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Elimination of Arbitrage States in Asymmetric Information Models

Abstract In a financial economy with asymmetric information and incomplete markets, we study how agents, having no model of how equilibrium prices are determined, may still refine their information by eliminating sequentially "*arbitrage state(s)*", namely, the state(s) which would grant the agent an arbitrage, if realizable.

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1 Introduction

In a financial economy with asymmetric information and incomplete markets, will agents be able to learn from prices about their partners' private information when they have no prior "*model*" or "*expectations*" of how equilibrium prices are determined? This paper, which complements an earlier one on arbitrage and price revelation under asymmetric information [Cornet-De Boisdeffre (2002)], answers positively and introduces a sequential and decentralized process of inferences, where agents learn from prices by analyzing arbitrage opportunities on financial markets. Refinement of information is then achieved in a decentralized manner by each agent eliminating sequentially her "*arbitrage state(s)*", namely, the state(s) which would grant the agent an arbitrage, if realizable. The paper also shows that a coarse refinement of information, which precludes arbitrage, may always be attained, alternatively, in a decentralized way through prices, or by a similar sequential elimination process when no price is given.

Section 2 presents the framework and recalls the basic concepts of information structures, refinements and arbitrage with asymmetric information. Section 3 introduces the "*no-arbitrage principle*", by which agents who only know their own characteristics may refine their information in successive steps, by eliminating "*arbitrage state(s)*". Section 4 defines a concept of no-arbitrage prices with asymmetric information and explains how such prices may reveal information. Section 5 describes the refinement process without prices.

2 The model

We consider the basic model of a two time-period economy with private information and nominal assets: the simplest tractable model which allows us to present arbitrage. The economy is finite, in the sense that there are finite sets I, S, and J, respectively, of consumers, states of nature, and nominal assets.

In what follows, the first period will also be referred to as $t = 0$ and the second period, as $t = 1$. There is an a priori uncertainty at the first period ($t = 0$) about which of the states of nature $s \in S$ will prevail at the second period ($t = 1$). The non-random state at the first period is denoted by $s = 0$ and if $\Sigma \subset S$, Σ' will stand for $\{0\} \cup \Sigma$.

Agents may operate financial transfers across states in S' (i.e., across the two periods and across the states of the second period) by exchanging a finite number of nominal assets $j \in J$, which define the financial structure of the model. The nominal assets are traded at the first period ($t = 0$) and yield payoffs at the second period ($t = 1$), contingent on the realization of the state of nature. We denote by V_s^j the payoff of asset $j \in J$, when state $s \in S$ is realized, by V the $S \times J$ -return matrix $V := (V_s^j)$, which does not depend upon the asset price $q \in \mathbb{R}^J$, and by $V[s]$ its row vector in state s (for each $s \in S$). A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies the quantities $|z_j|$ ($j \in J$) of each asset j (with the convention that it is bought if $z_j > 0$ and sold if $z_j < 0$) and Vz is thus its financial return across states at time $t = 1$.

At the first period, each agent $i \in I$ has some private information $S_i \subset S$ about which states of the world may occur at the next period: either this information is kept, or it is possible to infer that the true state will be in a smaller set $\Sigma_i \subset S_i$. In both cases, agents are assumed to receive no wrong information signal, that is, the true state always belongs to the set $\cap_{i \in I} S_i$ or $\cap_{i \in I} \Sigma_i$, hence assumed to be non-empty. A collection $(S_i)_{i \in I}$ of subsets of S, such that $\cap_{i \in I} S_i \neq \emptyset$, is called an (*information*) *structure* and a structure (Σ_i), such that $\Sigma_i \subset S_i$ for every i, is called a *refinement* of (S_i).

We summarise by $[(I, S, J), V, (S_i)_{i \in I}]$ the financial and information characteristics of the economy, which are fixed throughout the paper and referred to as the (*financial and information*) *structure.*

We recall the following standard definitions.

Given the return matrix V and a nonempty set $\Sigma \subset S$, the price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price for the couple (V, Σ), or the couple (V, Σ) to be q-arbitrage-free, if one of the following equivalent assertions, (i) or (ii), holds:
(i) there is no portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$ and $V[s] \cdot z \geq 0$ for every $s \in \Sigma$, with at least one strict inequality;
(ii) there exists $\lambda = (\lambda(s)) \in \mathbb{R}^{\Sigma}_{++}$, such that $q = \sum_{s \in \Sigma} \lambda(s) V[s]$.

We denote by $Q[V, \Sigma]$ the set of no-arbitrage prices associated to (V, Σ). By convention, we shall also say that the couple $(V, \emptyset)$ is q-arbitrage-free for every $q \in \mathbb{R}^J$, that is, we let $Q[V, \emptyset] = \mathbb{R}^J$.

3 Sequential elimination of arbitrage states

Given V, a nonempty subset Σ of S and $q \in \mathbb{R}^J$, we now define the sets:

$$\mathbf{A}(V, \Sigma, q) = \{\tilde{s} \in \Sigma : \exists z \in \mathbb{R}^J, -q \cdot z \geq 0, V[\tilde{s}] \cdot z > 0, V[s] \cdot z \geq 0, \forall s \in \Sigma\}$$
$$\mathbf{S}^1(V, \Sigma, q) := \Sigma \backslash \mathbf{A}(V, \Sigma, q),$$

with the convention that $\mathbf{A}(V, \emptyset, q) := \emptyset$ and $\mathbf{S}^1(V, \emptyset, q) := \emptyset$, for all $q \in \mathbb{R}^J$.

Given V and an information structure (S_i), the set $\mathbf{A}(V, S_i, q)$ consists in the so-called "*arbitrage state(s)*" (of the second period $t = 1$), that is, states which grant agent i an arbitrage when his beliefs are represented by the set S_i. The first stage of elimination of arbitrage states leads to the set $\mathbf{S}^1(V, S_i, q)$.

However, the refined set $\mathbf{S}^1(V, S_i, q)$ may display new arbitrage states, that is, there may exist states $s \in \mathbf{A}(V, \mathbf{S}^1(V, S_i, q), q)$ such that $s \notin \mathbf{A}(V, S_i, q)$. Thus, the elimination process may need to carry on. It is defined sequentially, hereafter, in two slightly different ways, which will be shown to be equivalent.

Given V, an agent i with (nonempty) private information set $S_i \subset S$, we define, by induction on $k \in \mathbb{N}$ and for every $q \in \mathbb{R}^J$, the sets $S_i^k(q)$ as follows:

$S_i^0(q) = S_i$, and for $k \geq 1$

$S_i^{k+1}(q) = \mathbf{S}^1(V, S_i^k(q), q) := S_i^k(q) \backslash \mathbf{A}(V, S_i^k(q), q)$.

Similarly, we define by induction on $k \in \mathbb{N}$, the sets $S_i'^k(q)$ as follows:

$S_i'^0(q) = S_i$, and for $k \geq 1$

$$S_i'^{k+1}(q) = \begin{cases} S_i'^k(q), & \text{if } \mathbf{A}(V, S_i'^k(q), q) = \emptyset, \\ S_i'^k(q) \backslash \{s^k\} \text{ for some } s^k \in \mathbf{A}(V, S_i'^k(q), q) & \text{if } \mathbf{A}(V, S_i'^k(q), q) \neq \emptyset. \end{cases}$$

The two sequences of finite sets $(S_i^k(q))_{k \in \mathbb{N}}$ and $(S_i'^k(q))_{k \in \mathbb{N}}$ are decreasing, that is, $S_i^{k+1}(q) \subset S_i^k(q)$ and $S_i'^{k+1}(q) \subset S_i'^k(q)$ for every k. Hence, they must be constant for k large enough and we let:

$S_i^*(q) := \cap_{k \in \mathbb{N}} S_i^k(q)$ (in fact equal to $S_i^{k^*}(q)$ for some k^* large enough);

$S_i^{**}(q) := \cap_{k \in \mathbb{N}} S_i'^k(q)$ (in fact equal to $S_i'^{k^{**}}(q)$ for some k^{**} large enough).

The following result shows that for every price q, the successive elimination of arbitrage states leads agents to infer the same information sets, whether they rule out the states of arbitrage one by one (and then, whatever the chronology of inferences), or in bundles.

Theorem 1 *Let $[V, (S_i)]$ be a given structure and $q \in \mathbb{R}^J$. Then, for every $i \in I$, $S_i^*(q) = S_i^{**}(q)$, and this set is the (possibly empty) greatest subset Σ of S_i such that $A(V, \Sigma, q) = \emptyset$.*

The successive elimination of arbitrage states, may be interpreted as a rational behavior. This behavior, referred to as the "*no-arbitrage principle*", does not require any knowledge of the ex ante characteristics of the economy (endowments and preferences of the other consumers) or of a relationship between prices and the private information of other agents. This is the main difference between our model of asymmetric information and that of rational expectations.

We prepare the proof of Theorem 1 with two claims.

Claim 1 *Given $q \in \mathbb{R}^J$ and $\Sigma^1 \subset \Sigma^2 \subset S$, then, $\mathbf{S}^1(V, \Sigma^1, q) \subset \mathbf{S}^1(V, \Sigma^2, q)$.*

Proof of Claim 1 By contraposition. Suppose that there exists some $\tilde{s} \in \mathbf{S}^1(V, \Sigma^1, q) \subset \Sigma^1 \subset \Sigma^2$, such that $\tilde{s} \notin \mathbf{S}^1(V, \Sigma^2, q)$. Then, $\tilde{s} \in \mathbf{A}(V, \Sigma^2, q)$, that is, there exists $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0$, $V[s] \cdot z \geq 0$ for every $s \in \Sigma^2$ and $V[\tilde{s}] \cdot z > 0$. Since $\tilde{s} \in \Sigma^1 \subset \Sigma^2$, we deduce that $\tilde{s} \in \mathbf{A}(V, \Sigma^1, q)$, which contradicts the fact that $\tilde{s} \in \mathbf{S}^1(V, \Sigma^1, q)$. $\square$

Claim 2 *For every $\Sigma \subset S_i$, such that $A(V, \Sigma, q) = \emptyset$, and every $k \in \mathbb{N}$, the following inclusions hold: $\Sigma \subset S_i^k(q) \subset S_i'^k(q)$.*

Proof of Claim 2 By induction on k. The above inclusions are true for $k = 0$, since $S_i^0(q) = S_i'^0(q) := S_i$. Assume now that the inclusions hold up to rank k. Let $\Sigma \subset S_i$ be such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. Then, from Claim 1:

$$\mathbf{S}^1(V, \Sigma, q) \subset \mathbf{S}^1(V, S_i^k(q), q) \subset \mathbf{S}^1(V, S_i'^k(q), q).$$

Since $\mathbf{A}(V, \Sigma, q) = \emptyset$, we deduce that $\mathbf{S}^1(V, \Sigma, q) = \Sigma$ and, from the definitions of $S_i^k(q)$ and $S_i'^k(q)$, that $S_i^{k+1}(q) := \mathbf{S}^1(V, S_i^k(q), q)$ and $\mathbf{S}^1(V, S_i'^k(q), q) := S_i'^k(q) \backslash \mathbf{A}(V, S_i'^k(q), q) \subset S_i'^{k+1}(q)$. Consequently, $\Sigma \subset S_i^{k+1}(q) \subset S_i'^{k+1}(q)$. $\square$

Proof of Theorem 1 Let $i \in I$ and $\Sigma \subset S_i$ be such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. Taking k large enough yields, from Claim 2: $\Sigma \subset S_i^*(q) \subset S_i^{**}(q)$. We deduce from the definitions of $S_i^*(q)$ and $S_i^{**}(q)$ that $\mathbf{A}(V, S_i^*(q), q) = \mathbf{A}(V, S_i^{**}(q), q) = \emptyset$. These relations imply, first, that $S_i^*(q) = S_i^{**}(q)$ (take $\Sigma = S_i^{**}(q)$ above) and, second, that $S_i^*(q) = S_i^{**}(q)$ is the greatest element (for the inclusion) among the subsets Σ of S_i, such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. $\square$

4 Sequential procedures and price revelation

Given the structure $[V, (\Sigma_i)]$, Theorem 1 shows the existence of a unique set, denoted $\tilde{\mathbf{S}}_i(q)$, which is the greatest subset Σ of S_i such that $\mathbf{A}(V, \Sigma, q) = \emptyset$. This section will compare the sets $\tilde{\mathbf{S}}_i(q)$ and $\mathbf{S}_i(q)$, the information set revealed to agent i by the price $q \in \mathbb{R}^J$, which was introduced in Cornet-De Boisdeffre (2002). We recall that $\mathbf{S}_i(q)$ is the unique (possibly empty) subset of S_i, which is the greatest subset of S_i that is q-arbitrage-free. It is immediate to see that $\mathbf{S}_i(q) \subset \tilde{\mathbf{S}}_i(q)$ and that both sets may be empty.

For arbitrary prices, the families $(\mathbf{S}_i(q))$, $(\tilde{\mathbf{S}}_i(q))$ may not be information structures, that is, one may have $\cap_i \mathbf{S}_i(q) = \emptyset$ or $\cap_i \tilde{\mathbf{S}}_i(q) = \emptyset$. To get information structures, we now need to consider no-arbitrage prices, as in Cornet-De Boisdeffre (2002), and we recall the following definitions. Given the structure $[V, (\Sigma_i)]$, the price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price for agent i if it is a no-arbitrage price for the couple (V, Σ_i), and a common no-arbitrage price of the structure $[V, (\Sigma_i)]$ if it is a no-arbitrage price for every agent $i \in I$, that is, if it belongs to the set $Q_c[V,(\Sigma_i)] = \cap_i Q[V,\Sigma_i]$. Alternatively, the structure $[V,(\Sigma_i)]$ is said to be arbitrage-free (resp. q-arbitrage-free) if it admits a common no-arbitrage price, that is, if $Q_c[V,(\Sigma_i)] \neq \emptyset$ (resp. $q \in Q_c[V,(\Sigma_i)]$). The price $q \in \mathbb{R}^J$ is said to be a no-arbitrage price of $[V,(S_i)]$ if q is a common no-arbitrage price for some information structure (Σ_i) refining (S_i). We denote by $Q[V,(S_i)]$ the set of no-arbitrage prices of the structure $[V,(S_i)]$.

Theorem 2 *Let $[V,(S_i)]$ be a given structure. Then, for every $q \in Q[V,(S_i)]$ and every $i \in I$, $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q) = S_i^*(q) = S_i^{**}(q)$.*

The proof of Theorem 2 is a direct consequence of Theorem 1 and the following proposition, which is also of interest for itself.

Proposition 1 *Given a structure $[V,(S_i)]$ and a price $q \in \mathbb{R}^J$, the following three conditions are equivalent:*
(i) q is a no-arbitrage price, that is, $q \in Q[V,(S_i)]$;
(ii) $(\mathbf{S}_i(q))$ is an information structure, (i.e., $\cap_i S_i(q) \neq \emptyset$);
(iii) $(\tilde{\mathbf{S}}_i(q))$ is an information structure (i.e., $\cap_i \tilde{\mathbf{S}}_i(q) \neq \emptyset$), and $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage-free for every agent i at the first period ($t = 0$), in the sense that there is no portfolio $z \in \mathbb{R}^J$ such that $-q \cdot z > 0$, $V[s] \cdot z \geq 0$ for every $s \in \tilde{\mathbf{S}}_i(q)$.
Moreover, if one of the above conditions holds, $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q)$ for every i.

Proof The equivalence $[(i) \Leftrightarrow (ii)]$ is proved in Cornet-De Boisdeffre (2002).

$[(ii) \Rightarrow (iii)]$ From (ii) we first deduce that $\emptyset \neq \cap_i \mathbf{S}_i(q) \subset \cap_i \tilde{\mathbf{S}}_i(q)$. Since $\mathbf{S}_i(q) \neq \emptyset$, from the definition, $q \in Q[V,\ \mathbf{S}_i(q)]$. This implies $q \in Q[V, \tilde{\mathbf{S}}_i(q)]$, since $\mathbf{S}_i(q) \subset \tilde{\mathbf{S}}_i(q)$. Hence $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage-free at $t = 0$.

$[(iii) \Rightarrow (ii)]$ It is clearly sufficient to show that $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q)$ for every i. Let $i \in I$ be given. By definition of $\tilde{\mathbf{S}}_i(q)$, for every $\tilde{s} \in \tilde{\mathbf{S}}_i(q)$, there is no $z \in \mathbb{R}^J$ such that $-q \cdot z \geq 0, V[s] \cdot z \geq 0$, for every $s \in \tilde{\mathbf{S}}_i(q)$, and $V[\tilde{s}] \cdot z > 0$. That condition, together with the fact that $\tilde{\mathbf{S}}_j(q)$ is q-arbitrage free for every agent $j \in I$ at the first period, implies that $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage-free for agent i. Consequently, $\tilde{\mathbf{S}}_i(q) \subset \mathbf{S}_i(q)$, from the definition of $\mathbf{S}_i(q)$, whereas the inclusion $\mathbf{S}_i(q) \subset \tilde{\mathbf{S}}_i(q)$ is immediate. Hence, $\mathbf{S}_i(q) = \tilde{\mathbf{S}}_i(q)$. This completes the proof. □

We point out that the above assertions (ii) and (iii) of Proposition 1 may not be equivalent if we do not assume in (iii) that $\tilde{\mathbf{S}}_i(q)$ is q-arbitrage free for every agent i at the first period, as shown by the following counter-example.

Example. Consider two agents, five states ($S = \{1,2,3,4,5\}$), private information sets $S_1 = \{1,2,3,5\}$, $S_2 = \{1,4,5\}$, and the payoff matrix:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, for $q = (1, 1, 0)$, $\mathbf{S}_1(q) = \tilde{\mathbf{S}}_1(q) = \{2, 5\}$, $\mathbf{S}_2(q) = \emptyset$, while $\tilde{\mathbf{S}}_2(q) = \{4, 5\}$. Thus, $\emptyset = \mathbf{S}_1(q) \cap \mathbf{S}_2(q) \subset \tilde{\mathbf{S}}_1(q) \cap \tilde{\mathbf{S}}_2(q) = \{5\}$. $\square$

5 Reaching the coarsest arbitrage-free refinement

We denote by $\mathcal{S}$ the set of arbitrage-free refinements of (S_i). Given the structure $[V, (S_i)]$, we recall that there exists a unique coarsest element in $\mathcal{S}$, denoted by $(\overline{\mathbf{S}}_i[V, (S_i)])$ or simply $(\overline{\mathbf{S}}_i)$ when no confusion is possible, that is $(\overline{\mathbf{S}}_i) \in \mathcal{S}$ and $(\Sigma_i) \in \mathcal{S}$ implies $\Sigma_i \subset \overline{\mathbf{S}}_i$, for every i. We refer to [1] for this definition and, also, below for an alternative proof of the existence of $(\overline{\mathbf{S}}_i)$.

The purpose of this section is to provide an alternative process of inferences, which does not rely on prices and leads agents to infer the coarsest refinement $(\overline{\mathbf{S}}_i)$ of (S_i). Thus, given the structure $[V, (S_i)]$ and a refinement (Σ_i) of (S_i), we let, for each $i \in I$:

$$\begin{aligned}
\mathbf{A}_i(V, (\Sigma_i)) &:= \{\widetilde{s}_i \in \Sigma_i : \exists (z_j) \in (\mathbb{R}^J)^I, \textit{ such that } V[\widetilde{s}_i] \cdot z_i > 0, \textstyle\sum_{j \in I} z_j = 0 \\
&\qquad \textit{and } V[s_j] \cdot z_j \geq 0, \forall j \in I, \forall s_j \in \Sigma_j\}; \\
\mathbf{S}^1_i(V, (\Sigma_i)) &:= \Sigma_i \backslash \mathbf{A}_i(V, (\Sigma_i)).
\end{aligned}$$

Then, we define, similarly as in the previous section, two alternative inference processes, by induction on the integer $k \in \mathbb{N}$. Namely, for each $i \in I$, we let:

$S^0_i := S_i$, and, for every $k \geq 0$, $S^{k+1}_i := \mathbf{S}^1_i(V, (S^k_i))$;
similarly, we let:
$S'^0_i := S_i$, and, for every $k \geq 0$,

$$S'^{k+1}_i := \left\{ \begin{array}{l} S'^k_i, \text{ if } \mathbf{A}_i(V, (S'^k_i)) = \emptyset, \text{ and, otherwise,} \\ S'^k_i \backslash \{s^k_i\}, \text{ for some arbitrary } s^k_i \in \mathbf{A}_i(V, (S'^k_i)) \neq \emptyset \end{array} \right\}$$

Again, both sequences $(S^k_i)_k$ and $(S'^k_i)_k$ are decreasing in the finite set S_i, hence, constant for k large enough. We denote their limits by (S^*_i) and (S^{**}_i).

Theorem 3 *Let $[V, (S_i)]$ be a given structure. Then, for every $i \in I$, $S^*_i = S^{**}_i$ and (S^*_i) is the coarsest arbitrage-free refinement of (S_i).*

The proof of Theorem 3 is similar to that of Theorem 1 and left to readers.

Again, Theorem 3 shows that the chronology of all agents' inferences will not change the outcome. Whatever the individual paths of inferences, they always lead to the same limit, namely the coarsest arbitrage-free refinement $(\overline{\mathbf{S}}_i)$.

Reference

[1] Cornet, B., De Boisdeffre, L.: Arbitrage and price revelation with asymmetric information and incomplete markets. Journal of Mathematical Economics **38**, 4, 393-410 (2002)