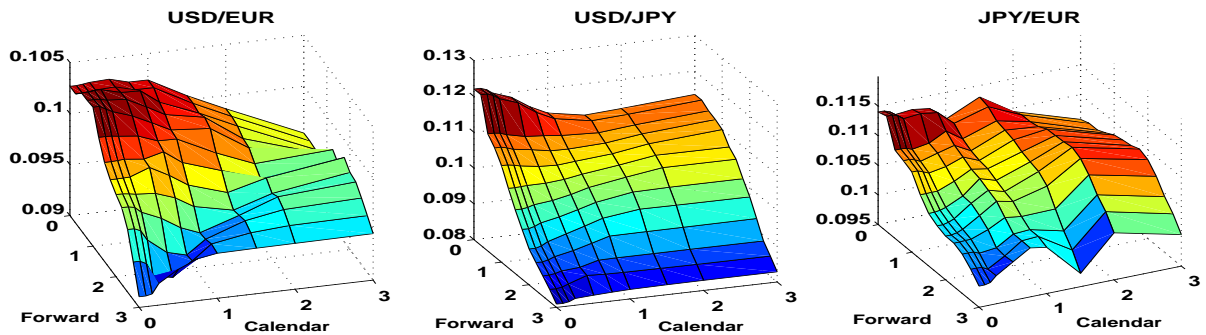


Figure 5: The (aggregated) exchange rate volatility surfaces.

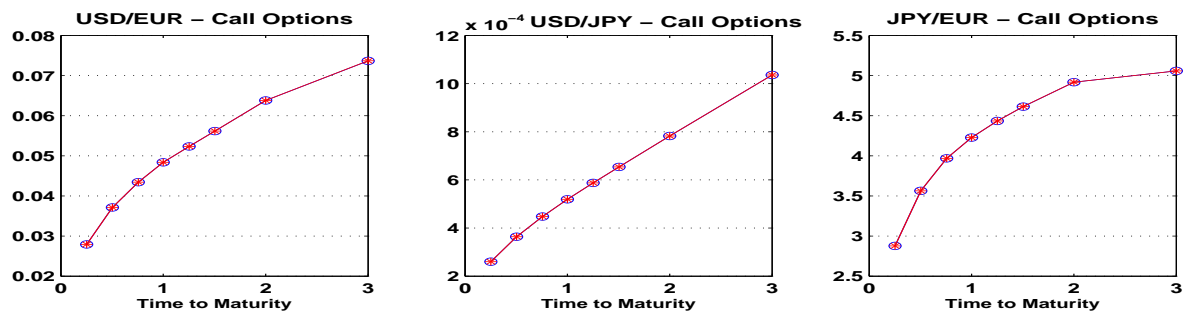


Figure 6: The market (blue) and model (red) prices for options on the FX forwards.

The fact that no IR/FX cross-correlations were fitted appears in our opinion less severe as long as no cross-correlation dependent market instruments are available. Historically estimated cross-correlations are typically much more unstable than, for instance, estimated correlations for forwards of different maturities within a market.

Figure 5 shows the finally calibrated FX rate volatilities and in Figure 6 the high quality of fit to market instruments is demonstrated.

3.4 Summary

The main problem of calibrating a multi-currency LMM is to fit a large number of parameters subject to many loss criteria. The attempt to do this in a single step leads to an infeasible optimization problem. A possible solution might be to split the whole problem into smaller sub-problems, such that

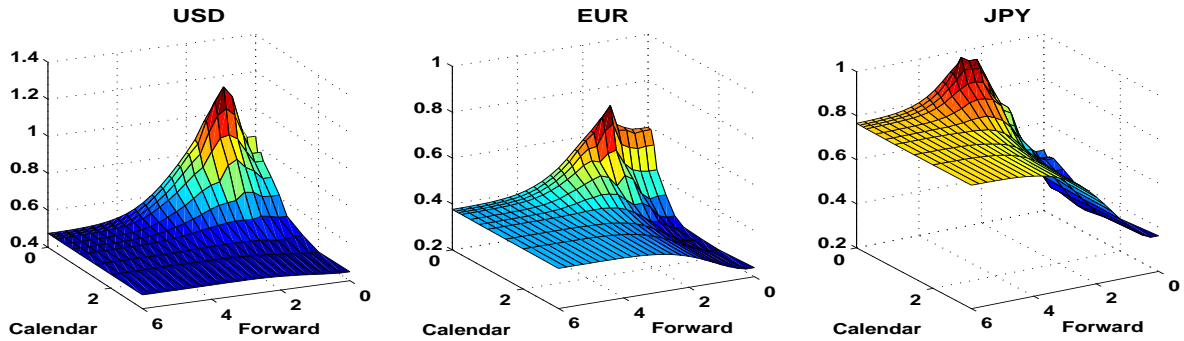


Figure 7: The calibrated forward rate volatilities V^{USD} , V^{EUR} and V^{JPY} for the calibration date 20th May 2009.

- the sub-problems are computationally manageable, and
- can be arranged in a way that solutions to succeeding problems do not affect solutions of preceding problems.

There are two no-arbitrage conditions introduced by jointly modelling more than one market. The first one is the multi-currency LMM no-arbitrage condition, which arises from linking stochastic interest rates and stochastic exchange rate forwards. It is fairly general in the sense that it has to be true for all arbitrage free LMM models. In particular it does not depend on whether the forward Libor volatilities are modeled deterministically or stochastically. In return, even for deterministic volatilities the no-arbitrage condition includes stochastic forward Libors, which means that for later calendar time intervals the condition can only be approximately enforced in the calibration (e.g. by the “frozen coefficient” approach (8)).

The second condition is the triangle relation, which results from jointly modeling three exchange rates referring to three currencies. The closed-form is established by assuming time-dependent but not strike-dependent volatilities. In return, the condition can be directly applied by calculating the volatility matrix $\Lambda^{\text{JPY/EUR}}$ of the secondary currencies by differencing the volatility matrices $\Lambda^{\text{USD/EUR}}$ and $\Lambda^{\text{USD/JPY}}$ involving the major currency.

4 Examples

In this section we present further examples on the calibration of the multi-currency LMM for May 20, 2009 and March 15, 2010. Thus given the example in the previous section for May 8, 2008, calibrations in the lead-up to, during and after the financial crisis are provided.

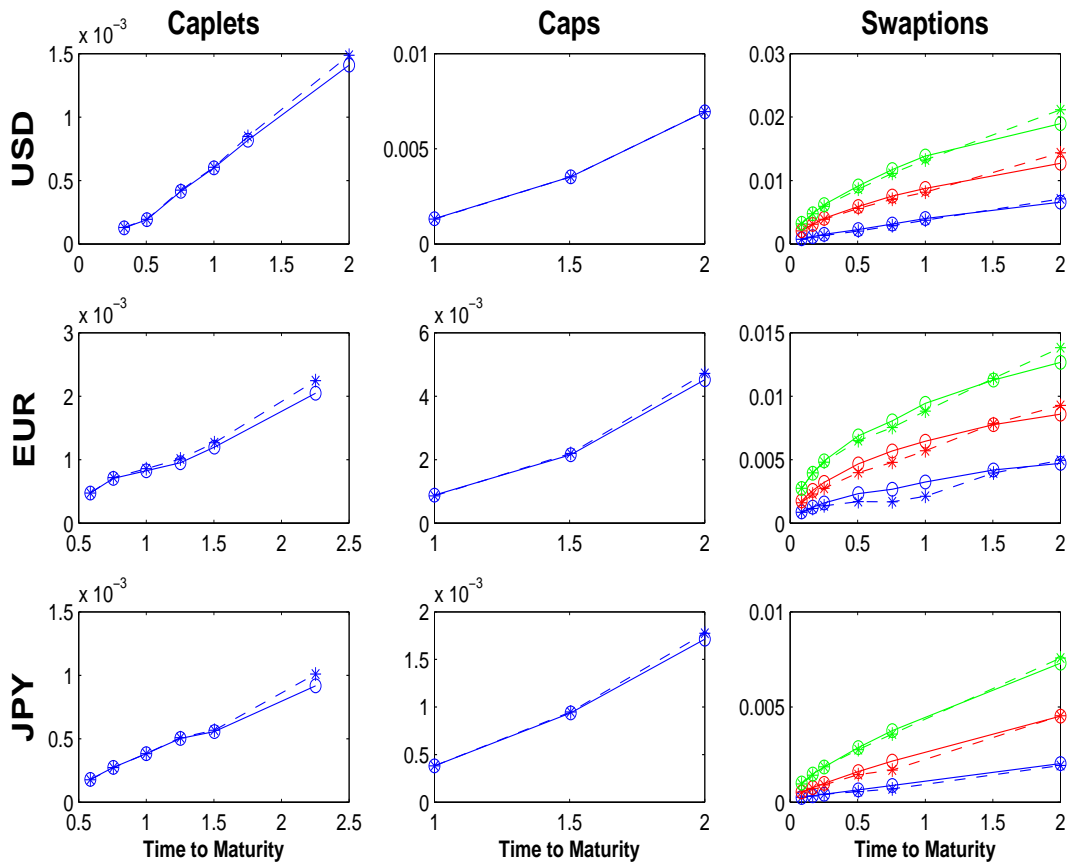


Figure 8: Fit of model prices (stars) to market prices (circles). The swaptions tenors are 1Y (blue), 2Y (red) and 3Y (green).

4.1 Calibration for May 20, 2009

The calibration procedure is the same as described in the previous section. The calendar time vector has been extended by 1Y3M in order to better match the FX market instruments, hence the calendar time vector is the year fraction equivalent of

$$t_c = [0M, 1M, 2M, 3M, 6M, 9M, 1Y, 1Y3M, 1Y6M, 2Y, 3Y].$$

The scale parameters for quality of fit and smoothness of the volatility surface were set to 10^{-2} and 10^{-3} , respectively. Putting more weight to the quality of fit and increasing the magnitude of both parameters result in a better fit. Figures 7 and 8 show the volatility surfaces and the market and fitted model prices.

The cross-correlations are matched quite reasonably with the same parameters as used in the previous example in Section 3.1. Figure 9 shows the cross-correlation surface for the first calendar time month compared to the exogenously given cross-correlations (resulting from a historical estimation). Figure 10 below depicts the differences between model and exogenous cross-correlations for the worst fit.

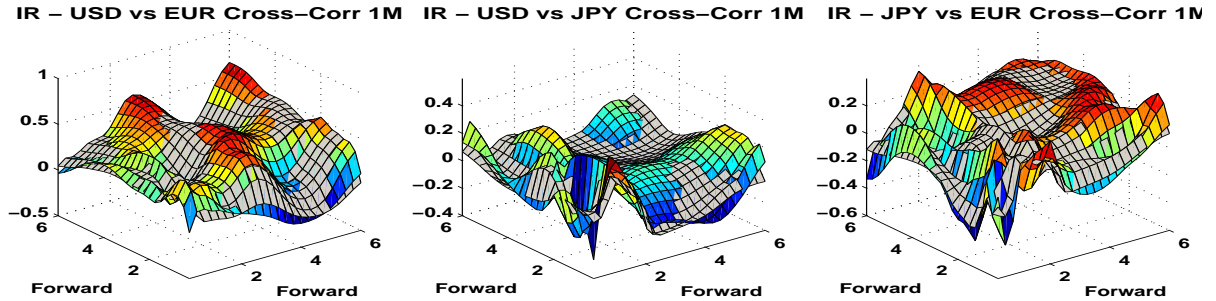


Figure 9: Target (gray) and fitted (colored) cross-correlation matrices for calendar time $t_1 = 1M$.

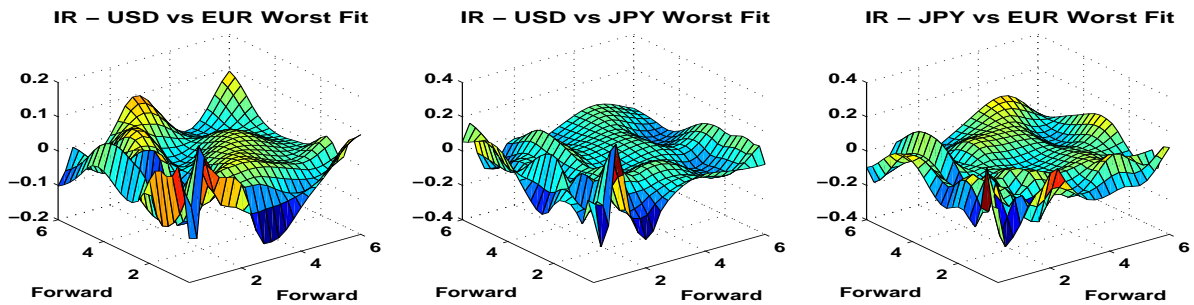


Figure 10: The differences between fitted and target cross-correlations for the worst fit (w.r.t. Frobenius norm) over all calendar times.

Finally, the FX market prices are also matched very well as Figure 12 demonstrates, with reasonably smooth volatility matrices, see Figure 11. No changes in the FX scale parameters compared to Section 3.1 were necessary.

4.2 Calibration for March 15, 2010

The last example is for the March 15, 2010, a certain time after the peak of the financial crisis. All objective function weights and time vectors are defined as in the calibration for May 20, 2009. The quality of all fits as well as the smoothness characteristics are similar to the calibration results before, which illustrates the stability of the calibration method proposed in this paper.

The interest rate volatility matrices and the corresponding fit of model to market prices are shown in Figures 13 and 14.

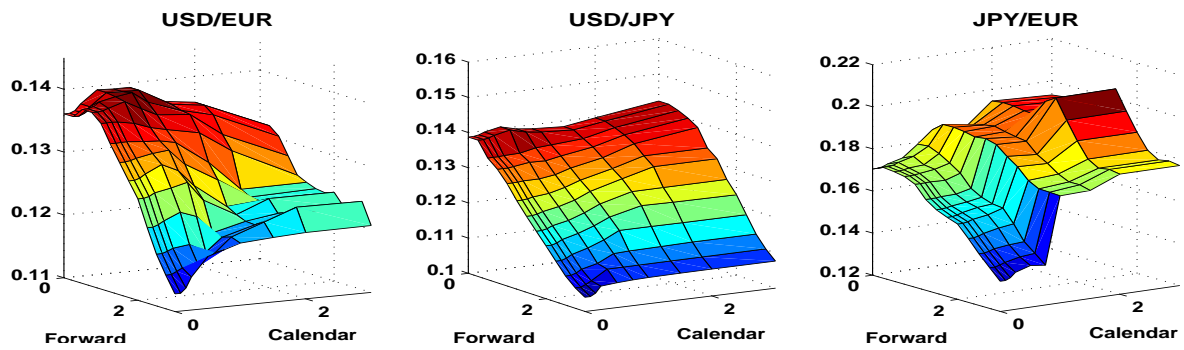


Figure 11: The (aggregated) exchange rate volatility surfaces.

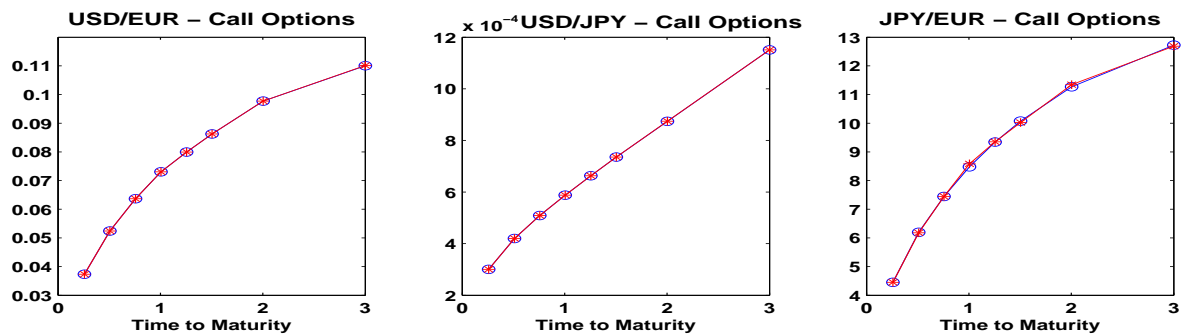


Figure 12: The market (blue) and model (red) prices for options on the FX forwards.

Model and exogenous cross-correlations for the first calendar time month and the fitting error of the worst fit from all calendar times are given in Figures 15 and 16.

Finally, the FX volatilities are shown in Figure 18 and the high quality of fit between market and model prices is demonstrated in Figure 17.

A Pedersen (1998) Calibration

This is a method to calibrate the LMM volatility functions to market prices of caps (or caplets) and/or swaptions. The LMM volatility function $\lambda(\cdot, \cdot)$ is chosen to be effectively non-parametric, piecewise constant on a discretisation of both time to maturity and calendar time. Calibration is achieved

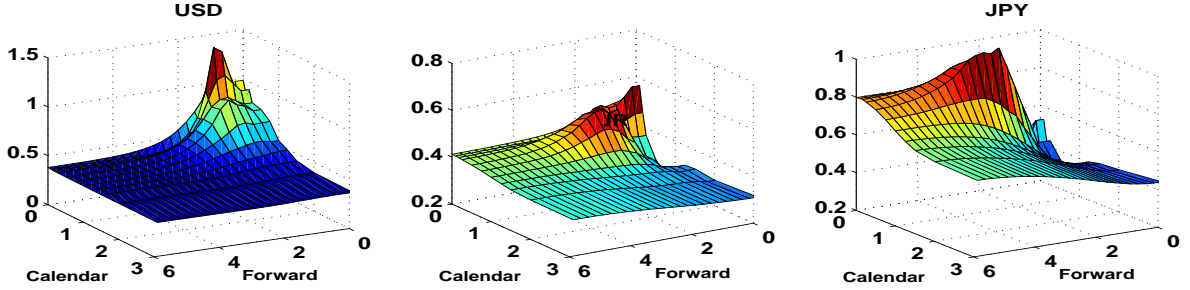


Figure 13: The calibrated forward rate volatilities V^{USD} , V^{EUR} and V^{JPY} for the calibration date 15th March 2010.

through unconstrained non-linear optimisation of a weighted sum of quality-of-fit and smoothness criteria. Correlation is exogenous to this calibration procedure: It is assumed to be constant in time and estimated from historical data. Reduction of the dimension of the optimisation problem is achieved via principal components analysis.

The calibration method proceeds as follows. Suppose we have n_{fac} factors (the dimension of the driving Brownian motion) and discretise process time into n_{cal} segments, and forward time (maturities) into n_{fwd} segments. The i -th component of ($1 \leq i \leq n_{\text{fac}}$) of the volatility function $\lambda(t, T)$ will be given by

$$\lambda_i(t, x) = \lambda_{ijk}, \quad t \in [t_{j-1}, t_j], \quad x \in [x_{k-1}, x_k]$$

where $x = T - t$ is the *forward tenor*, t_j , $j > 0$, and x_k , $k > 0$, are the chosen process and forward times, respectively. For convenience set $t_0 = x_0 = 0$.

The objective of the optimisation is to find the λ_{ijk} to minimise

$$w_{\text{caps}} \text{QOF}_{\text{caps}} + w_{\text{swaptions}} \text{QOF}_{\text{swaptions}} + \text{smooth}$$

where the quality of fit to observed market prices is given by

$$\text{QOF} = \frac{1}{N} \sum_{i=1}^N \left(\frac{\overline{\text{PV}}_i}{\text{PV}_i} - 1 \right)^2$$

with $\overline{\text{PV}}_i$ and PV_i the model and market prices of the i -th instrument, respectively. The measure of “smoothness” of the calibrated volatilities is

$$\text{smooth} = \text{scale}_{\text{fwd}} \cdot \text{smooth}_{\text{fwd}} + \text{scale}_{\text{cal}} \cdot \text{smooth}_{\text{cal}}$$

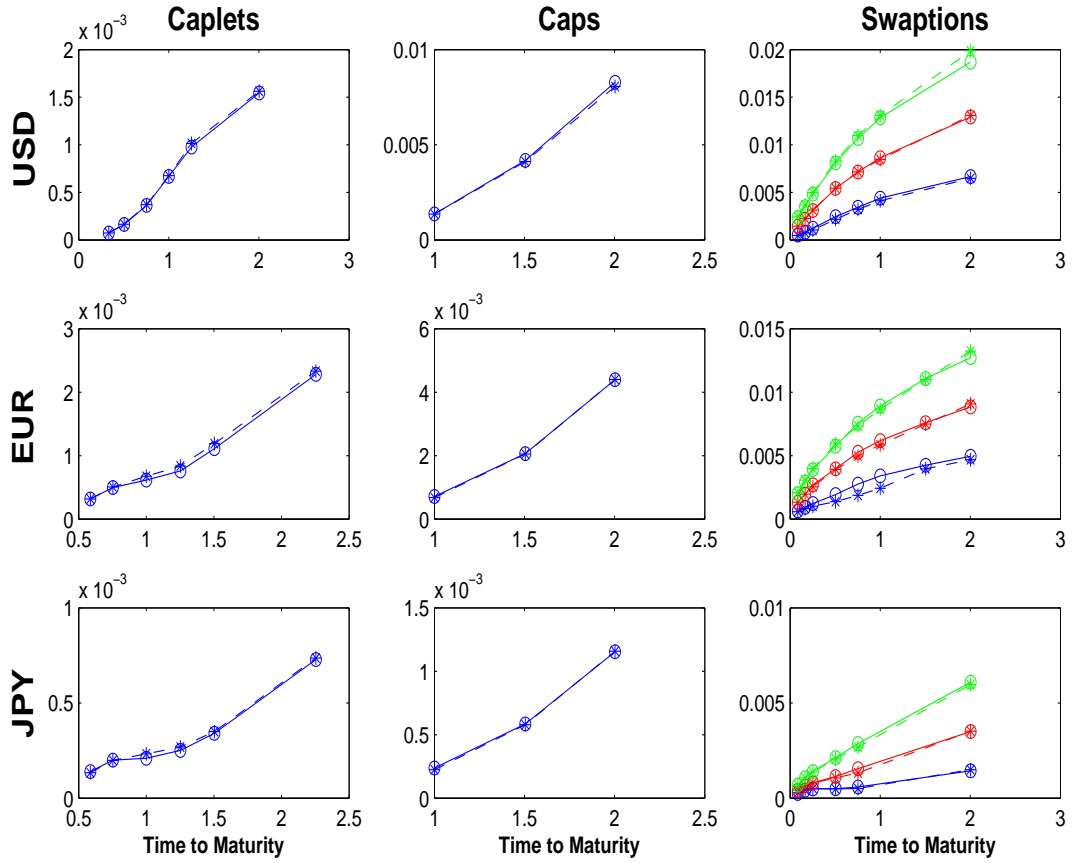


Figure 14: Fit of model prices (stars) to market prices (circles). The swaptions tenors are 1Y (blue), 2Y (red) and 3Y (green).

with

$$\text{smooth}_{\text{fwd}} = \sum_{i=1}^{n_{\text{cal}}} \sum_{j=2}^{n_{\text{fwd}}} \left(\frac{\text{vol}_{i,j}}{\text{vol}_{i,j-1}} - 1 \right)^2$$

$$\text{smooth}_{\text{cal}} = \sum_{i=2}^{n_{\text{cal}}} \sum_{j=1}^{n_{\text{fwd}}} \left(\frac{\text{vol}_{i,j}}{\text{vol}_{i-1,j}} - 1 \right)^2$$

where $\text{vol}_{i,j}$ is the *volatility level* on the calendar time segment $[t_{i-1}, t_i]$ and forward tenor (time to maturity) segment $[x_{j-1}, x_j]$. Thus $\text{vol}_{i,j}$ is the norm of the (i, j) -th row of Λ , i.e. the vector $\lambda_{:,i,j}$.

The original dimensionality of the problem is $n_{\text{fac}} \times n_{\text{cal}} \times n_{\text{fwd}}$. The Pedersen approach separates volatility levels and correlation, where the volatility levels $\text{vol}_{i,j}$ are the objects to be calibrated and the correlation structure is given exogenously to the calibration. Thus volatility levels given by the *volatility grid*

$$\text{vol}_{i,j}, \quad 1 \leq i \leq n_{\text{cal}}, \quad 1 \leq j \leq n_{\text{fwd}}$$

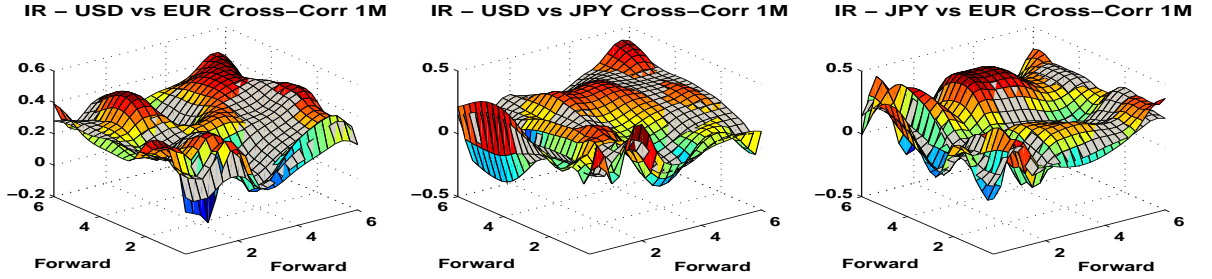


Figure 15: Target (gray) and fitted (colored) cross-correlation matrices for calendar time $t_1 = 1M$.

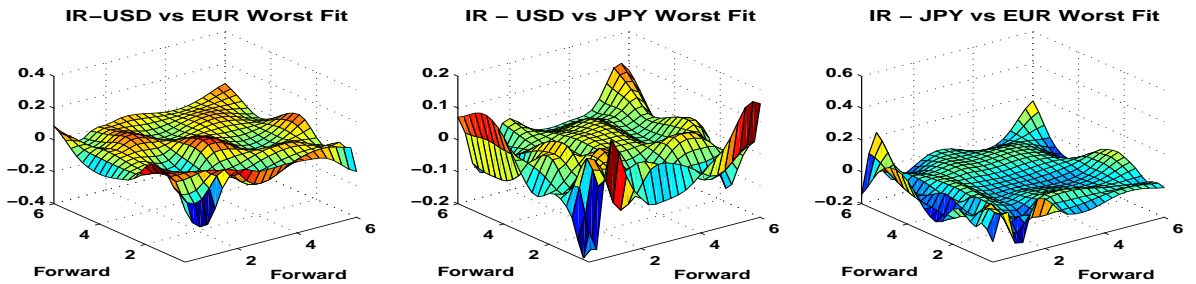


Figure 16: The differences between fitted and target cross-correlations for the worst fit (w.r.t. Frobenius norm) over all calendar times.

where $\text{vol}_{i,j}$ is the volatility level as seen at time t_{i-1} (assumed constant until t_i) of the basic period rate $L(\cdot, t_{i-1} + x_j)$ for the forward period beginning at time $t_{i-1} + x_j$.

Correlation and covariance are introduced via a principal components representation. Let vol be the vector of basic period forward rate volatilities as seen on time t_{j-1} . Let Corr be the corresponding correlation matrix. The covariance matrix is then computed as

$$\text{Cov} = \text{vol}^T \text{Corr} \text{vol}$$

Let Γ be the diagonal matrix containing the eigenvalues of Cov and V be the corresponding matrix of eigenvectors, i.e. we have the eigenvalue/eigenvector decomposition of Cov

$$\text{Cov} = V^T \Gamma V$$

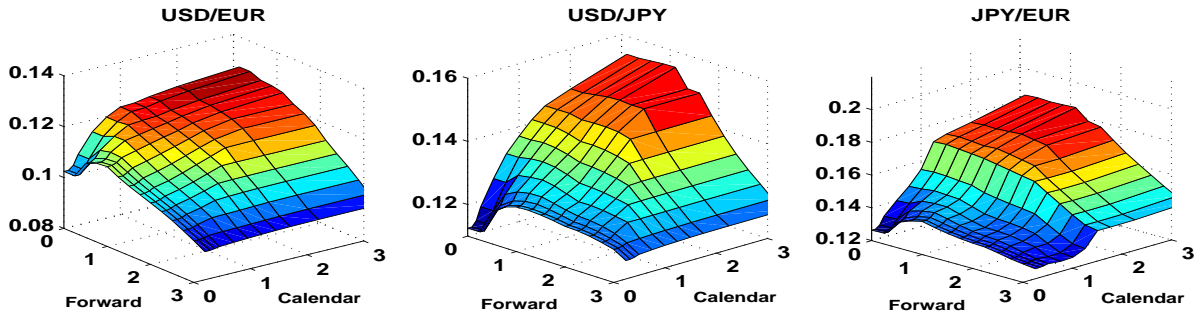


Figure 17: The (aggregated) exchange rate volatility surfaces.

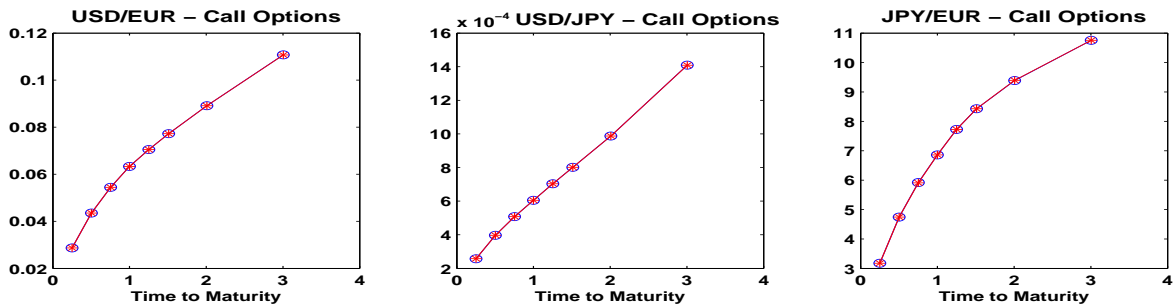


Figure 18: The market (blue) and model (red) prices for options on the FX forwards.

As Cov is positive semidefinite, all entries γ_k on the diagonal of Γ will be non-negative and we have

$$\text{Cov} = W^T W$$

where

$$w_{ik} = \sqrt{\gamma_k} v_{ik}$$

One can then extract the stepwise constant volatility function for forward LIBORs as

$$\lambda_{ijk} = w_{ik}$$

W will provide values for as many factors as the rank of the covariance matrix. For a given n_{fac} , we only use the rows of W corresponding to the n_{fac} largest eigenvalues.

B Absolute and Relative Volatilities

The purpose of this section is to demonstrate the difference between representing forward times in volatility matrices by times to maturity versus by maturity times, and how to convert one representation into the other. It is shown how to calculate integrated volatilities required for market instrument pricing for both cases.

We denote by $T = (T_0, T_1, \dots, T_{m_a})$ the maturity time and by $x = (x_0, x_1, \dots, x_{m_r})$ the times to maturity discretization. For both the first entry is $T_0 = x_0 = 0$ for technical reasons, but otherwise the discretization need not to coincide. The calendar times $t = (t_0, t_1, \dots, t_n)$ are set equal for both representations. Furthermore, we assume that volatilities are piecewise constant, i.e. as mentioned before the volatilities $\lambda_{i,j,k}$ and $v_{i,j}$ are constant and valid for calendar times $t_{i-1} < t \leq t_i$ and forward times $T_{j-1} < T \leq T_j$ and $x_{j-1} < T - t \leq x_j$, respectively.

Volatility matrices with times to maturity are integrated in a diagonal direction, because as calendar time t passes the time to maturity $x = T - t$ decreases. On the contrary, the absolute maturity time T is fixed over calendar time, hence integrating volatilities follows a vertical path in the matrix. This difference is schematically illustrated in Figure 19 below.

An option with time to maturity of, say, x_3 (at calendar time t_0) would collect a total volatility given by the green path going from x_3 at t_0 to x_1 at expiration (somewhat before t_3 in the figure). During the lifetime it passes five volatility regimes (given by the volatility entries for (t_1, x_3) , (t_2, x_3) , (t_3, x_3) , (t_3, x_2) and (t_3, x_1)).

The right illustration shows how the same integrated volatility would be calculated in an absolute maturity time volatility matrix. An option with, say, maturity time T_3 would collect volatilities along the red line, i.e. go through the three volatility regimes corresponding to (t_1, T_3) , (t_2, T_3) and (t_3, T_3) .

Remark 5 *If an absolute and a relative forward time matrix are given, for any maturity within the forward time range, the number of volatility regimes of the relative matrix is greater or equal to the number of volatility regimes of the absolute matrix. In other words, a relative forward time volatility matrix allows for a finer volatility modeling than an absolute forward time matrix.*

This demonstrates that getting an absolute maturity time volatility $\lambda_k(t, T)$ from a matrix with times to maturity $\Lambda = \{\lambda_{i,j,k}\} \in \mathbb{R}^{n_c, n_f, d}$ (where as before k is the index of the stochastic factor) simply means to a table lookup. Find the indices i and j such that $t_{i-1} < t \leq t_i$ and $x_{j-1} < T - t \leq x_j$, then $\lambda_k(t, T) = \lambda_{i,j,k}$ for all k .

Similarly integrated (or total) variances are calculated, and has also been described in Pedersen (1998). Given for calendar time $t_0 = 0$ a maturity time T or time to maturity $x = T - t_0$ the set of indices

$$\mathcal{I} = \{(i, j) : t_{i-1} < t \leq t_i, x_{j-1} < T - t \leq x_j, 0 \leq t \leq T\},$$

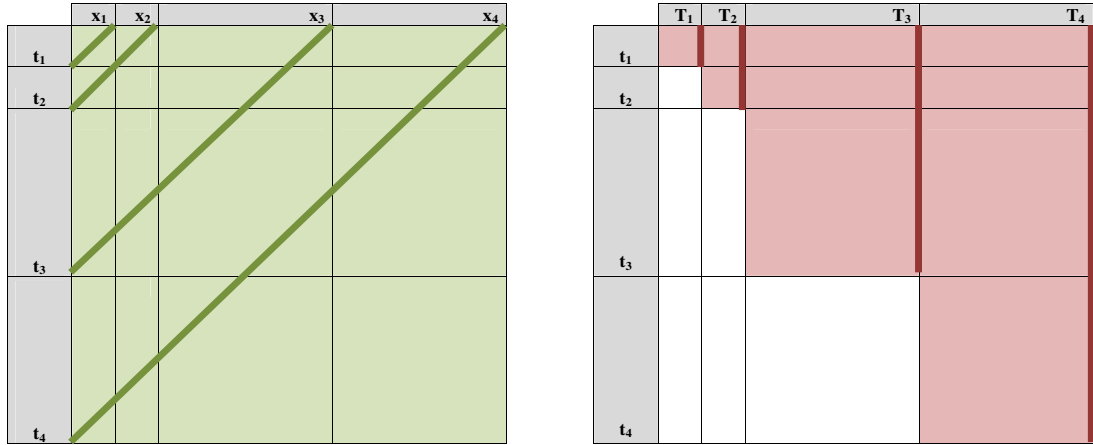


Figure 19: Schematic illustration of the integration paths for total volatilities when the matrix is given for times to maturity (left) and maturity times (right). Volatilities for the white time buckets in the right figure are meaningless, since there the calendar time is larger than the forward time.

describes exactly those volatility regimes, that are valid at some time during the lifetime of the underlying with maturity T . The regimes in \mathcal{I} are those that are crossed by the path corresponding to T as illustrated in Figure 19 above. For the case of relative times to maturity, the time spent in these regimes is determined by the intersections of the path with calendar or forward times and is given by

$$\kappa_{i,j} := \min\{t_i, T - x_{j-1}, T\} - \max\{t_{i-1}, T - x_j\},$$

which is always greater or equal to zero for $(i, j) \in \mathcal{I}$. Finally, the integrated variance can be calculated by

$$\int_0^T (\lambda(u, T))^\top \lambda(u, T) du = \sum_{(i,j) \in \mathcal{I}} \kappa_{i,j} \sum_{k=1}^d \lambda_{i,j,k}^2,$$

where k is the index of the stochastic factors.

The case of absolute maturity times is even simpler to handle, because for given T there is only an single T_j , such that $T_{j-1} < T \leq T_j$. Now, the corresponding set of indices is only regarding the calendar time index,

$$\mathcal{J}_j = \{i : t_{i-1} < t \leq t_i, 0 \leq t \leq T\},$$

and the sojourn times are $\theta_i := \min\{t_i - t_{i-1}, T - t_{i-1}\}$. The integrated variance becomes

$$\int_0^T (\lambda(u, T))^\top \lambda(u, T) du = \sum_{i \in \mathcal{J}_j} \theta_i \sum_{k=1}^d \lambda_{i,j,k}^2.$$

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