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DIFFERENCES IN OPINION AND RISK PREMIUM

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ABSTRACT. When people agree to disagree, this paper examines the impact of the disagreement among agents on market equilibrium and equity premium. Within the standard mean variance framework, we consider a market of two risky assets, a riskless asset and two (and then a continuum of) agents who have different preferences and heterogeneous beliefs in the means and variance/covariances of the asset returns. By constructing a consensus belief, we introduce a boundedly rational equilibrium (BRE) to characterize the market equilibrium and derive a CAPM under heterogeneous beliefs. When the differences in opinion are formed as mean-preserving spreads of a benchmark homogeneous belief, we analyze explicitly the impact on the market equilibrium and risk premium, showing that the risk tolerance, optimism/pessimism and confidence/doubt can jointly generate high risk premium and low risk-free rate.

JEL Classification: G12, D84.
Keywords: Asset prices; heterogeneous beliefs; boundedly rational equilibrium; zero-beta CAPM; risk premium.

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1. INTRODUCTION

To better explain market anomalies, puzzles and various market phenomena, economics and finance are witnessing an important paradigm shift, from a representative, rational agent approach towards a behavioral, agent-based approach in which economy and markets are populated with bounded rational agents who have heterogeneous beliefs (Conlisk (1996)). With common beliefs and perfect information, investors hold identical portfolios and trade mainly to balance portfolios, which contradicts the extremely large trading volume across stock markets. Recent literature (see, for example, Hong and Stein (2007), Berraday (2009)) have indicated that heterogeneity among agents plays an important role in explaining high volatility in asset price and high trading volume. In the literature, heterogeneity is used to characterize either asymmetric information or differences in opinion on the same information. In the first case, agents are in general Bayesians who hold the same prior probability but with asymmetric private information (Williams (1977), Grossman and Stiglitz (1980), Diamond and Verrecchia (1981), Kyle (1985), Grundy and McNichols (1989), Wang (1994), Detemple and Murthy (1994), Zapatero (1998), Angeletos and Werning (2006), Allen, Morris and Shin (2006), Weitzmann (2007), and Bakshia and Skoulakisb (2008)). In this framework, the trading volume driven by the portfolio rebalancing and unanticipated liquidity shocks seems to be far too small to account for the large trades observed in the financial markets. This suggests the second case as a more promising one in which the heterogeneity in beliefs is not due to asymmetric information but rather to intrinsic differences in how to view the world, that is people agree to disagree. The following discussion focuses on the second view about the heterogeneity.

Literature has made a significant contribution to the understanding of the market aggregation and the impact of heterogeneous beliefs amongst agents on market equilibrium (see, for example, Lintner (1969), Rubinstein (1976), Abel (1989, 2002), and more recently Calvet, Grandmont and Lemaire (2004), Wenzelburger (2004), Böhm and Chiarella (2005), Böhm and Wenzelburger (2005), Jouni and Napp (2006, 2007), Sharpe (2007), Gollier (2007), Chiarella, Deici and He (2010, 2010) and Horst and Wenzelburger (2008)). When agents have different opinions, the heterogeneity in beliefs among agents is often characterized by notions of different risk tolerance, optimism/pessimism, and confidence/doubt. Cao and Ou-Yang (2009) made clear that disagreement about the mean of future asset payoffs capture the investors' conditional optimism and pessimism about the asset value while disagreement about the precision of future asset payoffs capture the heterogeneity of the investors' confidence level in the payoffs. Along this line, some theoretical models have been developed to explain equity premium and risk-free rate puzzles (Abel (2002), Jouni and Napp

(2006, 2007), and Gollier (2007)). More recently, we also see some empirical studies on optimism/pessimism and confidence/doubt (Hvide (2002), Giordani and Soderlind (2006)), the heterogeneity as a pricing factor (Anderson, Ghysels and Juergens (2005)), and the stochastic discount factor to explain the equity premium and risk-free rate puzzles (Weitzmann (2007) and Bakshia and Skoulakisb (2008)). However, the theoretical implications are either inconsistent with empirical findings or not able to provide significant explanation on the empirical observations.

There are mainly two driving forces for the development of the literature in heterogeneous beliefs, one is to explain equity premium and risk-free rate puzzles (Mehra and Prescott (1985) and Weil (1989)), to which several theoretic explanations have been proposed recently, and the other is on the survival of irrational investors and their price impact (Friedman (1953), DeLong, Shleifer, Summers and Waldmann (1991), Sandroni (2000), Blume and Easley (2006), Kogan, Ross, Wang and Westerfield (2006), Dumas, Kurshev and Uppal (2009)). For example, Barberis, Huang and Santos (2001) adopt a non-standard utility function, motivated by prospect theory; Benartzi and Thaler (1995) consider myopic loss aversion. The notion of overconfidence has been explored in finance literature (DeLong, Shleifer, Summers and Waldmann (1990), Kyle and Wang (1997), and Daniel, Hirshleifer and Subrahmanyam (1998)). Several empirical studies of professionals' economic forecasts and psychological surveys indicate that agents have optimism and overconfidence[1] about their own (relative) abilities (Rabin (1998), Hirshleifer (2001), and Giordani and Soderlind (2006)). Deviating from rational expectation in the standard neoclassical paradigm, when beliefs are exogenously given, it has been found (Detemple and Murthy (1994), Abel (2002), Calvet et al. (2004), Jouini and Napp (2006), and Gollier (2007)) that a pessimistic bias and doubt in the subjective distribution of the growth rate of consumption and a positive correlation between risk tolerance and pessimism (doubt) leads to an increase of the market price of risk. To discipline the heterogeneity in beliefs and to understand how agents form their beliefs differently, in a static Nash equilibrium (Kyle (1989)) set up of two agents model when agents hold incorrect but strategic beliefs, Jouini and Napp (2009) provide a discipline for belief formation through a model of subjective beliefs

[1]As elaborated in Hvide (2002), experimental psychologic literature have applied, somewhat confusingly, two distinct meanings of the term 'overconfidence', overconfidence$_1$ and overconfidence$_2$ (called in Hvide (2002)). In the stock market, overconfidence$_1$ relates to a skewed first moment of a subjective probability distribution, while overconfidence$_2$ relates to a skewed second moment of a subjective probability distribution. In general, there is no clear relation between overconfidence$_1$ and overconfidence$_2$ since they reflect different underlying phenomena. In Abel (2002), a uniform pessimism is defined as (the subjective distribution being) a leftward translation of the objective distribution, doubt as a mean-preserving spread of the objective distribution. To avoid confusion, in our discussion, we adopt the notions of Abel (2002) and refer overconfidence$_1$ and overconfidence$_2$ to optimism and (over)confidence, respectively. The confidence in DeLong et al. (1990) and Kyle and Wang (1997) is actually referred to overconfidence$_2$.

in order to provide a rationale for belief heterogeneity. They find that optimism (confidence) as well as pessimism (doubt) emerge as optimal beliefs of agents' strategic behavior and there is a positive correlation between pessimism (doubt) and risk tolerance. This strategic explanation of heterogeneous beliefs is in contrast with rational approach to beliefs where agents try to reflect the 'world as it is' in their beliefs, and with approach in which forward-looking agents optimally distort beliefs and beliefs are of intrinsic value to agents, as with wishful thinking or fear of disappointment (see Brunnermeier and Parker (2005)).

Following the literature on the survival of irrational investors and their impact on the market, Friedman (1953) argued that irrational investors will consistently lose money and be driven out of the market by rational investors and therefore they have no price impact. Based on a partial equilibrium model, DeLong et al. (1991) suggest that traders with wrong beliefs may hold portfolios with higher growth rates and therefore can eventually outgrow the rational traders and survive in the long-run. In contrast, Sandroni (2000) and Blume and Easley (2006) use a general equilibrium approach with intermediate consumption and show that irrational traders do not survive in the long run. When investors only care about their terminal consumption and irrational investors have constant belief about the drift of the endowment process, Kogan et al. (2006) demonstrate that survival and price impact are two independent concepts. They show that survival is not a necessary condition for the irrational trader to influence long-run prices. Dumas et al. (2009) propose a general equilibrium difference-of-opinion sentiment model with overconfident agents overreacting to public signals but intertemporally optimizing, causing excessive volatility. It is often found in this literature that, irrational agents may not survive, but they become extinct after long time (in hundreds of years). Therefore, they can have impact on the market before becoming extinction.

Recently, some empirical tests on optimism/pessimism and confidence/doubt have been conducted. By introducing a concept of pragmatic beliefs, Hvide (2002) uses a simple game-theoretic example of a job market and shows optimism can be the equilibrium outcome if agents form beliefs pragmatically. The main justification for programmatic beliefs is dynamics in the sense that, without awareness about their own optimism, agents are gradually learning that a certain way of forming beliefs is more rewarding than other ways. Also, by quantifying the amount of pessimism and doubt in survey data on US consumption and income, Giordani and Soderlind (2006) find some evidence of pessimism, but individual forecasters clearly exhibit overconfidence rather than doubt. By showing that the average distribution shows no statistically significant sign of either overconfidence or doubt, they conclude that doubt is not a promising explanation of the equity premium puzzle and the amount of pessimism provides only a rather small improvement in the empirical performance of the model. To examine whether heterogeneous beliefs are a priced factor in traditional asset pricing models, Anderson et al. (2005) develop a new empirical measures based on the disagreement among analysts about expected earnings. They find that the inclusion of this factor does improve the fit of the factor models, especially for small firms. However, on average, dispersion only captures 9 to 26 basis points of excess return and therefore a factor for dispersion cannot fully compensate for the explanatory power of fundamental factors. To assess the impact of "structural uncertainty" about the volatility of consumption growth, Weitzmann (2007) and Bakshia and Skoulakisb (2008) propose a Bayesian learning model, resulting a well-specified stochastic discount factor and tractable solutions for equity premium and risk-free rate. However, to explaining the puzzles, the model requires implausible prior beliefs well outside typical calibration ranges for historical volatility of consumption growth and risk aversion.

It can be seen from the above discussion that most of the theoretical models that relate heterogeneous beliefs to equity premium and the risk-free rate assume that investors know the payoff of the risky assets in each state of the world, but disagree on the probability of each state occurring. This is of course a convenient way to formulate the problem because it limits the dimension of heterogeneity, however, in reality investors may need to make prediction about the entire joint probability distribution of asset returns, this complicates the problem immensely. To simplify the analysis, we consider in this paper a static financial market with two risky assets, one risk-free asset, and two agents with different risk preferences have heterogeneous beliefs about the joint probability distribution of asset returns. We also extend the analysis to allow a continuum of investors. We assume that both agents agree on the expected return and risk (measured by the standard deviation) for one risky asset but disagree on that of the other risky asset, they may also disagree on the correlation coefficient of the returns of the two assets. It is a common belief that when investors are "on average" unbiased, in aggregate the biases may 'cancel out' and hence have no effect on equilibrium prices. We will examine to what extent this statement is true. We impose heterogeneous beliefs as biased beliefs characterized by mean-preserving spreads of a benchmark homogeneous belief such that investors are on average unbiased. By assuming that agents maximize a primitive utility function (Sharpe (1991) and Levy and Markowitz (1979)), agents choose their optimal portfolios based on their subjective beliefs and risk tolerance. By constructing a consensus belief, the market equilibrium is characterized by the consensus belief. Different from the standard rational expectation equilibrium, the market equilibrium under the consensus belief reflects the bounded rationality of the agents in the sense that the market equilibrium is achieved when agents make their optimal decision based on their subjective beliefs. We call such equilibrium as a *boundedly rational equilibrium*, when the impact of investors' biased beliefs do "cancel out", the consensus belief conforms to the average belief which is assumed to be unbiased. We show that the "cancel out" effect holds when the different aspects of the heterogeneity, including the risk tolerance, optimism/pessemism and confidence/doubt, are uncorrelated. However, they do not cancel out when different aspects of heterogeneity are correlated and have significant effect on the endogenous variables such as the market risk premium (equity premium), risk-free rate, market volatility and the portfolio weights of the market portfolio. This paper aims to improve our understanding of the impact of differences in opinion on the market equilibrium, in particular, the market risk premium and the risk-free rate (rather than trying to address the equity premium and risk-free rate puzzles in the standard setup of maximization of intertemporal utility of consumption). It will become clear in our analysis that the impact of heterogeneity on market equilibrium with two risky assets is significantly different to the case with one risky asset. For example, when the more risk tolerant investor is less optimistic about the future return for one of the risky asset, the market indeed becomes pessimistic about the asset's future return, consistent with the findings in Jouni and Napp (2006, 2007). However, we show that this does not necessarily imply a higher market risk premium and lower risk-free rate as one would expected in the case with only one risky asset. The additional dimension of heterogeneity induced by having two correlated risky assets plays an important role on the market risk premium and the risk-free rate when combining with biases in expected asset returns.

The paper is structured as follows. In Section 2, we set up the economy and describe the aggregation problem when agents have heterogeneous preferences and beliefs. We show how the different risk tolerance and heterogeneous beliefs can be aggregated through a consensus belief in market equilibrium. In particular, we derive a CAPM under heterogeneous beliefs. In Section 3, as a benchmark of our analysis, we include the traditional CAPM under the homogeneous belief. In Section 4, by introducing different risk preference and biased beliefs among two agents, we examine the joint impact of heterogeneity on the equity premium and risk-free rate in market equilibrium both analytically and numerically. In particular, we explore the conditions on the biased beliefs to achieve high equity premium and low risk-free rate. The analysis with two agents is then extended to a continuum of agents. The paper concludes in Section 5.

2. Heterogeneous Beliefs and Boundedly Rational Equilibrium

In this section, we first set up the stylized economy with heterogeneous beliefs and then characterize the market equilibrium.

2.1. **The Economy.**

We consider a two-date economy in which there are two risky assets, indexed by $j = 1, 2$ and a riskless asset, furthermore there are two agents[2], indexed by $i = 1, 2$. Assume agents have different preferences and the end-of-period

[2]The discussion and results of this section for the general economy of many risky assets and many heterogeneous beliefs can be found in He and Shi (2009).

return of asset $j(j = 1,2)$ is denoted by $\tilde{r}_j$, return of the riskless asset is denoted by r_f. The probability distribution of the returns of the two risky assets are assumed to be jointly normal. Agents have heterogeneous beliefs in the expected returns and variance/covariances of asset returns. For agent $i\,(i = 1,2)$, let τ_i be his/her risk tolerance, and

$$\boldsymbol{\mu}_i = \begin{pmatrix} \mu_{i,1} \\ \mu_{i,2} \end{pmatrix} \qquad \text{and} \qquad V_i = \begin{pmatrix} \sigma_{i,1}^2 & \rho_i\sigma_{i,1}\sigma_{i,2} \\ \rho_i\sigma_{i,1}\sigma_{i,2} & \sigma_{i,2}^2 \end{pmatrix}$$

be his/her beliefs in the means and covariance matrix, respectively, where

$$\mu_{i,j} = 1 + \mathbb{E}_i(\tilde{r}_j), \qquad \sigma_{i,j}^2 = Var_i(\tilde{r}_j), \qquad \rho_i = Correl_i(\tilde{r}_1, \tilde{r}_2)$$

for $i, j = 1, 2$. We use $\mathcal{B}_i := (\mu_{i,1}, \mu_{i,2}, \sigma_{i,1}, \sigma_{i,2}, \rho_i)$ to denote the beliefs of agent i.

2.2. **Optimal Portfolio Problem.** The terminal wealth[3] of agent i is given by

$$\tilde{W}_i = W_{i0}(1 + \tilde{r}_{i,p}),$$

where $\tilde{r}_{i,p} = ((1 + r_f)(1 - \mathbf{1}^T\boldsymbol{\pi}_i) + \boldsymbol{\pi}_i^T(\mathbf{1} + \tilde{\mathbf{r}}))$ is the random return of agent i's portfolio, $\boldsymbol{\pi}_i = (\pi_{i,1}, \pi_{i,2})^T$ is the vector of portfolio weights (proportion of wealth invested in each risky asset) and $\mathbf{1} = (1,1)^T$. We assume that investors maximize a primitive utility function

$$U_i(\tilde{r}_{i,p}) = \mathbb{E}_i(\tilde{r}_{i,p}) - \frac{1}{2\tau_i}Var_i(\tilde{r}_{i,p}) = (1 + r_f)(1 - \mathbf{1}^T\boldsymbol{\pi}_i) + \boldsymbol{\pi}_i^T\boldsymbol{\mu}_i - \frac{1}{2\tau_i}\boldsymbol{\pi}_i^T V_i \boldsymbol{\pi}_i,$$

where the risk-tolerance τ_i measures the marginal rate of substitution of variance for expected return. This utility function has been used in Sharpe (1991), it is consistent with Markowitz portfolio selection criterion and also serves as a good approximation for other type of utility functions (Levy and Markowitz (1979)). Solving this standard portfolio optimization problem leads to the optimal portfolio weights given by

$$\boldsymbol{\pi}_i^* = \tau_i V_i^{-1}(\boldsymbol{\mu}_i - R_f\mathbf{1}), \tag{2.1}$$

where $R_f = 1 + r_f$ measures the marginal certainty equivalent rate of return (CER) per one percent investment in each asset. The CER is the same across both investors.

2.3. **Consensus Belief and Boundedly Rational Equilibrium.** We characterize market equilibrium by the concept of a consensus belief developed in Chiarella, Dieci and He (2010). Essentially, the consensus belief reflects the aggregation of the heterogeneous beliefs when the market is in equilibrium. It helps us to understand how heterogeneity or biases in agents' beliefs can affect the endogenous variables derived from market equilibrium such as the market risk premium, risk-free rate and market volatility.

[3]Within the two-period model, the terminal wealth and consumption of agents are the same.

Definition 2.1. *A belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ is called a market* **consensus belief** *of the two agent economy if the equilibrium price vector of the risky assets and the risk-free rate under the heterogeneous beliefs $\mathcal{B}_i := (\mu_{i,1}, \mu_{i,2}, \sigma_{i,1}, \sigma_{i,2}, \rho_i)$ $(i = 1, 2)$ is also the market equilibrium price vector of the risky assets and the risk-free rate under the homogeneous belief $\mathcal{B}_a$.*

In the following, we show that such consensus belief for the economy with heterogeneous beliefs can be constructed. Let W_{i0} be the initial wealth of agent i $(i = 1, 2)$. Then $W_0 = W_{10} + W_{20}$ corresponds to the total market wealth. Define the market wealth proportion $w_i = \frac{W_{i0}}{W_0}$ of agent i $(i = 1, 2)$, then the market clearing condition is given by

$$W_o \boldsymbol{\pi}_m = W_{10}\boldsymbol{\pi}_1^* + W_{20}\boldsymbol{\pi}_2^*; \qquad \text{that is,} \qquad \boldsymbol{\pi}_1^*\ w_1 + \boldsymbol{\pi}_2^*\ w_2 = \boldsymbol{\pi}_m,$$

where $\boldsymbol{\pi}_m$ denotes the market portfolio of risky assets. We also assume that the riskless asset is in net zero supply, this implies that the aggregate market must invest all its wealth in the risky assets, that is, $\mathbf{1}^T \boldsymbol{\pi}_m = 1$. Since the market equilibrium is obtained based on the fact that both agents make their optimal portfolio decision under their subjective beliefs, rather than the objective belief, we call such equilibrium as *Boundedly Rational Equilibrium (BRE)*, characterized by the following Proposition.

Proposition 2.2. *Let*

$$\tau_a := w_1\ \tau_1 + w_2\ \tau_2,$$

then the consensus belief $\mathcal{B}_a$ is given by

$$V_a^{-1} = \frac{1}{\tau_a}\big[w_1\tau_1\ V_1^{-1} + w_2\tau_2\ V_2^{-1}\big], \tag{2.2}$$

$$\boldsymbol{\mu}_a = \mathbb{E}_a(\mathbf{1} + \tilde{\mathbf{r}}) = \frac{1}{\tau_a}\big[w_1\tau_1(V_1^{-1}V_a)\boldsymbol{\mu}_1 + w_2\tau_2(V_2^{-1}V_a)\boldsymbol{\mu}_2\big]; \tag{2.3}$$

the equilibrium market portfolio is given by

$$\boldsymbol{\pi}_m = \tau_a V_a^{-1}(\mathbb{E}_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}); \tag{2.4}$$

the equilibrium CAPM relation is given by

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - r_f\mathbf{1} = \boldsymbol{\beta}[\mathbb{E}_a(\tilde{r}_m) - r_f], \tag{2.5}$$

and the equilibrium risk-free rate is

$$r_f = \frac{\mathbf{1}^T V_a^{-1}\mathbb{E}_a(\tilde{\mathbf{r}}) - \frac{1}{\tau_a}}{\mathbf{1}^T V_a^{-1}\mathbf{1}}. \tag{2.6}$$

Proposition 2.2 can be proved similarly to the proof of Proposition 1 in Chiarella, Dieci and He (2010). Based on Proposition 2.2, there is a consensus belief which is a weighted average of the heterogeneous beliefs. Different from the homogeneous belief of the representative agent in the standard rational paradigm, the consensus belief contains useful information of the heterogeneous beliefs of individual agents. In particular, we have the following observations. (i) The risk tolerance of the market is a weighted average of that of the two agents weighted by the market wealth share of the agent. Note that the wealth weighted risk tolerance $w_i\tau_i (i = 1, 2)$ also appears in the consensus beliefs defined in (2.2) and (2.3). Therefore, we can treat $w_i\tau_i$ $(i = 1, 2)$ as a risk tolerance, implying that wealthier investors are more risk tolerant. (ii) If we interpret the inverse of the variance/covariance matrix as the *precision matrix*, then consensus belief of the precision matrix is a risk-tolerance weighted average of investors' subjective beliefs of the precision matrices. (iii) The consensus belief of expected asset returns is also an average of agents' subjective beliefs weighted by their risk-tolerances and precision matrices.

The impact of the heterogeneity on the market equilibrium, CAPM relation, market risk premium, and risk free rate can be complicated in general. By focusing on the case of two assets and two agents in the following discussion, we are able to examine explicitly the impact of heterogeneity on the market equilibrium. To compare with the traditional CAPM, we first consider the homogeneous belief as the benchmark case in the next section.

3. A BENCHMARK CASE UNDER HOMOGENEOUS BELIEF

To examine the impact of the heterogeneity on the market equilibrium and compare with the market equilibrium under a homogeneous belief, we consider in this section a benchmark case under the standard rational expectation in which both agents may have different risk tolerance, but they have the same beliefs in returns[4], denoted by $\mathcal{B}_o = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, that is $\mathcal{B}_i = \mathcal{B}_o$ for $i = 1, 2$. For this benchmark case, we have from Proposition 2.2 that

$$V_a = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} := V_o, \qquad \boldsymbol{\mu}_a = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} := \boldsymbol{\mu}_o.$$

Consequently, the market portfolio is simply given by

$$\hat{\boldsymbol{\pi}}_m := \frac{1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \begin{pmatrix} \tau_a(\mu_1 - \mu_2) + \sigma_2(\sigma_2 - \rho\sigma_1) \\ \tau_a(\mu_2 - \mu_1) + \sigma_1(\sigma_1 - \rho\sigma_2 \end{pmatrix}, \tag{3.1}$$

[4]The benchmark beliefs $\mathcal{B}_o$ can be treated as either an objective belief or a benchmark homogeneous belief.

the market risk-premium, risk-free return and market variance are given, respectively, by

$$\begin{aligned}
\hat{\mathbb{E}}(\tilde{r}_m - r_f) &:= \frac{(\mu_1-\mu_2)^2\tau_a^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}{\tau_a(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)},\\
\hat{R}_f &:= \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\left(\frac{\mu_1}{\sigma_1^2}+\frac{\mu_2}{\sigma_2^2}-\frac{\rho(\mu_1+\mu_2)}{\sigma_1\sigma_2}-\frac{1}{\tau_a}(1-\rho^2)\right), \qquad (3.2)\\
\hat{\sigma}^2(\tilde{r}_m) &:= \frac{(\mu_1-\mu_2)^2\tau_a^2 + (1-\rho^2)\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)}.
\end{aligned}$$

It is easy to see that $\tau_a = \hat{\sigma}^2(\tilde{r}_m)/\hat{\mathbb{E}}(\tilde{r}_m - r_f)$, so the market risk-tolerance represents the marginal rate of substitution between market risk premium and market variance. Equations (3.1) and (3.2) show that the market risk premium and the risk-free rate are quite complex expressions of the benchmark belief $\mathcal{B}_o$. Next we use an numerical example to illustrate the limitation of the benchmark case in generating high risk premium and low risk-free rate.

Example 3.1. *Let the two risky assets in the economy have expected returns* $(\mu_1, \mu_2) = (1.06, 1.09)$*, standard deviations* $(\sigma_1, \sigma_2) = (0.08, 0.3)$*, and correlation coefficient* $\rho = 0.8$*. Both agents hold the benchmark belief, that is,* $\mathcal{B}_i = \mathcal{B}_o = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$*. For simplicity, we also assume that both agents have equal shares of the initial market wealth (so that* $w_1 = w_2 = 1/2$*)*[5].

By choosing a reasonable level of risk-tolerance, say $\tau_i = 0.5\,(i = 1, 2)$ (and hence $\tau_a = 0.5$), we have from $\tau_a = \hat{\sigma}^2(\tilde{r}_m)/\hat{\mathbb{E}}(\tilde{r}_m - r_f)$ that the market in equilibrium requires 2% expected excess return (above the risk-free rate) for 10% standard deviation. Consequently, we have from equations (3.1) and (3.2) that the market portfolio is given by $\boldsymbol{\pi}_m = (0.962,\ 0.038)^T$ and

$$\hat{r}_f = 4.62\%, \qquad \hat{\mathbb{E}}(\tilde{r}_m - r_f) = 1.49\%, \qquad \text{and} \qquad \hat{\sigma}_m = 8.63\%.$$

Note that the risk-free rate is rather high, the risk-premium and market volatility are rather low. Intuitively, because asset 2 has a much larger volatility relative to asset 1, but there is not enough compensation in terms of expected return, every investor knows about this (homogeneous beliefs), therefore the market portfolio consists little of asset 2, resulting in low market volatility and market risk premium. If both agents become more risk averse so that the risk tolerance τ_i reduces, say to $\tau_a = 0.1$, then

$$\hat{r}_f = 0\%, \qquad \hat{\mathbb{E}}(\tilde{r}_m - r_f) = 6\%, \qquad \text{and} \qquad \hat{\sigma}_m = 8\%.$$

[5]Alternatively, we can treat $w_i\tau_i$ as the risk tolerance of agent i and argue that wealthier agents are more risk tolerant. Therefore, in the rest of the paper, we always assume that $w_1 = w_2 = 1/2$.

The market portfolio becomes $\boldsymbol{\pi}_m = (1, \ 0)^T$ (asset 2 is redundant). In this case, reducing the risk tolerance can certainly increase the market risk premium and reduce the risk-free rate, however, it does not increase the proportion of total wealth invested in asset 2 or the market volatility. In order to achieve those, we introduce biases into investors' beliefs in the following discussion. If the biases "cancel out", then the consensus belief $\mathcal{B}_a$ would conform to the homogeneous benchmark belief $\mathcal{B}_o$, and heterogeneity would not matter in determining the endogenous variables when market is in equilibrium. However, we will show that this is generally not the case and heterogeneity in beliefs can have significant impact on the market equilibrium. In particular, we show that certain correlations among biased beliefs can generate high market risk premium and low risk-free rate without having to decrease the risk tolerance level.

4. THE IMPACT OF HETEROGENEITY

To examine the impact of the heterogeneity on the market equilibrium explicitly, we assume that both agents agree about the expected return and the variance of the first risky asset, asset 1 (which might be well informed) but disagree about the expected return, standard deviation of the second asset, asset 2 (which may be less informed) and the correlation coefficient of the returns of the two risky assets. To see whether biases in investors' beliefs indeed "cancel out" and have no effect on market equilibrium, disagreements are characterized by mean-preserving spread about the benchmark belief. We assume the beliefs in the expected return and the standard deviation of the first asset for both agents are given by the benchmark beliefs: $(\sigma_{i,1}, \mu_{i,1}) = (\sigma_1, \mu_1)$ for $i = 1, 2$, while the risk tolerance and the beliefs of the two agents in the expected return and standard deviation of the second asset, and the return correlation of the two assets are mean-preserving spreads of the benchmark belief $\mathcal{B}_o$ and risk tolerance τ_o. More precisely, we assume that the risk-tolerances of the two agents are given, respectively, by

$$\tau_1 = \tau_o(1 - \Delta), \qquad \tau_2 = \tau_o(1 + \Delta), \qquad \Delta \in (-1, 1); \tag{4.1}$$

the beliefs about the standard deviation of asset 2 are given by

$$\sigma_{1,2} = \sigma_2(1 - \delta), \qquad \sigma_{2,2} = \sigma_2(1 + \delta), \qquad \delta \in (-1, 1); \tag{4.2}$$

the beliefs about the correlation between asset returns are given by

$$\rho_1 = \rho(1 - \varepsilon), \qquad \rho_2 = \rho(1 + \varepsilon), \qquad \varepsilon \in (-1, 1); \tag{4.3}$$

and the beliefs of expected returns of asset 2 are given by

$$\mu_{1,2} = \mu_2(1 - \alpha), \qquad \mu_{2,2} = \mu_2(1 + \alpha), \qquad \alpha \in (-1, 1). \tag{4.4}$$

The mean-preserving spreads imply that, *on average*, risk-tolerance and belief in this heterogeneous economy is exactly the same as the benchmark homogeneous economy. However, the consensus belief is not necessarily the same as the benchmark belief. As a result, the market portfolio, market risk-premium, risk-free rate and the market volatility may also differ from the homogeneous benchmark economy. For this setup, the different aspects of the heterogeneity are characterized by Δ, δ, ε and α. To examine the joint impact of risk tolerance, optimism/pessimism, and confidence/doubt on the market, we consider four different combinations of these parameters in the following.

4.1. **Case 1: Impact of Risk Tolerance and Optimism/Pessimism.** We first consider the case where the two agents have different risk-tolerance and heterogeneous belief regarding the expected future return of asset 2, that is

$$\delta = 0, \qquad \varepsilon = 0, \qquad \Delta \in (-1, 1), \qquad \alpha \in (-1, 1). \tag{4.5}$$

This means that agent 1 is less (more) risk tolerant than agent 2 when $\Delta > (<)0$ and agent 1 is more pessimistic (optimistic) than agent 2 when $\alpha > (<)0$. In particular, when $\Delta\alpha > 0(< 0)$, the risk tolerance and optimism of the two agents are positively (negatively) correlated, meaning that the more risk-tolerant agent is optimistic, while the less risk-tolerant agent is pessimistic about future return of asset 2. Applying Proposition 2.2 to this case, we obtain the following result.

Corollary 4.1. *Under* (4.5)*, the consensus belief is given by*

$$\tau_a = \tau_o, \qquad V_a = V_o, \qquad \boldsymbol{\mu}_a = (\mu_1,\ \mu_2(1 + \alpha\Delta))^T. \tag{4.6}$$

Consequently, comparing to the benchmark case,

(i) *the change in market portfolio is given by*

$$\boldsymbol{\pi}_m - \hat{\boldsymbol{\pi}}_m = \frac{\alpha\Delta\ \tau_o\mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\mathbf{1}; \tag{4.7}$$

(ii) *the change in risk-premium is given by*

$$(\mathbb{E}(\tilde{r}_m) - r_f) - (\hat{\mathbb{E}}(\tilde{r}_m) - \hat{r}_f) = \alpha\Delta\mu_2\frac{\sigma_1(\rho\sigma_2 - \sigma_1) + \tau_o(\mu_2 - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.8}$$

(iii) *the change in risk-free rate is given by*

$$\hat{r}_f - r_f = \alpha\Delta\sigma_1\mu_2\frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.9}$$

(iv) *the change in market volatility is given by*

$$\sigma_m^2 - \hat{\sigma}_m^2 = \alpha\Delta\tau_o^2\mu_2\frac{(\mu_2 - \mu_1) + (\mu_2(1 + \alpha\Delta) - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.10}$$

(v) *the changes in the beta coefficients are given by*

$$\begin{cases} \beta_1 &= \hat{\beta}_1 + \alpha\Delta\mu_2\tau_o\frac{\rho\sigma_2-\sigma_1}{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}, \\ \beta_2 &= \hat{\beta}_2 + \alpha\Delta\mu_2\tau_o\frac{\sigma_2-\rho\sigma_1}{\sigma_1^2-2\rho\sigma_1\sigma_2+\sigma_2^2}, \end{cases} \tag{4.11}$$

where $(\hat{\beta}_1,\ \hat{\beta}_2)^T = V_o\hat{\boldsymbol{\pi}}_m/\hat{\sigma}_m^2$ *are the asset betas under the homogeneous benchmark case.*

Proof. Substitute (4.5) into (4.1)-(4.4) yields risk tolerance and beliefs for both agents, then the consensus belief in (4.6) can be computed by applying Proposition 2.2, equation (2.2) and (2.3). The market portfolio $\boldsymbol{\pi}_m$ can be computed by equation (2.4) and the risk-free rate r_f by (2.6). Asset betas $\boldsymbol{\beta}$, market volatility σ_m^2 and the market premium $\mathbb{E}(\tilde{r}_m - r_f)$ can be easily calculated subsequently once the market portfolio has been computed.

□

Corollary 4.1 characterizes explicitly the impact of the heterogeneity on the market. It is easy to see that if both agents have either the same risk preference (so that $\Delta = 0$) or the same benchmark belief in the expected return of both assets (so that $\alpha = 0$), then $\alpha\Delta = 0$ and the results for the heterogeneous beliefs are reduced to that for the benchmark homogeneous case. Consequently, the heterogeneity among the two agents are cancelled out. Otherwise, the impact of the heterogeneity in this case (4.5) depends on the sign of $\alpha\Delta$ and the return correlation ρ in the benchmark belief, to which Corollary 4.1 leads to following three implications.

Firstly, when risk-tolerance and optimism about future returns are positively (negatively) correlated[6], that is $\alpha\Delta > (<) 0$, it follows from (4.6) that the aggregate market is optimistic (pessimistic) about the expected return of the second asset. Consequently, the aggregate market, indicated by the market portfolio in (4.7), invests more (less) into asset 2 and less (more) into asset 1 and that the market volatility measured by σ_m is high (low) following (4.10). This observation that the market becomes pessimistic when the risk tolerance and pessimism are positively correlated is also found in Jouini and Napp (2006).

Secondly, comparing with the benchmark belief case, we have from (4.8) that the market with biased beliefs among the two agents increase the market risk premium when either

$$\alpha\Delta > 0 \qquad \text{and} \qquad \mu_2 - \mu_1 > \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o \tag{4.12}$$

or

$$\alpha\Delta < 0 \qquad \text{and} \qquad \mu_2 - \mu_1 < \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o. \tag{4.13}$$

[6]In the sense that more risk-tolerant agent is optimistic while less risk-tolerant agent is pessimistic.

Similarly, from (4.9), the risk-free rate under the biased belief is reduced when either

$$\alpha\Delta > 0 \qquad \text{and} \qquad \rho > \sigma_1/\sigma_2, \tag{4.14}$$

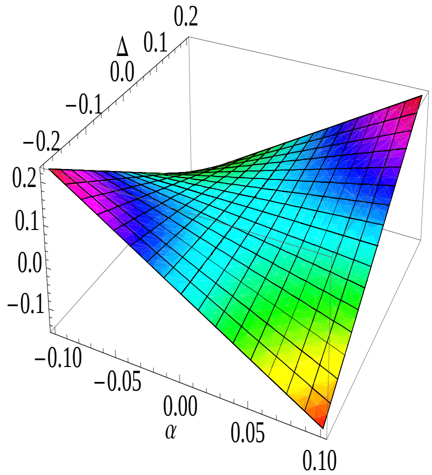
or

$$\alpha\Delta < 0 \qquad \text{and} \qquad \rho < \sigma_1/\sigma_2. \tag{4.15}$$

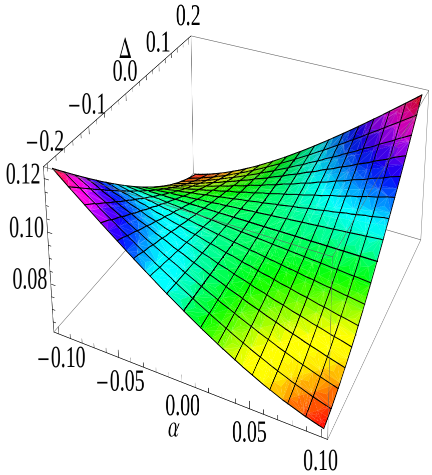
This observation implies that the biased beliefs can increase the market premium and reduce the risk-free rate either (i) when the risk tolerance and optimism of agent are positively correlated, the returns of the two assets are highly positively correlated (so that $\rho > \sigma_1/\sigma_2$), and the disagreement in asset expected returns is large ($\mu_2 - \mu_1 > \sigma_1(\sigma_1 - \rho\sigma_2)/\tau_o$); or (ii) when the risk tolerance and pessimism of agent are positively correlated, the returns of the two assets are less (even negatively) correlated (so that $\rho < \sigma_1/\sigma_2$), and the disagreement in asset expected returns is small ($\mu_2-\mu_1 < \sigma_1(\sigma_1-\rho\sigma_2)/\tau_o$). Within the framework of heterogeneous beliefs, Abel (2002) and Jouini and Napp (2006) argue that a positive correlation between the risk tolerance and pessimism is sufficient to generate high equity premium and low risk-free rate. However, our analysis shows that correlation between risk tolerance and optimism/pessism may not be sufficient, depending on the disagreement dispersion and return correlation. In particular, we will show in Section 4.3 that, in certain situation, the biased belief in return correlation can generate significant high market equity premium and low risk-free rate.

Thirdly, observation of equation (4.11) implies that the standard CAPM relation under the benchmark belief is no longer held, though the CAPM under the heterogeneous beliefs still holds under the consensus belief. It is clear that, when the disagreement in the expected return disappears, the betas become the standard betas under the benchmark belief. Under the biased beliefs, the betas β_j in equation (4.11) can be decomposed into the betas $\hat{\beta}_j$ under the benchmark belief and a term related to the biases in the beliefs, which becomes a risk factor under the heterogeneous CAPM relation. As part of the β_j, the risk factor related to the biased beliefs becomes part of the systematic risk which is missing in the standard CAPM relation under rational expectation. In general, the systematic risk factor due to the biased beliefs can be either positive or negative. Foe instance, if asset 2 is riskier than asset 1 (in the sense of $\mu_2 > \mu_1, \sigma_2 > \sigma_1$) and the return correlation is high (so that $\rho > \sigma_1/\sigma_2$), then a positive (negative) correlation between the risk tolerance and optimism increases (decreases) the systematic risk of the two risky assets ($\beta_i > (<)\hat{\beta}_i$) and the market risk premium, but decreases (increases) the risk-free rate. This indicates that, when the biases among agents are correlated, they become part of the systematic risk of risky assets.

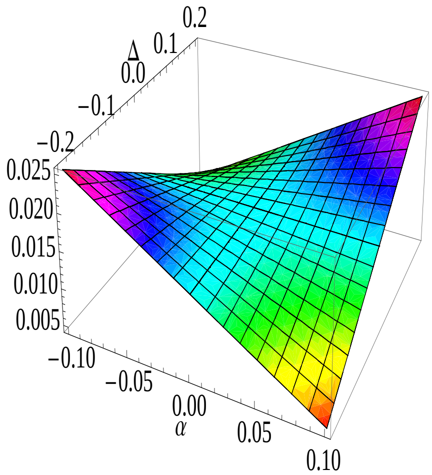
To assess the exact impact, we now conduct a numerical analysis. Based on the numerical values provided in Example 3.1, we show graphically in Figure 4.1 the impact of heterogeneity in terms of α and Δ on the change of market portfolio (in terms of the



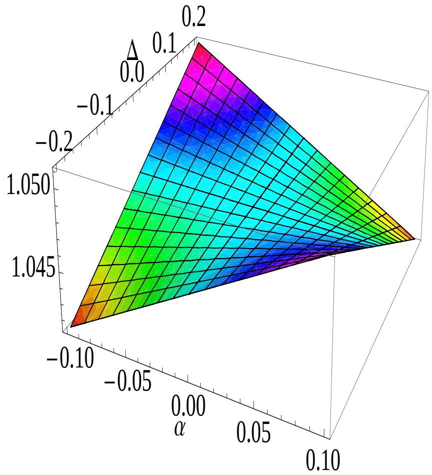
(a1) Market Proportion of asset 2



(a2) Market volatility



(a3) Risk premium



(a4) Risk-free rate

FIGURE 4.1. Effect of heterogeneity in risk-tolerance Δ and beliefs of expected return α on the market proportion of asset 2 (a1), market volatility (a2), market risk-premium (a3) and the risk-free rate (a4).

second risky asset in the market portfolio) in Figure 4.1(a), the market volatility in Figure 4.1(b), the expected market return in Figure 4.1(c), and the risk-free rate in market equilibrium in Figure 4.1(d). For the numerical values, we have $\rho > \sigma_1/\sigma_2$. The plots are symmetric reflecting the fact the effect of heterogeneity depends on the product $\alpha\Delta$ rather than individually. We see that, when the product $\alpha\Delta$ increases, the market portfolio consists more of asset 2, which leads to higher market return and volatility; at the same time the risk-free rate reduces and the risk-premium increases. In addition, the Sharpe ratio of the market portfolio increases, suggesting that heterogeneity of $\alpha\Delta$ improves the mean-variance efficiency of the aggregate market.

To quantify the impact on the market, base on the numerical values provided in Example 3.1, we choose $(\Delta, \delta, \varepsilon, \alpha) = (0.2, 0, 0, 0.1)$. The results are reported in

Cases	$(\Delta, \delta, \varepsilon, \alpha)$	$\pi_{m,2}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark	$(0,0,0,0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(0.2,0,0,0.1)$	0.2258	12.3%	2.54%	4.14%	0.2061
Case 2	$(0,-0.2,0,0.1)$	0.7511	24.15%	3.88%	4.37%	0.1606
Case 3	$(0,0,0.2,0.1)$	0.5415	19.31%	5.69%	1.94%	0.2947
Case 4	$(-0.2,0.2,0,0)$	0.1124	10.00%	1.77%	4.57%	0.1770

TABLE 4.1. Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$), and the Sharpe ratio ($\mathbb{E}(\tilde{r}_m - r_f)/\sigma_m$ for the four cases, compared with the benchmark homogeneous case. Numerical values for the benchmark belief and risk tolerance are given in Example 3.1.

Table 4.1. Comparing with the benchmark homogeneous belief, the results for Case 1 in Table 4.1 show that heterogeneity in the risk tolerance and the expected return helps to increase the market risk premium and reduce the risk-free rate when $\alpha\Delta > 0$. However, the overall effect is not significant for the chosen parameters. Risk premium increases moderately by 1% and the risk-free rate is merely reduced by less than half of a percent. This is mainly due to that the market becomes over-optimistic with respect to asset 2's future return which offsets the increase in aggregate volatility.

4.2. **Case 2: The Impact of Optimism/Pessimism and Confidence/Doubt.** In the second case, we focus on the impact of the optimism/pessimism (measured by α) and confidence/dobut (measured by δ) for asset 2 on the market in equilibrium by letting $\Delta = 0,\ \varepsilon = 0$. Measured by the beliefs in the standard deviation, agent 1 becomes confident (doubt) when $\delta > 0$. Applying Proposition 2.2, we obtain the following result.

Corollary 4.2. *For the second case when* $\Delta = 0,\ \varepsilon = 0$ *and* $\delta,\ \alpha \in (-1,1)$*, the consensus belief* $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ *is given by* $\tau_a = \tau_o$*,*

$$\mu_{a,1} = \mu_1 - \alpha\delta\mu_2 \frac{\rho\sigma_1}{\sigma_2(1+\delta^2-\rho^2)}, \qquad \mu_{a,2} = \mu_2\big(1 - \frac{\alpha\delta(2-\rho^2)}{1-\rho^2+\delta^2}\big) \tag{4.16}$$

and

$$\sigma_{a,1}^2 = \sigma_1^2\big[1 - \frac{\delta^2\rho^2}{1+\delta^2-\rho^2}\big], \quad \sigma_{a,2}^2 = \sigma_2^2 \frac{(1-\delta^2)^2(1-\rho^2)}{1+\delta^2-\rho^2},$$
$$\rho_a = \rho\big[1 - \frac{\rho^2\delta^2}{1+\delta^2-\rho^2}\big]\frac{\sigma_1\sigma_2}{\sigma_{a,1}\sigma_{a,2}}. \tag{4.17}$$

Proof. Substitute $\Delta = 0$ and $\epsilon = 0$ into (4.1)-(4.4) yields risk tolerance and beliefs for both agents, then the consensus belief $\mathcal{B}_a$ can be computed by applying Proposition 2.2, equations (2.2) and (2.3). □

Corollary 4.2 gives the explicit impact of the biased beliefs in the expected return and the standard deviation for the second asset among the two agents. One special case is particular interesting. This is when there is no biased beliefs in the standard deviation of the second asset (that is $\delta = 0$). In this case, we see from (4.16) and (4.17) that there is no difference between the heterogeneous case with biased beliefs in expected return on the second asset and the benchmark case; so effect from the biased beliefs in the expected return of the asset 2 is canceled out and has no impact on the market. In general, based on (4.16) and (4.17), we see that the biased beliefs in the expected returns of asset 2 has impact on the market expected return, but not the standard deviations and correlation. However, the biased beliefs on the standard deviation of the return of asset 2 affects the expected returns, standard deviation, and correlation of both assets when the asset returns are correlated. This effect vanishes when $\rho = 0$. Corollary 4.2 reflects a joint impact of the optimism/pessimism and confidence/dobut on the market. From equations (4.17), one can see that the aggregate market becomes over-confident when agents have biased beliefs regarding the variance of asset 2's return so that, for $0 < \delta < 1$, we have $\sigma_{a,1} < \sigma_1,\ \sigma_{a,2} < \sigma_2$ and $\rho_a\sigma_{a,1}\sigma_{a,2} < \rho\sigma_1\sigma_2$. From (4.16), when $\alpha\delta < 0$, that is when the optimistic (pessimistic) agent is confident (doubtful) about the future return of asset 2, the market perceives a higher expected return for both assets.

To examine the impact on the market, we let $\delta = -0.2$ and $\alpha = 0.1$. This means that the second (first) agent is optimistic (pessimistic) and confident (doubt) on the future return of the second asset, so that $\alpha\delta < 0$. The numerical results in Table 4.1 show a dramatic increase in the market's holding of asset 2. Therefore the market gains in risk premium but also becomes much more volatile. This is due to the fact that the increase in expected return is much higher for asset 2 than for asset 1 and the value of $(\rho\sigma_1/\sigma_2)$ is small relative to $(2 - \rho^2)$, see (4.16). The risk-free rate reduces only slightly. Intuitively, although the market consists much more of the riskier asset, but the market also becomes over-confident and over-optimistic which drives up the risk-free rate. This observation is consistent with the survey result in Giordani and Soderlind (2006) that doubt is not promising explanation of the high equity premium and the amount of pessimism provides only a rather small improvement. The Sharpe ratio drops comparing to the benchmark case, suggesting that the gain in risk premium cannot compensate for the higher volatility.

4.3. **Case 3: The Impact of Optimism/Pessimism and Biased Belief in the Correlation.** In the third case, we examine the joint impact of heterogeneity in the expected return of asset 2 and the correlation coefficient by letting $\Delta = 0, \delta = 0$ and considering the effect of (ε, α). When $\epsilon > (<)0$, agent 1 believes that the return correlation is lower (higher) while agent 2 believes that the return correlation is higher (lower). Applying Proposition 2.2 to this case, we obtain the following result.

Corollary 4.3. *For the case that* $\Delta = 0, \delta = 0$ *and* $\varepsilon, \alpha \in (-1, 1)$*, the consensus belief* $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ *is given by* $\tau_a = \tau_o$*,*

$$\mu_{a,1} = \mu_1 - \alpha\varepsilon \frac{\rho\sigma_1}{(1-\rho^2)\sigma^2}\mu_2, \qquad \mu_{a,2} = \mu_2\big[1 + \alpha\varepsilon\frac{\rho^2}{1-\rho^2}\big], \tag{4.18}$$

$$\sigma_{a,1}^2 = \sigma_1^2\big[1 - \frac{\varepsilon^2\rho^2}{1-\rho^2}\big], \quad \sigma_{a,2}^2 = \sigma_2^2\big[1 - \frac{\varepsilon^2\rho^2}{1-\rho^2}\big], \quad \frac{\rho_a\sigma_{a,1}\sigma_{a,2}}{\rho\sigma_1\sigma_2} = 1 + \frac{\varepsilon^2\rho^2}{1-\rho^2}. \tag{4.19}$$

Proof. Analogous to the proof of Corollary 4.2. □

Corollary 4.3 shows the impact of the optimism/pessimism and the biased beliefs in the correlation on the market. The biased beliefs in the expected return of asset 2 affect the market expected returns of both assets, but not the variances and covariances. However, the biased beliefs in the return correlation affect both the first and second moments of the market returns of both assets as well as the return correlation. It is easy to see that, for $0 < \varepsilon < 1$, we have $\sigma_{a,1} < \sigma_1, \sigma_{a,2} < \sigma_2$ and $\rho_a\sigma_{a,1}\sigma_{a,2} > \rho\sigma_1\sigma_2$. This indicates that in aggregate the market becomes more confident about the future returns of the both assets but perceives a higher return covariance comparing to the benchmark case. For $\alpha\varepsilon > 0$, that is when the optimistic agent also believes in a higher correlation between asset returns, we see from equation (4.18) that the market perceives a higher (lower) expected return for asset 2 (asset 1) when $\rho > 0$ and vice versa when $\rho < 0$. Intuitively, when $\rho > 0$ and $\alpha\epsilon > 0$, the market invests more into asset 2 because of the higher perceived expected return. As a result, the aggregate market expected return and volatility increase. However, different from the previous cases, because (4.19) indicates that $\rho_a > \rho$, hence there is less diversification effect and consequently one should expect a significant reduction in the risk-free rate.

To examine the impact of the heterogeneity in the expected return and correlation, we choose $\epsilon = 0.2$ and $\alpha = 0.1$ so that $\alpha\varepsilon > 0$. The results are given for Case 3 in Table 4.1, showing the most desirable result with a high market risk premium and low risk-free rate. The risk-free rate in this case is reduced significantly by nearly 3% while the risk premium increased significantly by more than 4%. Most noticeably, the Sharpe ratio in this case becomes 0.2497, the highest amongst all cases including the homogeneous benchmark by far, implying that the aggregate market becomes the most mean-variance efficient when $\alpha\varepsilon > 0$.

4.4. **Case 4: The Impact of Risk-tolerance and Confidence/Doubt.** In the fouth case, we examine the joint impact of heterogeneity in the risk-tolerance (measure by

Δ) and confident/doubt (measure by δ by letting $\alpha = 0$ and $\varepsilon = 0$. Applying Proposition 2.2 to this case, we obtain the following result.

Corollary 4.4. *For the case that $\alpha = 0, \varepsilon = 0$ and $\Delta, \delta \in (-1, 1)$, the consensus belief $\mathcal{B}_a = (\mu_{a,1}, \mu_{a,2}, \sigma_{a,1}, \sigma_{a,2}, \rho_a)$ is given by $\tau_a = \tau_o$, $\boldsymbol{\mu}_a = (\mu_1, \mu_2)^T$ and*

$$\sigma_{a,1}^2 = \sigma_1^2 \left[\frac{(1 + \delta^2 - 2\Delta\delta)(1 - \rho^2)}{(1 + \delta^2 - 2\Delta\delta) - (1 - \Delta\delta)^2\rho^2} \right],$$

$$\sigma_{a,2}^2 = \sigma_2^2 \left[\frac{(1 - \delta^2)^2(1 - \rho^2)}{(1 + \delta^2 - 2\Delta\delta) - (1 - \Delta\delta)^2\rho^2} \right],$$

$$\rho_a = \rho \left[\frac{(1 - \delta^2)^2(1 - \Delta\delta)(1 - \rho^2))}{(1 + \delta^2 - 2\Delta\delta) - (1 - \Delta\delta)^2\rho^2} \right] \frac{\sigma_1\sigma_2}{\sigma_{a,1}\sigma_{a,2}}.$$

Proof. Analogous to the proof of Corollary 4.2. □

Corollary 4.4 shows that the joined impact of the risk-tolerance and confidence/doubt on the market is rather complicated. When there is no biased belief in the standard deviation (so that $\delta = 0$), the consensus belief is reduced to the benchmark belief. Jouni and Napp (2006) argue that a positive correlation between risk tolerance and doubt can contribute to high equity premium and low risk-free rate. In our example, if we choose $\Delta = -0.2$ and $\delta = 0.2$ so that $\delta\Delta < 0$, that is the more risk-tolerant agent is more confident about the future return for asset 2, we report the numerical results in Tab 4.1 for Case 4. We can see that the market risk premium increases and risk-free rate reduces though the magnitudes of the changes are not very significant. On the one hand, this result is inconsistent with the result suggested by Jouni and Napp (2006), suggesting that results from the single risky asset case do not necessarily carry over to the case with two risky assets, typically impact of heterogeneity depend on the correlation structure of the asset returns. On the other hand, consisting with the survey result in Giordani and Soderlind (2006), this illustrates that doubt may not be a promising explanation of the equity premium puzzle.

4.5. **Biased Beliefs in the "Safe" Stock.** In the previous cases, agents are assumed to have disagreement over the distribution of the terminal return of the "risky" stock, in the sense that the stock has a higher expected return and higher risk. We now show that, when agents have heterogeneous beliefs in the "safe" stock which has lower expected return and lower risk, the impact on the market can be different. The is illustrated by considering the following numerical example[7].

Example 4.5. *Let the two risky assets in the economy have expected returns $(\mu_1, \mu_2) = (1.09, 1.06)$ and standard deviations $(\sigma_1, \sigma_2) = (0.3, 0.08)$ and correlation coefficient $\rho = 0.8$. Agents have heterogeneous beliefs about terminal return of asset 2, and*

[7] Basically, we swap the two risky assets and still consider the biased beliefs in the second asset.

different risk tolerance. Their heterogeneity is characterized by parameters Δ, δ, ε and α as described earlier in this section.

Cases	$(\Delta, \delta, \varepsilon, \alpha)$	$\pi_{m,2}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark	$(0, 0, 0, 0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(-0.2, 0, 0, 0.1)$	0.2207	12.2%	4.63%	2.04%	0.3794
Case 2	$(0, 0.2, 0, 0.1)$	0.3854	12.99%	6.11%	1.04%	0.4710
Case 3	$(0, 0, -0.2, 0.1)$	0.2525	10.11%	1.87%	4.9%	0.1846
Case 4	$(0.2, 0.2, 0, 0)$	0.0637	9.09%	1.48%	4.71%	0.1633

TABLE 4.2. Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$) and the Sharpe ratio $(\mathbb{E}(\tilde{r}_m) - r_f)/\sigma_m$ for the four cases, compared with the benchmark homogeneous case.

We redo the numerical analysis in Table 4.1 for the four cases and present the results in Table 4.3. The interesting cases are (i) Case 1 when $(\Delta, \delta, \varepsilon, \alpha) = (-0.2, 0, 0, 0.1)$ and (ii) Case 2 when $(\Delta, \delta, \varepsilon, \alpha) = (0, 0.2, 0, 0.1)$. In both cases, there is a significant increase in the market risk premium and reduction in the risk-free rate. Now we provide some explanations for these results. For (i), we have $\alpha\Delta < 0$, suggesting that there is a positive correlation between the risk-tolerant and pessimism. This leads the aggregate market to perceiving a lower expected return for asset 2 (see (4.6)), therefore investing more into asset 1 (risky stock), driving up the aggregate market expected return and volatility. However, in contrast with Case 1 in Table 4.1, the risk-free also reduces significantly because the market is pessimistic rather than optimistic about expected equity returns overall and more willing to invest in the risk-free security. In (ii), it follows from (4.16) and $\alpha\delta > 0$ (the more optimistic agent is less confident) that the aggregate market becomes pessimistic about future return for both asset. As a result, the market is even more willing to invest in the risk-free security than in the case (i), thus the reduction in risk-free rate is greater in this case. Therefore, the combined effect of heterogeneity in the beliefs of the expected and variance of terminal return leads to the most desirable result of a low risk-free rate and high market risk premium. This case also produces the highest Sharpe ratio amongst all the cases considered.

4.6. **Extension to a continuum of investors.** As the number of investors increase in the above model, increasing dimensionality can make the model infeasible. In order to have a parsimonious model to incorporate the realism of large number of investors, we extend the previous model of two agents to a model with a continuum of investors. In this case, we are able to characterize investors' heterogeneity in risk tolerance, initial wealth share and beliefs by probability distributions and obtain similar results to that of the two agent economy.

Consider a continuum of investors indexed by $e \in [0,1]$, we assume that the initial wealth shares of investors are equal, thats is $w_e = 1$ for all e[8]. The economy is defined by a measurable function $(\tau_F, \sigma_{1,F}, \sigma_{2,F}, \rho_F, \mu_{1,F}, \mu_{2,F}) : [0,1] \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \in [0,1] \times \mathbb{R} \times \mathbb{R}$, where $(\tau_e, \sigma_{e,1}, \sigma_{e,2}, \rho_e, \mu_{e,1}, \mu_{e,2})$ is the risk tolerance, belief of standard deviations, return correlation and the belief of expected returns for investor e. The consensus belief $\mathcal{B}_a$ in this economy is the limits of equation (2.2) and (2.3) as the number of investors approaches infinity, thus we can rewrite the consensus belief as

$$\tau_a = \int_0^1 \tau_e \; de, \qquad V_a^{-1} = \frac{1}{\tau_a} \int_0^1 \tau_e \; V_e^{-1} \; de, \qquad \boldsymbol{\mu}_a = \frac{V_a}{\tau_a} \int_0^1 \tau_e \; V_e^{-1} \boldsymbol{\mu}_e \; de,$$

where

$$\boldsymbol{\mu}_e = \begin{pmatrix} \mu_{e,1} \\ \mu_{e,2} \end{pmatrix} \qquad \text{and} \qquad V_e = \begin{pmatrix} \sigma_{e,1}^2 & \rho_e \sigma_{e,1} \sigma_{e,2} \\ \rho_e \sigma_{e,1} \sigma_{e,2} & \sigma_{e,2}^2 \end{pmatrix}. \tag{4.20}$$

We assume that investors' risk tolerance τ_e and beliefs of future asset returns $\mathcal{B}_e = (V_e, \boldsymbol{\mu}_e)$ are i.i.d random variables for each investor e. This allows us to characterize the consensus belief by simply taking expectations across all investors. In the same spirit of Admati (1985) (together with necessary technical requirements, such as the uniform boundedness of the second moments of the random variables, to ensure that the law of large number applies, see p.636 in Admati (1985) for details.), we define the integral of a random vector $\int_0^1 \tilde{\mathbf{X}}_e \; dw_e \equiv \mathbf{0}$ if for every sequence $\{e_i\}$ of distinct indices from $[0,1]$, $(1/I)\sum_i^I \tilde{\mathbf{X}}_{e_i} \to \mathbf{0}$ as $I \to \infty$. By this definition[9], we can write the consensus belief $\mathcal{B}_a$ as $V_a = \tau_a \mathbb{E}[\tilde{\tau}\tilde{V}^{-1}]^{-1}$, $\boldsymbol{\mu}_a = \tau_a^{-1} V_a \mathbb{E}[\tilde{\tau}\tilde{V}^{-1}\tilde{\boldsymbol{\mu}}]$. Note that the expectation operator is not taken over all possible outcomes of future asset return, but all possible beliefs among the investors. We will see that in some cases, it is possible to write down explicitly the consensus belief in terms of the first two moments of the random variables, in other, we may require Monte-Carlo simulations.

In the sprit of the case for two agents, we make the following assumptions about the distributions of the heterogeneous beliefs of the continuum agent $e \in [0,1]$. There are two risky assets and a risk-free asset in the economy. We assume that agents agree on the expected and standard deviation of future returns for the first risky asset with $\mu_{e,1} = \mu_1$ and $\sigma_{e,1} = \sigma_1$, but disagree on that for the second risky asset. For agent e and the second risky asset, let the expected return, the standard deviation, the return correlation, and the risk tolerance are given by, respectively,

$$\mu_{e,2} = \mu_2(1 + \tilde{\alpha}_e), \qquad \sigma_{e,2} = \sigma_2(1 + \tilde{\delta}_e), \qquad \rho_e = \rho(1 + \tilde{\epsilon}_e), \qquad \tau_e = \tau_o(1 + \tilde{\Delta}_e), \tag{4.21}$$

[8]Another way is to treat the product $w_e \tau_e$ as investor e's risk tolerance as in the two agent economy

[9]Suppose that $\tilde{\mathbf{X}}_e = \tilde{\mathbf{Y}}_e - \mathbb{E}[\tilde{\mathbf{Y}}_e]$, since $\int_0^1 \tilde{\mathbf{X}}_e \; de = \mathbf{0}$, it is natural to define $\int_0^1 \tilde{\mathbf{X}}_e + \tilde{\mathbf{X}}_e' \; de \equiv \int_0^1 \tilde{\mathbf{X}}_e' \; de$. This means that $\int_0^1 \tilde{\mathbf{Y}}_e \; de = \int_0^1 \mathbb{E}[\tilde{\mathbf{Y}}_e] \; de$ which equals $\mathbb{E}[\tilde{\mathbf{Y}}]$ if $\tilde{\mathbf{Y}}_e$ are i.i.d.

where, $\tilde{\delta}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\delta}))$, $\tilde{\alpha}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\alpha}))$, $\tilde{\epsilon}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\epsilon}))$, and $\tilde{\Delta}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\Delta}))$, all truncated between -1 and 1. We also denote correlations between random variables $\tilde{\Delta}_e$, $\tilde{\delta}_e$, $\tilde{\epsilon}_e$ and $\tilde{\alpha}_e$ by $\rho(\tilde{\Delta}, \tilde{\alpha})$, $\rho(\tilde{\delta}, \tilde{\alpha})$, $\rho(\tilde{\epsilon}, \tilde{\alpha})$ and $\rho(\tilde{\Delta}, \tilde{\delta})$, the correlations are independent of e.

In the case where investors' beliefs in the variance/covariances matrix is homogeneous, that is,

$$\sigma^2(\tilde{\delta}) = 0 \qquad \text{and} \qquad \sigma^2(\tilde{\epsilon}) = 0. \tag{4.22}$$

We can extend Corollary 4.1 as follows.

Corollary 4.6. *Under* (4.22)*, the consensus belief is given by*

$$\tau_a = \tau_o, \qquad V_a = V_o, \qquad \boldsymbol{\mu}_a = (\mu_1,\ \mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta}))^T. \tag{4.23}$$

where $Cov(\tilde{\alpha}, \tilde{\Delta}) = \rho(\tilde{\alpha}, \tilde{\Delta})\sigma(\tilde{\Delta})\sigma(\tilde{\alpha})$. *Consequently, comparing with the benchmark case,*

(i) *the change in market portfolio is given by*

$$\boldsymbol{\pi}_m - \hat{\boldsymbol{\pi}}_m = \frac{Cov(\tilde{\alpha}, \tilde{\Delta})\ \tau_o \mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\mathbf{1}; \tag{4.24}$$

(ii) *the change in risk-premium is given by*

$$(\mathbb{E}(\tilde{r}_m) - r_f) - (\hat{\mathbb{E}}(\tilde{r}_m) - \hat{r}_f) = Cov(\tilde{\alpha}, \tilde{\Delta})\mu_2 \frac{\sigma_1(\rho\sigma_2 - \sigma_1) + \tau_o(\mu_2 - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.25}$$

(iii) *the change in risk-free rate is given by*

$$\hat{r}_f - r_f = Cov(\tilde{\alpha}, \tilde{\Delta})\sigma_1\mu_2 \frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.26}$$

(iv) *the change in market volatility is given by*

$$\sigma_m^2 - \hat{\sigma}_m^2 = Cov(\tilde{\alpha}, \tilde{\Delta})\tau_o^2\mu_2 \frac{(\mu_2 - \mu_1) + (\mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})) - \mu_1)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}; \tag{4.27}$$

(v) *the changes in the beta coefficients are given by*

$$\begin{aligned} \beta_1 &= \hat{\beta}_1 + Cov(\tilde{\alpha}, \tilde{\Delta})\mu_2\tau_o \frac{\rho\sigma_2 - \sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \\ \beta_2 &= \hat{\beta}_2 + Cov(\tilde{\alpha}, \tilde{\Delta})\mu_2\tau_o \frac{\sigma_2 - \rho\sigma_1}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}. \end{aligned} \tag{4.28}$$

where $(\hat{\beta}_1,\ \hat{\beta}_2)^T = V_o\hat{\boldsymbol{\pi}}_m/\hat{\sigma}_m^2$ *are the asset betas under the homogeneous benchmark case.*

Proof. From (4.21) and (4.22), the consensus belief of expected asset returns according to equation (4.20) is given by $\boldsymbol{\mu}_a = (\mu_1,\ \mu_{a,2})^T$ where $\mu_{a,2} = \tau_a^{-1}(\mathbb{E}(\tilde{\tau})\mathbb{E}(\tilde{\mu}_2) + Cov(\tilde{\tau}, \tilde{\mu}_2))$ where $\tau_a = \mathbb{E}(\tilde{\tau}) = \tau_o$. Since $Cov(\tilde{\tau}, \tilde{\mu}_2) = Cov(\tau_o(1 + \tilde{\Delta}), \mu_2(1 + \tilde{\alpha}))$, we obtain $\mu_{a,2} = \mu_2(1 + Cov(\tilde{\alpha}, \tilde{\Delta})$. The rest of the proof is analogous to that of Corollary 4.1 by simply replacing $\alpha\Delta$ with $Cov(\tilde{\alpha}, \tilde{\Delta})$. □

Corollary 4.6 shows that we can derive expression for the equilibrium market portfolio, market risk premium, risk-free rate and market volatility analogous to Corollary 4.1, simply replacing $\alpha\Delta$ with $Cov(\tilde{\Delta}, \tilde{\alpha})$. Hence, it can be seen that results in the two-agent economy extend to the infinite-agent economy when we characterize investors' beliefs by i.i.d random variables. In other case when $\sigma(\tilde{\delta})$ or $\sigma(\tilde{\epsilon})$ is positive, it seems difficult to derive analytically tractable expressions for the endogenous variable in market equilibrium. Corresponding to the four cases in Table 4.1, we approximate the continuum of agents by Monte Carlo simulations with $100,000$ investors and summarize the results in Table 4.3[10].

Cases	$(\sigma_\Delta, \sigma_\delta, \sigma_\varepsilon, \sigma_\alpha)$	$\pi_{m,2}$	$\sigma(\tilde{r}_m)$	$\mathbb{E}(\tilde{r}_m - r_f)$	r_f	$\frac{\mathbb{E}(\tilde{r}_m)-r_f}{\sigma_m}$
Benchmark	$(0, 0, 0, 0)$	0.038	8.63%	1.49%	4.62%	0.1727
Case 1	$(0.2, 0, 0, 0.1)$	0.2108	11.99%	2.45%	4.18%	0.2044
Case 2	$(0, 0.2, 0, 0.1)$	0.8737	27.03%	4.17%	4.45%	0.1543
Case 3	$(0, 0, 0.2, 0.1)$	0.5297	19.04%	5.50%	2.09%	0.2889
Case 4	$(0.2, 0.2, 0, 0)$	0.1446	10.63%	1.85%	4.58%	0.1744

TABLE 4.3. Effects of heterogeneity on the market proportion of asset 2 ($\pi_{m,2}$), market volatility ($\sigma(\tilde{r}_m)$), market risk-premium ($\mathbb{E}(\tilde{r}_m - r_f)$), the risk-free rate ($r_f$) and the Sharpe ratio $(\mathbb{E}(\tilde{r}_m) - r_f)/\sigma_m$ for the four cases, compared with the benchmark homogeneous case.

We can see that the results are fairly similar to that of Table 4.1. The increase in market risk premium and reduction in the risk-free rate is most significant when beliefs in expected future asset returns is positively correlation with beliefs in the return correlation, see Tab. 4.3 case 3. In this case, the Sharpe Ratio is also the highest. Based on this observation, we could argue that the model with two agents, which is simple to analyze, can provide useful insight into the model with continuum of agents.

5. CONCLUSION

Heterogeneity, reflecting diversity and disagreement among agents, is very common in financial markets and it has significant impact on the market. In this paper, we have

[10]In calculation, we take $(\mu_1, \mu_2) = (1.06, 1.09)$, $(\sigma_1, \sigma_2) = (0.08, 0.3)$ and correlation coefficient $\rho = 0.8$. Also we assume that $\tilde{\delta}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\delta}))$ is truncated between -1 and 1 and $\tilde{\epsilon}_e \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2(\tilde{\epsilon}))$ is truncated between -0.25 and 0.25. In addition, to compare with the results in Table 4.1 for two agents, we assume the beliefs in expected future asset returns are positively correlated with both risk tolerance and beliefs in return correlation with $\rho(\tilde{\Delta}, \tilde{\alpha}) = 0.9$ and $\rho(\tilde{\epsilon}, \tilde{\alpha}) = 0.9$ and negatively correlated with beliefs in the volatility of asset returns with $\rho(\tilde{\delta}, \tilde{\alpha}) = -0.9$. Furthermore the beliefs in the volatility is negatively correlated risk tolerance with $\rho(\tilde{\Delta}, \tilde{\delta}) = -0.9$.

examined the impact of heterogeneity among investor in a market with two risky assets on the market equilibrium, in particular, the market risk premium, risk-free rate, market volatility and the Sharpe Ratio of the market portfolio. Within a mean-variance setting, investors' heterogeneity is represented by their different risk tolerance, beliefs about the expected and variance of future asset returns and the return correlation between two risky assets. Furthermore, we assume that they agree on the expected and variance of future return for one asset, but not the other. We show that, when investors are on average unbiased with respect to the homogeneous benchmark belief characterizing by mean-preserving spreads of the homogenous belief, the belief of the market equilibrium, represent by the consensus belief, is in general biased with respect to the benchmark belief. We show that the impact of heterogeneity on the market with two risky assets is very different from that with one risky asset. For the market with only one risky asset, a negative correlation between risk tolerance and beliefs in expected returns, that is the more risk tolerant investor is less optimistic about future returns, makes the aggregate market less optimistic about the risky asset and increases the market risk premium while reducing the risk-free rate. On the contrary, for the market with two risky assets, we found an increase in the risk premium and a reduction in the risk-free rate when the investor who is more optimistic about future asset returns is more risk tolerant or more confident about future asset returns. More interestingly, we found that this effect becomes even stronger when the more optimistic agent also perceives a high correlation between asset returns. Therefore, we can conclude that the impact of heterogeneity on the market equilibrium is very different when there is more than one risky asset in the market. In general, depending on whether the heterogeneity is greater for the more risky asset or the less risky asset, its impact on the market can be different. We have also extended our model to a case with a continuum of investors by using i.i.d random variables to characterize heterogeneous risk tolerance and beliefs of investors. The analytical and numerical results obtained are very much in line with those in the two-agent case.

The biased beliefs may depend on the market conditions. Intuitively, there may be more disagreement among agents on the risky stocks when markets are moving downwards. The empirical implications of the results obtained in this paper, in particular when the biased beliefs become part of the systematic risk in the CAPM, would be of very interesting (such as in Anderson et al. (2005)). The disagreement in this paper is characterized by mean-preserving spreads about a benchmark homogeneous belief. It would also be interesting to extend the analysis to situations with skewed distribution about the heterogeneous beliefs such as in Abel (2002). In addition, extension to a dynamical model to examine the profitability and survivability of agents with different beliefs and the impact on the market equilibrium and trading volume in long-run (such as in Kogan et al. (2006) and Hong and Stein (2007)) would also be interesting. We leave these to future research.

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