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THE REPRESENTATION OF AMERICAN OPTIONS PRICES UNDER STOCHASTIC VOLATILITY AND JUMP-DIFFUSION DYNAMICS

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Abstract. This paper considers the problem of pricing American options when the dynamics of the underlying are driven by both stochastic volatility following a square root process as used by Heston (1993), and by a Poisson jump process as introduced by Merton (1976). Probability arguments are invoked to find a representation of the solution in terms of expectations over the joint distribution of the underlying process. A combination of Fourier transform in the log stock price and Laplace transform in the volatility is then applied to find the transition probability density function of the underlying process. It turns out that the price is given by an integral dependent upon the early exercise surface, for which a corresponding integral equation is obtained. The solution generalises in an intuitive way the structure of the solution to the corresponding European option pricing problem in the case of a call option and constant interest rates obtained by Scott (1997).

Keywords: American options, stochastic volatility, jump-diffusion processes, Volterra integral equations, free boundary problem, method of lines.

JEL Classification: C61, D11.

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1. Introduction

Derivative securities are commonly written on underlying assets with return dynamics that are not sufficiently well described by the geometric Brownian motion process proposed by Black & Scholes (1973) and Merton (1973). There have been numerous efforts to develop alternative asset return models that are capable of capturing the leptokurtic features found in financial market data, and subsequently use these models to develop option prices that accurately reflect the volatility smiles and skews found in market traded options. There are two classical ways of developing option pricing models that are capable of generating such behaviour; the first is to add jumps into the price process for the underlying asset, as originally proposed by Merton (1976); the second is to allow the volatility to evolve stochastically, for instance according to the square-root process introduced by Heston (1993).

While both alternative models have proven valuable in capturing the leptokurtosis found in realised market returns, Cont & Tankov (2004*b*) indicate that a model combining both jump-diffusion and stochastic volatility features can lead to even better results. Such a model is proposed by Bates (1996), combining the features of the models by Merton (1976) and Heston (1993). A similar model is considered by Scott (1997), generalised to allow for stochastic interest rates. Scott explores the pricing of European options under these dynamics, but American options are not considered.

There seems to have been very little research on American option pricing under stochastic volatility models with jump-diffusion, despite the fact that many traded options contain early exercise features. In this paper we consider the problem of pricing American options under the combined stochastic volatility and jump-diffusion model of Bates (1996). We focus here on the representation of the solution. We use change of measure and probabilistic arguments to obtain the general form of the American option price as well as the associated integro-partial differential equation. Implementation of this form requires knowledge of the joint transition probability density function for the log stock price and volatility, which we obtain by solving the corresponding Kolmogorov backward equation using integral transform methods.

With regard to the probability approach, this was developed by Karatzas (1988) when the underlying follows pure diffusion dynamics and extended by Pham (1997) to the case of jump-diffusion dynamics. Here we shall extend this approach to the situation when the underlying follows jump-diffusion dynamics.

The remainder of this paper is structured as follows. Section 2 outlines the free boundary problem that arises from pricing an American call option under stochastic volatility and jump-diffusion dynamics and discusses change of measure results. Section 3 derives a representation of the American option price. Section 4 applies transform techniques to solve the underlying Kolmogorov integro-partial differential equation (IPDE) for the transition probability density function. Section 5 applies this to the representation of Section 3 to obtain the expressions for the option price and free boundary. Here a more detailed discussion on the incompleteness of the model is found. Finally Section 6 concludes. Most of the lengthy mathematical derivations are given in appendices.

2. Problem Statement - The Merton-Heston-Bates Model

In this section we derive the representation of the option value by using the probabilistic arguments originally applied to the American option pricing problem by Karatzas (1988) and extended to the stochastic volatility case by Touzi (1999) and the jump-diffusion case by Pham (1997).

Assume a filtered probability measure space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\mathbb{P}$ is the market measure, and the filtration $\{\mathcal{F}_t\}$ generates all the relevant processes required in our model. Let C^E be the price of a European call option at time t written on a stock the price of which at time t is denoted S_t and strike price K. The price of its American counterpart is denoted by C^A. For the underlying dynamics, we assume that the stochastic differential equation (SDE) for S is given by the jump-diffusion process proposed by Merton (1976), in conjunction with the square root volatility process by Heston (1993). Thus the dynamics for S_t are governed by the SDE system

$$dS_t = \mu S_{t-}dt + \sqrt{v_t}S_{t-}dZ_{1,t} + S_{t-}\int_{\mathbb{R}}(e^y - 1)(p(dy,dt) - \lambda m_{\mathbb{P}}(dy)dt), \tag{1}$$

$$dv_t = \kappa_v(\theta - v_t)dt + \sigma\sqrt{v_t}dZ_{2,t}. \tag{2}$$

In (1), μ is the instantaneous return per unit time, v_t is the instantaneous squared volatility per unit time, and $Z_{1,t}$ is a standard Wiener process under the measure $\mathbb{P}$. There is a Poisson marked point process $(\{Y_i\}_{i=1}^{N_t}, N_t)$ with the associated Poisson arrival process N_t and the associated counting measure $p(dy, dt)$ under the measure $\mathbb{P}$. Its compensator under the $\mathbb{P}$-measure is $\lambda m_{\mathbb{P}}(dy)dt$. The arrival intensity of N_t is λ under $\mathbb{P}$. Conditional on the jump event occurring the distribution of the return jump-size is $m_{\mathbb{P}}(dy)$. The (return) jump-sizes Y_i arriving at different times are assumed to be independently and identically distributed with density $m_{\mathbb{P}}(dy_i) = m_{\mathbb{P}}(dy)$. The expected jump-size increment under the measure $\mathbb{P}$ is

$$\kappa = \int_{\mathbb{R}} (e^y - 1) m_{\mathbb{P}}(dy),$$

so that we can also write (1) as

$$dS_t = (\mu - \lambda\kappa) S_{t-} dt + \sqrt{v_t} S_{t-} dZ_{1,t} + S_{t-} \int_{\mathbb{R}} (e^y - 1) p(dy, dt). \tag{3}$$

We denote the moment generating function of the return jump-size in the $\mathbb{P}$-measure by

$$M_{\mathbb{P},Y}(u) = \mathbb{E}_{\mathbb{P}}[e^{uY}].$$

For now we do not make any assumptions on the distribution of the return jump-sizes except that its moment generating function exists. Note that N_t, Y and the Wiener components are otherwise uncorrelated.

In (2), θ is the long-run mean for v_t, κ_v is the rate of mean reversion, σ is the instantaneous volatility of the variance process v_t per unit time, and $Z_{2,t}$ is a standard Wiener process correlated with $Z_{1,t}$ such that $dZ_{1,t}.dZ_{2,t} = \rho dt$. This is basically a CIR square-root process (Cox, Ingersoll & Ross (1985)).

Let r be the risk-free rate of interest, and assume that the stock pays a continuously compounded dividend yield at rate q. Here we assume that r and q are both constant, although the results which follow can be readily generalised to facilitate the case where r and q are deterministic functions of time with some boundedness conditions. It will also be convenient to introduce the correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

We introduce the vector notation $\boldsymbol{Z}_t = (Z_{1,t}, Z_{2,t})^\top$ for the Wiener processes, and $\boldsymbol{\zeta}_t = (\zeta_{1,t}, \zeta_{2,t})^\top$ for the market prices of Z_1 and Z_2 risk. Note that $\zeta_{2,t}$ is referred to as the market price of variance risk. We provide assumptions on the parameters of the model (1) and (2) so that the components of $\boldsymbol{\zeta}_t = (\zeta_{1,t}, \zeta_{2,t})^\top$ are strictly positive and do not explode in finite time and that there will exist solutions to (1) and (2). Hence the first assumption:

Assumption 2.1. *The parameters μ, λ, κ_v, θ and σ are all positive constants. The parameters in (2) satisfy*

$$2\kappa_v\theta \geq \sigma^2 \tag{4}$$

and the instantaneous correlation between the Wiener components satisfy

$$-1 < \rho < \min\left(\frac{\kappa_v}{\sigma}, 1\right). \tag{5}$$

We will show in Appendix 1 that (4) allows us to conclude that the variance process v_t neither explodes nor makes excursions to zero in finite time under the various measures that we consider in the model. We will also show that (5) ensures that the solution to (1) takes the form

$$S(t) = S(0)\exp\left((\mu - \lambda\kappa)t - \frac{1}{2}\int_0^t v_u du + \int_0^t \sqrt{v_u} dZ_{1,u} + \sum_{i=1}^{N_t} Y_i\right), \tag{6}$$

where

$$\exp\left(-\frac{1}{2}\int_0^t v_u du + \int_0^t \sqrt{v_u} dZ_{1,u} - \lambda\kappa t + \sum_{i=1}^{N_t} Y_i\right) \tag{7}$$

is a strictly positive martingale under $\mathbb{P}$. The condition (5) also ensures that an equivalent solution to (1) under a suitable risk-neutral measure also exists and that the discounted stock yield process under this measure is a martingale. The assumption of the constant parameters in the SDEs in Assumption 2.1 basically means that (S_t, v_t) is jointly Markov.

The model in (1) and (2) is inherently incomplete in the Harrison & Pliska (1981) sense. Even without the jumps, the Heston (1993) stochastic volatility model is incomplete since there are two sources of Wiener risk and one traded asset, and this incompleteness can lead to situations where there is only an equivalent strictly local martingale measure, or even multiple option prices (see Sin (1998) and Heston, Loewenstein & Willard (2007)). The jump-component introduces another source of randomness into the model. In order to facilitate the analysis, a Radon-Nikodým derivative is needed for the transformation of measures from the original market measure $\mathbb{P}$ to some equivalent measure $\mathbb{Q}$. Because of the incompleteness, the parameters in the Radon-Nikodým derivative will either have to be calibrated or chosen with specific financial economic scenarios in mind. For instance, one could choose the parameters in the Radon-Nikodým derivative that minimizes the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ subject to the martingale condition for the discounted stock yield processes being met, this corresponds to the minimal entropy martingale measure found in Miyahara (2001). Alternatively, one could seek to find the values of the parameters that would minimize the divergence $\mathbb{E}_{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right]$ in the manner of Jeanblanc, Klöppel & Miyahara (2007). Yet another approach would be to choose the unspecified parameters by calibrating the model to market data, with the aim of minimizing the relative entropy of the calibrated risk-neutral measure relative to the original measure, as done in Cont & Tankov (2004*a*). It is not the aim of this paper to discuss how these parameters in the Radon-Nikodým derivative are chosen, we shall simply assume that they have been selected by one or other of the various possible methods.

In Heston et al. (2007) and Lewis (2000), it is pointed out if the volatility goes to zero or to infinity within the lifespan of the option, then there are multiple option prices. The minimal price is the risk-neutral price

$$\mathbb{E}_{\mathbb{Q}}\left[\left.\frac{C^E(S_T, v_T, T)}{e^{r(T-t)}}\right| \mathcal{F}_t\right], \tag{8}$$

where $\mathbb{Q}$ is a measure under which both the discounted stock yield process $\left\{\frac{S_t e^{qt}}{e^{rt}}\right\}$ and the discounted risk-neutral option price process are martingales. We show in Appendix 1 that condition (5) ensures that the discounted stock yield process is a strictly positive martingale under $\mathbb{Q}$. Since (S_t, v_t) is jointly Markov, we can express the risk-neutral price of a European option with non-path dependent payoff as

$$\bar{C}^E(S,v,t) = \mathbb{E}_{\mathbb{Q}}\left[\left.\frac{C^E(S_T,v_T,T)}{e^{r(T-t)}}\right| S_t = s, v_t = v\right], \tag{9}$$

that is, it allows us to write the option prices as some function of S_t, v_t and time t. The possibility of multiple option prices under a specific measure arises due to the possibility of so-called option bubbles, which are related to the probability of the volatility process exploding during the option lifespan (see Lewis (2000), Cox & Hobson (2005), Heston et al. (2007)). Thus any admissible option price $C^E(S,v,t)$ will satisfy

$$C^E(S,v,t) \geq \bar{C}^E(S,v,t). \tag{10}$$

We need (4) in Assumption 2.1 since this will ensure that the volatility neither goes to zero nor explodes under the original market measure $\mathbb{P}$ as well as under any other equivalent measure $\mathbb{Q}$. In the pure diffusion case, that is the Heston (1993) model, it is already well-known from Cox & Ross (1976) that condition (4) is sufficient to ensure that v_t is strictly positive. Also for the pure diffusion case, conditions similar to (4) and (5) are given that are sufficient for the existence of an equivalent (risk-neutral) martingale measure $\mathbb{Q}$ in the Heston (1993) model (see Wong & Heyde (2004), Andersen & Piterbarg (2007)). In Appendix 1, we provide details as to why conditions (4) and (5) are also sufficient to ensure the existence of appropriate Radon-Nikodým derivatives of the form

$$L_t = L_t^D \times L_t^J, \tag{11}$$

where

$$L_t^D = e^{-\int_0^t (\Sigma^{-1}\boldsymbol{\zeta}_u)^\top d\boldsymbol{Z}_u - \frac{1}{2}\int_0^t \boldsymbol{\zeta}_u^\top \Sigma^{-1}\boldsymbol{\zeta}_u du}, \tag{12}$$

and

$$L_t^J = e^{\sum_{i=1}^{N_t}(\gamma J_i + \nu) - \lambda\kappa' t}, \tag{13}$$

with

$$\kappa' = e^{\nu} M_{\mathbb{P},Y}(\gamma) - 1,$$

$\gamma \in \mathbb{R}$, $\nu \in \mathbb{R}$ and $\boldsymbol{\zeta}_u \in \mathbb{R}^2$. In our application, the parameters $\boldsymbol{\zeta}_u$ are adapted to the filtration and will be chosen so that they are always independent of L_t^J. Radon-Nikodým derivatives of the form (11) facilitate the measure transformation from the original measure to suitable equivalent martingale measures.

We also require the second assumption.

Assumption 2.2. *The option prices $C^E(S,v,t)$, $C^A(S,v,t)$ are at least twice differentiable in the first two variables with continuous second order partial derivatives, and at least once in the time variable with continuous first order partial derivatives with respect to time.*

This assumption allows us to apply the Itô formula or the Feynman-Kac theorem for jump-diffusion processes (see Protter (2004)). In this respect, we are dealing with a European style option on the underlying that does not have path-dependent final payoff and the final payoff function is Lipshitz, for instance, a European or American style call option.

The following theorem is standard.

Theorem 2.1. *Consider the probability measure space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ such that $\{\mathcal{F}_t\}$ is the natural filtration generated by correlated Wiener components (Z_1, Z_2) and a compound Poisson process $\sum_{n=0}^{N_t} Y_n$. Suppose L_t given by (11) is a strict martingale under $\mathbb{P}$ and that $\mathbb{E}_{\mathbb{P}}[L_t] = 1$. Then L_t is a Radon-Nikodým derivative of some equivalent measure $\mathbb{Q}$ with respect to $\mathbb{P}$, that is*

$$L_t = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = e^{-\int_0^t (\Sigma^{-1}\boldsymbol{\zeta}_u)^\top d\boldsymbol{Z}_u - \frac{1}{2}\int_0^t \boldsymbol{\zeta}_u^\top \Sigma^{-1}\boldsymbol{\zeta}_u du} e^{\sum_{i=1}^{N_t}(\gamma Y_i + \nu) - \lambda\kappa' t}. \tag{14}$$

Then the Wiener components Z_i have drift $\zeta_{i,t}$ under the measure $\mathbb{Q}$ and the compound Poisson process $\sum_{n=0}^{N_t} Y_n$ has a new intensity rate $\widetilde{\lambda} = \lambda(1+\kappa')$ and a new distribution for the jump sizes given by the moment generating function

$$M_{\mathbb{Q},Y}(u) = \frac{M_{\mathbb{P},Y}(u+\gamma)}{M_{\mathbb{P},J}(\gamma)}, \tag{15}$$

Proof. Since the Wiener part and the jump part of the Radon-Nikodým derivative (14) are independent, following Wong & Heyde (2004) or Andersen & Piterbarg (2007), the conditions in Assumption 2.1 ensures that L_t^D is a positive martingale under $\mathbb{P}$. By construction, the jump part L_t^J is already a positive martingale. Once we have established that L_t is a strictly positive martingale under $\mathbb{P}$, then we are able to determine the distribution of the Wiener components and the jump components (see Cont & Tankov (2004*b*) (Chapter 9) or Runggaldier (2003)).

■

The jump part of the Radon-Nikodým derivative (13) can be written in a more general form. In the notation of Runggaldier (2003), the jump part of the Radon-Nikodým derivative (14) takes the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{(\{Y_i\}_{i=1}^{N_t}, N_t)} = \exp\left[\int_0^t \int_{\mathbb{R}} (1 - \psi_u h_u(y))\lambda m_{\mathbb{P}}(dy)du\right] \prod_{i=1}^{N_t} \psi_{T_i} h_{T_i}(Y_i). \qquad (16)$$

A comparison between (13) and (16) allows us to make the equivalence $h_t(y) = e^{\gamma y}/M_{\mathbb{P},Y}(\gamma)$ and $\psi_t = e^{\nu} M_{\mathbb{P},Y}(\gamma) = 1 + \kappa'$. Following Runggaldier (2003), the market price of jump-risk in our model is $[\psi_t h_t - 1]m_{\mathbb{P}}(dy) = [e^{\gamma y + \nu} - 1]m_{\mathbb{P}}(dy)$ per jump-size of magnitude dy. Under $\mathbb{Q}$, the return jump-size distribution is

$$m_{\mathbb{Q}}(dy) = h_t(y) m_{\mathbb{P}}(dy) = \frac{e^{\gamma y}}{M_{\mathbb{P},Y}(\gamma)} m_{\mathbb{P}}(dy). \qquad (17)$$

We denote the associated counting measure of $N(dy, dt)$ under $\mathbb{Q}$ as $q(dy, dt)$ and its compensator is $\widetilde{\lambda} m_{\mathbb{Q}}(dy)dt$, since the arrival intensity of N_t is now $\tilde{\lambda}$ and the jump-sizes have density $m_{\mathbb{Q}}(dy)$. The choice of constant parameters γ and ν in the Radon-Nikodým derivative (14) ensures that the Poisson process N_t remains a homogeneous Poisson process and that the return jump-size distribution of the jumps arriving at different times are identically distributed under the measure transformation.

Some specific choices of the new measure $\mathbb{Q}$ can be determined by specific values of γ and ν in the Radon-Nikodým derivative (11). The choice of $\gamma = \nu = 0$ is analogous to the case in Merton's (1976) jump-diffusion model where the jump-risk is unpriced. If $\gamma = 0$ but $\nu \neq 0$, then there is a change to the jump-arrival intensity but not to the jump-size distribution under the measure transformation. Lastly, if $\nu = -\ln M_{\mathbb{P},Y}(\gamma)$, then the jump-arrival intensity does not change although the distribution of the jump-sizes changes under the transformation of measure.

We note the equivalent martingale measure $\mathbb{Q}$ is chosen so that the value of the stock position measured in units of the money market account, namely the discounted stock yield process, $\left\{\frac{S_t e^{qt}}{e^{rt}}\right\}$ is martingale under $\mathbb{Q}$ once ν, γ and $\zeta_{2,t}$ in the Radon-Nikodým derivative (14) in Theorem 2.1 are chosen. Having assured ourselves that the volatility process does not make an excursion to zero nor explodes to infinity, and the existence of an equivalent martingale measure $\mathbb{Q}$, we can conclude that there is a European call option price

$$C^E(S,v,t) = \bar{C}^E(S,v,t), \tag{18}$$

which is the risk-neutral valuation price of the discounted final payoff, under a risk-neutral measure $\mathbb{Q}$ that corresponds to a selected pair of parameters γ and ν in the Radon-Nikodým derivative (14). Here we see that the jump components also contribute to the incompleteness of the model since now the choices of γ and ν will determine the price.

For convenience, define $C_{t-} = C^E(S,v,t-)$ as the pre-jump option value evaluated at the pre-jump stock price $S_{t-}(=S)$. Standard application of Itô's Lemma for jump-diffusion processes (see Protter (2004)) yields the option price dynamics

$$\begin{aligned} dC_t = \Bigg[& \frac{\partial C_{t-}}{\partial t} + (\mu - \lambda\kappa) S_{t-} \frac{\partial C_{t-}}{\partial s} + \kappa_v(\theta - v_t)\frac{\partial C_{t-}}{\partial v} + \frac{v_t S_{t-}^2}{2}\frac{\partial^2 C_{t-}}{\partial s^2} \\ & + \rho\sigma v_t S_{t-}\frac{\partial^2 C_{t-}}{\partial s \partial v} + \frac{\sigma^2 v_t}{2}\frac{\partial^2 C_{t-}}{\partial v^2}\Bigg] dt + \sqrt{v_t} S_{t-}\frac{\partial C_{t-}}{\partial s} dZ_{1,t} \\ & + \sigma\sqrt{v_t}\frac{\partial C_{t-}}{\partial v} dZ_{2,t} + \int_{\mathbb{R}} \Big[C(S_{t-}e^y, v, t) - C_{t-} \Big] p(dy, dt). \end{aligned} \tag{19}$$

If $\zeta_{2,t}$ is specified then, by Girsanov's theorem for Wiener processes, there exists

$$d\widetilde{Z}_{2,t} = \zeta_{2,t} dt + dZ_{2,t}$$

such that $\widetilde{Z}_{2,t}$ is a standard Wiener process under $\mathbb{Q}$. Therefore, the dynamics for the variance process (2) become

$$dv_t = \kappa_v(\theta - v_t)dt - \zeta_{2,t}\sqrt{v_t}\sigma dt + \sigma\sqrt{v_t} d\widetilde{Z}_{2,t}. \tag{20}$$

If we choose

$$\zeta_{2,t} = \frac{\lambda_v \sqrt{v_t}}{\sigma}$$

to coincide with Heston's choice of the market price of volatility risk, then (21) becomes

$$dv_t = [\kappa_v \theta - (\kappa_v + \lambda_v) v_t]\, dt + \sigma \sqrt{v_t} d\widetilde{Z}_{2,t}. \tag{21}$$

Here we assume $\lambda_v \geq 0$ in line with standard financial arguments that investors require positive premiums for bearing volatility risk. This choice of $\zeta_{2,t}$ ensures that both the historical measure $\mathbb{P}$ and $\mathbb{Q}$ will be equivalent since the condition (4) results in positive values of v_t and together with condition (5), prevents v_t exploding in a finite time-horizon under $\mathbb{P}$ and also under $\mathbb{Q}$ since

$$2(\kappa_v + \lambda_v)\frac{\kappa_v \theta}{(\kappa_v + \lambda_v)} = 2\kappa_v \theta \geq \sigma^2.$$

This will be demonstrated using the usual Feller tests in Appendix 1.

From Proposition 2.1, the Poisson arrival process N_t has intensity $\widetilde{\lambda} = \lambda(1+\kappa')$ and the jump-size J has moment generating function given by (15), so that expected relative jump-size increment under the $\mathbb{Q}$-measure is

$$\widetilde{\kappa} = \int_{\mathbb{R}} (e^y - 1) m_{\mathbb{Q}}(dy).$$

Next we note that the dynamics of $S_t e^{qt}$, measured in units of the money market account, are given by

$$d\left(\frac{S_t e^{qt}}{e^{rt}}\right) = \left(\frac{S_{t-} e^{qt}}{e^{rt}}\right)\Big[(\mu + q - r - \lambda\kappa + \widetilde{\lambda}\widetilde{\kappa})dt + \sqrt{v_t} dZ_{1,t} - \widetilde{\lambda}\widetilde{\kappa} dt + \int_{\mathbb{R}} [e^y - 1] q(dy, dt)\Big]. \tag{22}$$

Thus if we set the market price of Z_1 risk as

$$\zeta_{1,t} = \frac{(\mu + q - r - \lambda\kappa + \widetilde{\lambda}\widetilde{\kappa})}{\sqrt{v_t}},$$

which is the risk premium $\mu + q - r$ of the stock less jump risk $(\lambda\kappa - \widetilde{\lambda}\widetilde{\kappa})$ per unit of stock volatility $\sqrt{v_t}$, then (22) becomes

$$d\left(\frac{S_t e^{qt}}{e^{rt}}\right) = \left(\frac{S_{t-} e^{qt}}{e^{rt}}\right)\left[\sqrt{v_t}d\widetilde{Z}_{1,t} - \widetilde{\lambda}\widetilde{\kappa}dt + \int_{\mathbb{R}}[e^y - 1]q(dy, dt)\right], \tag{23}$$

where

$$d\widetilde{Z}_{1,t} = \zeta_{1,t}dt + dZ_{1,t},$$

with $\widetilde{Z}_{1,t}$ being a standard Wiener process under the $\mathbb{Q}$-measure by Girsanov's theorem. We know that the discounted European option price process $\left\{\frac{C^E(S,v,t)}{e^{rt}}\right\}$ should be driftless under the equivalent martingale measure $\mathbb{Q}$. The following steps will enable us to express this fact as an IPDE for the option price.

We note that (23) implies that

$$\frac{dS_t}{S_{t-}} = (r - q - \widetilde{\lambda}\widetilde{\kappa})dt + \sqrt{v_t}d\widetilde{Z}_{1,t} + \int_{\mathbb{R}}(e^y - 1)q(dy, dt), \tag{24}$$

and again we will show in Appendix 1 that condition (4) and (5) ensures that (24) has a solution

$$S(t) = S(0)\exp\left((r - q - \widetilde{\lambda}\widetilde{\kappa})t - \frac{1}{2}\int_0^t v_u du + \int_0^t \sqrt{v_u}d\widetilde{Z}_{1,u} + \sum_{i=1}^{N_t} Y_i\right), \tag{25}$$

where

$$\exp\left(-\frac{1}{2}\int_0^t v_u du + \int_0^t \sqrt{v_u}d\widetilde{Z}_{1,u} - \widetilde{\lambda}\widetilde{\kappa}t + \sum_{i=1}^{N_t} Y_i\right) \tag{26}$$

is a positive martingale under $\mathbb{Q}$. A consequence of (26) being a $\mathbb{Q}$-martingale is that the stock price S_t itself can be used as numéraire and (4) prevents the value of v_t from hitting zero or exploding in a finite time-horizon under the measure associated with S_t as the numéraire (see Appendix 1).

From Itô's lemma for jump diffusion processes, the stochastic differential equation for the option price is given by

$$\begin{aligned} dC_t = \Bigg[\frac{\partial C_{t-}}{\partial t} + (r - q - \widetilde{\lambda}\widetilde{\kappa}) S_{t-} \frac{\partial C_{t-}}{\partial s} + \Big(\kappa_v(\theta - v_t) - \lambda_v v_t\Big) \frac{\partial C_{t-}}{\partial v} \\ + \frac{v_t S_{t-}^2}{2} \frac{\partial^2 C_{t-}}{\partial s^2} + \rho\sigma v_t S_{t-} \frac{\partial^2 X_{t-}}{\partial s \partial v} + \frac{\sigma^2 v_t}{2} \frac{\partial^2 C_{t-}}{\partial v^2} \\ + \widetilde{\lambda} \mathbb{E}_{\mathbb{Q}}^Y \Big[C_t(S_{t-}e^Y, v_t) - C_{t-} \Big] \Bigg] dt + \sqrt{v_t} S_{t-} \frac{\partial C_{t-}}{\partial s} d\widetilde{Z}_{1,t} \\ + \sigma \sqrt{v_t} \frac{\partial C_{t-}}{\partial v} d\widetilde{Z}_{2,t} \\ + \int_{\mathbb{R}} \Big[C_t(S_{t-}e^y, v_t) - C_{t-} \Big] (q(dy, dt) - \widetilde{\lambda} m_{\mathbb{Q}}(dy) dt), \end{aligned} \tag{27}$$

where we use the notation

$$\mathbb{E}_{\mathbb{Q}}^Y \Big[C(S_{t-}e^Y, v, t) - C_{t-} \Big] = \int_{\mathbb{R}} \Big[C(S_{t-}e^y, v, t) - C_{t-} \Big] m_{\mathbb{Q}}(dy). \tag{28}$$

In order for $\left\{ \frac{C^E(S,v,t)}{e^{rt}} \right\}$ to be driftless, the coefficient of dt in (27) in the square brackets must be zero. Hence we find that the option price must satisfy the integro-partial differential equation (IPDE)

$$\begin{aligned} \frac{\partial C_{t-}}{\partial t} + (r - q - \widetilde{\lambda}\widetilde{\kappa}) S_{t-} \frac{\partial C_{t-}}{\partial s} + \Big(\kappa_v \theta - (\kappa_v + \lambda_v) v\Big) \frac{\partial C_{t-}}{\partial v} \\ + \frac{v_t S_{t-}^2}{2} \frac{\partial^2 C_{t-}}{\partial s^2} + \rho\sigma v_t S_{t-} \frac{\partial^2 C_{t-}}{\partial s \partial v} + \sigma^2 \frac{v_t}{2} \frac{\partial^2 C_{t-}}{\partial v^2} \\ + \widetilde{\lambda} \mathbb{E}_{\mathbb{Q}}^Y \Big[C(S_{t-}e^Y, v, t) - C_{t-} \Big] = r C_{t-}. \end{aligned} \tag{29}$$

The final time conditions of the above IPDE is determined by the nature of the option of interest, for instance for a European style call option with final payoff we would have

$$C^E(S, v, T) = (S_T - K)^+, \tag{30}$$

and as discussed earlier, the European option price $C^E(S, v, t)$ is given by the risk-neutral valuation of the final payoff

$$C^E(S, v, t) = \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ e^{-r(T-t)} | \mathcal{F}_t], \tag{31}$$

in the absence of option bubbles. An application of the Feynman-Kac formula for jump-diffusion processes re-expresses (31) as (29) with final time condition (30). It will be shown in Appendix 1 that (31) can be expressed as

$$C^E(S,v,t) = S_t e^{-q(T-t)}\hat{\mathbb{Q}}(\mathcal{A}|S_t = S, v_t = v) - Ke^{-r(T-t)}\mathbb{Q}(\mathcal{A}|S_t = S, v_t = v) \quad (32)$$

where $\hat{\mathbb{Q}}$ is the measure corresponding to S_t as the numéraire, $\mathbb{Q}$ is the risk-neutral measure corresponding to the parameters γ and ν, and $\mathcal{A}$ is the event that the call option is in the money at maturity. This representation (32) is analogous to the representation obtained by Geman, El-Karoui & Rochet (1995). In later sections we see that by inversion of the Fourier and Laplace transforms of the solution to IPDE (29), the solution for the European call price is exactly (32).

Of course we are interested here in American option pricing to which we turn in the next section.

3. Representation of The American Option Price

In the case of an American style call option, the option price is

$$C^A(S,v,t) = \text{ess} \sup_{u\in[t,T]} \mathbb{E}_{\mathbb{Q}}[(S_u - K)^+ e^{-r(u-t)}|\mathcal{F}_t], \quad (33)$$

where u is a stopping time.

A quantity of interest in American option pricing is the early exercise boundary which in the current context will depend on both stochastic variance v and time t. Hence we use $S = b(v,t)$ to denote the early exercise surface at time t and variance v, though for ease of notation we shall occasionally simply write b_t.

For any option, whether American or European, let the discounted option prices be

$$\widetilde{C}^E_t = \frac{C^E(S,v,t)}{e^{rt}}, \text{ and } \widetilde{C}^A_t = \frac{C^A(S,v,t)}{e^{rt}}.$$

For the discounted option prices evaluated at the pre-jump stock prices $S_{t-}(= S)$, we write

$$\widetilde{C}^E_{t-} = \frac{C^E(S,v,t-)}{e^{rt}}, \text{ and } \widetilde{C}^A_{t-} = \frac{C^A(S,v,t-)}{e^{rt}}.$$

Applying Ito's Lemma for jump-diffusions to $\widetilde{C}_t^A$, we obtain

$$
\begin{aligned}
d\widetilde{C}_t^A = \Bigg[& \frac{\partial \widetilde{C}_{t-}^A}{\partial t} + (r - q - \widetilde{\lambda}\widetilde{\kappa}) S_{t-} \frac{\partial \widetilde{C}_{t-}^A}{\partial s} + (\kappa_v(\theta - v_t) - \lambda_v v_t) \frac{\partial \widetilde{C}_{t-}^A}{\partial v} \\
& + \frac{1}{2} v_t S_{t-}^2 \frac{\partial^2 \widetilde{C}_{t-}^A}{\partial s^2} + \rho\sigma v_t S_{t-} \frac{\partial^2 \widetilde{C}_{t-}^A}{\partial s \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 \widetilde{C}_{t-}^A}{\partial v^2} \Bigg] dt \\
& + \sqrt{v_t} S_{t-} \frac{\partial \widetilde{C}_{t-}^A}{\partial S} d\widetilde{Z}_{1,t} + \sigma \sqrt{v_t} \frac{\partial \widetilde{C}_{t-}^A}{\partial v} d\widetilde{Z}_{2,t} \\
& + \int_{\mathbb{R}} [\widetilde{C}^A(S_{t-}e^y, v, t) - \widetilde{C}^A(S_{t-}, v, t)] q(dy, dt). \qquad (34)
\end{aligned}
$$

Integrating (34) from t to T, and then multiplying (34) by the integrating factor e^{rt} yields

$$
\begin{aligned}
\frac{C_T^A}{e^{r(T-t)}} = & C_t^A + \int_t^T e^{-r(u-t)} \hat{\mathcal{L}} C_{u-}^A du + \int_t^T e^{-r(u-t)} \frac{\partial C_{u-}^A}{\partial s} \sqrt{v_u} S_{u-} d\widetilde{Z}_{1,u} \\
& + \int_t^T e^{-r(u-t)} \frac{\partial C_{u-}^A}{\partial v} \sigma \sqrt{v_u} d\widetilde{Z}_{2,u} \\
& + \int_t^T e^{-r(u-t)} \int_{\mathbb{R}} [C^A(S_{u-}e^y, v, u) - C^A(S_{u-}, v, u)][q(dy, du) - \widetilde{\lambda} m_{\mathbb{Q}}(dy) du],
\end{aligned} \qquad (35)
$$

where

$$
\begin{aligned}
\hat{\mathcal{L}} C_{u-}^A = & \frac{\partial C_{u-}^A}{\partial u} - r C_{u-}^A + (r - q - \widetilde{\lambda}\widetilde{\kappa}) S_{u-} \frac{\partial C_u^A}{\partial s} + (\kappa_v(\theta - v_t) - \lambda_v v_t) \frac{\partial C_{u-}^A}{\partial v} \\
& + \frac{v_t S_{u-}^2}{2} \frac{\partial^2 C_{u-}^A}{\partial s^2} + \rho\sigma v_t S_{t-} \frac{\partial^2 C_{u-}^A}{\partial s \partial v} + \frac{\sigma^2 v_t}{2} \frac{\partial^2 C_{u-}^A}{\partial v^2} \\
& + \widetilde{\lambda} \int_{\mathbb{R}} [C^A(S_{u-}e^y, v, u) - C^A(S_{u-}, v, u)] m_{\mathbb{Q}}(dy). \qquad (36)
\end{aligned}
$$

Note that in (35) it must be the case that the integro-partial differential operator satisfies

$$
\hat{\mathcal{L}} C_{u-}^A < 0 \qquad (37)
$$

in the early exercise region $S_{u-} \geq b_u$ since the American option is a strict supermartingale in that region. In the continuation region $S_{u-} < b_u$, it is not optimal to exercise the American option and hence it behaves like a European option there, thus

$$
\hat{\mathcal{L}} C_{u-}^A = 0. \qquad (38)
$$

Thus the American option price is the solution to the integro-partial differential equation (29) subject to the early exercise boundary condition

$$C^A(S_u, v, u) = S_u - K \quad \text{where} \quad S_u \geq b_u, \tag{39}$$

together with the smooth pasting condition

$$\lim_{s \to b_u} \frac{\partial C^A(S, v, u)}{\partial s} = 1. \tag{40}$$

In addition, the conditions

$$\frac{\partial C^A(S, v, u)}{\partial u} = 0, \quad \text{where} \quad S > b_u \tag{41}$$

and

$$\frac{\partial C^A(S, v, u)}{\partial v} = 0 \quad \text{where} \quad S > b_u, \tag{42}$$

must also be satisfied.

Figure 1 shows the relation between the payoff, price profile and early exercise boundary for the American call under consideration.

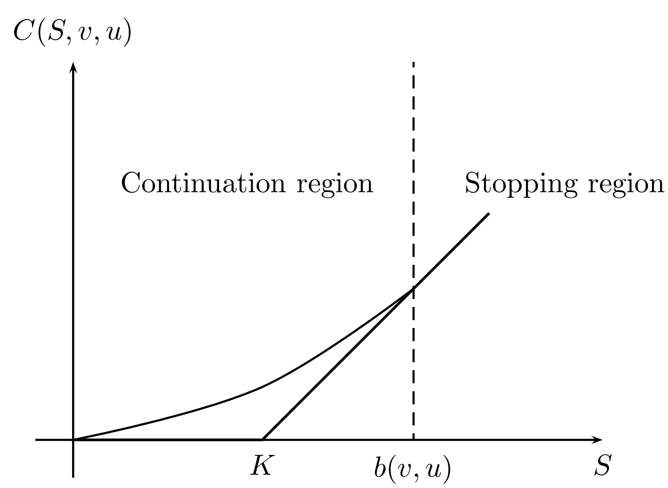

FIGURE 1. The continuation region, the stopping region and the early exercise point for the American call option, for a given value of v and time.

The following proposition gives a decomposition of the American option price in terms of its European counterpart and an early exercise premium.

Proposition 3.1. *The American call option value decomposes as*

$$C^A(S,v,t) = C^E(S,v,t) + C_P(S,v,t), \tag{43}$$

where the first term on the right hand side is the European call option price and the second term is the early exercise premium. The early exercise premium term can be written as

$$C_P(S,v,t) = C_P^D(S,v,t) - \widetilde{\lambda} C_P^J(S,v,t), \tag{44}$$

where

$$C_P^D(S,v,t) = \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}}\Big[(qS_u - rK)\mathbb{1}_{\{S_u \geq b_u\}} | \mathcal{F}_t\Big] du, \tag{45}$$

and

$$C_P^J(S,v,t) = \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}^Y\big[C^A(S_{u-}e^Y, v, u) - (S_{u-}e^Y - K)\big]\mathbb{1}_{B(u)}\Big|\mathcal{F}_t\Big] du, \tag{46}$$

where the event $B(u)$ is defined by

$$B(u) = \{b_u \leq S_{u-} < b_u e^{-Y}\}$$

with b_u being the early exercise boundary at time u and the notation $\mathbb{E}_{\mathbb{Q}}^Y$ denotes taking expectations over the return jump-sizes only.

Remark 3.1. Each component of the early exercise premium in (44) has a distinct financial interpretation. The first part, C_P^D given by (45), denotes the component of the early exercise premium arising from the diffusion part of the dynamics for the stock. Specifically, C_P^D is the expected present value of the portfolio $qS - rK$ held whenever S is in the stopping region. The second term, $\widetilde{\lambda} C_P^J$ as given in (46), arises from the presence of jumps[1], and is the expected present value of the cost incurred by the option holder whenever S jumps from the stopping region back into the continuation region. The explanation of this rebalancing cost was first given by Gukhal (2001) and a more detailed discussion is given by Chiarella & Ziogas (2008).

[1]Note that when there are no jumps in the model, $\widetilde{\lambda} = 0$ and C_P^J no longer contributes towards the early exercise premium.

Proof. Taking the conditional expectation of (35) under the martingale measure $\mathbb{Q}$ and conditioning on $\mathcal{F}_t$, the discounted conditional expectation of the final payoff of the American option is

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{C_T^A}{e^{r(T-t)}}\bigg|\mathcal{F}_t\right] = C_t^A + \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-r(u-t)}\hat{\mathcal{L}}C_{u-}^A du\bigg|\mathcal{F}_t\right]. \tag{47}$$

Note that the conditional expectation of the other remaining terms in (35) are zero since they are all (local) martingales with zero drift.

At maturity time T, the final payoffs of the American and European calls are the same, that is, $C^A(S,v,T) = C^E(S,v,T) = (S_T - K)^+$. Hence (47) simplifies to

$$C_t^A = C_t^E - \int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[\hat{\mathcal{L}}C_{u-}^A \mid \mathcal{F}_t]du. \tag{48}$$

Decomposing the integral in the second term of (48) as integrals over the early exercise region and over the continuation region, the American option price satisfies

$$\begin{aligned} C_t^A &= C_t^E - \int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[\hat{\mathcal{L}}C_{u-}^A \mathbb{1}_{\{S_{u-}\geq b_u\}} + \hat{\mathcal{L}}C_{u-}^A \mathbb{1}_{\{S_{u-}<b_u\}} \mid \mathcal{F}_t]du \\ &= C_t^E - \int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[\hat{\mathcal{L}}C_{u-}^A \mathbb{1}_{\{S_{u-}\geq b_u\}} \mid \mathcal{F}_t]du, \end{aligned} \tag{49}$$

where we have made use of (37) and (38).

Using the boundary and smooth pasting conditions (40) to (42) in the early exercise region, the last line (49) becomes

$$\begin{aligned} C_t^A = C_t^E &- \int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[(-r(S_{u-} - K) + (r - q - \widetilde{\lambda}\widetilde{\kappa})S_{u-})\mathbb{1}_{\{S_{u-}\geq b_u\}} \mid \mathcal{F}_t]du \\ &- \widetilde{\lambda}\int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}^Y(C^A(S_{u-}e^Y, v, u) - (S_{u-} - K))\mathbb{1}_{\{S_{u-}\geq b_u\}} \mid \mathcal{F}_t]du \\ = C_t^E &+ \int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[(qS_{u-} - rK)\mathbb{1}_{\{S_{u-}\geq b_u\}} \mid \mathcal{F}_t]du \\ &- \widetilde{\lambda}\int_t^T e^{-r(u-t)}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}^Y[C^A(S_{u-}e^Y, v, u) - (S_{u-}e^Y - K)]\mathbb{1}_{\{S_{u-}\geq b_u\}} \mid \mathcal{F}_t]du. \end{aligned} \tag{50}$$

Note that in (50), the post-jump option price less the post-jump intrinsic value if $S_{u-}e^Y \geq b_u$ satisfies

$$C^A(S_{u-}e^Y, v, u) - (S_{u-}e^Y - K) = 0,$$

and if $S_{u-}e^{Y} < b_u$ satisfies

$$C^A(S_{u-}e^Y, v, u) - (S_{u-}e^Y - K) > 0.$$

Hence (50) can be written as

$$\begin{aligned} C_t^A =& C_t^E + \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}}[(qS_{u-} - rK)\mathbb{1}_{\{S_{u-} \geq b_u\}} \mid \mathcal{F}_t]du \\ & - \widetilde{\lambda} \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}^Y[C^A(S_{u-}e^Y, v, u) - (S_{u-}e^Y - K)]\mathbb{1}_{\{b_u < S_{u-} < b_u e^{-Y}\}} \mid \mathcal{F}_t]du. \end{aligned}$$

The last equation is the result of the proposition and we define $C_P(S, v, t)$ according to (44).

■

In order to evaluate the option price from equation (43) we need an expression for the early exercise boundary $b(v, t)$. Using the fact that at the early exercise boundary we have

$$C^A(b_t, v, t) = b_t - K \tag{51}$$

we can obtain the integral equation that determines the early exercise boundary.

Proposition 3.2. *The early exercise boundary $b(v, t)$ is determined by the integral equation*

$$b(v, t) - K = C^E(b_t, v, t) + C_P(b_t, v, t) \tag{52}$$

Proof. A simple matter of setting $S = b_t$ in (43) and making use of (51).

■

It should be kept in mind that the term $C_P(b_t, v, t)$ in (52) involves the unknown option price though its second term (the one arising due to the presence of jumps). Hence (43) and (52) need to be solved as a linked system, this is in contrast to the corresponding system of equations in the no jump case which can be solved sequentially, that is first the integral equation for the free boundary is solved and then that is used to evaluate the option price.

4. Calculation of the Transition Density Function

In order to compute the terms on the right hand side of (44) we need to be able to calculate the expectations $\mathbb{E}_{\mathbb{Q}}$ and $\mathbb{E}_{\mathbb{Q}}^{Y}$. To calculate the expectation restricted to the jump-sizes $\mathbb{E}_{\mathbb{Q}}^{Y}$ we need to specify a jump-size distribution, below we shall assume that the Y are normally distributed. The calculation of $\mathbb{E}_{\mathbb{Q}}$ requires knowledge of the joint transition density of S and v given that their dynamics under the risk-neutral measure $\mathbb{Q}$ are driven by equations (24) and (21). As is well known the joint transition density satisfies the Kolmogorov backward equation associated with (24) and (21), which here will be an IPDE because of the jump term. In terms of the time variable t we use $G(S, v, t; S_T, v_T, T)$ to denote the transition probability density for passage from S, v at current time t to S_T, v_T at maturity time T. In order to express the IPDE for G as an initial value problem (which is standard for the transform techniques we shall use) it if more convenient in this section to express G in terms of time-to-maturity $\tau \equiv T - t$. Thus in this section we solve for $G(S, v, \tau; S_0, v_0, 0)$[2] which is now interpreted as the transition probability density for passage from S, v at time-to-maturity τ to S_0, v_0 at maturity. Quite frequently we will simply write $G(S, v, \tau; S_0, v_0)$ and in order to simplify the notation we shall suppress the dependence on S_0, v_0 unless it is required. The Kolmogorov IPDE associated with the system (24) and (21), in terms of time-to-maturity τ, is

$$\begin{aligned}\frac{\partial G}{\partial \tau} = \frac{vS^2}{2}\frac{\partial^2 G}{\partial S^2} + \rho\sigma v S\frac{\partial^2 G}{\partial S \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 G}{\partial v^2} \\ + (r - q - \widetilde{\lambda}\widetilde{\kappa})S\frac{\partial G}{\partial S} + \big(\kappa_v\theta - (\kappa_v + \lambda_v)v\big)\frac{\partial G}{\partial v} \\ + \widetilde{\lambda}\int_0^\infty [G(Se^y, v, \tau) - G(S, v, \tau)]Q(y)dy. \end{aligned} \tag{53}$$

In the last term on the RHS we have set $m_{\mathbb{Q}}(y) = Q(y)dy$ where $Q(y)$ is the density function of the jump-size distribution under the risk neutral measure. Equation (53) is

[2]Of course there is an abuse of notation here, strictly speaking we should introduce a different name for this function since it is in fact $G(S, v, T - \tau; S_T, v_T, T)$. However in order to reduce the notational burden we shall continue to use G and convert back to the time t notation once we have obtained the solution.

to be solved subject to the initial conditions

$$G(S, v, 0; S_0, v_0) = \delta(S - S_0)\delta(v - v_0), \tag{54}$$

where $\delta(\cdot)$ denotes the Dirac delta function.

Here we shall use the Fourier transform to reduce the two-dimensional IPDE (53) to a one-dimensional PDE whose solution is already known, thereby allowing us to readily find the solution to (53). We begin by making a change of variable. Let $S = e^x$ and with a slight abuse of notation we continue to refer to the density function as $G(x, v, \tau; x_0, v_0)$. Thus equation (53) becomes

$$\begin{aligned} \frac{\partial G}{\partial \tau} = \frac{v}{2}\frac{\partial^2 G}{\partial x^2} + \rho\sigma v \frac{\partial^2 G}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 G}{\partial v^2} + \left(r - q - \widetilde{\lambda}\widetilde{\kappa} - \frac{v}{2}\right)\frac{\partial G}{\partial x} + (\alpha - \beta v)\frac{\partial G}{\partial v} \\ + \widetilde{\lambda}\int_0^\infty [G(x + y, v, \tau) - G(x, v, \tau)]Q(y)dy, \end{aligned} \tag{55}$$

which is to be solved in the region $-\infty < x < \infty$, $0 \leq v < \infty$, $0 \leq \tau \leq T$, subject to the initial condition

$$G(x, v, 0) = \delta(x - x_0)\delta(v - v_0),$$

with $\alpha \equiv \kappa_v \theta$, $\beta \equiv \kappa_v + \lambda_v$ and $x_0 = \ln S_0$. Since x does not appear in the coefficients of any of the terms in (55), we are now able to take the Fourier transform of the IPDE with respect to x, which we present in Proposition 4.1 below.

It should be noted that integral transforms require knowledge of the behaviour of the functions being transformed at the extremities of the domain. In applying the Fourier transform, we require that $G(x, v, \tau)$ and $\partial G/\partial x$ tend to zero as $x \to \pm\infty$. These conditions will be assumed to be satisfied as they seem reasonable to impose on density functions under jump-diffusion dynamics with the jump-size distribution we assume here.

Proposition 4.1. *Let $\mathcal{F}_x\{G(x, v, \tau)\}$ be the Fourier transform of $G(x, v, \tau)$ taken with respect to x, defined as*

$$\mathcal{F}_x\{G(x, v, \tau)\} = \int_{-\infty}^{\infty} e^{i\phi x} G(x, v, \tau)dx = \hat{G}(\phi, v, \tau). \tag{56}$$

The Fourier transform of (55) is the solution of the differential equation

$$\frac{\partial \hat{G}}{\partial \tau} = \frac{\sigma^2 v}{2}\frac{\partial^2 \hat{G}}{\partial v^2} + (\alpha - \Theta v)\frac{\partial \hat{G}}{\partial v} + \left(\frac{\Lambda}{2}v - i\phi\Psi\right)\hat{G}, \tag{57}$$

where

$$\Theta = \Theta(\phi) \equiv \beta + \rho\sigma i\phi,$$

$$\Lambda = \Lambda(\phi) \equiv i\phi - \phi^2,$$

$$\Psi = \Psi(\phi) \equiv r - q - \widetilde{\lambda}\widetilde{\kappa} - \frac{\widetilde{\lambda}}{i\phi}(A(\phi) - 1),$$

and

$$A(\phi) \equiv \int_0^\infty Q(y)e^{-i\phi y}dy. \tag{58}$$

We note that the initial condition $\hat{G}(\phi, v, 0) \equiv \hat{g}(\phi, v) = e^{i\phi x_0}\delta(v - v_0)$*, is obtained by calculating directly* $\mathcal{F}_x\{G(x, v, 0)\}$*.*

Proof: Refer to Appendix 2.

■

The general solution of the two-dimensional PDE (57) has already been derived by Feller (1951) using a Laplace transform approach in the v direction.[3] The solution procedure in the case of stochastic volatility only is given in Adolfsson, Chiarella & Ziogas (2009). For completeness we will outline the main steps, noting that the main difference here is the Ψ term in (57) that contains additional terms arising from the jump process.

In applying the Laplace transform in the v direction we must make certain assumptions about the behaviour of $\hat{G}(\phi, v, \tau)$, specifically that $e^{-sv}\hat{G}(\phi, v, \tau)$ and $e^{-sv}\partial\hat{G}(\phi, v, \tau)/\partial v$ tend to zero as $v \to \infty$, where s is the Laplace transform variable[4]. Proposition 4.5 provides the Laplace transform of (57) with respect to v.

[3]Feller (1951) in essence obtained the transitional probability density function for the process for v. Whilst the problem considered here is more general, involving the dynamics of S with jumps, the main steps in solving for the transitional probability density function by transform methods are essentially the same

[4]Again these conditions can be reasonably assumed to be satisfied by the density function under consideration here.

Proposition 4.2. *Let $\mathcal{L}_v\{\hat{G}(\phi,v,\tau)\}$ be the Laplace transform of $\hat{G}(\phi,v,\tau)$ taken with respect to v, defined as*

$$\mathcal{L}_v\{\hat{G}(\phi,v,\tau)\} = \int_0^\infty e^{-sv}\hat{G}(\phi,v,\tau)dv \equiv \tilde{G}(\phi,s,\tau). \tag{59}$$

Applying the Laplace transform to (57) we find the $\tilde{G}$ satisfies the PDE

$$\frac{\partial\tilde{G}}{\partial\tau} + \left(\frac{\sigma^2}{2} - \Theta s + \frac{\Lambda}{2}\right)\frac{\partial\tilde{G}}{\partial s} = \left[(\alpha-\sigma^2)s + \Theta - i\phi\Psi\right]\tilde{G} + f_L(\tau), \tag{60}$$

with initial condition $\tilde{G}(\phi,s,0) \equiv \tilde{g}(\phi,v) = e^{-isv_0+i\phi x_0}$, and where $f_L(\tau) \equiv (\sigma^2/2 - \alpha)\hat{G}(\phi,0,\tau)$ is determined such that

$$\lim_{s\to\infty}\tilde{G}(\phi,s,\tau) = 0. \tag{61}$$

Proof: Refer to Appendix 3.

■

Equation (60) is now a first order PDE which can be solved using the method of characteristics. The unknown function $f_L(\tau)$ on the right hand side of (60) is then found by applying the condition (61). In this way we are able to solve (60) for $\tilde{G}(\phi,s,\tau)$, and the result is given in Proposition 4.3.

For future reference we recall that $\Gamma(n;z)$ is a (lower) incomplete gamma function, defined as

$$\Gamma(n;z) = \frac{1}{\Gamma(n)}\int_0^z e^{-\xi}\xi^{n-1}d\xi, \tag{62}$$

and $\Gamma(n)$ is the (complete) gamma function given by

$$\Gamma(n) = \int_0^\infty e^{-\xi}\xi^{n-1}d\xi. \tag{63}$$

Proposition 4.3. *Using the method of characteristics, and subsequently applying condition (61) to determine* $f_L(\tau)$*, the solution to the first order PDE (60) is*

$$\begin{aligned}\tilde{G}(\phi, s, \tau) = &\exp\left\{ \left[\frac{(\alpha - \sigma^2)(\Theta - \Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] \tau \right\} \\ &\times \left(\frac{2\Omega}{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega} \right)^{2 - \frac{2\alpha}{\sigma^2}} \\ &\times e^{i\phi x_0 - \left(\frac{\Theta - \Omega}{\sigma^2}\right) v_0} \exp\left\{ \frac{-2\Omega v_0 (\sigma^2 s - \Theta + \Omega) e^{\Omega\tau}}{\sigma^2[(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega]} \right\} \\ &\times \Gamma\left(\frac{2\alpha}{\sigma^2} - 1; \frac{2\Omega v_0 e^{\Omega\tau}}{\sigma^2(e^{\Omega\tau} - 1)} \times \frac{2\Omega}{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega} \right),\end{aligned} \tag{64}$$

where

$$\Omega = \Omega(\phi) \equiv \sqrt{\Theta^2(\phi) - \sigma^2 \Lambda(\phi)}. \tag{65}$$

Proof: Refer to Appendix 4.

■

Having determined $\tilde{G}(\phi, s, \tau)$, we now seek to return to the original variables S and v, and thus obtain the expression for $G(S, v, \tau)$. We begin this process by inverting the Laplace transform in Proposition 4.4, again using the techniques of Feller (1951).

Proposition 4.4. *The inverse Laplace transform of* $\tilde{G}(\phi, s, \tau)$ *in (64) is*

$$\begin{aligned}\hat{G}(\phi, v, \tau) = &e^{i\phi x_0 + \frac{(\Theta - \Omega)}{\sigma^2}(v - v_0 + \alpha\tau)} e^{-i\phi\Psi\tau} \\ &\times \frac{2\Omega e^{\Omega\tau}}{\sigma^2(e^{\Omega\tau} - 1)} \left(\frac{v_0 e^{\Omega\tau}}{v} \right)^{\frac{\alpha}{\sigma^2} - \frac{1}{2}} \exp\left\{ \frac{-2\Omega}{\sigma^2(e^{\Omega\tau} - 1)} (v_0 e^{\Omega\tau} + v) \right\} \\ &\times I_{\frac{2\alpha}{\sigma^2} - 1}\left(\frac{4\Omega}{\sigma^2(e^{\Omega\tau} - 1)} (v_0 v e^{\Omega\tau})^{\frac{1}{2}} \right),\end{aligned} \tag{66}$$

where $I_k(z)$ *is the modified Bessel function of the first kind, defined as*

$$I_k(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+k}}{\Gamma(n + k + 1) n!}. \tag{67}$$

Proof: Refer to Appendix 5.

■

Next we take the inverse Fourier transform of (66) and return to the original variables, to obtain the form of the transition probability density function $G(S, v, \tau; S_o, v_0)$ as featured in Proposition 4.5.

Proposition 4.5. *Given the definition of $\mathcal{F}_x$ in (56), the inverse Fourier transform is*

$$\mathcal{F}_x^{-1}\{\hat{G}(\phi, v, \tau)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x}\hat{G}(\phi, v, \tau)d\phi = G(x, v, \tau). \tag{68}$$

Substitution of (66) into (68) yields (after transforming back to the original S variable)

$$\begin{aligned} &G(S, v, \tau; S_0, v_0) \\ &= \sum_{n=0}^{\infty} \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\phi \ln S} e^{\frac{(\Theta-\Omega)}{\sigma^2}(v_0 - v + \alpha\tau)} e^{-i\phi(r-q-\widetilde{\lambda}\widetilde{k})\tau} \right. \\ &\qquad \times e^{-i\phi \ln S_0 X_n} \frac{2\Omega}{\sigma^2(e^{\Omega\tau}-1)} \left(\frac{v e^{\Omega\tau}}{v}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \\ &\qquad \times \exp\left\{ \frac{-2\Omega}{\sigma^2(e^{\Omega\tau}-1)} \left(v e^{\Omega\tau} + v\right) \right\} \\ &\qquad \left. \times I_{\frac{2\alpha}{\sigma^2}-1}\left(\frac{4\Omega}{\sigma^2(e^{\Omega\tau}-1)} \left(v v_0 e^{\Omega\tau}\right)^{\frac{1}{2}} \right) d\phi \right\}, \end{aligned} \tag{69}$$

with $X_n \equiv e^{y_1}e^{y_2}\dots e^{y_n}$, $X_0 \equiv 1$, and

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}^{(n)}[f(X_n)] &= \int_0^{\infty} f(X_n)Q(X_n)dX_n \\ &= \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} f(e^{y_1}e^{y_2}\dots e^{y_n})Q(y_1)Q(y_2)\dots Q(y_n)dy_1dy_2\dots dy_n. \end{aligned} \tag{70}$$

Note that each y_j ($j = 0, \dots, n$) is an independent jump drawn from the density $Q(y)$, and it is assumed that the density $Q(y)$ is of a form which facilitates the reduction from an n-dimensional integral to a one-dimensional integral in (70)[5].

Proof: Refer to Appendix 6.

■

[5]This holds true for certain popular types of distributions, such as the lognormal (Merton, 1976) and the double exponential (Kou, 2002).

5. The American Option Value

Having found the transition probability density function $G(S, v, \tau; S_0, v_0)$ we can now find both the European call price and the early exercise premium term for the American call option that occur in the representation given by equation (43). Firstly, we evaluate the European call price defined at equation (31), and express it in a form that is analogous to the solution of Heston (1993).

Note that in the following discussion it is convenient to denote the initial and final stock price, volatility and time by S, v, t and S_T, v_T, T respectively and also to express G in terms of time t. Thus (31) can be expressed as[6]

$$C^E(S, v, t) = \int_0^\infty \int_0^\infty (S_T - K)^+ G(S, v, t; S_T, v_T, T) dS_T dv_T. \tag{71}$$

Proposition 5.1. *The European call price, $C^E(S, v, t)$, given in (71) can be expressed as*

$$C^E(S, v, t) = \sum_{n=0}^{\infty} \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Big\{ S X_n e^{-\widetilde{\lambda}\widetilde{\kappa}\tau} e^{-q\tau} P_1^H(S X_n e^{-\widetilde{\lambda}\widetilde{\kappa}\tau}, v, \tau; K) \\ - K e^{-r\tau} P_2^H(S X_n e^{-\widetilde{\lambda}\widetilde{\kappa}\tau}, v, \tau; K) \Big\}, \tag{72}$$

where $\tau = T - t$ and

$$P_j^H(S, v, \tau; K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left(\frac{f_j(S, v, \tau; \phi) e^{-i\phi \ln K}}{i\phi} \right) d\phi, \tag{73}$$

for $j = 1, 2$, with

$$f_j(S, v, \tau; \phi) = \exp\{B_j(\phi, \tau) + D_j(\phi, \tau) + i\phi \ln S\}, \tag{74}$$
$$B_j(\phi, \tau) = i\phi(r - q)\tau + \frac{\alpha}{\sigma^2} \left\{ (\Theta_j + \Omega_j)\tau - 2 \ln \left(\frac{1 - Q_j e^{\Omega_j \tau}}{1 - Q_j} \right) \right\},$$
$$D_j(\phi, j) = \frac{(\Theta_j + \Omega_j)}{\sigma^2} \left(\frac{1 - e^{\Omega_j \tau}}{1 - \Theta_i e^{\Omega_j \tau}} \right),$$

and $Q_j = (\Theta_j + \Omega_j)/(\Theta_j - \Omega_j)$, where we define $\Theta_1 = \Theta(i - \phi)$, $\Omega_1 = \Omega(i - \phi)$, and $\Theta_2 = \Theta(-\phi)$, $\Omega_2 = \Omega(-\phi)$. The random variable X_n has been defined in Proposition 4.5.

[6] Note that we now switch back to the time t notation.

Proof: Refer to Appendix 7.

■

Next we use (44) to determine the early exercise premium for the American call.

Proposition 5.2. *By use of equation (69), the early exercise premium for the American call, $C_P(S,v,t)$ in (44) can be expressed as*

$$C_P(S,v,t) = C_P^D(S,v,t) - \widetilde{\lambda} C_P^J(S,v,t). \tag{75}$$

The term $C_P^D(S,v,t)$ is given by

$$\begin{aligned} C_P^D(S,v,t) \equiv \sum_{n=0}^{\infty} \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^{\tau} \int_0^{\infty} (\tau-\xi)^n e^{-\widetilde{\lambda}(\tau-\xi)} \\ \times \Bigg[qSX_n e^{-\widetilde{\lambda}(\tau-\xi)} e^{-q(\tau-\xi)} P_1^A(SX_n e^{-\widetilde{\lambda}(\tau-\xi)}, v, \tau-\xi; v_T, b(v_T,\xi)) \\ - rKe^{-r(\tau-\xi)} P_2^A(SX_n e^{-\widetilde{\lambda}(\tau-\xi)}, v, \tau-\xi; v_T, b(v_T,\xi)) \Bigg] dv_T d\xi \Bigg\}, \end{aligned} \tag{76}$$

where $\tau = T - t$ and

$$P_j^A(S,v,\tau-\xi;v_T,b(v_T,\xi)) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{g_j(S,v,\tau-\xi;\phi,v_T)e^{-i\phi\ln b(v_T,\xi)}}{i\phi}\right) d\phi, \tag{77}$$

for $j = 1,2$, with

$$\begin{aligned} g_j(S,v,\tau-\xi;\phi,v_T) = e^{\frac{(\Theta_j-\Omega_j)}{\sigma^2}(v-v_T+\alpha(\tau-\xi))} e^{i\phi(r-q)(\tau-\xi)} e^{i\phi\ln S} \\ \times \frac{2\Omega_j}{\sigma^2(e^{\Omega j(\tau-\xi)}-1)} \left(\frac{v_T e^{\Omega_j(\tau-\xi)}}{v}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \\ \times \exp\left\{\frac{-2\Omega_j}{\sigma^2(e^{\Omega j(\tau-\xi)}-1)}(v_T e^{\Omega_j(\tau-\xi)}+v)\right\} \\ \times I_{\frac{2\alpha}{\sigma^2}-1}\left(\frac{4\Omega_j}{\sigma^2(e^{\Omega_j(\tau-\xi)}-1)}(v_T v e^{\Omega j(\tau-\xi)})^{\frac{1}{2}}\right), \end{aligned} \tag{78}$$

where $I_k(z)$ given by (67), and Θ_j and Ω_j are given in Proposition 5.1.

The term $C_P^J(S,v,\tau)$ *is given by*

$$C_P^J(S,v,t) = \sum_{n=0}^{\infty} \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Big\{ \int_0^{\tau} \int_0^{\infty} (\tau-\xi)^n e^{-\widetilde{\lambda}(\tau-\xi)} e^{-r(\tau-\xi)} \times \int_0^1 G(Y) \int_{\ln b(v_T,\xi)}^{\ln[b(v_T,\xi)/Y]} [C(zY,v_T,\xi) - (zY-K)] \times \bar{Q}_J(z,v_T,\tau-\xi;SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)},v) dz dY dv_T d\xi \Big\}, \tag{79}$$

where

$$\bar{Q}_J(z,v_T,\tau-\xi;SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)},v) \equiv \frac{1}{2\pi z} \int_{-\infty}^{\infty} e^{\frac{(\Theta-\Omega)}{\sigma^2}(v-v_T+\alpha(\tau-\xi))} e^{-i\phi(r-q)(\tau-\xi)} \times e^{-i\phi \ln(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}/z)} \frac{2\Omega}{\sigma^2(e^{\Omega(\tau-\xi)}-1)} \left(\frac{v_T e^{\Omega(\tau-\xi)}}{v}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \times \exp\left\{\frac{-2\Omega}{\sigma^2(e^{\Omega\tau}-1)}(v_T e^{\Omega(\tau-\xi)}+v)\right\} \times I_{\frac{2\alpha}{\sigma^2}-1}\left(\frac{4\Omega}{\sigma^2(e^{\Omega(\tau-\xi)}-1)}(v_T v e^{\Omega(\tau-\xi)})^{\frac{1}{2}}\right) d\phi. \tag{80}$$

Proof: Refer to Appendix 8.

■

The interpretations of the terms C_P^D and C_P^J that occur here have already been given in the remark after the statement of Proposition 3.1.

We also note that the functions $f_j(S,v,\tau;\phi)$ in (74) and $g_j(S,v,\tau;\phi,w)$ in (78) are related according to

$$f_j(S,v,\tau;\phi) = \int_0^{\infty} g_j(S,v,\tau;\phi,w)dw.$$

Having derived integral representations for the European call price and early exercise premium, we now obtain the integral equation for the American call price. We can also readily derive the corresponding integral equation for the early exercise boundary, and thus determine the linked system of integral equations for $C(S,v,t)$ and $b(v,t)$.

Proposition 5.3. *The price of an American call, $C^A(S,v,t)$, written on S is given by*

$$C^A(S,v,t) = C^E(S,v,t) + C_P^D(S,v,t) - \widetilde{\lambda} C_P^J(S,v,t), \tag{81}$$

where $C^E(S,v,t)$ is given by (72), and the terms $C_P^D(S,v,t)$ and $C_P^J(S,v,t)$ are given respectively by (76) and (79). Equation (81) depends upon the early exercise boundary, $b(v,\tau)$, which is the solution to the integral equation

$$C(b(v,t),v,t) = b(v,t) - K. \tag{82}$$

Proof: Substituting (72) and (75) into (43) gives (81). Evaluating (81) at $S = b(v,\tau)$ and applying the boundary condition (39) produces (82).

■

We should point out that the decomposition (81) is a representation of (46) where we have effectively evaluated the various expectation operations by the use of transform methods. Equations (81)-(82) both contain integrals involving $C(S,v,\tau)$ and $b(v,\tau)$. The dependence upon C arises because of the presence of the jump terms. This means that one cannot solve sequentially for $b(v,\tau)$ and $C(S,v,\tau)$, as in the corresponding situation when jumps are not present. While it is possible to develop numerical methods that reduce this dependence, as demonstrated by Chiarella & Ziogas (2008) for the case of American options under jump-diffusion dynamics, such approaches involve an exponentially increasing computational burden as the number of underlying stochastic factors in the model increases.

Tzavalis & Wang (2003) provide an integration method for pricing American call options under stochastic volatility. One of the features of this method is that the free boundary is approximated as an exponential-linear function of v, which in turn provides a reduction of the dimensions for the integration in $C_P^D(S,v,\tau)$. This will not hold true once jumps are introduced, as the term $C_P^J(S,v,\tau)$ cannot be simplified in this manner. We note, however, that depending on the functional form of $Q(Y)$, we may be able to complete the integration with respect to Y analytically in $C_P^J(S,v,\tau)$, after interchanging the order of integration for z and Y. This, combined with the need to evaluate the infinite sums arising from the Poisson arrival process for the jumps, results in the fact that the system (81)-(82) would be very cumbersome to solve. In a related paper Chiarella, Kang, Meyer & Ziogas (2009) discuss the solution of the IPDE (29) subject to the free boundary value conditions (39) and (40) using the method lines, which they find to be a relatively efficient method.

6. Conclusion

This paper studies the evaluation of American call options under stochastic volatility and jump-diffusion dynamics. Using change of measure results we have derived the integro-partial differential equation that determines the option price. We have also shown how the American option price may be represented as the sum of the corresponding European option and an early exercise premium. In order to operationalise the representation it is necessary to obtain the transition probability density function for the stock price and variance under the risk neutral jump-diffusion dynamics. This is done by solving the associated Kolmogorov IPDE using a combination of the Fourier transform (in the log stock price dimension) and the Laplace transform (in the variance dimension).

The resulting transition probability density function is then used to express the American option price in integral form involving the early exercise surface, for which an integral equation is obtained. It turns out that the solution for the American call involves a linked system of integral equations for the option price and early exercise surface. The difficulties involved in solving this system of integral equations are also discussed.

Here we have focused on the American call option, but knowledge of the transition probability density function allows similar representations to be found for a range of other payoffs such as the put, strangles and various other positions. The knowledge of the transition probably density function may also be used to develop efficient Monte Carlo schemes for option payoffs for which analytic representations may not be so readily found, such as barrier type options.

Appendix 1. Explosion of the Volatility Process

As discussed in Section 2, we want to ensure that the discounted stock yield process is a martingale, and the existence of a risk-neutral valuation of the option price

$$\bar{C}^E(S, v, t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+|\mathcal{F}_t], \tag{83}$$

which we want to be the only option price $C^E(S, v, t)$ if the market price of jump-risk is determined a priori by the parameters γ and ν in the Radon-Nikodým derivative (14). For the option price at time $t = 0$, this is then

$$C^E(S, v, 0) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S(T) - K)\mathbb{1}_{\mathcal{A}}], \tag{84}$$

where $\mathcal{A}$ denotes the event that the option is in the money at maturity.

Following Geman et al. (1995), we need conditions that will enable us to express the call option price at time $t = 0$ as

$$C^E(S, v, 0) = Se^{-qT}\hat{\mathbb{Q}}(\mathcal{A}) - Ke^{-rT}\mathbb{Q}(\mathcal{A}), \tag{85}$$

and equivalently at time t as

$$C^E(S_t, v_t, t) = S_t e^{-q(T-t)}\hat{\mathbb{Q}}(\mathcal{A}|S_t, v_t) - Ke^{-r(T-t)}\mathbb{Q}(\mathcal{A}|S_t, v_t), \tag{86}$$

where $\hat{\mathbb{Q}}$ is the measure corresponding to the stock price S_t as the numéraire. Thus the conditions that we require should not only allow us to establish the existence of the Radon-Nikodým derivatives $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_t$ and $\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_t$ but also $\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}\Big|_t$. This is only possible if the variance process v_t neither explodes nor makes excursions to zero in finite time under the measures $\mathbb{P}$, $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ (analogous to similar results for the pure-diffusion stochastic volatility models in Sin (1998), Andersen & Piterbarg (2007) and Heston et al. (2007)). In this Appendix, we show that the conditions (4) and (5) in Assumption 2.1 are sufficient.

We first state a result about the explosion and the excursion to zero in the Heston (1993) stochastic volatility model. Similar results can be found in Andersen & Piterbarg (2007).

Theorem A.1.1. *Consider a filtered probability measure space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \hat{\mathbb{P}})$. *Suppose*

$$dv_t = a(b - v_t)dt + \sigma\sqrt{v_t}dB_t, \tag{87}$$

where B_t *is a standard Wiener process under the measure* $\hat{\mathbb{P}}$ *and adapted to the filtration that we are considering, and the inequality*

$$2ab \geq \sigma^2 \tag{88}$$

holds. Then the process v_t *neither explodes nor makes an excursion to zero in finite time under* $\hat{\mathbb{P}}$.

Proof. In order to apply the Feller test (see Lewis (2000)) to show that v_t neither explodes nor makes excursions to zero in finite time, we need to examine the scale measure

$$S(c,d) = \int_c^d e^{\frac{2ax}{\sigma^2}} x^{-\frac{2ab}{\sigma^2}} dx. \tag{89}$$

The conditions

$$\lim_{c \downarrow 0} S(c,d) = \infty \text{ and } \lim_{d \uparrow \infty} S(c,d) = \infty \tag{90}$$

must be satisfied so that v_t neither explodes nor makes excursions to zero in finite time. It is clear from (89) that (88) is sufficient for (90) to hold since a and b are always assumed to be positive.

■

In Section 2, under the original market measure $\mathbb{P}$, the variance SDE (87) takes the form

$$a = \kappa_v, \quad b = \theta \tag{91}$$

and $B_t = Z_{2,t}$. Given Assumption 2.1, the condition (88) in Theorem A.1.1 is satisfied and the Feller condition indicates that the variance process in (2) neither explodes nor makes an excursion to zero under $\mathbb{P}$. Under a risk-neutral measure $\mathbb{Q}$, the variance SDE (87) takes the form

$$a = \kappa_v + \lambda_v, \quad b = \frac{\kappa_v \theta}{(\kappa_v + \lambda_v)} \tag{92}$$

and $B_t = \widetilde{Z}_{2,t}$. Given Assumption 2.1, the condition (88) in Theorem A.1.1 under this measure is satisfied since

$$2(\kappa_v + \lambda_v)\frac{\kappa_v \theta}{(\kappa_v + \lambda_v)} = 2\kappa_v \theta \geq \sigma^2.$$

and the Feller condition indicates that the variance process in (21) neither explodes nor makes an excursion to zero under $\mathbb{Q}$. Thus we are able to conclude that the Radon-Nikodým derivative $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t$ exists and it is a strictly positive martingale under $\mathbb{P}$. Similarly $\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_t$ also exists and it is a strictly positive martingale under $\mathbb{Q}$.

Now it remains to establish that (7) and (26) are martingales under $\mathbb{P}$ and $\mathbb{Q}$ respectively. We will show that condition (5) in Assumption 2.1 is sufficient. Furthermore, if (26) is strictly positive martingale under $\mathbb{Q}$, then we have established the existence of a martingale measure $\hat{\mathbb{Q}}$ which corresponds to the stock price S_t given by (25) as the numéraire. This facilitates change of numéraire techniques analogous to Geman et al. (1995).

We now demonstrate that condition (5) is sufficient to ensure that (26) is a martingale under $\mathbb{Q}$. The steps needed to show that (7) is a martingale under $\mathbb{P}$ are similar. By expansion of the expectation in (84), we can write

$$\begin{aligned} C^E(S,v,0) \quad = \quad & Se^{-qT}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\frac{1}{2}\int_0^T v_u du + \int_0^T \sqrt{v_u}d\widetilde{Z}_{1,u} - \widetilde{\lambda}\widetilde{\kappa}T + \sum_{i=1}^{N_T} Y_i\right)\mathbb{1}_{\mathcal{A}}\right] \\ & -Ke^{-rT}\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\mathcal{A}}]. \end{aligned} \tag{93}$$

In order to interpret the term

$$\exp\left(-\frac{1}{2}\int_0^T v_u du + \int_0^T \sqrt{v_u}d\widetilde{Z}_{1,u} - \widetilde{\lambda}\widetilde{\kappa}T + \sum_{i=1}^{N_T} Y_i\right) \tag{94}$$

as a Radon-Nikodým derivative of some risk-adjusted measure $\hat{\mathbb{Q}}$ with respect to $\mathbb{Q}$, we have to ensure that the variance process v_t neither explodes nor makes excursions to zero under $\hat{\mathbb{Q}}$. Note that if

$$\left.\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}\right|_T = \exp\left(-\frac{1}{2}\int_0^T v_u du + \int_0^T \sqrt{v_u}d\widetilde{Z}_{1,u} - \widetilde{\lambda}\widetilde{\kappa}T + \sum_{i=1}^{N_T} Y_i\right), \tag{95}$$

then the application of Theorem 2.1 allows us to conclude that

$$d\widetilde{Z}_{1,t} = -\sqrt{v_t} + d\hat{Z}_{1,t} \tag{96}$$

and

$$d\widetilde{Z}_{2,t} = -\rho\sqrt{v_t} + d\hat{Z}_{2,t} \tag{97}$$

where $(\hat{Z}_{1,t}, \hat{Z}_{2,t})$ are standard Wiener process components under $\hat{\mathbb{Q}}$ with instantaneous correlation ρ. Hence under $\hat{\mathbb{Q}}$, the SDE for the stock price process is

$$\frac{dS_t}{S_{t-}} = (r - q + \sqrt{v_t} - \hat{\lambda}\hat{\kappa})dt + \sqrt{v_t}d\hat{Z}_{1,t} + \int_{\mathbb{R}} (e^y - 1)\hat{q}(dy, dt), \tag{98}$$

where N_t has the associated Poisson counting measure $\hat{q}(dy, dt)$ with intensity

$$\hat{\lambda} = \widetilde{\lambda}(1 + \widetilde{\kappa}),$$

return jump-size density

$$m_{\hat{\mathbb{Q}}}(dy) = \frac{e^y}{M_{\mathbb{Q},Y}(1)} m_{\mathbb{Q}}(dy),$$

and the expected jump-size increment is

$$\hat{\kappa} = \int_{\mathbb{R}} [e^y - 1] m_{\hat{\mathbb{Q}}}(dy).$$

The SDE for the variance process is

$$dv_t = (\kappa_v + \lambda_v - \rho\sigma)\left[\frac{\kappa_v\theta}{(\kappa_v + \lambda_v - \rho\sigma)} - v_t\right] dt + \sigma\sqrt{v_t}d\hat{Z}_{2,t}. \tag{99}$$

Analogous to the pure-diffusion situation in Sin (1998) and Wong & Heyde (2004), there is a solution to the variance process SDE (99) that neither explodes nor makes an excursion to zero if and only if (94) is a strictly positive martingale. Wong & Heyde (2006) have a condition that is slightly different to (5) in Assumption 2.1. This difference is due to the fact that they allow the market price of variance risk to also assume negative values.

We thus examine an auxiliary variance process v_t of the form (87) where

$$a = \kappa_v + \lambda_v - \rho\sigma, b = \frac{\kappa_v\theta}{(\kappa_v + \lambda_v - \rho\sigma)}, \tag{100}$$

where ρ is the instantaneous correlation $dZ_{1,t}dZ_{2,t}$ in (1) and (2), σ is the volatility of the variance process v_t, and $B_t = \hat{Z}_{2,t}$ where $\hat{Z}_{2,t}$ is a standard Wiener process under $\hat{\mathbb{Q}}$. The condition (5) in Assumption 2.1 ensures that the parameters (100) are positive. Furthermore the condition (88) under the measure $\hat{\mathbb{Q}}$ is satisfied since

$$2(\kappa_v + \lambda_v - \rho\sigma)\frac{\kappa_v\theta}{(\kappa_v + \lambda_v - \rho\sigma)} = 2\kappa_v\theta \geq \sigma^2.$$

From the Feller condition, the auxiliary variance process v_t neither explodes nor makes an excursion to zero under $\hat{\mathbb{Q}}$. Hence $\hat{\mathbb{Q}}$ is an equivalent measure to $\mathbb{Q}$ and the Radon-Nikodým derivative $\left.\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}\right|_T$ given by (95) is a strictly positive martingale under $\mathbb{Q}$. Condition (5) in Assumption 2.1 is not surprising since it is well known in some pure-diffusion stochastic volatility models (for example, the Hull & White (1987) model) that arbitrage-free models established through the existence of an equivalent martingale measure only exist when the correlation ρ is strictly negative (see Sin (1998)).

Appendix 2. Proof of Proposition 4.1 – Fourier Transform of the IPDE(55)

By use of the definition (56), we can readily show that

$$\mathcal{F}_x\left\{\frac{\partial G}{\partial x}\right\} = -i\phi\hat{G},\ \mathcal{F}_x\left\{\frac{\partial^2 G}{\partial x^2}\right\} = -\phi^2\hat{G},\ \mathcal{F}_x\left\{\frac{\partial^2 G}{\partial x\partial v}\right\} = -i\phi\frac{\partial\hat{G}}{\partial v},\ \mathcal{F}_x\left\{\frac{\partial G}{\partial\tau}\right\} = \frac{\partial\hat{G}}{\partial\tau}.$$

All that remains is to evaluate the transform of the integral term. Applying the definition of the transform (56), we have

$$F_x\left\{\int_0^\infty G(x+J,v,\tau)Q(J)dJ\right\} = \int_0^\infty\int_{-\infty}^\infty e^{i\phi x}G(x+J,v,\tau)Q(J)dxdJ.$$

Making the change of integration variable $z = x + J$ this becomes

$$\begin{aligned} F_x\left\{\int_0^\infty G(x+J,v,\tau)Q(J)dJ\right\} &= \int_0^\infty Q(J)\int_{-\infty}^\infty e^{i\phi(z-J)}G(z,v,\tau)dzdJ \\ &= \int_0^\infty Q(J)e^{-i\phi J}dJ\int_{-\infty}^\infty e^{i\phi z}G(z,v,\tau)dz \\ &= A(\phi)\hat{G}(\phi,v,\tau), \end{aligned}$$

where $A(\phi)$ is defined in (58). Simple factorisation then yields the two-dimensional PDE (57) in Proposition 4.1.

Appendix 3. Proof of Proposition 4.2 – Laplace Transform of the PDE (57)

Taking the Laplace transform (59) of (57), we require the following results. Firstly,

$$\begin{aligned}
\mathcal{L}_v\left\{\frac{\Lambda v}{2}\hat{G} - i\phi\Psi\hat{G}\right\} &= \frac{\Lambda}{2}\int_0^\infty ve^{-sv}\hat{G}dv - i\phi\Psi\tilde{G} \\
&= -\frac{\Lambda}{2}\frac{\partial}{\partial s}\int_0^\infty e^{-sv}\hat{G}dv - i\phi\Psi\tilde{G} \\
&= -\frac{\Lambda}{2}\frac{\partial\tilde{G}}{\partial s} - i\phi\Psi\tilde{G}.
\end{aligned}$$

For the first order derivative with respect to v we have

$$\begin{aligned}
\mathcal{L}_v\left\{(\alpha - \Theta v)\frac{\partial\hat{G}}{\partial v}\right\} &= \int_0^\infty (\alpha - \Theta v)e^{-sv}\frac{\partial\hat{G}}{\partial v}dv \\
&= \alpha\left\{-\hat{G}(\phi,0,\tau) + s\int_0^\infty e^{-sv}\hat{G}dv\right\} + \Theta\frac{\partial}{\partial s}\int_0^\infty e^{-sv}\frac{\partial\hat{G}}{\partial v}dv \\
&= \alpha(-\hat{G}(\phi,0,\tau) + s\tilde{G}) + \Theta\frac{\partial}{\partial s}\left\{-\hat{G}(\phi,0,\tau) + s\tilde{G}\right\} \\
&= -\alpha\hat{G}(\phi,0,\tau) + \alpha s\tilde{G} + \Theta\left(s\frac{\partial\tilde{G}}{\partial s} + \tilde{G}\right) \\
&= \Theta s\frac{\partial\tilde{G}}{\partial s} + (\alpha s + \Theta)\tilde{G} - \alpha\hat{G}(\phi,0,\tau).
\end{aligned}$$

Finally, for the second order derivative term we have

$$\begin{aligned}
\mathcal{L}_v\bigg\{\frac{\sigma^2 v}{2}\frac{\partial^2\hat{G}}{\partial v^2}\bigg\} &= -\frac{\sigma^2}{2}\frac{\partial}{\partial s}\int_0^\infty e^{-sv}\frac{\partial^2\hat{G}}{\partial v^2}dv \\
&= -\frac{\sigma^2}{2}\frac{\partial}{\partial s}\bigg\{s\int_0^\infty e^{-sv}\frac{\partial\hat{G}}{\partial v}dv\bigg\} \\
&= -\frac{\sigma^2}{2}\frac{\partial}{\partial s}\bigg\{s[-\tilde{G}(\phi,0,\tau)] + s^2\tilde{G}\bigg\} \\
&= -\frac{\sigma^2}{2}\bigg(- \hat{G}(\phi,0,\tau) + 2s\tilde{G} + s^2\frac{\partial\tilde{G}}{\partial s}\bigg) \\
&= \frac{-\sigma^2 s^2}{2}\frac{\partial\tilde{G}}{\partial s} - \sigma^2 s\tilde{G} + \frac{\sigma^2}{2}\hat{G}(\phi,0,\tau).
\end{aligned}$$

Thus the Laplace transform of (57) satisfies the partial differential equation

$$\frac{\partial\tilde{G}}{\partial\tau} + \left(\frac{\sigma^2}{2}s^2 - \Theta s + \frac{\Lambda}{2}\right)\frac{\partial\tilde{G}}{\partial s} = [(\alpha-\sigma^2)s + \Theta - i\phi\Psi]\tilde{G} + \left(\frac{\sigma^2}{2} - \alpha\right)\hat{G}(\phi,0,\tau).$$

Finally we set $f_L(\tau) = (\sigma^2/2-\alpha)\hat{G}(\phi,0,\tau)$, and note that since $\tilde{G}(\phi,s,\tau)$ must be finite for all $s > 0$, $f_L(\tau)$ must be determined such that $\tilde{G}(\phi,s,\tau) \to 0$ as $s \to \infty$.

Appendix 4. Proof of Proposition 4.3 – Solving the PDE (60)

First we express the solution in terms of the so far unknown function $f_L(t)$. Since (60) is a first order PDE, it may be solved using the method of characteristics. The characteristic equation for (60) is

$$d\tau = \frac{ds}{\left(\frac{\sigma^2}{2}s^2 - \Theta s + \frac{\Lambda}{2}\right)} = \frac{d\tilde{G}}{[(\alpha-\sigma^2)s + \Theta - i\phi\Psi]\tilde{G} + f_L(\tau)}. \tag{101}$$

We break the calculation of $\tilde{G}(\phi,s,\tau)$ from (101) into a number of steps:-

(i) Taking the first equality in (101), we have

$$\int d\tau = \frac{2}{\sigma^2}\int\frac{ds}{s^2 - \frac{2\Theta}{\sigma^2}s + \frac{\Lambda}{\sigma^2}},$$

so that [7]

$$\tau + c_1 = \frac{1}{\Omega}\int\left(\frac{1}{s-(\frac{\Theta+\Omega}{\sigma^2})} - \frac{1}{s-(\frac{\Theta-\Omega}{\sigma^2})}\right)ds,$$

where we use the notation c_j to denote an undetermined constant term. Integrating with respect to s gives a relation between the transform variable s and time-to-maturity τ, namely

$$\Omega\tau + c_2 = \ln\left(\frac{\sigma^2 s - \Theta - \Omega}{\sigma^2 s - \Theta + \Omega}\right),$$

and hence[8]

$$c_3 = \frac{(\sigma^2 s - \Theta - \Omega)e^{-\Omega\tau}}{\sigma^2 s - \Theta + \Omega}. \tag{102}$$

We also note that (102) may be re-expressed as

$$s = \frac{(\Theta-\Omega)}{\sigma^2} - \frac{2\Omega e^{-\Omega\tau}}{\sigma^2(c_3 - e^{-\Omega\tau})}. \tag{103}$$

(ii) We next consider the second equality in (101), which can be rearranged to give the first order ODE

$$\frac{d\tilde{G}}{d\tau} + [(\sigma^2-\alpha)s - \Theta + i\phi\Psi]\tilde{G} = f_L(\tau). \tag{104}$$

The integrating factor, $R(\tau)$, for this ODE is the solution to

$$\frac{dR}{d\tau} = [(\sigma^2-\alpha)s - \Theta + i\phi\Psi]R.$$

Note that in step (i) we have found s as a function of τ in (103). Using this expression for s and integrating with respect to τ gives

$$\ln R = \left[\frac{(\sigma^2-\alpha)(\Theta-\Omega)}{\sigma^2} - \Theta + i\phi\Psi\right]\tau - (\sigma^2-\alpha)\int\frac{2\Omega e^{-\Omega\xi}}{\sigma^2(c_3 - e^{-\Omega\xi})}d\xi,$$

which can be expressed as[9]

[7]Note that $x^2 - \frac{2\Theta}{\sigma^2}x + \frac{\Lambda}{\sigma^2} = 0$ has solution $x = \frac{\Theta\pm\Omega}{\sigma^2}$ where we define $\Omega = \Omega(\phi) \equiv \sqrt{\Theta^2 - \Lambda\sigma^2}$.
[8]Note that $c_3 = e^{c_2} = e^{\Omega c_1}$.
[9]Using the change of integration variable $u = c_3 - e^{-\Omega\xi}$, the integral in the second term on the right-hand side of the last equation can be evaluated as $\int\frac{e^{-\Omega\xi}}{c_3-e^{-\Omega\xi}}d\xi = \frac{1}{\Omega}\ln|u|$.

$$R(\tau) = \exp\left\{ \left[\frac{(\sigma^2-\alpha)(\Theta-\Omega)}{\sigma^2} - \Theta + i\phi\Psi \right] \tau \right\} \left| \frac{1}{c_3 - e^{-\Omega\tau}} \right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)}. \tag{105}$$

Thus applying the method of variation of parameters to solve the ordinary differential equation (104) we find that $\tilde{G}(\phi, s, \tau)$ is given by

$$R(\tau)\tilde{G}(\phi, s, \tau) = \int_0^\tau R(\xi) f_L(\xi) d\xi + c_4,$$

which on use of the expression (105) for $R(\tau)$ becomes

$$\begin{aligned} \tilde{G}(\phi, s, \tau) = \exp&\left\{ \left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] \tau \right\} |c_3 - e^{-\Omega\tau}|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} \\ &\times \left\{ c_4 + \int_0^\tau f_L(\xi) \exp\left\{ \left[\frac{(\sigma^2-\alpha)(\Theta-\Omega)}{\sigma^2} - \Theta + i\phi\Psi \right] \xi \right\} \right. \\ &\qquad \left. \times \left| \frac{1}{c_3 - e^{-\Omega\xi}} \right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} d\xi \right\}. \end{aligned} \tag{106}$$

(iii) Next we determine the constant c_4 that appears in (106). We anticipate that we will find a function A such that $c_4 = A(c_3)$, where c_3 is given by (102). When $\tau = 0$, we have from (102) and (106) that

$$\tilde{G}(\phi, s, 0) = \left| \frac{\sigma^2 s - \Theta - \Omega}{\sigma^2 s - \Theta + \Omega} - 1 \right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} A(c_3). \tag{107}$$

Note from (103) that at $\tau = 0$ we have

$$s = \frac{\Theta - \Omega}{\sigma^2} - \frac{2\Omega}{(c_3-1)\sigma^2}. \tag{108}$$

Thus substituting this last expression into (107) we find that $A(c_3)$ is given by

$$A(c_3) = |c_3 - 1|^{-\frac{2}{\sigma^2}(\sigma^2-\alpha)} \tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} - \frac{2\Omega}{(c_3-1)\sigma^2}, 0 \right).$$

(iv) Next we substitute out the term c_3 appearing in the expression for $\tilde{G}(\phi, s, \tau)$ in (106). To this end, in (106) consider the term

$$\begin{aligned} &|c_3 - e^{-\Omega\tau}|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} A(c_3). \\ &= \left| \frac{2\Omega e^{-\Omega\tau}}{(\sigma^2 s - \Theta + \Omega)(1 - e^{-\Omega\tau}) + 2\Omega e^{-\Omega\tau}} \right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} \tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} - \frac{2\Omega}{(c_3-1)\sigma^2}, 0 \right), \end{aligned}$$

and note from (102) that

$$\frac{2\Omega}{(c_3-1)\sigma^2} = \frac{(\sigma^2 s - \Theta + \Omega)2\Omega}{\sigma^2[(e^{-\Omega\tau}-1)(\sigma^2 s - \Theta + \Omega) - 2\Omega e^{-\Omega\tau}]}.$$

Also, we note that for $0 \leq \xi \leq \tau$, we have

$$\begin{aligned}\left|\frac{c_3 - e^{-\Omega\tau}}{c_3 - e^{-\Omega\xi}}\right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} &= \left|\frac{(\sigma^2 s - \Theta - \Omega)e^{-\Omega\tau} - (\sigma^2 s - \Theta + \Omega)e^{-\Omega\tau}}{(\sigma^2 s - \Theta - \Omega)e^{-\Omega\tau} - (\sigma^2 s - \Theta + \Omega)e^{-\Omega\xi}}\right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} \\ &= \left|\frac{2\Omega e^{-\Omega\tau}}{(\sigma^2 s - \Theta + \Omega)(e^{-\Omega\xi} - e^{-\Omega\tau}) + 2\Omega e^{-\Omega\tau}}\right|^{\frac{2}{\sigma^2}(\sigma^2-\alpha)},\end{aligned}$$

and make the observation that all real arguments in $|\cdot|$ are positive. Thus by substituting the last equation into (106) we have

$$\begin{aligned}\tilde{G}(\phi, s, \tau) = \exp&\left\{\left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi\right]\tau\right\} \\ &\times \tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} - \frac{(\sigma^2 s - \Theta + \Omega)2\Omega}{\sigma^2[(e^{-\Omega\tau}-1)(\sigma^2 s - \Theta + \Omega) - 2\Omega e^{-\Omega\tau}]}, 0\right) \\ &\times \left(\frac{2\Omega e^{-\Omega\tau}}{(\sigma^2 s - \Theta + \Omega)(1 - e^{-\Omega\tau}) + 2\Omega e^{-\Omega\tau}}\right)^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} \\ &+ \int_0^\tau f_L(\xi)\exp\left\{\left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi\right](\tau-\xi)\right\} \\ &\qquad \times \left(\frac{2\Omega e^{-\Omega\tau}}{(\sigma^2 s - \Theta + \Omega)(e^{-\Omega\xi} - e^{-\Omega\tau}) + 2\Omega e^{-\Omega\tau}}\right)^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} d\xi. \quad (109)\end{aligned}$$

(v) We must now determine the function $f_L(\xi)$. We achieve this by applying to (109) the condition (61) that $\lim_{s\to\infty}\tilde{G}(\phi, s, \tau) = 0$. Taking the limit of (109) as $s \to \infty$, a little algebra reveals that we require that

$$\begin{aligned}\int_0^\tau f_L(\xi)\exp&\left\{-\left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi\right]\xi\right\}\left(\frac{1-e^{-\Omega\tau}}{e^{-\Omega\xi} - e^{-\Omega\tau}}\right)^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} d\xi \\ &= -\tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} - \frac{2\Omega}{\sigma^2(e^{-\Omega\tau}-1)}, 0\right). \quad (110)\end{aligned}$$

In (110) make the change of variable

$$\zeta^{-1} = 1 - e^{-\Omega\xi}, \qquad z^{-1} = 1 - e^{-\Omega\tau}, \quad (111)$$

so that

$$\int_z^\infty g(\zeta)(\zeta - z)^{\frac{2}{\sigma^2}(\alpha-\sigma^2)} d\zeta = -\Omega \tilde{G}\left(\phi, \frac{\Theta - \Omega}{\sigma^2}, + \frac{2\Omega z}{\sigma^2}, 0\right), \tag{112}$$

where

$$g(\zeta) = f_L(\xi) \exp\left\{ - \left[\frac{(\alpha - \sigma^2)(\Theta - \Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] \xi \right\} \frac{\zeta^{\frac{2}{\sigma^2}(\sigma^2-\alpha)}}{\zeta(\zeta - 1)}. \tag{113}$$

Thus our task is to solve (112) for $g(\zeta)$, and hence we will obtain the function $f_L(\xi)$.

Firstly, by definition (59) for the Laplace transform,

$$\tilde{G}\left(\phi, \frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2}, 0\right) = \int_0^\infty \hat{G}(\phi, w, 0) \exp\left\{ - \left(\frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2} \right) w \right\} dw.$$

Introducing a gamma function, as defined by (63), we have[10]

$$\begin{aligned}
&\tilde{G}\left(\phi, \frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2}, 0\right) \\
&= \frac{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \int_0^\infty \hat{G}(\phi, w, 0) \exp\left\{ - \left(\frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2} \right) w \right\} dw \\
&= \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \int_0^\infty \int_0^\infty e^{-a} a^{\frac{2\alpha}{\sigma^2}-2} \hat{G}(\phi, w, 0) \exp\left\{ - \left(\frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2} \right) w \right\} da dw.
\end{aligned}$$

The change of integration variable $a = (2\Omega w/\sigma^2)y$ gives

$$\begin{aligned}
&\tilde{G}\left(\phi, \frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2}, 0\right) \\
&\qquad = \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \int_0^\infty \hat{G}(\phi, w, 0) \exp\left\{ - \left(\frac{\Theta - \Omega}{\sigma^2} \right) w \right\} \\
&\qquad\qquad \times \left[\int_0^\infty e^{-\frac{2\Omega w}{\sigma^2} y} \left(\frac{2\Omega w}{\sigma^2} y \right)^{\frac{2\alpha}{\sigma^2}-2} e^{-\frac{2\Omega z}{\sigma^2} w} dy \right] \left(\frac{2\Omega w}{\sigma^2} \right) dw \\
&\qquad = \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \int_0^\infty \hat{G}(\phi, w, 0) \exp\left\{ - \left(\frac{\Theta - \Omega}{\sigma^2} \right) w \right\} \left(\frac{2\Omega w}{\sigma^2} \right)^{\frac{2\alpha}{\sigma^2}-1} \\
&\qquad\qquad \times \left(\int_0^\infty e^{-\frac{2\Omega w}{\sigma^2}(y+z)} y^{\frac{2\alpha}{\sigma^2}-2} dy \right) dw.
\end{aligned}$$

[10]The choice of the term $\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)$ two lines down may seem arbitrary and it seems we could have chosen $\Gamma(\beta)$ for β arbitrary. However it turns out that to make (114) match with (112) we would need to take $\beta = \frac{2\alpha}{\sigma^2} - 1$.

Making one further change of variable, namely $\zeta = z + y$, we have

$$\begin{aligned}
&\tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2}, 0\right) \\
&\qquad = \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)} \int_0^\infty \hat{G}(\phi, w, 0)\ \exp\left\{-\left(\frac{\Theta-\Omega}{\sigma^2}\right) w\right\} \left(\frac{2\Omega w}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}-1} \\
&\qquad\quad \times \int_z^\infty e^{-\frac{2\Omega w}{\sigma^2}\zeta} (\zeta - z)^{\frac{2\alpha}{\sigma^2}-2} d\zeta dw \\
&\qquad = \int_z^\infty (\zeta - z)^{\frac{2}{\sigma^2}(\alpha-\sigma^2)} \left[\int_0^\infty \frac{\hat{G}(\phi, w, 0)}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)} \left(\frac{2\Omega w}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}-1}\right. \\
&\qquad\quad \left. \times \exp\left\{-\left(\frac{\Theta-\Omega}{\sigma^2} + \frac{2\Omega\zeta}{\sigma^2}\right) w\right\} dw\right] d\zeta. \qquad (114)
\end{aligned}$$

Comparing (114) with (112) and recalling that $\hat{G}(\phi, w, 0) = e^{i\phi x_0}\delta(w - v_0)$, we can conclude that

$$g(z) = \frac{-\Omega}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)} \left(\frac{2\Omega v_0}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}-1} \exp\left\{i\phi x_0 - \left(\frac{\Theta-\Omega}{\sigma^2} + \frac{2\Omega z}{\sigma^2}\right) v_0\right\}, \qquad (115)$$

and hence $f_L(\xi)$ can be readily found by expressing $f_L(\xi)$ as a function of $g(\zeta)$ using (113).

(vi) Having found $f_L(\xi)$, all that remains is to substitute for $f_L(\xi)$ in (109), which requires us to consider the following expressions. First we have

$$\begin{aligned}
J_1 =& \exp\left\{\left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Gamma - i\phi\Psi\right]\tau\right\} \\
&\times \tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} + \frac{(\sigma^2 s - \Theta + \Omega)2\Omega}{\sigma^2[(1-e^{-\Omega\tau})(\sigma^2 s - \Theta + \Omega) - 2\Omega e^{-\Omega\tau}]}, 0\right) \\
&\times \left(\frac{2\Omega e^{-\Omega\tau}}{(\sigma^2 s - \Theta + \Omega)(1-e^{-\Omega\tau}) + 2\Omega e^{-\Omega\tau}}\right)^{\frac{2}{\sigma^2}(\sigma^2-\alpha)} \\
=& \exp\left\{\left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi\right]\tau\right\} \\
&\times \tilde{G}\left(\phi, \frac{\Theta-\Omega}{\sigma^2} + \frac{2\Omega(\sigma^2 s - \Theta + \Omega)z}{\sigma^2[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)]}, 0\right) \\
&\times \left(\frac{2\Omega(z-1)}{(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)}\right)^{2-\frac{2\alpha}{\sigma^2}}. \qquad (116)
\end{aligned}$$

Next we consider

$$J_2 = \int_z^\infty f_L(\xi) \exp\left\{ \left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] (\tau - \xi) \right\} \frac{1}{\Omega\zeta^2 e^{-\Omega\xi}} \times \left(\frac{2\Omega(z-1)\zeta}{(\sigma^2 s - \Theta + \Omega)(\zeta - 2) + 2\Omega(z-1)\zeta} \right)^{2-\frac{2\alpha}{\sigma^2}} d\zeta. \tag{117}$$

By use of (113) J_2 becomes

$$J_2 = \frac{1}{\Omega} \exp\left\{ \left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\Psi\phi \right] \tau \right\} \times \int_z^\infty g(\zeta) \left(\frac{2\Omega(z-1)}{(\sigma^2 s - \Theta + \Omega)(\zeta - z) + 2\Omega(z-1)\zeta} \right)^{2-\frac{2\alpha}{\sigma^2}} d\zeta,$$

and substituting for $g(\zeta)$ using (115) we have

$$\begin{aligned} J_2 &= \frac{1}{\Omega} \exp\left\{ \left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] \tau \right\} \\ &\quad \times \int_z^\infty \frac{-\Omega}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \left(\frac{2\Omega v_0}{\sigma^2} \right)^{\frac{2\alpha}{\sigma^2}-1} \exp\left\{ i\phi x_0 - \left(\frac{\Theta - \Omega}{\sigma^2} + \frac{2\Omega\zeta}{\sigma^2} \right) v_0 \right\} \\ &\quad \times \left(\frac{2\Omega(z-1)}{(\sigma^2 s - \Theta + \Omega)(\zeta - z) + 2\Omega(z-1)\zeta} \right)^{2-\frac{2\alpha}{\sigma^2}} d\zeta \\ &= \frac{-[2\Omega(z-1)]^{2-\frac{2\alpha}{\sigma^2}}}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \exp\left\{ \left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] \tau \right\} \\ &\quad \times \left(\frac{2\Omega v_0}{\sigma^2} \right)^{\frac{2\alpha}{\sigma^2}-1} \exp\left\{ i\phi x_0 - \left(\frac{\Theta - \Omega}{\sigma^2} \right) v_0 \right\} J_3(v_0), \end{aligned} \tag{118}$$

where for convenience we set

$$J_3(w) = \int_z^\infty e^{-\frac{2\Omega w}{\sigma^2}\zeta} [(\sigma^2 s - \Theta + \Omega)(\zeta - z) + 2\Omega(z-1)\zeta]^{\frac{2\alpha}{\sigma^2}-2} d\zeta.$$

Before proceeding further, we perform some manipulations on $J_3(w)$. Firstly, make the change of integration variable $y = (\sigma^2 s - \Theta\Omega)(\zeta - z) + 2\Omega(z-1)\zeta$ to give

$$\begin{aligned} J_3(w) &= \int_{2\Omega(z-1)z}^{\infty} \exp\left\{ \frac{-2\Omega w}{\sigma^2} \left(\frac{y + (\sigma^2 s - \Theta + \Omega)z}{(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)} \right) \right\} y^{\frac{2\alpha}{\sigma^2}-2} \\ &\qquad \times \frac{dy}{(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)} \\ &= \frac{1}{(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)} \exp\left\{ \frac{-2\Omega w(\sigma^2 s - \Theta + \Omega)z}{\sigma^2\left[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)\right]} \right\} \\ &\qquad \times \int_{2\Omega(z-1)z}^{\infty} \exp\left\{ \frac{-2\Omega wy}{\sigma^2\left[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)\right]} \right\} y^{\frac{2\alpha}{\sigma^2}-2} dy. \end{aligned}$$

By making a further change of integration variable, namely

$$\xi = \frac{2\Omega wy}{\sigma^2\left[(\sigma^2 s - \Gamma + \Omega) + 2\Omega(z-1)\right]},$$

we have

$$\begin{aligned} J_3(w) = {} & \frac{\sigma^2}{2\Omega w} \left(\frac{\sigma^2\left[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)\right]}{2\Omega w} \right)^{\frac{2\alpha}{\sigma^2}-2} \\ & \times \exp\left\{ \frac{-2\Omega w(\sigma^2 s - \Theta + \Omega)z}{\sigma^2\left[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)\right]} \right\} \int_{\frac{4\Omega^2(z-1)zw}{\sigma^2\left[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)\right]}}^{\infty} e^{-\xi} \xi^{\left(\frac{2\alpha}{\sigma^2}-1\right)-1} d\xi, \end{aligned}$$

which in terms of the gamma functions defined in (62) and (63) may be written

$$\begin{aligned} J_3(w) = {} & \left[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)\right]^{\frac{2\alpha}{\sigma^2}-2} \exp\left\{ \frac{-2\Omega w(\sigma^2 s - \Theta + \Omega)z}{\sigma^2[(\sigma^2 s - t + \Omega) + 2\Omega(z-1)]} \right\} \\ & \times \left(\frac{\sigma^2}{2\Omega w} \right)^{\frac{2\alpha}{\sigma^2}-1} \left[\Gamma\left(\frac{2\alpha}{\sigma^2} - 1 \right) - \int_0^{\frac{4\Omega^2(z-1)zw}{\sigma^2[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)]}} e^{-\xi} \xi^{\left(\frac{2\alpha}{\sigma^2}-1\right)-1} d\xi \right]. \end{aligned} \tag{119}$$

Substituting (119) into (118) we find that

$$\begin{aligned} J_2 = {} & \frac{-1}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)} \exp\left\{ \left[\frac{(\alpha - \sigma^2)(\Theta - \Omega)}{\sigma^2} + \Theta - i\phi\Psi \right] \tau \right\} \\ & \times \left(\frac{2\Omega(z-1)}{(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)} \right)^{2-\frac{2\alpha}{\sigma^2}} \\ & \times e^{i\phi x_0 - \left(\frac{\Theta-\Omega}{\sigma^2}\right)v_0} \exp\left\{ \frac{-2\Omega v_0(\sigma^2 s - \Theta + \Omega)z}{\sigma^2[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)]} \right\} \\ & \times \Gamma\left(\frac{2\alpha}{\sigma^2} - 1 \right) \left[1 - \Gamma\left(\frac{2\alpha}{\sigma^2} - 1; \frac{4\Omega^2(z-1)zv_0}{\sigma^2[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)]} \right) \right]. \end{aligned}$$

(vii) Finally by comparing (109), (116) and (117) we note that $\tilde{G}(\phi, s, \tau) = J_1 + J_2$, and hence we have

$$\begin{aligned}\tilde{G}(\phi, s, \tau) = &\exp\left\{ \left[\frac{(\alpha - \sigma^2)(\Theta - \Omega)}{\sigma^2} + \Theta - i\phi\Psi\right] \tau \right\} \\ &\times \left(\frac{2\Omega(z-1)}{(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)} \right)^{2 - \frac{2\alpha}{\sigma^2}} \\ &\times e^{i\phi x_0 - \left(\frac{\Theta - \Omega}{\sigma^2}\right) v_0} \exp\left\{ \frac{-2\Omega v_0 (\sigma^2 s - \Theta + \Omega) z}{\sigma^2[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)]} \right\} \\ &\times \Gamma\left(\frac{2\alpha}{\sigma^2} - 1; \frac{4\Omega^2 (z-1) z v_0}{\sigma^2[(\sigma^2 s - \Theta + \Omega) + 2\Omega(z-1)]} \right),\end{aligned}$$

which, after substituting for z from (111) becomes

$$\begin{aligned}\tilde{G}(\phi, s, \tau) = &\exp\left\{ \left[\frac{(\alpha - \sigma^2)(\Theta - \Omega)}{\sigma^2} + \Theta - i\phi\Psi\right] \tau \right\} \\ &\times \left(\frac{2\Omega}{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega} \right)^{2 - \frac{2\alpha}{\sigma^2}} \\ &\times e^{i\phi x_0 - \left(\frac{\Theta - \Omega}{\sigma^2}\right) v_0} \exp\left\{ \frac{-2\Omega v_0 (\sigma^2 s - \Theta + \Omega) e^{\Omega\tau}}{\sigma^2[(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega]} \right\} \\ &\times \Gamma\left(\frac{2\alpha}{\sigma^2} - 1; \frac{2\Omega v_0 e^{\Omega\tau}}{\sigma^2 (e^{\Omega\tau} - 1)} \times \frac{2\Omega}{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega} \right),\end{aligned}$$

which is the result in Proposition 4.3.

Appendix 5. Proof of Proposition 4.4 – Inverting the Laplace Transform

The inverse Laplace transform of (64) is most easily found by using the new variables

$$A = \frac{2\Omega v_0}{\sigma^2(1 - e^{-\Omega\tau})}, \qquad z = \frac{1}{2\Omega}\{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau} - 1) + 2\Omega\}. \tag{120}$$

If we set

$$h(\phi, v_0, \tau) = \exp\left\{ \left[\frac{(\alpha - \sigma^2)(\Theta - \Omega)}{\sigma^2} + \Theta - i\phi\Psi\right] \tau \right\} e^{i\phi x_0 - \left(\frac{\Theta - \Omega}{\sigma^2}\right) v_0}, \tag{121}$$

then under the change of variables (120) equation (64) becomes

$$\begin{aligned}\tilde{G}(\phi, s(z), \tau) =& h\,(\phi, v_0, \tau)\ \exp\left\{\frac{-(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau}-1)A}{2\Omega z}\right\} \\ & \times z^{\frac{2\alpha}{\sigma^2}-2}\Gamma(\frac{2\alpha}{\sigma^2}-1; Az).\end{aligned}$$

Making use of (62), the last equation becomes

$$\begin{aligned}\widetilde{G}(\phi, s(z), \tau) =& h(\phi, v_0, \tau) \exp\left\{-\frac{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau}-1)A}{2\Omega z}\right\} \\ & \times \frac{z^{\frac{2\alpha}{\sigma^2}-2}}{\Gamma(\frac{2\alpha}{\sigma^2}-1)}\int_0^{A/z} e^{-\beta}\beta^{\frac{2\alpha}{\sigma^2}-2}d\beta.\end{aligned}$$

Changing the integration variable according to $\xi = 1 - \frac{z}{A}\beta$, we have

$$\begin{aligned}\tilde{G}(\phi, s(z), \tau) =& h\,(\phi, v_0, \tau) \\ & \times e^{-A}\frac{A^{\frac{2\alpha}{\sigma^2}-1}}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)}\int_0^1 (1-\xi)^{\frac{2\alpha}{\sigma^2}-2}\ z^{-1}e^{\frac{A\xi}{z}}d\xi,\end{aligned} \tag{122}$$

where we have made use of the fact that

$$\frac{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau}-1)}{2\Omega z} - \frac{A}{z} = \frac{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau}-1)A + 2\Omega A}{(\sigma^2 s - \Theta + \Omega)(e^{\Omega\tau}-1) + 2\Omega} = A.$$

The crucial observation is that the Laplace transform variable s, though the change of variables, is now represented by the variable z. In equation (59), the Laplace transform is defined with respect to the parameter s. In order to invert (122), we must first establish the relationship between the Laplace transform with respect to parameter s, and the inverse Laplace transform with respect to the parameter z which is a function of s as defined in the second part of (120).

From (120) we see that

$$s = \frac{2\Omega(z-1)}{\sigma^2(e^{\Omega\tau}-1)} + \frac{\Theta - \Omega}{\sigma^2}.$$

Substituting this into (59) gives

$$\mathcal{L}_v\{\hat{G}(\phi, v, \tau)\} = \int_0^\infty \exp\left\{-\left[\frac{2\Omega(z-1)}{\sigma^2(e^{\Omega\tau}-1)} + \frac{\Theta-\Omega}{\sigma^2}\right]v\right\}\hat{G}(\phi, v, \tau)dv.$$

By letting

$$y = \frac{2\Omega v}{\sigma^2(e^{\Omega\tau} - 1)}, \tag{123}$$

we have

$$\mathcal{L}_v\{\hat{G}(\phi, v(y), \tau)\} = \frac{\sigma^2(e^{\Omega\tau} - 1)}{2\Omega} \int_0^\infty e^{-zy} \exp\left\{-\left(\frac{(\Theta - \Omega)(e^{\Omega\tau} - 1)}{2\Omega} - 1\right) y\right\} \times \hat{G}(\phi, v(y), \tau) dy,$$

which, by use of (59) can be written as

$$\mathcal{L}_v\{\hat{G}(\phi, v(y), \tau)\} = \frac{\sigma^2(e^{\Omega\tau} - 1)}{2\Omega} \mathcal{L}_y \left\{\exp\left\{-\left(\frac{(\Theta - \Omega)(e^{\Omega\tau} - 1)}{2\Omega} - 1\right) y\right\} \hat{G}(\phi, v(y), \tau)\right\}.$$

Thus we find that

$$\mathcal{L}_v^{-1}\{\tilde{G}(\phi, s(z), \tau)\} = \frac{2\Omega}{\sigma^2(e^{\Omega\tau} - 1)} \exp\left\{\left[\frac{(\Theta - \Omega)(e^{\Omega\tau} - 1)}{2\Omega} - 1\right] y\right\} \times \mathcal{L}_y^{-1}\left\{\tilde{G}(\phi, s(z), \tau)\right\}, \tag{124}$$

where

$$\mathcal{L}_y\{f(y)\} = \int_0^\infty e^{-zy} f(y) dy, \tag{125}$$

and we recall that y is given by (123), and z is defined by (120).

Applying the inverse transform (124) to (122), we have

$$\hat{G}(\phi, v(y), \tau) = h(\phi, v_0, \tau) \times e^{-A} \frac{A^{\frac{2\alpha}{\sigma^2} - 1}}{\Gamma\left(\frac{2\alpha}{\sigma^2} - 1\right)} \frac{2\Omega}{\sigma^2(e^{\Omega\tau} - 1)} \exp\left\{\left[\frac{(\Theta - \Omega)(e^{\Omega\tau} - 1)}{2\Omega} - 1\right] y\right\} \times \int_0^1 (1 - \xi)^{\frac{2\alpha}{\sigma^2} - 2} \mathcal{L}_y^{-1}\left\{z^{-1} e^{\frac{A\xi}{z}}\right\} d\xi.$$

Referring to Abramowitz & Stegun (1970) we find that

$$\mathcal{L}_y\left\{I_0(2\sqrt{A\xi y})\right\} = \frac{1}{z} e^{\frac{A\xi}{z}},$$

where $I_k(x)$ is the modified Bessel function defined by (67). Thus the inverse Laplace transform of $\tilde{G}(\phi, s, \tau)$ becomes

$$\begin{aligned}\hat{G}(\phi, v(y), \tau) &= h(\phi, v_0, \tau) \\ &\quad \times e^{-A} \frac{A^{\frac{2\alpha}{\sigma^2}-1}}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)} \frac{2\Omega}{\sigma^2(e^{\Omega\tau}-1)} \exp\left\{\left[\frac{(\Theta-\Omega)(e^{\Omega\tau}-1)}{2\Omega}-1\right]y\right\} \\ &\quad \times \int_0^1 (1-\xi)^{\frac{2\alpha}{\sigma^2}-2}\, I_0(2\sqrt{A\xi y})d\xi.\end{aligned}$$

We can further simplify this result by noting that[11]

$$\int_0^1 (1-\xi)^{\frac{2\alpha}{\sigma^2}-2}\, I_0(2\sqrt{A\xi y})d\xi = \Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)(Ay)^{\frac{1}{2}-\frac{\alpha}{\sigma^2}}\, I_{\frac{2\alpha}{\sigma^2}-1}(2\sqrt{Ay}),$$

and therefore

$$\begin{aligned}\hat{G}(\phi, v, \tau) &= h(\phi, v_0, \tau)\ \frac{2\Omega}{\sigma^2(e^{\Omega\tau}-1)} \\ &\quad \times e^{-A-y}\left(\frac{A}{y}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \exp\left\{\frac{(\Theta-\Omega)(e^{\Omega\tau}-1)}{2\Omega}y\right\} I_{\frac{2\alpha}{\sigma^2}-1}(2\sqrt{Ay}).\end{aligned}$$

Recalling the definitions for A and y, from (120) and (123) respectively, the last equation becomes

$$\begin{aligned}\hat{G}(\phi, v, \tau) &= \exp\left\{\left[\frac{(\alpha-\sigma^2)(\Theta-\Omega)}{\sigma^2}+\Theta-i\phi\Psi\right]\tau\right\} e^{i\phi x_0-\left(\frac{\Theta-\Omega}{\sigma^2}\right)v_0} \\ &\quad \times \frac{2\Omega}{\sigma^2(e^{\Omega\tau}-1)}\ \exp\left\{-\frac{2\Omega}{\sigma^2(e^{\Omega\tau}-1)}(v_0 e^{\Omega\tau}+v)\right\} \\ &\quad \times \left(\frac{v_0 e^{\Omega\tau}}{v}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \exp\left\{\frac{(\Theta-\Omega)}{\sigma^2}v\right\} I_{\frac{2\alpha}{\sigma^2}-1}\left(\frac{4\Omega}{\sigma^2(e^{\Omega\tau}-1)}(v_0 v e^{\Omega\tau})^{\frac{1}{2}}\right).\end{aligned}$$

Further manipulations yield the result in Proposition 4.4.

Appendix 6. Proof of Proposition 4.5 – Inverting The Fourier Transform

Applying the Fourier inversion theorem we have

$$G(x, v, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x}\hat{G}(\phi, v, \tau)d\phi,$$

[11]This result is simply obtained by expanding both terms in the integral in power series.

which upon use of (66) becomes

$$G(x,v,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\phi x_0} e^{\frac{(\Theta-\Omega)}{\sigma^2}(v-v_0+\alpha\tau)} e^{-i\phi(r-q-\widetilde{\lambda}\widetilde{\kappa})\tau} e^{\widetilde{\lambda}[A(\phi)-1]\tau} e^{-i\phi x}$$
$$\times \frac{2\Omega}{\sigma^2(e^{\Omega\tau}-1)}\left(\frac{v_0 e^{\Omega\tau}}{v}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \exp\left\{\frac{-2\Omega}{\sigma^2(e^{\Omega\tau}-1)}(v_0 e^{\Omega\tau}+v)\right\}$$
$$\times I_{\frac{2\alpha}{\sigma^2}-1}\left(\frac{4\Omega}{\sigma^2(e^{\Omega\tau}-1)}(v_0 v e^{\Omega\tau})^{\frac{1}{2}}\right) d\phi.$$

Next we expand the term $e^{\widetilde{\lambda}A(\phi)\tau}$ using a Taylor series expansion, and find that

$$\begin{aligned} e^{\widetilde{\lambda}A(\phi)\tau} &= \sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n}{n!}[A(\phi)]^n \\ &= \sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n}{n!}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{-i\phi(y_1+y_2,\ldots+y_n)} \\ &\qquad\qquad \times Q(y_1)Q(y_2)\ldots Q(y_n)dy_1dy_2\ldots dy_n, \end{aligned}$$

where each y_j $(j = 0,\ldots,n)$ is an independent jump drawn from the density $Q(y)$. We define $X_n \equiv e^{J_1+J_2+\ldots+J_n}$, and $X_0 \equiv 1$, and assume that $Q(J)$ is of a form that allows us to make a simplification of the form

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} f(e^{y_1}+e^{y_2}+\ldots+e^{y_n})Q(y_1)Q(y_2)\ldots Q(y_n)dy_1dy_2\ldots dy_n \\ &\quad= \int_0^{\infty} f(X_n)Q(X_n)dX_n \equiv \mathbb{E}_{\mathbb{Q}}^{(n)}[f(X_n)], \end{aligned}$$

where $f(X_n)$ is some general function of X_n. Thus we have

$$e^{\widetilde{\lambda}A(\phi)\tau} = \sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}[e^{-i\phi\ln X_n}],$$

and the result of the proposition then follows after some further algebra and converting back to the stock price variable S.

Appendix 7. Proof of Proposition 5.1 – Deriving the price for The European Call

Since the payoff function does not depend on v_T it is simplest to first perform the outer integration in (71) with respect to this variable. Thus

$$\begin{aligned}
&\int_0^\infty G(S_T, v_T, T; S, v, t)dv_T \\
&\quad = \sum_{n=0}^{\infty} \frac{(\widetilde{\lambda}(T-t))^n e^{-\widetilde{\lambda}(T-t)}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\phi x} e^{-i\phi \ln(S X_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)})} \\
&\qquad \times \Bigg[\int_0^\infty e^{\frac{(\Theta-\Omega)}{\sigma^2}(v - v_T + \alpha(T-t))} e^{-i\phi(r-q)(T-t)} \frac{2\Omega}{\sigma^2(e^{\Omega(T-t)}-1)} \left(\frac{w e^{\Omega(T-t)}}{v} \right)^{\frac{\alpha}{\sigma^2} - \frac{1}{2}} \\
&\qquad \times \exp\left\{ - \frac{2\Omega}{\sigma^2(e^{\Omega(T-t)}-1)} \left(v_T e^{\Omega(T-t)} + v \right) \right\} \\
&\qquad \times I_{\frac{2\alpha}{\sigma^2}-1} \left(\frac{4\Omega}{\sigma^2(e^{\Omega(T-t)}-1)} \left(v_T v e^{\Omega(T-t)} \right)^{\frac{1}{2}} \right) dv_T \Bigg] d\phi \Bigg\},
\end{aligned}$$

where we have set $x = \ln S_T$. Carrying out the integration with respect to v_T, we have

$$\begin{aligned}
\int_0^\infty G(S, v, t; S_T, v_T, T)dv_T &= \sum_{n=0}^{\infty} \frac{(\widetilde{\lambda}(T-t))^n e^{-\widetilde{\lambda}(T-t)}}{n!} \\
&\quad \times \mathbb{E}_{\mathbb{Q}}^{(n)} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\phi x} e^{-i\phi \ln S X_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)}} e^{B_2(-\phi,(T-t)) + D_2(-\phi,(T-t))v} d\phi \right\},
\end{aligned}$$

where

$$B_2(\phi, (T-t)) = i\phi(r-q)(T-t) + \frac{\alpha}{\sigma^2} \left\{ (\Theta_2 + \Omega_2)(T-t) - 2\ln\left(\frac{1 - Q_2 e^{\Omega_2(T-t)}}{1 - Q_2} \right) \right\},$$

and

$$D_2(\phi, (T-t)) = \frac{(\Theta_2 + \Omega_2)}{\sigma^2} \left[\frac{1 - e^{\Omega_2(T-t)}}{1 - Q_2 e^{\Omega_2(T-t)}} \right],$$

where we define $\Theta_2 = \Theta_2(\phi) \equiv \Theta(-\phi)$, $\Omega_2 = \Omega_2(\phi) \equiv \Omega(-\phi)$, and $Q_2 = Q_2(\phi) \equiv (\Theta_2 + \Omega_2)/(\Theta_2 - \Omega_2)$.

Referring to the results given by Adolfsson et al. (2009), it follows that

$$
\begin{aligned}
C^E(S,v,t) &= e^{-r(T-t)}\int_{\ln K}^{\infty}(e^x-K)\Bigg[\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}(T-t))^n e^{-\widetilde{\lambda}(T-t)}}{n!} \\
&\qquad \times \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\phi x}e^{-i\phi\ln(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)})}e^{B_2(-\phi,(T-t))+D_2(-\phi,(T-t))v}d\phi\right\}\Bigg]dy \\
&= \frac{e^{-r(T-t)}}{2\pi}\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}(T-t))^n e^{-\widetilde{\lambda}(T-t)}}{n!} \\
&\qquad \times \mathbb{E}_{\mathbb{Q}}^{(n)}\left\{\int_{-\infty}^{\infty} f_2(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)},v,(T-t);-\phi)\int_{\ln K}^{\infty}(e^x-K)e^{i\phi x}dxd\phi\right\},
\end{aligned}
$$

where we set

$$
f_2(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)},v,(T-t);\phi) \equiv e^{B_2(\phi,(T-t))+D_2(\phi(T-t))v+i\phi\ln(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)})}.
$$

Further use of the results from Adolfsson et al. (2009) allows us to express $C^E(S,v,(T-t))$ as

$$
\begin{aligned}
C^E(S,v,t) = \sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}(T-t))^n e^{-\widetilde{\lambda}(T-t)}}{n!}\mathbb{E}_{\mathbb{Q}}^{(n)}\{&SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)}e^{-q(T-t)}P_1^H(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)},v,(T-t);K) \\
&- Ke^{-r(T-t)}P_2^H(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)},v,(T-t);K)\},
\end{aligned}
$$

where $P_j^H(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(T-t)},v,(T-t);K)$, for $j=1,2$, is as defined in (73).

Appendix 8. Proof of Proposition 5.2 – Deriving the Early Exercise Premium

Using the notation established earlier, equation (44) may be written in the form

$$
C_P(S,v,t) = C_P^D(S,v,t) - \widetilde{\lambda}C_P^J(S,v,t),
$$

where

$$
C_P^D(S,v,t) = \int_0^{\tau} e^{-r(\tau-\xi)}\int_0^{\infty}\int_{\ln b(v_T,\xi)}^{\infty}(qe^x - rK)G(x,v_T,\tau-\xi;S,v)dxdv_T d\xi, \quad (126)
$$

and

$$C_P^J(S,v,t) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_{\ln b(v_T,\xi)}^\infty \int_0^{b(v_T,\xi)e^{-x}} [C(e^x Y, v_T, \tau) - (e^x Y - K)]G(Y)dY \times G(x, v_T, \tau-\xi; S, v)dx dv_T d\xi, \tag{127}$$

and we have set $x = \ln S_T$ and $\tau = T - t$.

Firstly we consider $C_P^D(S,v,t)$. Substituting for G we have

$$C_P^D(S,v,t) = \sum_{n=0}^\infty \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^\tau (\tau-\xi)^n e^{-\widetilde{\lambda}(\tau-\xi)} e^{-r(\tau-\xi)} \int_0^\infty \int_{\ln b(v_T,\xi)}^\infty [qe^x - rK] \times \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\phi x} e^{\frac{(\Theta-\Omega)}{\sigma^2}(v - v_T + \alpha(\tau-\xi))} e^{-i\phi(r-q)(\tau-\xi)} \times e^{-i\phi \ln SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}} \frac{2\Omega}{\sigma^2(e^{\Omega(\tau-\xi)}-1)} \left(\frac{v_T e^{\Omega(\tau-\xi)}}{v}\right)^{\frac{\alpha}{\sigma^2}-\frac{1}{2}} \times \exp\left\{ - \frac{2\Omega}{\sigma^2(e^{\Omega(\tau-\xi)}-1)}(v_T e^{\Omega(\tau-\xi)} + v)\right\} \times I_{\frac{2\alpha}{\sigma^2}-1}\left(\frac{4\Omega}{\sigma^2(e^{\Omega(\tau-\xi)}-1)}(v_T v e^{\Omega(\tau-\xi)})^{\frac{1}{2}}\right) d\phi dx dv_T d\xi \Bigg\}.$$

Using results from Adolfsson et al. (2009), we can readily show that

$$C_P^D(S,v,t) = \sum_{n=0}^\infty \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^\tau \int_0^\infty (\tau-\xi)^n e^{-\widetilde{\lambda}(\tau-\xi)} \times [qSX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)} e^{-q(\tau-\xi)} P_1^A(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}, v, \tau-\xi, v_T, b(v_T,\xi)) - rKe^{-r(\tau-\xi)} P_2^A(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}, v, \tau-\xi, v_T, b(v_T,\xi))] dv_T d\xi \Bigg\},$$

where $P_j^A(SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}, v, \tau-\xi, v_T, b(v_T,\xi))$ is given by (77).

Next we examine the $C_P^J(S,v,\tau)$ term. Interchanging the order of integration with respect to Y and x, we have

$$C_P^J(S,v,t) = \int_0^\tau e^{-r(\tau-\xi)} \int_0^\infty \int_0^1 Q(Y) \int_{\ln b(v_T,\xi)}^{\ln \frac{b(v_T,\xi)}{Y}} [C(e^x Y, v_T, \tau) - (e^x Y - K)] \times G(x, v_T, \tau-\xi; S, v) dx dY dv_T d\xi.$$

Making the change of integration variable $z = e^x$, we have

$$C_P^J(S,v,t) = \int_0^\tau \int_0^\infty e^{-r(\tau-\xi)} \int_0^1 Q(Y) \int_{b(v_T,\xi)}^{\frac{b(v_T,\xi)}{Y}} \frac{[C(zY,v_T,\xi) - (zY-K)]}{z} \times G(\ln z, v_T, \tau-\xi; S, v) dz dY dv_T d\xi$$

$$= \sum_{n=0}^\infty \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!} \mathbb{E}_{\mathbb{Q}}^{(n)} \Bigg\{ \int_0^\tau \int_0^\infty (\tau-\xi)^n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)} e^{-r(\tau-\xi)} \times \int_0^1 G(Y) \int_{b(v_T,\xi)}^{b(v_T,\xi)/Y} [C(zY,v_T,\xi) - (zY-K)] \times Q_J(z, v_T, \tau-\xi; SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}, v) dz dY dv_T d\xi \Bigg\},$$

where $Q_J(z, v_T, \tau-\xi; SX_n e^{-\widetilde{\lambda}\widetilde{\kappa}(\tau-\xi)}, v)$ is defined in (80).

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