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THE EVALUATION OF BARRIER OPTION PRICES UNDER STOCHASTIC VOLATILITY

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Abstract. This paper considers the problem of numerically evaluating barrier option prices when the dynamics of the underlying are driven by stochastic volatility following the square root process of Heston (1993). We develop a method of lines approach to evaluate the price as well as the delta and gamma of the option. The method is able to efficiently handle both continuously monitored and discretely monitored barrier options and can also handle barrier options with early exercise features. In the latter case, we can calculate the early exercise boundary of an American barrier option in both the continuously and discretely monitored cases.

Keywords: barrier option, stochastic volatility, continuously monitored, discretely monitored, free boundary problem, method of lines, Monte Carlo simulation.
JEL Classification: C61, D11.

1. Introduction

Barrier options are path-dependent options and are very popular in foreign exchange markets. They have a payoff that is dependent on the realized asset path via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low during the option's life. For example, an up-and-out call option pays off the usual $\max(S-K,0)$ at expiry unless at any time during the life of the option the underlying asset has traded at a value H or higher. In this example, if the asset reaches this level (from below, obviously) then it is said to "knock out" and become worthless. Apart from "out" options like this, there are also "in" options which only receive a payoff if a certain level is reached, otherwise they expire worthless. Barrier options are popular for a number of reasons. The purchaser can use them to hedge very specific cash flows with similar properties.

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Usually, the purchaser has very precise views about the direction of the market. If he or she wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying will be limited prior to expiry, then he may choose to buy an up-and-out call. It will be cheaper than a similar vanilla call, since the upside is severely limited. If he is right and the barrier is not triggered he gets the payoff he wanted. The closer that the barrier is to the current asset price then the greater the likelihood of the option being knocked out, and thus the cheaper the contract.

Barrier options are common path-dependent options traded in the financial markets. The derivation of the pricing formula for barrier options was pioneered by Merton (1973) in his seminal paper on option pricing. A list of pricing formulas for one-asset barrier options and multi- asset barrier options both under the geometric Brownian motion (GBM) framework can be found in the articles by Rich (1994) and Wong & Kwok (2003), respectively. Gao, Huang & Subrahmanyam (2000) analyzed option contracts with both knock-out barrier and American early exercise features.

Derivative securities are commonly written on underlying assets with return dynamics that are not sufficiently well described by the GBM process proposed by Black & Scholes (1973). There have been numerous efforts to develop alternative asset return models that are capable of capturing the leptokurtic features found in financial market data, and subsequently to use these models to develop option prices that better reflect the volatility smiles and skews found in market traded options. One of the classical ways to develop option pricing models that are capable of generating such behaviour is to allow the volatility to evolve stochastically, for instance according to the square-root process introduced by Heston (1993). The evaluation of barrier option prices under Heston stochastic volatility model has been extensively discussed by Griebsch (2008) in her thesis.

There are certain drawbacks in the evaluation of the Barrier option prices under SV using either tree or finite difference methods, for instance the convergence is rather slow and it takes more effort to obtain accurate hedge ratios. It turns out that another well known method, the method of lines is able to overcome those disadvantages. In this paper, we introduce a unifying approach to price both continuously and discretely monitored barrier options without or with early exercise features. Specifically, except for American style knock-in options, we are able to price all other kinds of European or American barrier options using the framework developed here.

The remainder of the paper is structured as follows. Section 2 outlines the problem of both continuously and discretely monitored barrier options where the underlying asset follows stochastic volatility dynamics. In Section 3 we outline the basic idea of the method of lines approach and implement it to find the price profile of the barrier option. A number of numerical examples that demonstrate the computational advantages of the method of lines approach are provided in Section 4. Finally we discuss the impact of stochastic volatility on the prices of the barrier option in Section 5 before we draw some conclusions in Section 6.

2. Problem Statement-Barrier Option with Stochastic Volatility

Let $C(S, v, \tau)$ denote the price of an up-and-out (UO) call option with time to maturity τ[1] written on a stock of price S and variance v that pays a continuously compounded dividend yield q. The option has strike price K and a barrier H.

Analogously to the setting in Heston (1993), the dynamics for the share price S under the risk neutral measure are governed by the stochastic differential equation (SDE) system [2]

$$dS = (r - q)Sdt + \sqrt{v}SdZ_1, \tag{1}$$

$$dv = \kappa_v(\theta_v - v)dt + \sigma\sqrt{v}dZ_2, \tag{2}$$

where Z_1, Z_2 are standard Wiener processes and $\mathbb{E}(dZ_1 dZ_2) = \rho dt$ with $\mathbb{E}$ the expectation operator under the risk neutral measure. In (1), r is the risk free rate of interest. In (2) the parameter σ is the so called vol-of-vol (in fact, $\sigma^2 v$ is the variance of the variance process v). The parameters κ_v and θ_v are respectively the rate of mean reversion and long run variance of the process for the variance v. These are under the risk-neutral measure and are relate to the corresponding quantities by a parameter that appears in the market price of volatility risk.[3]

[1]Note that $\tau = T - t$, where T is the maturity date of the option and t is the time.

[2]Of course, since we are using a numerical technique we could in fact use more general processes for S and v. The choice of the Heston processes is driven partly by the fact that this has become a very traditional stochastic volatility model and partly because a companion paper on the evaluation of European compound options under stochastic volatility uses techniques based on a knowledge of the characteristic function for the stochastic volatility process, which is known for the Heston process (see Chiarella, Griebsch & Kang (2009)), and can be used for comparison purpose.

[3] In fact, if it is assumed that the market price of risk associated with the uncertainty driving the variance process has the form $\lambda\sqrt{v}$, where λ is a constant (this was the assumption in Heston (1993)) and $\kappa_v^{\mathbb{P}}, \theta_v^{\mathbb{P}}$ are the corresponding parameters under the physical measure, then $\kappa_v = \kappa_v^{\mathbb{P}} + \lambda\sigma, \theta_v = \frac{\kappa_v^{\mathbb{P}}\theta_v^{\mathbb{P}}}{\kappa_v^{\mathbb{P}} + \lambda\sigma}$.

We are also able to write down the above system (1)-(2) using independent Wiener processes. Let $W_1 = Z_2$ and $Z_1 = \rho W_1 + \sqrt{1-\rho^2} W_2$ where W_1 and W_2 are independent Wiener processes under the risk neutral measure. Then, the dynamics of S and v can be rewritten in terms of independent Wiener processes as

$$dS = (r-q)Sdt + \sqrt{v}S(\rho dW_1 + \sqrt{1-\rho^2}dW_2), \tag{3}$$

$$dv = \kappa_v(\theta_v - v)dt + \sigma\sqrt{v}dW_1. \tag{4}$$

The price of a barrier option under stochastic volatility at time t, $C(S, v, \tau)$, can be formulated as the solution to a partial differential equation (PDE) problem. We need to solve the PDE for the value of the barrier option $C(S, v, \tau)$ given by

$$\frac{\partial C}{\partial \tau} = \mathcal{K}C - rC, \tag{5}$$

on the interval $0 \leq \tau \leq T$, where the Kolmogorov operator $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{vS^2}{2}\frac{\partial^2}{\partial S^2} + \rho\sigma v S\frac{\partial^2}{\partial S \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2}{\partial v^2} + (r-q)\, S\frac{\partial}{\partial S} + (\kappa_v(\theta_v - v) - \lambda v)\frac{\partial}{\partial v}, \tag{6}$$

where λ is the constant appearing in the equation for the market price of volatility risk, which as stated in Footnote 3 is of the form $\lambda\sqrt{v}$.

Both the terminal and boundary conditions need to be specified depending on the detailed specifications of the barrier options:

- A continuously Monitored Barrier Option with or without early exercise features, has the terminal condition

$$C(S, v, 0) = (S-K)^+ \mathbb{1}_{\{\max_t S(t) < H\}}, \tag{7}$$

- A discretely Monitored Barrier Option with or without early exercise features, with N monitoring dates $t \leq t_1 < t_2 <, \cdots, < t_N \leq T$, has the terminal condition

$$C(S, v, 0) = (S-K)^+ \mathbb{1}_{\{S(t_1)<H, S(t_2)<H, \cdots, S(t_N)<H\}}, \tag{8}$$

- A continuously Monitored Barrier Option with early exercise features, has the free (early exercise) boundary condition

$$C(b(v,\tau), v, \tau) = b(v,\tau) - K, \text{ when } b(v,\tau) < H \tag{9}$$

  where $S = b(v, \tau)$ is the early exercise boundary for the barrier option at time to maturity τ and variance v, and there also hold the smooth-pasting conditions

$$\lim_{S \to b(v,\tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S \to b(v,\tau)} \frac{\partial C}{\partial v} = 0. \tag{10}$$

- A discretely Monitored Barrier Option with early exercise features, has the free (early exercise) boundary condition

$$C(b(v,\tau), v, \tau) = b(v,\tau) - K, \tag{11}$$

where $b(v,\tau)$ is the early exercise boundary for the barrier option at time to maturity τ and variance v, and the smooth-pasting conditions

$$\lim_{S \to b(v,\tau)} \frac{\partial C}{\partial S} = 1, \quad \lim_{S \to b(v,\tau)} \frac{\partial C}{\partial v} = 0. \tag{12}$$

Before going into detail of the valuation, the following relations between the payoffs of barrier options and vanilla options are pointed out. The in-out parity for European barrier options, namely

$$\text{knock-in } + \text{ knock-out} = \text{vanilla};$$

allows us to consider only the family of knock-out options for the valuation using the method of lines (MOL) since we are able to price vanilla option under Heston model using the method of lines already. In next section, we are going to discuss the detail of computing the up-and-out barrier option prices by implementing the MOL.

3. Method of Lines (MOL) Approach

In this section,we will provide the details of the implementation of the Method of Lines. The key idea behind the method of lines is to approximate a PDE with a system of ordinary differential equations (ODEs), the solution of which can be obtained with ODE techniques. When volatility is constant, the system of ODEs is obtained by discretizing time. For the PDE (5), we must in addition discretize the variance, v. S is retained as independent variable. We begin by setting $v_m = m\Delta v$, where $m = 0, 1, 2, ..., M$. Typically we will set the maximum variance to be $v_M = 100\%$. Furthermore, we disctretise the time to expiry according to $\tau_n = n\Delta\tau$, where $n = 0, 1, 2, \ldots, N$ and $\tau_N = T$. We denote the option price along the variance line v_m and time line τ_n by $C(S, v_m, \tau_n) = C_m^n(S)$, and set

$$V(S, v_m, \tau_n) \equiv \frac{\partial C(S, v_m, \tau_n)}{\partial S} = V_m^n(S), \tag{13}$$

which is of course the option delta at the particular grid point.

We now select finite difference approximations for the derivative terms with respect to v. For the second order term, at the grid point (S, v_m, τ_n) we use the standard central difference scheme

$$\frac{\partial^2 C}{\partial v^2} = \frac{C^n_{m+1} - 2C^n_m + C^n_{m-1}}{(\Delta v)^2}. \tag{14}$$

Similarly for the cross-derivative term at the grid point (S, v_m, τ_n), we use the central difference approximation

$$\frac{\partial^2 C}{\partial S \partial v} = \frac{V^n_{m+1} - V^n_{m-1}}{2\Delta v}. \tag{15}$$

Since the coefficients of the second order derivative terms go to zero as $v \to 0$, we use an upwinding finite difference scheme for the first order derivative term (see Duffy (2006)), such that, at the grid point (S, v_m, τ_n) we have

$$\frac{\partial C}{\partial v} = \begin{cases} \frac{C^n_{m+1} - C^n_m}{\Delta v} & \text{if} \quad v \le \frac{\alpha}{\beta}, \\ \frac{C^n_m - C^n_{m-1}}{\Delta v} & \text{if} \quad v > \frac{\alpha}{\beta}, \end{cases} \tag{16}$$

where $\alpha = \kappa_v \theta_v$ and $\beta = \kappa_v + \lambda_v$. Since the second order derivative terms both vanish as $v \to 0$, upwinding helps to stabilise the finite difference scheme with respect to v.

Next we must select a discretisation for the time derivative. Initially we use a standard backward difference scheme, given at the grid point (S, v_m, τ_n) by

$$\frac{\partial C}{\partial \tau} = \frac{C^n_m - C^{n-1}_m}{\Delta \tau}. \tag{17}$$

This approximation is only first order accurate with respect to time. For the case of the standard American put option, it is known from Meyer (2009) that the accuracy of the method of lines increases considerably by using a second order approximation for the time derivative, specifically

$$\frac{\partial C}{\partial \tau} = \frac{3}{2}\frac{C^n_m - C^{n-1}_m}{\Delta \tau} - \frac{1}{2}\frac{C^{n-1}_m - C^{n-2}_m}{\Delta \tau}. \tag{18}$$

Thus we initiate the method of lines solution by using (17) for the first several time steps, and then switch to (18) for all subsequent time steps.

Applying (14)-(18) to the PDE (5), we now need to solve a system of second order ODEs at each time step and variance grid point. For the first few time steps, the

ODE system at the grid point $v = v_m$ and $\tau = \tau_n$ is

$$\begin{aligned}
&\frac{v_m S^2}{2}\frac{d^2 C_m^n}{dS^2} + \rho\sigma v_m S \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta v} + \frac{\sigma^2 v_m}{2}\frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{(\Delta v)^2} \\
&\quad + \frac{\alpha - \beta v_m}{2}\frac{C_{m+1}^n - C_{m-1}^n}{\Delta v} + \frac{|\alpha - \beta v_m|}{2}\frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{\Delta v} \\
&\quad + (r-q)S\frac{dC_m^n}{dS} - rC_m^n - \frac{C_m^n - C_m^{n-1}}{\Delta\tau} = 0,
\end{aligned} \tag{19}$$

and for all subsequent time steps the ODE system has the form

$$\begin{aligned}
&\frac{v_m S^2}{2}\frac{d^2 C_m^n}{dS^2} + \rho\sigma v_m S \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta z} + \frac{\sigma^2 v_m}{2}\frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{(\Delta v)^2} \\
&\quad + \frac{\alpha - \beta v_m}{2}\frac{C_{m+1}^n - C_{m-1}^n}{\Delta v} + \frac{|\alpha - \beta v_m|}{2}\frac{C_{m+1}^n - 2C_m^n + C_{m-1}^n}{\Delta v} \\
&\quad + (r-q)S\frac{dC_m^n}{dS} - rC_m^n - \frac{3}{2}\frac{C_m^n - C_m^{n-1}}{\Delta\tau} + \frac{1}{2}\frac{C_m^{n-1} - C_m^{n-2}}{\Delta\tau} = 0.
\end{aligned} \tag{20}$$

We require two boundary conditions in the v direction, one at v_0 and the other at v_M. For large values of v, the rate of change of the option price with respect to v converges to zero. So for sufficiently large values of v, one can treat this rate of change as zero without any impact on the accuracy of the solution at other values of v. Thus we set $\partial C/\partial v = 0$ along the variance boundary $v = v_M$. To handle the boundary condition at v is zero,we fit a quadratic polynomial through the option prices at v_1, v_2 and v_3, and then use this to extrapolate an approximation of the price at v_0. It turns out that this provides a satisfactory estimate of the price along v_0 for the purpose of generating a stable solution for small values of v[4].

After taking the boundary conditions into consideration, at each grid point (τ_n, v_m) we must solve a system of $M-1$ second order ODEs along a line in the S direction. We solve this system of ODEs iteratively for increasing values of v, using the latest available estimates for C_{m+1}^n, C_{m-1}^n, V_{m+1}^n and V_{m-1}^n. The initial estimates for C_m^n and V_m^n are simply C_m^{n-1} and V_m^{n-1}, then we use the latest estimates for C_m^n and V_m^n found during the current iteration through the variance lines. At a grid value of S we continue to iterate through the (v, τ) grid until the price profile converges to a desired level of accuracy, and then proceed to the next value of S.

[4]See Chiarella, Kang, Meyer & Ziogas (2009) for more discussion and justification for this procedure for handling the boundary conditions at $v = 0$ for stochastic volatility models.

The system of ODEs (19) and (20), after rearrangement, maybe cast into the generic first order system form

$$\frac{dC_m^n}{dS} = V_m^n, \tag{21}$$

$$\frac{dV_m^n}{dS} = A_m(S)C_m^n + B_m(S)V_m^n + P_m^n(S), \tag{22}$$

where $P_m^n(S)$ is a function of C_{m+1}^n, C_{m-1}^n, V_{m+1}^n, V_{m-1}^n, C_m^{n-1} and C_m^{n-2}. We solve the system (21)-(22) using the Riccati transform, full details of which are provided by Meyer (2009). Note that we are only able to apply the Riccati transform to the system (21)-(22) provided that both equations are treated as ODEs. We use an iterative technique in which the price (C_m^n) and the derivative (V_m^n) terms are updated until the price profile converges.

The Riccati transformation is given by

$$C_m^n(S) = R_m(S)V_m^n(S) + W_m^n(S), \tag{23}$$

where R and W are solutions to the initial value problems

$$\frac{dR_m}{dS} = 1 - B_m(S)R_m(S) - A_m(S)(R_m(S))^2, \quad R_m(0) = 0, \tag{24}$$

$$\frac{dW_m^n}{dS} = -A_m(S)R_m(S)W_m^n - R_m(S)P_m^n(S), \quad W_m^n(0) = 0, \tag{25}$$

Since R_m is independent of τ, we begin by solving (24) and storing the solution. The terminal condition of the final value ODE for V_m^n

$$\frac{dV_m^n}{dS} = A_m(S)(R_m(S)V_m^n + W_m^n(S)) + B_m(S)V_m^n + P_m^n(S), \tag{26}$$

will depend on the properties and the specifications of the barrier options:

- For continuously monitored barrier options without early exercise opportunities, we solve (25) for increasing values of S, ranging from $0 < S < S_{max}$. Using the fact that $C_m^n(H) = 0$ we obtain from (23) the terminal condition

$$V_m^n(H) = -\frac{W_m^n(H)}{R_m(H)}. \tag{27}$$

- For continuously monitored barrier options with early exercise opportunity [5], we solve (26) from the early exercise boundary point at which

$$V(b_m^n) = 1, \tag{28}$$

[5]Technically, for the knock-out event and the exercise date to be well defined, the option contract is defined in a way such that when the asset price first touches the barrier, the option holder has the option to either exercise or let the option be knocked out.

where we denote the free boundary by $S = b(v_m, \tau_n)$ which at grid point (v_m, τ_n) becomes $b(v_m, \tau_n) = b_m^n$. We solve (25) for increasing values of S, ranging from $0 < S < S_{\max}$, where we select $S_{\max}$ sufficiently large such that $S_{\max} > b_m^n$ will be guaranteed. We continue stepping forward in S, solving (25), until we encounter the value S^* such that

$$S^* - K = R_m(S^*) + W_m^n(S^*), \tag{29}$$

and thus $b_m^n = \min(S^*, H)$ is the value of the free boundary at grid point (v_m, τ_n)[6]. Once b_m^n has been determined we then solve (26) starting at $S = b_m^n$ and sweeping back to $S = 0$.

- For discretely monitored barrier options without early exercise features, the procedures to solve the PDE are similar to those for the continuously monitored counterpart, but we should change back to standard Euler backward time difference for a number of steps after each monitoring time and then switch to the second order scheme before the next monitoring time. The time difference in the Riccati equation should be adjusted in a similar manner as well.
- For discretely monitored options with early exercise features, we solve R from the Riccati equation (24) and solve W from the forward sweep (25) as usual. We find the free boundary point S^* in the standard way as for the continuously monitored option but let $b_m^n = \min\{S^*, H\}$ at each of the monitoring dates and update the corresponding option value as well. At the non-monitoring dates, we set $b_m^n = S^*$ as the early exercise boundary value which is used as the terminal value from which to work backward to solve equation (26) from $S = b_m^n$ to $S = 0$. In this case, we need to change back to the standard Euler backward time difference for a number of steps after each monitoring time and then switch to the second order scheme before the next monitoring time. The time difference in the Riccati equation should be adjusted in a similar manner.

In Figure 1 we illustrate one sweep through the grid points on the $v - \tau$ plane. In Figure 2 we show the stencil for the typical grid point in Figure 1, this essentially shows the grid point values of C that enter the right-hand side of (22). Figure 3 then illustrates the solution of (25) along a line in the S direction from a typical grid point in the $v - \tau$ plane.

[6]We remind the reader that at S^* the first of the free boundary conditions (28) becomes $V_m^n(S^*) = 1$.

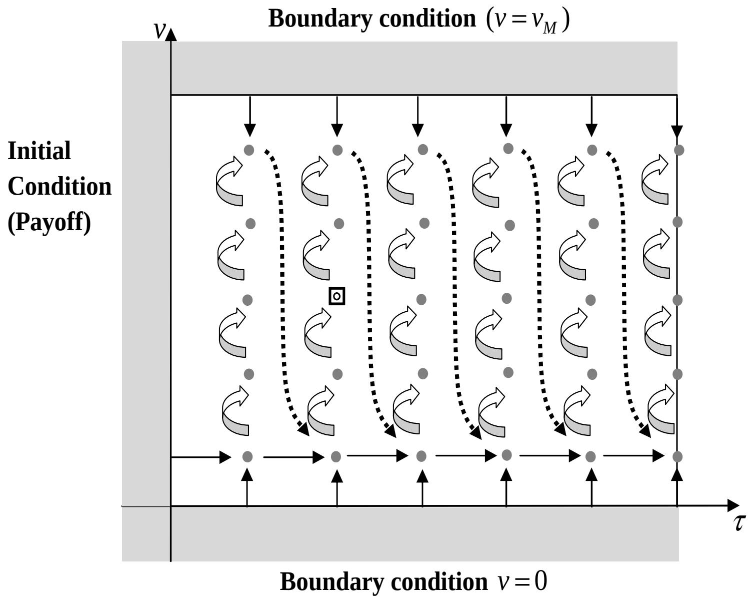

FIGURE 1. One sweep of the solution scheme on the $v-\tau$ grid. The stencil for the typical point $\boxed{\text{O}}$ is displayed in Figure 2.

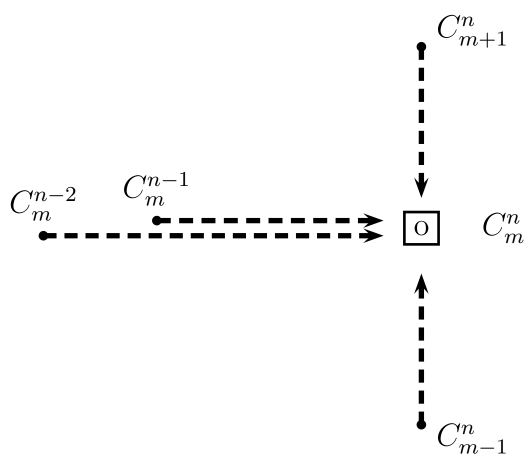

FIGURE 2. Stencil for the typical grid point $\boxed{\text{O}}$ of Figure 1. The stencil for C_m^n depends on $(C_{m-1}^n, C_m^n, C_{m+1}^n, C_m^{n-1}, C_m^{n-2})$.

4. Numerical Examples

To demonstrate the performance of method of lines outlined in Section 3 we implement the method for a given set of parameter values shown in Table 1, chosen to be

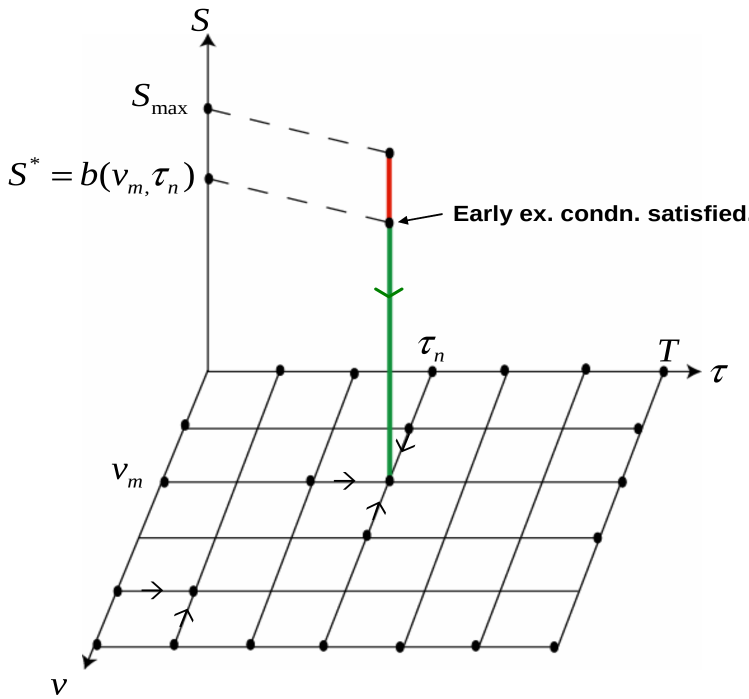

FIGURE 3. Solving for the option prices along a (v_m, τ_n) line.

consistent with the stochastic volatility parameters being used by Heston (1993) and which have been standard in many papers undertaking numerical studies of stochastic volatility models [7].

Also in order to check and benchmark the results and to demonstrate the performance of the MOL, we use several available methods, such as Finite Difference (FD) method (see Kluge (2002)), Fourier Cosine Expansion (COS) method (see Fang & Oosterlee (2009*a*) and Fang & Oosterlee (2009*b*)) together with the Monte Carlo Simulation method (see Ibáñez & Zapatero (2004)) to work out the prices of different kinds of the Barrier options to compare the prices from the MOL.

From Tables 2 − 9 we can see that the MOL is very efficient in producing the barrier option prices and it is also more important to note that the MOL produces hedge ratios, such as deltas, gammas to the same level of accuracy as the prices themselves. Figures 4 − 11 demonstrate that the MOL is able to produce both smooth option prices and option deltas which is a part of the solution we have after running the MOL.

[7]The source code for all methods was implemented using NAG Fortran with the IMSL library running on the UTS, Faculty of Business F&E HPC Linux Cluster which consists of 8 nodes running Red Hat Enterprise Linux 4.0 (64bit) with 2 × 3.33 GHz, 2 × 6 MB cache Quad Core Xeon X5470 Processors with 1333MHz FSB 8GB DDR2-667 RAM.

Parameter	Value	SV Parameter	Value
r	0.03	θ	0.1
q	0.05	κ_v	2.00
T	0.5	σ	0.1
K	100	λ_v	0.00
ρ	± 0.50	H	130

TABLE 1. Parameter values used for the barrier option. The stochastic volatility (SV) parameters are those used in Heston's original paper.

In fact, Tables $2 - 9$ show that

- the prices of continuously monitored European up-and-call option produced from the MOL are close to those prices generated from the finite difference method but the MOL provides better hedge ratios;
- the prices of discretely monitored European up-and-call option produced from the MOL are close to those prices generated from the Fourier Cosine Expansion method but the MOL is more efficient since the runtime of COS method shown in Tables 6 and 8 are the time to produce only 5 prices while the runtime of the MOL is the time to have prices of all grid points;
- the prices of both continuously and discretely monitored American up-and-call option produced from the MOL are close to those prices generated from the Monte Carlo simulation[8] which ran considerably longer than the MOL.

$\rho = -0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL (50,100,1140)	0.9045	1.8807	2.5978	2.4859	1.4858	9
MOL (100,200,6400)	0.9044	1.8781	2.5908	2.4769	1.4782	268
FD $(200, 100, 200)$	0.9029	1.8778	2.5903	2.4760	1.4775	162
MC $(400, 20, 1,000,000)$	0.9355	1.9579	2.7407	2.6706	1.6773	485
MC upper bound	0.9389	1.9628	2.7464	2.6762	1.6820	
MC lower bound	0.9321	1.9530	2.7351	2.6649	1.6726	

TABLE 2. Prices of the continuously monitored barrier option without early exercise features computed using method of lines (MOL), finite difference (FD) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

[8]The specification of each Monte Carlo simulation in the tables are the numbers in the parenthesis after MC which mean (No. of time steps, No. of volatility levels, No. of simulations) for the options without early exercise opportunities and (No. of time steps, No. of volatility levels, No. of early exercise opportunities, No. of simulations) for the options with early exercise opportunities, respectively.

$\rho = -0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL (50,150,1140)	1.4009	3.9350	8.2981	14.4015	21.8229	26
MOL (100,200,2440)	1.4012	3.9364	8.3003	14.4033	21.8219	123
MOL (100,200,6400)	1.4012	3.9363	8.3003	14.4032	21.8216	318
MOL (200,400,9100)	1.4015	3.9371	8.3014	14.4037	21.8201	2668
MC $(100, 20, 50, 1,000,000)$	1.3994	3.9238	8.2302	14.1086	20.9401	155600
MC upper bound	1.4058	3.9347	8.2454	14.1261	20.9568	
MC lower bound	1.3930	3.9129	8.2151	14.0909	20.9234	

Table 3. Prices of the continuously monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

$\rho = 0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL (50,100,1140)	0.8397	1.6226	2.2501	2.2539	1.4371	9
MOL (100,200,6400)	0.8387	1.6200	2.2452	2.2472	1.4303	270
FD $(200, 100, 200)$	0.8375	1.6200	2.2452	2.2472	1.4300	160
MC $(400, 20, 1,000,000)$	0.8683	1.6958	2.3771	2.4225	1.6089	536
MC upper bound	0.8729	1.7022	2.3846	2.4301	1.6154	
MC lower bound	0.8636	1.6894	2.3696	2.4149	1.6025	

Table 4. Prices of the continuously monitored barrier option without early exercise features computed using method of lines (MOL), finite difference (FD) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = 0.50$ and $v = 0.1$.

5. Impact of Stochastic Volatility on the prices of the barrier option

In this section, we explore the impact of stochastic volatility on the price profiles of Barrier options with various features. We consider two models for the underlying asset price: (i) the geometric Brownian motion (GBM) model of Black & Scholes (1973) and Merton (1973); (ii) the stochastic volatility (SV) model of Heston (1993). Here we aim to observe the impact that stochastic volatility has on the shape of the price profile, where the variance of S is consistent for both models.

Setting the spot variance to $v = 0.1$ (corresponding to a volatility - standard deviation - of 33%) in the SV model, we determine the time-averaged variance s^2 for $\ln S$ over the life of the option by using the characteristic function for the marginal density of $x = \ln S$ given in Cheang, Chiarella & Ziogas (2009).

$\rho = 0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL (50,150,1140)	1.6147	4.1178	8.3417	14.2937	21.6674	33
MOL (100,200,2440)	1.6153	4.1193	8.3438	14.2954	21.6672	125
MOL (100,200,6400)	1.6153	4.1192	8.3438	14.2953	21.6670	311
MOL (200,300,8100)	1.6156	4.1199	8.3447	14.2959	21.6662	1252
MC $(100, 20, 50, 1,000,000)$	1.6147	4.0763	8.2146	13.9252	20.8682	712578
MC upper bound	1.6201	4.0844	8.2259	13.9378	20.8800	
MC lower bound	1.6093	4.0682	8.2033	13.9126	20.8563	

Table 5. Prices of the continuously monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = 0.50$ and $v = 0.1$.

$\rho = -0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL(50,100,1140)	1.0764	2.5173	4.0895	4.9894	4.8291	10
MOL (100,200,6400)	1.0807	2.5289	4.1116	5.0235	4.8706	301
COS $(100, 200, 100)$	1.0809	2.4871	4.0454	4.9779	4.8646	498
MC $(400, 20, 1,000,000)$	1.0780	2.5257	4.1033	5.0166	4.8605	510
MC upper bound	1.0834	2.5339	4.1135	5.0279	4.8718	
MC lower bound	1.0726	2.5175	4.0930	5.0054	4.8492	

Table 6. Prices of the discretely monitored barrier option without early exercise features computed using method of lines (MOL), Fourier Cosine expansion (COS) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

$\rho = -0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL(50,100,1140)	1.4008	3.9339	8.3010	14.4446	22.0389	11
MOL (100,250,2400)	1.4012	3.9364	8.3025	14.4182	21.8719	204
MOL (150,250,6400)	1.4014	3.9368	8.3028	14.4157	21.8615	622
MC $(100, 20, 50, 1,000,000)$	1.4002	3.9338	8.2967	14.4285	21.9274	155179
MC upper bound	1.4066	3.9449	8.3123	14.4473	21.9459	
MC lower bound	1.3938	3.9228	8.2810	14.4097	21.9089	

Table 7. Prices of the discretely monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = -0.50$ and $v = 0.1$.

$\rho = 0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL(50,100,1140)	1.0889	2.3442	3.7654	4.7280	4.7939	10
MOL (100,200,6400)	1.0935	2.3554	3.7867	4.7612	4.8355	305
COS $(100, 200, 100)$	1.0881	2.3784	3.8342	4.8015	4.8380	511
MC $(400, 20, 1,000,000)$	1.0995	2.3556	3.7860	4.7406	4.8087	513
MC upper bound	1.1050	2.3636	3.7959	4.7515	4.8199	
MC lower bound	1.0919	2.3476	3.7760	4.7297	4.7976	

TABLE 8. Prices of the discretely monitored barrier option without early exercise features computed using method of lines (MOL), Fourier Cosine expansion (COS) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = 0.50$ and $v = 0.1$.

$\rho = 0.50, v = 0.1$			S			Runtime
Method (N, M, S_{pts})	80	90	100	110	120	(sec)
MOL(50,100,1140)	1.6157	4.1226	8.3647	14.3766	21.9039	12
MOL (100,250,2400)	1.6162	4.1249	8.3688	14.3813	21.9079	217
MOL (150,250,6400)	1.6164	4.1254	8.3696	14.3822	21.9086	598
MC $(100, 20, 50, 1,000,000)$	1.6148	4.1250	8.3540	14.3172	21.7583	155311
MC upper bound	1.6222	4.1371	8.3704	14.3653	21.7771	
MC lower bound	1.6073	4.1130	8.3376	14.2980	21.7395	

TABLE 9. Prices of the discretely monitored barrier option with early exercise features computed using method of lines (MOL) and Monte Carlo simulation (MC). Parameter values are given in Table 1, with $\rho = 0.50$ and $v = 0.1$.

By requiring that s^2 be equal for both the models, we then determine the necessary parameter volatility σ for the BGM to ensure that they both have consistent variance over the time period of interest. To match the time-averaged variance for the GBM and SV models for a 6-month option, the global volatilities, s, are 31.48% for $\rho = 0.50$, and 31.80% for $\rho = -0.50$. The value of v in the SV model is 10%. Hence, the constant volatility σ in GBM is chosen to be 31.48% for $\rho = 0.50$, and 31.80% for $\rho = -0.50$ in all the following comparisons.

Figures (12) and (13) demonstrate the difference between the continuously monitored European up-and-out call option prices under Heston stochastic volatility model and those option prices under the standard Geometric Brownian Motion.

Figures (14) and (15) demonstrate the difference between the discretely monitored European up-and-out call option prices under Heston stochastic volatility model and those option prices under the standard Geometric Brownian Motion.

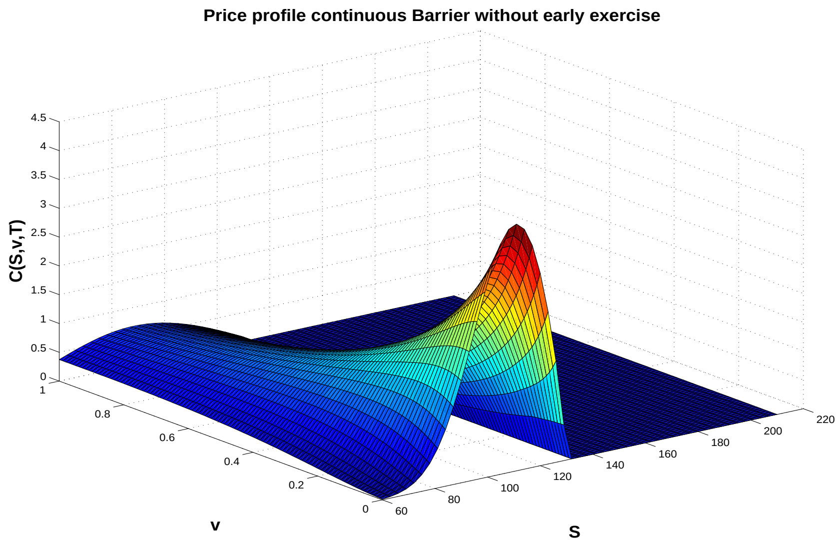

FIGURE 4. Price profile of a continuously monitored up-and-out call option without early exercise opportunities.

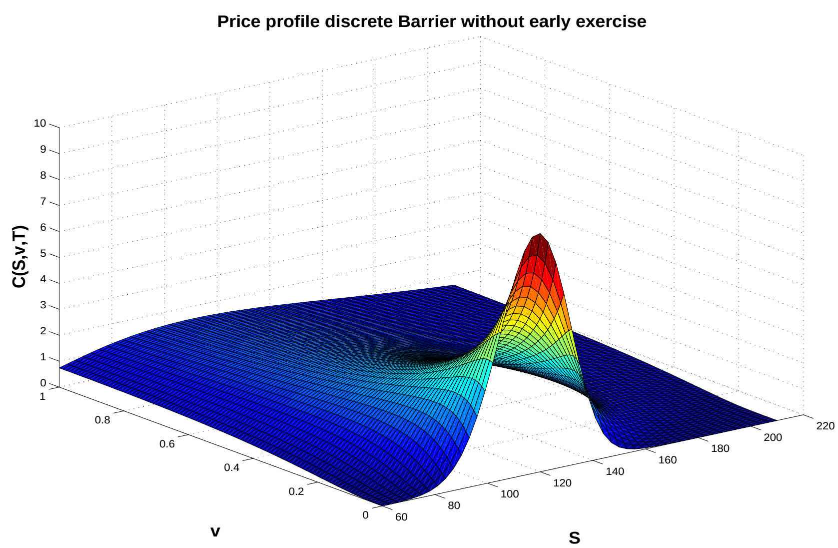

FIGURE 5. Price profile of a discretely monitored up-and-out call option without early exercise opportunities.

Figures (16) and (17) demonstrate the difference between the continuously monitored American up-and-out call option prices under Heston stochastic volatility model and those option prices under the standard Geometric Brownian Motion.

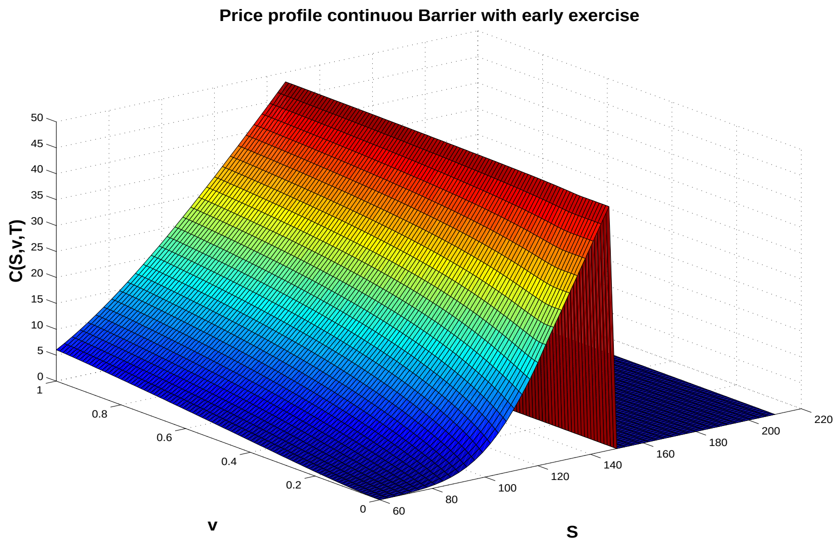

FIGURE 6. Price profile of a continuously monitored up-and-out call option with early exercise opportunities.

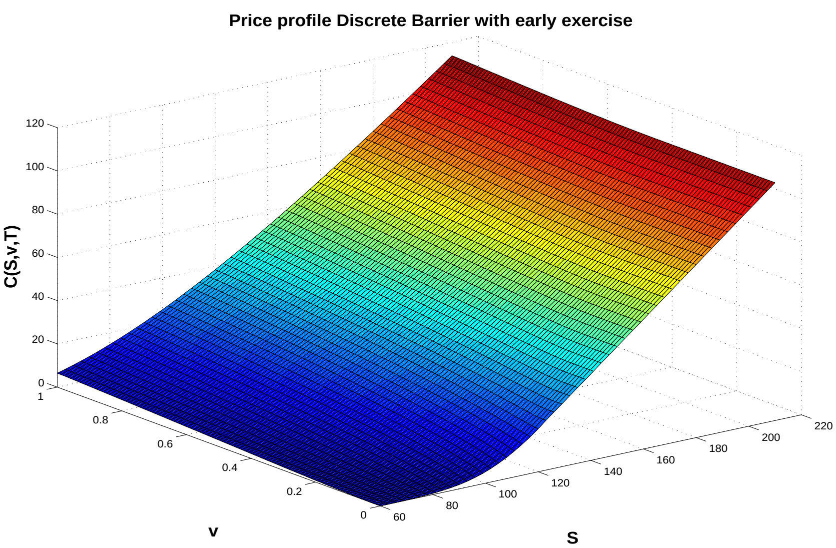

FIGURE 7. Price profile of a discretely monitored up-and-out call option with early exercise opportunities.

Figures (18) and (19) demonstrate the difference between the discretely monitored American up-and-out call option prices under Heston stochastic volatility model and those option prices under the standard Geometric Brownian Motion.

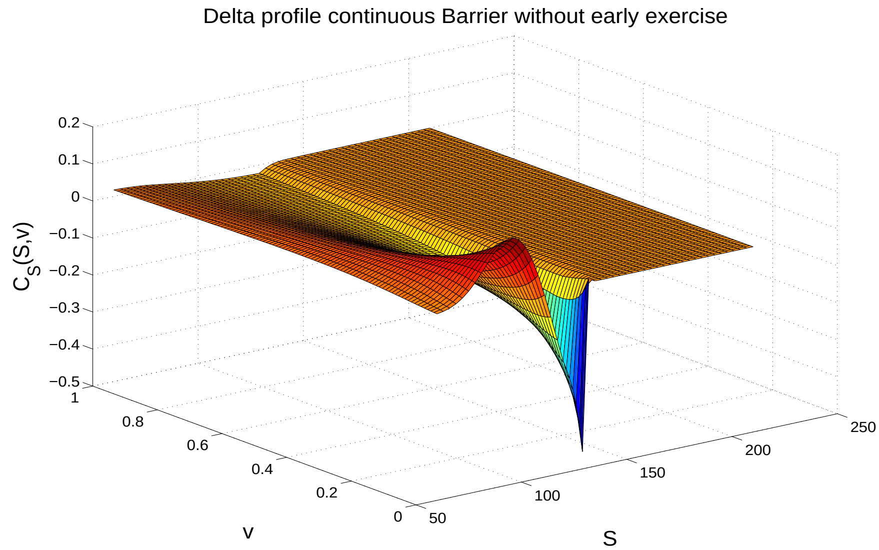

FIGURE 8. Delta profile of a continuous monitoring up-and-out call option without early exercise opportunities.

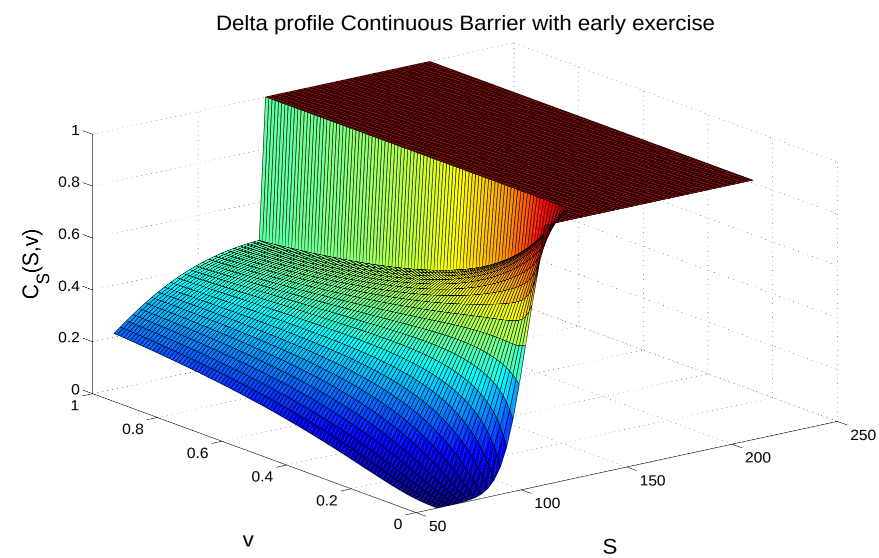

FIGURE 9. Delta profile of an up-and-out call option with early exercise opportunities.

6. Conclusion

We have studied the pricing of Barrier options under stochastic volatility using the Method of Lines. We also provide the Barrier option pricing results from Finite Difference method, Fourier Cosine Expansion method and Monte Carlo Simulation approach as benchmarks to the MOL.

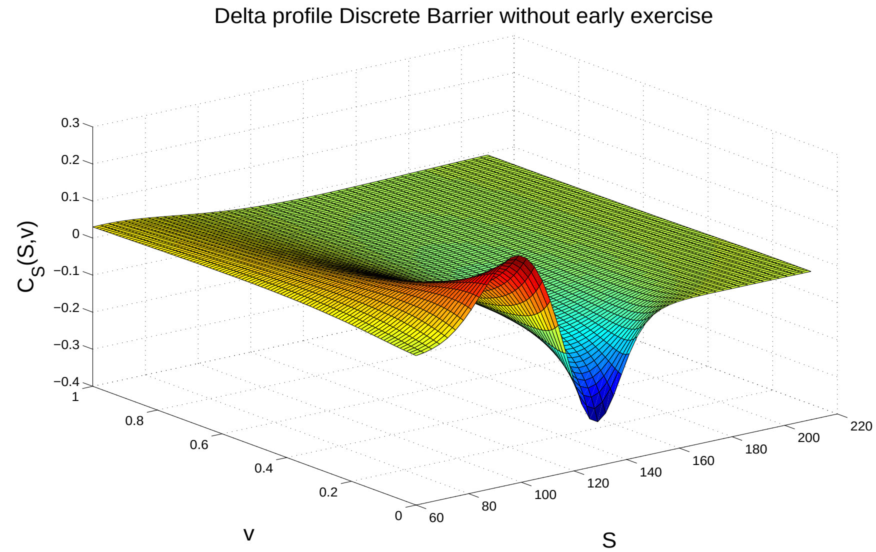

FIGURE 10. Delta profile of a discrete monitoring up-and-out call option without early exercise opportunities.

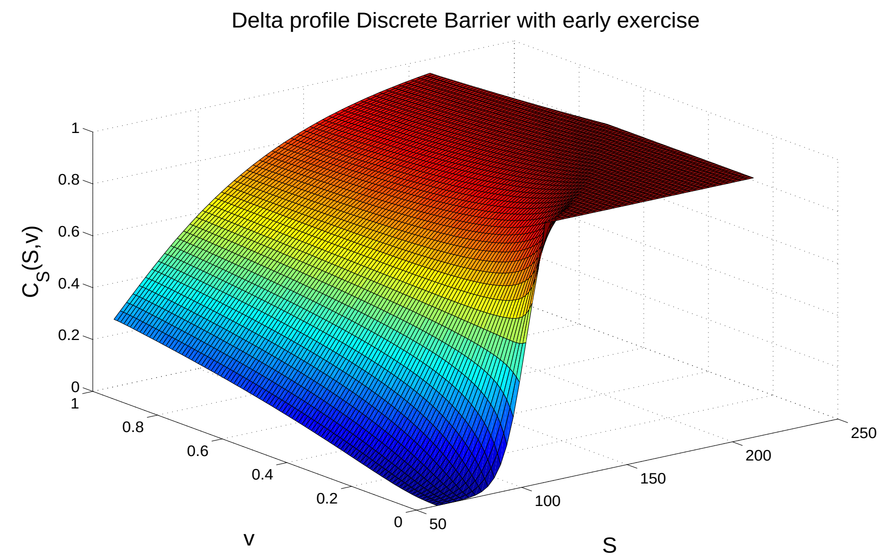

FIGURE 11. Delta profile of a discrete monitoring up-and-out call option with early exercise opportunities.

It turns out that the MOL is able to handle both continuously and discretely monitored options with or without early exercise opportunities. Hence we believe this provides a unified framework to efficiently price various kinds of Barrier options with different kinds of properties. One main advantage of the MOL is that it produces the hedge ratios of the option, namely the deltas and gammas, to the same accuracy as the prices themselves within the same time frame.

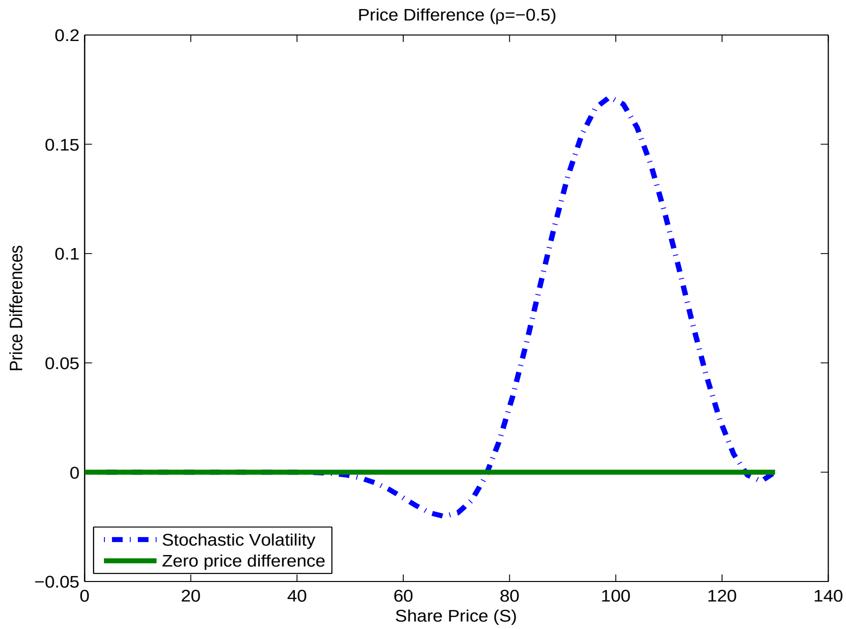

FIGURE 12. The effect of stochastic volatility on continuously monitored European up-and-out call (UOC) option. The correlation is $\rho = -0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 2.4197.

In future research, the knock-in option under stochastic volatility with early exercise features should be further investigated.

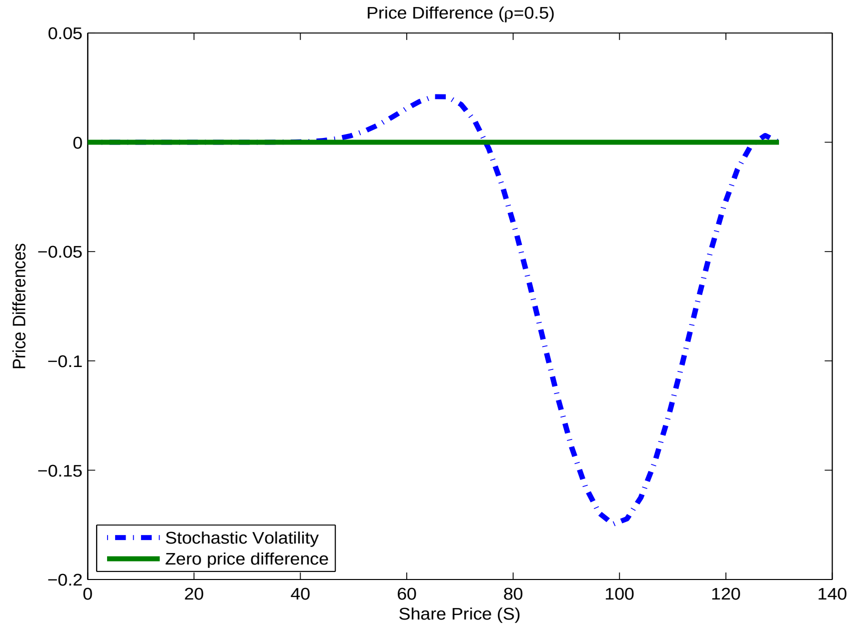

FIGURE 13. The effect of stochastic volatility on continuously monitored European up-and-out call option. The correlation is $\rho = 0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 2.4197.

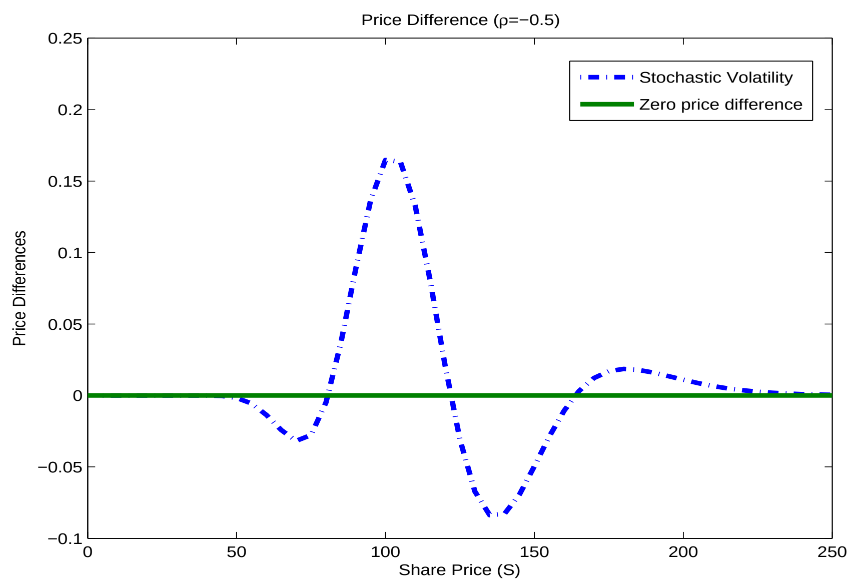

FIGURE 14. The effect of stochastic volatility on discretely monitored European up-and-out call option. The correlation is $\rho = -0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 3.9487.

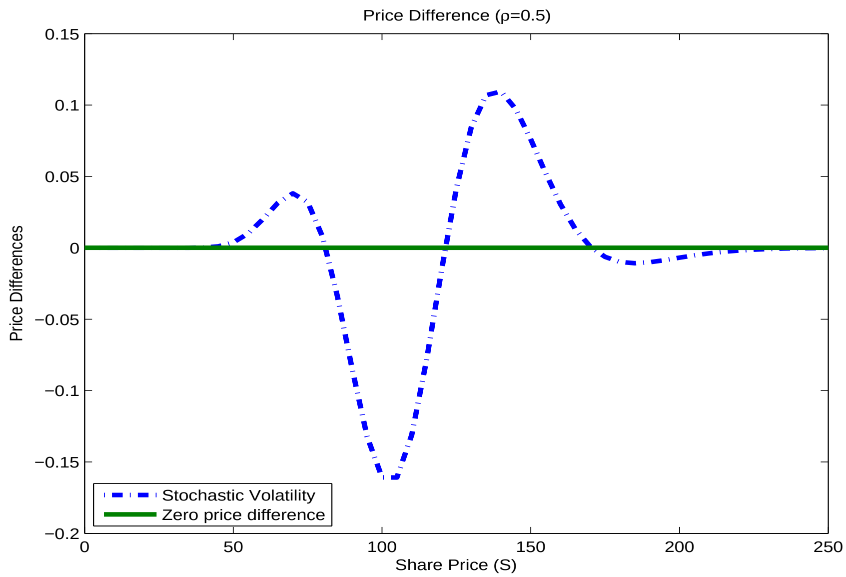

FIGURE 15. The effect of stochastic volatility on discretely monitored European up-and-out call option. The correlation is $\rho = 0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 3.9487.

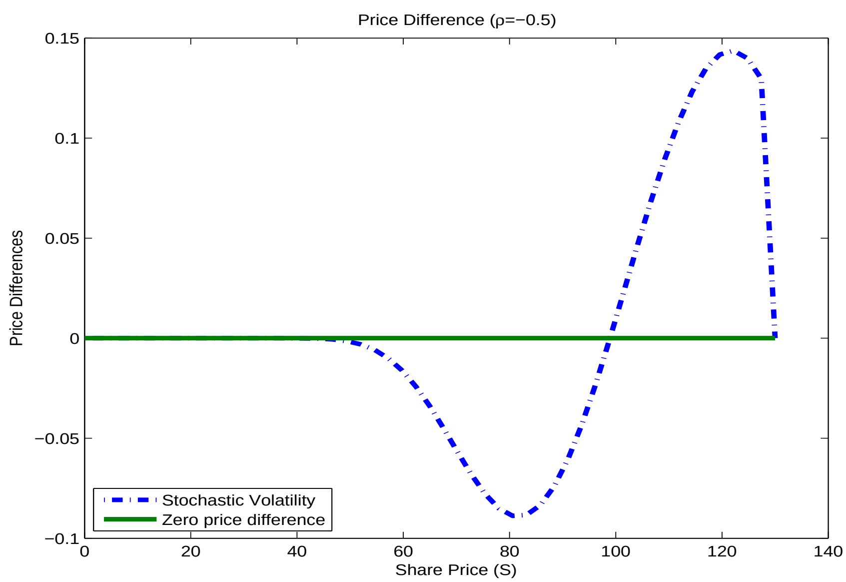

FIGURE 16. The effect of stochastic volatility on the continuously monitored American up-and-out call option. The correlation is $\rho = -0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 8.2917.

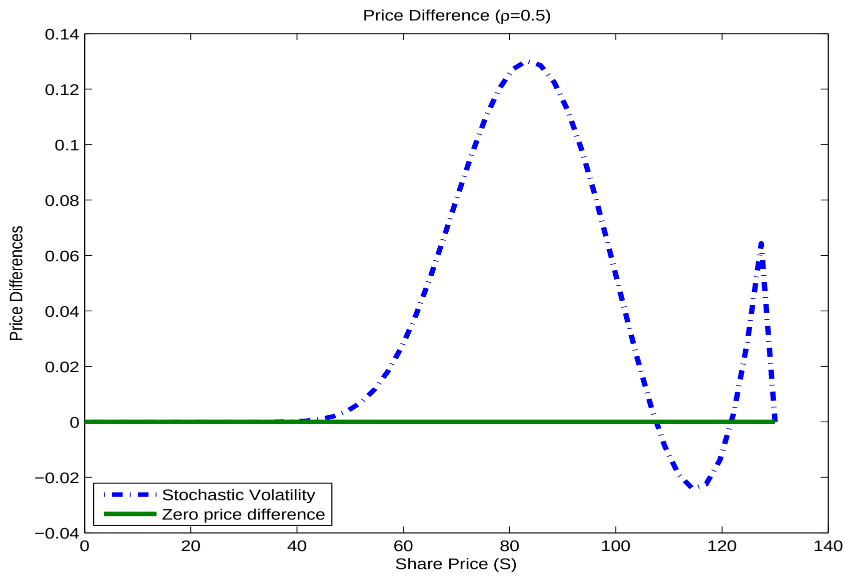

FIGURE 17. The effect of stochastic volatility on the continuously monitored American up-and-out call option. The correlation is $\rho = 0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 8.2917.

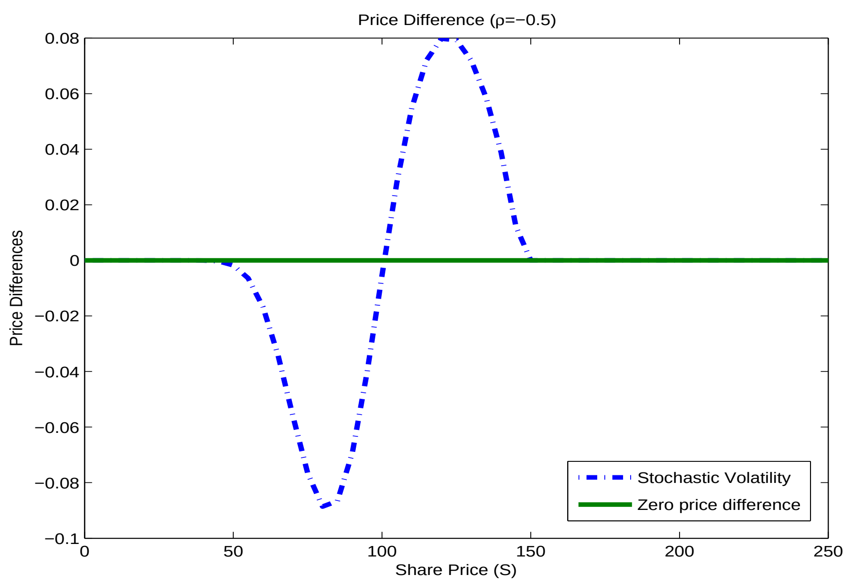

FIGURE 18. The effect of stochastic volatility on discretely monitored American up-and-out call option. The correlation is $\rho = -0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 8.3125.

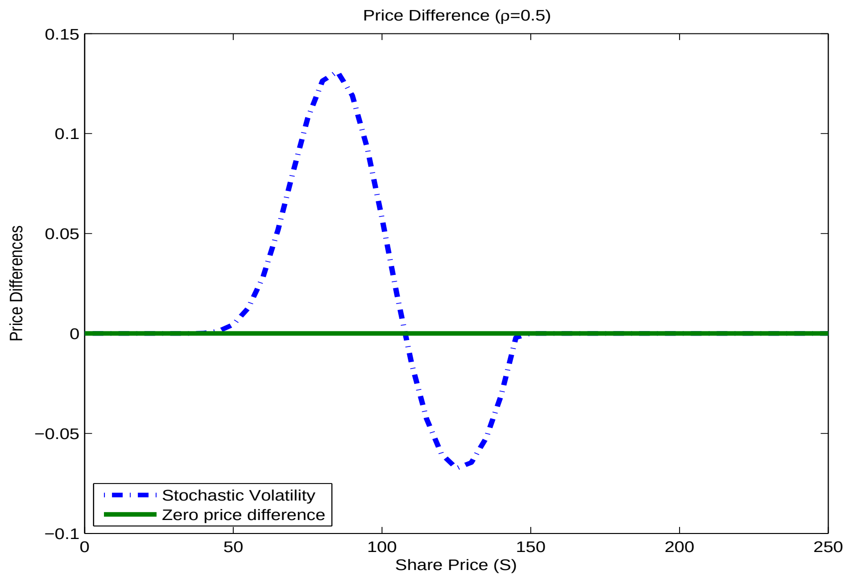

FIGURE 19. The effect of stochastic volatility on discretely monitored American up-and-out call option. The correlation is $\rho = 0.5$ and all other parameter values are as listed in Table 1. The at the money UOC price under GBM is 8.3125.

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