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Research Paper 284 August 2010

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ISSN 1441-8010 www.qfrc.uts.edu.au

# Using Dynamic Copulae for Modeling Dependency in Currency Denominations of a Diversified World Stock Index

This version: 25 August 2010

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**Abstract**

The aim of this paper is to model the dependency among log-returns when security account prices are expressed in units of a well diversified world stock index. The paper uses the equi-weighted index EWI104s, calculated as the average of 104 world industry sector indices. The log-returns of its denominations in different currencies appear to be Student-t distributed with about four degrees of freedom. Motivated by these findings, the dependency in log-returns of currency denominations of the EWI104s is modeled using time-varying copulae, aiming to identify the best fitting copula family. The Student-t copula turns generally out to be superior to e.g. the Gaussian copula, where the dependence structure relates to the multivariate normal distribution. It is shown that merely changing the distributional assumption for the log-returns of the marginals from normal to Student-t leads to a significantly better fit. Furthermore, the Student-t copula with Student-t marginals is able to better capture dependent extreme values than the other models considered. Finally, the paper applies copulae to the estimation of the Value-at-Risk and the expected shortfall of a portfolio, constructed of savings accounts of different currencies. The proposed copula-based approach allows to split market risk into general and specific market risk, as defined in regulatory documents. The paper demonstrates that the approach performs clearly better than the RiskMetrics approach.

*Key words:* Diversified world stock index, Student-t distribution, time-varying copula, Value-at-Risk, expected shortfall

*AMS Subject Classification:* 62F03, 62P20, 91B28

## 1 Introduction

When modeling the expected return of an asset portfolio, one has to take into account several empirical findings, such as e.g. the stylized fact that asset returns are more correlated when markets are more volatile, see Longin and Solnik (2006) and Ang and Chen (2002), and appear to be more correlated during market recessions than during upturns, see Patton (2006). Thus, not only the expected return and the risk of the individual assets, but also the dependence structure among the assets in a portfolio affect the distribution of portfolio returns and, thus, investment decisions, see Embrechts et al. (2001). Finding a model that would be able to capture better the dependency in the multivariate log-return data than the multivariate normal one, is an important task, which is crucial for portfolio and risk management.

The paper studies the dependency between different currency denominations of a *benchmark*, a proxy of the *growth optimal portfolio* (GOP). The GOP maximizes expected logarithmic utility from terminal wealth, and is the portfolio with the almost sure maximum long-run growth rate. It is also known as the *Kelly portfolio*, see Kelly (1956). An *equi-weighted index* (EWI), see Platen and Heath (2006), is the simplest diversified portfolio that can be used as a proxy for the GOP, see Platen (2005). The paper will use for its empirical analysis the EWI104s, which is calculated in Lee and Platen (2006) and Platen and Rendek (2008) as an equi-weighted portfolio of 104 world industry sector indices. Platen and Rendek (2008) have applied the maximum likelihood ratio test to the large class of generalized hyperbolic distributions and have shown that the estimated log-return distribution for different currency denominations of the EWI104s appears with high significance to be the Student-t distribution with about four degrees of freedom.

Measuring dependency in log-returns of a multi-dimensional time series, where a diversified index is expressed in different currencies, calls for a nonlinear (i.e. non-Gaussian) dependency and, thus, requires a modeling framework different from the multivariate normal one. In this paper a copula approach is applied to analyze the log-returns of the EWI104s denominated in different currencies. The data represent high-dimensional time series with clearly non-Gaussian behavior. Based on the findings in Platen and Rendek (2008) regarding the distributional properties of the log-returns of the marginals, the current paper now analyzes the dependencies in the log-returns of currency denominations of the EWI104s using time-varying copulae. It aims to identify the best fitting copula family in the considered class.

Breymann et al. (2003), Dias and Embrechts (2004) and Dias and Embrechts (2010) analyzed the dependence structure within two-dimensional high-frequency time series of exchange rate returns. They have shown that the data can be fitted best by using the Student-t copula, which can be thought of as representing the dependence structure implicit in a multivariate Student-t distribution, see Embrechts et al. (2001). This is different to the study below of log-returns of a well diversified equity index denominated in different currencies. The paper aims to show for log-returns of different currency denominations of the EWI104s that the empirical fit of the Student-t copula is generally superior to that of e.g. the Gaussian copula, where the dependence structure is that of the multivariate normal distribution. Compared to the Gaussian copula, the Student-t copula turns out to capture better the dependent extreme values, observed in the log-return data. The study will consider currency pairs, as well as, multivariate portfolios and will identify the best performing copulae family among those considered, which captures best the dependency among the log-returns of currency denominations of the EWI104s.

Finally, the copula methodology is applied to the estimation of the *Value-at-Risk* (VaR) and the *expected shortfall* (ES) of a portfolio which is constructed of savings accounts of different currencies. Therefore, the paper either uses in its inference a benchmarked portfolio, which is expressed in units of the EWI104s, or a portfolio denominated in units of different currencies.

The suggested modeling approach allows to split market risk into *general* and *specific market risk*, as motivated in Platen and Stahl (2003) and Platen and Heath (2006). For an equity portfolio the general market risk measures the exposure of the portfolio to the movements of the market as a whole, captured by an appropriately chosen *benchmark*. The specific market risk is the risk which is not captured by the general market risk, and relates to the risk of portfolio movements against the benchmark. The separation of market risk into its general and specific component is essential when determining the amount of regulatory capital required to cover the market risk of a portfolio of traded financial instruments, see Morgan/Reuters (1996).

As prices are relative, one should define an appropriate reference unit to be used as a benchmark, or numeraire in order to establish an appropriate metric for measuring risk. The paper consistently uses the EWI104s as benchmark when measuring the risk of a portfolio. Furthermore, it applies backtesting to evaluate the performance of the copula-based VaR when comparing the fit of the Student-t copula to the one of the Gaussian copula. It is shown that when starting from the Gaussian copula, changing the distributional assumption for the log-returns of the marginals from normal to Student-t leads to a significantly better fit. The Gaussian copula with Student-t marginals clearly outperforms statistically the Gaussian copula with normal marginals. The latter represents the well-known *RiskMetrics* approach, a widely used methodology for VaR estimation. Modeling the dependency with a Student-t rather than a Gaussian copula, leads to further improvements, producing the smallest relative squared errors, and thus, the best backtesting results in the class of copulae considered.

The paper is organized as follows. Section 2 briefly reviews some facts on copulae, including definitions, examples and the estimation techniques to be employed. Section 3 presents an empirical analysis of the log-returns of the EWI104s when denominated in different currencies. It studies the marginals and fits static and time-varying copulae to the data. Finally, Section 4 discusses VaR and expected shortfall applications. Section 5 summarizes the findings.

## 2 Specification of the Copula Family

When modeling a joint distribution of a random vector of risk factors, one has to take into account both, the marginal behavior of individual risk factors, as well as their dependence structure. The copula approach provides a way to measure the dependence structure by separating it from modeling the marginal distribution. Copulae define the multivariate distribution functions on the unit cube $[0,1]^d$ which allow to connect their one-dimensional uniform-(0,1) marginals to the joint cumulative distribution function, see Nelsen (1998) for the formal definition.

According to *Sklar's theorem*, every distribution function can be decomposed into its marginal distribution and a copula, and every distribution function can be obtained by coupling marginal distributions with the dependence structure given by a copula. More precisely, Sklar's theorem states that if $F$ is a $d$-dimensional distribution function with marginals $F_1 \dots, F_d$, then there exists a copula $C$ with

$$F(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\} \tag{1}$$

for every $x_1, \dots, x_d \in \overline{\mathbb{R}} = [-\infty, \infty]$. If $F_1, \dots, F_d$ are continuous, then $C$ is unique. On the other hand, if $C$ is a copula and $F_1, \dots, F_d$ are distribution functions, then the function $F$ defined in (1) is a joint distribution function with marginals $F_1, \dots, F_d$. A proof of Sklar's theorem can be found e.g. in Joe (1997).

Thus, for a random vector $X = (X_1, \dots, X_d)^\top$ with joint distribution $X \sim F_X$ and continuous marginals $X_j \sim F_{X_j}$, $j \in \{1, \dots d\}$, one defines the multivariate copula using Sklar's theorem, as the distribution function $C_X$ of $u = (u_1, \dots, u_d)^\top \in [0,1]^d$, where $u_j = F_{X_j}(x_j)$:

$$C_X(u_1, \dots, u_d) = F_X\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\}. \tag{2}$$

For an absolutely continuous copula $C$ one can define the copula density as

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}. \tag{3}$$

If $X = (X_1, \dots, X_d)^\top$ has an absolute continuous distribution function $F$ and a copula $C_X$, then the density $c_X$ is obtained by differentiating $C_X$ in (2):

$$c_X(u_1, \dots, u_d) = \frac{f\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\}}{\prod_{j=1}^d f_j\{F_{X_j}^{-1}(u_j)\}}, \tag{4}$$

where $f$ is the joint density of $F_X$, and $f_j$ is the density of $F_{X_j}$. The density function of $X$ is then given by

$$f(x_1, \dots, x_d) = c_X(u_1, \dots, u_d) \cdot \prod_{j=1}^{d} f_j(x_j)$$

with $x_j = F_{X_j}^{-1}(u_j)$.

Sklar's identity (1) can also be applied to the so-called, *survival copula* $C^*$ related to $C$. If $F$ is the distribution function of the random vector $X = (X_1, \dots, X_d)^\top$ with marginals $F_1 \dots, F_d$, then there exist a copula $C^*$ with

$$\overline{F}(x_1, \dots, x_d) = C^*\{\overline{F}_1(x_1), \dots, \overline{F}_d(x_d)\}, \tag{5}$$

where $\overline{F}(x_1, \ldots, x_d) = P(X_1 > x_1, \ldots, X_d > x_d)$ and $\overline{F}_i(x_i) = P(X_i > x_i)$, $i \in \{1, \ldots d\}$. In particular, for the bivariate case the survival copula can be defined as follows:

$$C^*(u_1, u_2) = 1 - u_1 - u_2 + C(1 - u_1, 1 - u_2), \tag{6}$$

see Nelsen (1998).

### *2.1 Dependence and Tail Dependence*

Common approaches to quantify dependency among random variables include the *Pearson correlation coefficient* $r$, *Spearmann's* $\rho$ and *Kendall's* $\tau$. While Pearson's linear correlation depends on the distribution of the univariate marginals (i.e., keeping the dependence structure constant, different marginals might lead to different values for the joint distribution, see Dias (2004)), the other two rank correlations are independent of the univariate marginal distributions. For properties of dependence measures one can refer to Embrechts et al. (2001).

While the above coefficients consider only linear dependence among random variables, *tail dependence* coefficients allow to measure the extreme dependence in the tails of the multivariate distribution. These appear to be particularly useful in insurance and risk management when modeling the joint (dependent) risk, see e.g. Wang (1997) and Embrechts et al. (2001). The concepts of *lower* and *upper* tail dependence refer to the study of the dependence between extreme values in the lower and in the upper tails. The notion of tail dependence in relation to copulae first appeared in Joe (1997). For the bivariate case, the upper and the lower tail dependence coefficients can be defined as follows: If $(U_1, U_2)$ is a pair of uniform variables on the unit square $[0, 1]^2$, then the upper tail dependence coefficient $\lambda_u \in [0, 1]$ is defined as

$$\lambda_u = \lim_{u \to 1-} P(U_1 > u | U_2 > u) = \lim_{u \to 1-} \frac{C^*(u, u)}{1 - u}. \tag{7}$$

Similarly, the lower tail dependence coefficient $\lambda_l \in [0, 1]$ is defined as

$$\lambda_l = \lim_{u \to 0+} P(U_1 \leq u | U_2 \leq u) = \lim_{u \to 0+} \frac{C(u, u)}{u}. \tag{8}$$

If the upper tail dependence coefficient $\lambda_u$ falls into the interval $(0, 1]$, then $U_1$ and $U_2$ are said to be asymptotically dependent in the upper tail, and if $\lambda_u = 0$, then $U_1$ and $U_2$ are said to be asymptotically independent in the upper tail. Similarly, if $\lambda_l \in (0, 1]$ or $\lambda_l = 0$, then $U_1$ and $U_2$ are said to be asymptotically dependent, or independent, respectively, in the lower tail. For properties of the lower and the upper tail dependence coefficients the reader is referred to Embrechts et al. (2001) and McNeil et al. (2005). Hu (2006) reviews dependence and tail dependence measures for mixture copula models. The following results for the Student-t copula, as well as, the derivations for other copula models can be found in Embrechts et al. (2001) and McNeil et al. (2005).

### 2.2 Copulae Examples

The paper will concentrate mostly on two popular copula families: the *elliptical copulae family* and the *Archimedean copulae family*. Some $d$-dimensional copulae from these parametric copulae families, with copula parameter controlling the degree of dependence, are presented below. Further copula models, in particular, the Ali-Mikhail-Haq and Plakett copulae, can be found e.g. in Joe (1993) and Nelsen (1998).

Elliptical copulae families, see Lindskog et al. (2001), have a dependence structure generated by elliptical distributions, which include among others the normal and Student-t distributions, as well as the stable distribution class discussed in e.g. Rachev and Mittnik (2000) and Rachev and Han (2000). For the modeling of dependencies using elliptical distributions one can refer to Hult and Lindskog (2002), Fang et al. (2002) and Frahm et al. (2003). Its applications in finance and risk management are discussed, for instance, in Breymann et al. (2003), McNeil et al. (2005) and Dias and Embrechts (2010). The following considers among others the *Gaussian copula* and the *Student-t copula*.

The Archimedean copulae family includes the *Gumbel*, *Clayton* and *Frank* copulae, which have a simple closed form and have been studied in relation to the modeling of portfolio credit risk in McNeil et al. (2005), Dias (2004) and Wu et al. (2006).

In addition to the one-parametric copulae, the current paper will consider some *mixture copula models* of the Archimedean copulae as introduced in Joe (1993). Here the distribution function has the form of a convex combination of two or more copulae. Denoting by $C^A$ and $C^B$ copulae with dependence parameters $\theta_1$ and $\theta_2$, respectively, the mixture copulae model has the form:

$$C_X(u_1, \dots, u_d, \theta) = \theta_3 C_X^A(u_1, \dots, u_d, \theta_1) + (1 - \theta_3) C_X^B(u_1, \dots, u_d, \theta_2), \tag{9}$$

where $\theta_3 \in [0, 1]$. The following three subsections will consider four mixture models studied in Dias (2004). These include the *Clayton & survival Clayton*, *Clayton & Gumbel*, *survival Clayton & survival Gumbel* and *Gumbel & survival Gumbel* copulae. Dias (2004), Canela and Collazo (2006) and Hu (2006) study mixture models for modeling dependencies of log-returns across international financial markets.

#### 2.2.1 Gaussian Copula

The Gaussian copula expresses the dependence structure of the multivariate normal distribution, i.e. normal marginal distributions are combined with a Gaussian copula to form a multivariate normal distribution. If $Y_j \sim N(0,1)$ and $Y = (Y_1, \dots, Y_d)^\top \sim N_d(0, \Psi)$, where $\Psi$ denotes a correlation matrix, an explicit expression for the Gaussian copula is given by

$$C_\Psi^{Ga}(u_1, \dots, u_d) = F_Y\{\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\} \tag{10}$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \dots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} \exp\left(-\frac{1}{2} r^\top \Psi^{-1} r\right) dr_1 \dots dr_d.$$

Defining $\zeta_j = \Phi^{-1}(u_j)$, $\zeta = (\zeta_1, \dots, \zeta_d)^\top$, the density of the Gaussian copula can be written as

$$c_\Psi^{Ga}(u_1, \dots, u_d) = |\ \Psi\ |^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^\top(\Psi^{-1} - \mathcal{I}_d)\zeta\right\}. \tag{11}$$

### *2.2.2 Student-t Copula*

The Student-t copula inherits the dependence structure from the multivariate Student-t distribution. Let $X = (X_1, \dots, X_d)^\top \sim t_d(\nu, \mu, \Sigma)$ have a multivariate Student-t distribution with $\nu$ degrees of freedom, mean vector $\mu$ and positive-definite dispersion or scatter matrix $\Sigma$. The copula remains invariant under a standardization of the marginal distributions[1]. This means that the copula of a $t_d(\nu, \mu, \Sigma)$ distribution is identical to that of a $t_d(\nu, 0, \Psi)$ distribution, where $\Psi$ is the correlation matrix associated with $\Sigma$. The unique Student-t copula is the copula $C_X = C^t_{\nu,\Psi}$. For $u = (u_1, \dots, u_d)^\top \in [0,1]^d$, it is given by

$$C^t_{\nu,\Psi}(u_1, \dots, u_d) = t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)\}, \tag{12}$$

where $t_\nu^{-1}$ is the quantile function of the univariate $t$-distribution. The density of the $t$-copula is given as

$$c^t_{\nu,\Psi}(u_1, \dots, u_d) = \frac{t_{\nu,\Psi}\{t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)\}}{\prod_{j=1}^d t_{\nu,\Psi}\{t_\nu^{-1}(u_j)\}}. \tag{13}$$

With $\zeta_j = t_\nu^{-1}(u_j)$ the density of the $t$-copula can be expressed as:

$$c^t_{\nu,\Psi}(u_1, \dots, u_d) = |\ \Psi\ |^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu+d}{2})\left\{\Gamma(\frac{\nu}{2})\right\}^{d-1}\left(1 + \frac{1}{\nu}\zeta^\top\Psi^{-1}\zeta\right)^{-\frac{\nu+d}{2}}}{\left\{\Gamma(\frac{\nu+1}{2})\right\}^d \prod_{j=1}^d \left(1 + \frac{1}{\nu}\zeta_j^2\right)^{-\frac{\nu+1}{2}}}. \tag{14}$$

The Student-t copula generates symmetric tail dependence. Its tail dependence coefficients, as defined in equations (7) and (8), take the following relatively simple form:

$$\lambda_u = \lambda_l = 2t_{\nu+1}\sqrt{(\nu+1)(1-\rho)/(1+\rho)}, \tag{15}$$

where $t_\nu$ denotes the Student-t distribution function. Here $\nu$ is the number of degrees of freedom, and $\rho$ is the correlation coefficient. Figure 1 shows 1000 bivariate realizations simulated using two different dependence structures. Both models use identical marginal distributions, the Gamma(3,1) distribution in the upper panel and the Student-t distribution with four degrees of freedom in the lower panel, and for both identical correlation $\theta = 0.7$. However, the Gaussian copula is applied to model the dependency in the left panel and the Student-t copula is used to model the dependency in the right panel. One observes that the second model favors the simultaneous occurrence of extreme values, which is from the point of view of risk management the more dangerous case. Since extreme losses have a tendency to occur simultaneously in log-returns, it appears also to be a more realistic model just from this observation alone. However, the standard

[1] In fact, it remains invariant under any series of strictly increasing transformations of the components of the random vector $X$, see Nelsen (1998).

*RiskMetrics* methodology, which assumes a Gaussian copula, does only have very few extreme losses appearing jointly.

### *2.2.3 Clayton Copula*

The Clayton copula with the dependence parameter $\theta \in (0, \infty)$ is defined as

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left( \sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-1/\theta} \tag{16}$$

with density:

$$c_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} \{1 + (j-1)\theta\} u_j^{-(\theta+1)} \left( \sum_{j=1}^{d} u_j^{-\theta} - d + 1 \right)^{-(1/\theta + d)} . \tag{17}$$

As the copula parameter $\theta$ tends to infinity, the dependence becomes maximal and, as $\theta$ tends to zero, one has independence. The Clayton copula can mimic lower tail dependence but no upper tail dependence.

### *2.2.4 Gumbel Copula*

The Gumbel copula with the dependence parameter $\theta \in [1, \infty)$ is of the form:

$$C_\theta(u_1, \ldots, u_d) = \exp \left[ - \left\{ \sum_{j=1}^{d} (-\log u_j)^\theta \right\}^{1/\theta} \right] . \tag{18}$$

For $\theta > 1$ this copula generates an upper tail dependence, while for $\theta = 1$ it reduces to the product copula (i.e. independence): $C_\theta(u_1, \ldots, u_d) = \prod_{j=1}^{d} u_j$. Maximal dependence is achieved when $\theta$ tends to infinity.

## *2.3 Copula Estimation*

The remainder of this section focuses on the estimation of parametric copulae, including the estimation of the marginal parameters, as well as the dependence structure characterized by the dependence parameter. Consider a vector of random variables $X = (X_1, ..., X_d)^\top$ with parametric univariate marginal distributions $F_{X_j}(x_j, \delta_j)$, $j \in \{1, ..., d\}$. Furthermore, let a copula belong to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. From Sklar's Theorem the distribution of $X$ can be expressed as

$$F_X(x_1, \ldots, x_d) = C\{F_{X_1}(x_1; \delta_1), \ldots, F_{X_d}(x_d; \delta_d); \theta\} \tag{19}$$

and its density as

$$f(x_1,\dots,x_d;\delta_1,\dots,\delta_d,\theta) = c\{F_{X_1}(x_1;\delta_1),\dots,F_{X_d}(x_d;\delta_d);\theta\}\prod_{j=1}^{d} f_j(x_j;\delta_j), \tag{20}$$

where

$$c(u_1,\dots,u_d) = \frac{\partial^d C(u_1,\dots,u_d)}{\partial u_1 \dots \partial u_d} \tag{21}$$

is a copula density. For a sample of observations $\{x_t\}_{t=1}^{T}$, where $x_t = (x_{1,t},\dots,x_{d,t})^\top$, and a vector of parameters $\alpha = (\delta_1,\dots,\delta_d,\theta)^\top \in \mathbb{R}^{d+1}$ the likelihood function is given by the product

$$L(\alpha;x_1,\dots,x_T) = \prod_{t=1}^{T} f(x_{1,t},\dots,x_{d,t};\delta_1,\dots,\delta_d,\theta) \tag{22}$$

and the corresponding log-likelihood function by the sum

$$\ell(\alpha;x_1,\dots,x_T) = \sum_{t=1}^{T}\ln\left[c\{F_{X_1}(x_{1,t};\delta_1),\dots,F_{X_d}(x_{d,t};\delta_d);\theta\}\right] + \sum_{t=1}^{T}\sum_{j=1}^{d}\ln\left[f_j(x_{j,t};\delta_j)\right]. \tag{23}$$

The objective is to maximize this log-likelihood function numerically. The estimation can be performed, for instance, in the following three different ways, employing the *exact maximum likelihood* (EML), the *inference for marginals* (IFM) and the *canonical maximum likelihood* (CML) method.

### *2.3.1 Exact Maximum Likelihood (EML)*

The exact maximum likelihood (EML) method is straightforward and estimates the parameter $\alpha$ in one step through

$$\tilde{\alpha}_{EML} = \arg\max_{\alpha} \ell(\alpha). \tag{24}$$

The estimates $\tilde{\alpha}_{EML} = (\tilde{\delta}_1,\dots,\tilde{\delta}_d,\tilde{\theta})^\top$ solve the first order condition

$$(\partial\ell/\partial\delta_1,\dots,\partial\ell/\partial\delta_d,\partial\ell/\partial\theta) = 0. \tag{25}$$

The drawback of the EML method is that with an increasing scale of the problem the algorithm becomes computationally challenging.

### *2.3.2 Inference for Marginals (IFM)*

In the inference for marginals (IFM) method the parameters for marginals and copulae are estimated separately, which represents a sequential two-step maximum likelihood method, see McLeish and Small (1988) and Joe (1997). The parameters $\delta_j$ from the marginal distributions are estimated in the first step and the dependence parameter $\theta$ is estimated in the second step, after the estimated marginal distributions have been substituted into the copula. For $j \in \{1,\dots,d\}$ the log-likelihood function for each of the marginal distributions is given by

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \ln f_j\left[x_{j,t};\delta_j\right] \tag{26}$$

and the estimated marginal parameter is obtained as

$$\hat{\delta}_j = \arg\max_{\delta} \ell_j(\delta_j). \tag{27}$$

The pseudo log-likelihood function

$$\ell(\theta,\hat{\delta}_1,\ldots,\hat{\delta}_d) = \sum_{t=1}^{T} \ln\left[c\{F_{X_1}(x_{1,t};\hat{\delta}_1),\ldots,F_{X_d}(x_{d,t};\hat{\delta}_d);\theta\}\right] \tag{28}$$

is maximized over $\theta$ to obtain the estimator $\hat{\theta}$ for the dependence parameter $\theta$. The estimates $\hat{\alpha}_{IFM} = (\hat{\delta}_1,\ldots,\hat{\delta}_d,\hat{\theta})^\top$ solve the first order condition

$$(\partial\ell_1/\partial\delta_1,\ldots,\partial\ell_d/\partial\delta_d,\partial\ell/\partial\theta) = 0. \tag{29}$$

### *2.3.3 Canonical Maximum Likelihood (CML)*

In contrast to the EML and IFM methods, where one has to make an assumption about the parametric form of the marginal distributions, the canonical maximum likelihood (CML) method maximizes the pseudo log-likelihood function with empirical marginal distributions in the form:

$$\ell(\theta) = \sum_{t=1}^{T} \ln\left[c\{\hat{F}_{X_1}(x_{1,t}),\ldots,\hat{F}_{X_d}(x_{d,t});\theta\}\right]. \tag{30}$$

Here the empirical marginal cumulative distribution function is given by

$$\hat{F}_{X_j}(x) = \frac{1}{T+1}\sum_{t=1}^{T} \mathbf{1}_{\{X_{j,t}\leq x\}}, \tag{31}$$

see Genest and Rivest (2002). Using this method, the parameter can be estimated in one step by using the estimate

$$\hat{\theta}_{CML} = \arg\max_{\theta} \ell(\theta). \tag{32}$$

## 3 Empirical Analysis

As argued in Platen and Stahl (2003), a diversified world stock index is very suitable for capturing the general market risk, see Basel (1996). Such indices are also widely used as benchmarks when evaluating investment strategies. The study of diversified world stock indices is motivated by the *benchmark approach*, see Platen and Heath (2006), which puts a well diversified portfolio into the center of financial modeling. By the *diversification theorem* in Platen (2005), a diversified portfolio can be interpreted as a good proxy for the *growth optimal portfolio* (GOP), which is in many ways the best performing portfolio: it can be characterized as the portfolio maximizing the expected logarithmic utility from terminal wealth, and is the portfolio with almost surely the maximal long-run growth rate, which outperforms that of any other strictly positive portfolio. The *equi-weighted index* (EWI) of a given investment universe can be regarded, under minor regularity assumptions, as one of the best performing diversified portfolios and, thus, as a good proxy for the GOP.

### *3.1 Data*

An equi-weighted index is an almost ideally diversified index, all fractions equal $\pi_{\delta_{EWI},t} = 1/d$ with $\sum_{j=1}^{d} \pi_{\delta_{EWI},t} = 1$, $j \in \{1, 2, ..., d\}$. The EWI104s used in the following analysis is based on 104 world industry sector indices as constituents provided by *Datastream Thomson Financial* and is constructed in Lee and Platen (2006) and Platen and Rendek (2008). Figure 2 plots the EWI104s denominated in USD for the time span from 02 January, 1973 to 10 March, 2006. The analysis of this paper uses 20 currency denominations, the selection of respective currencies is determined by the availability of currency data for the entire period. These include the currencies of the following 20 countries: Australia(Aus), Austria(Au), Belgium(Be), Canada(Ca), Denmark(Dk), Finland(Fi), France(Fr), Germany(Ge), India(Ind), Ireland(Ire), Italy(It), Japan(J), Malaysia(Ma), Netherlands(Nth), Norway(Nor), Portugal(Por), Spain(Sp), Sweden(Swe), UK, USA. The paper concentrates its inference on the same data set as used in Platen and Rendek (2008) to make the results complementing the earlier ones. The inclusion of the data of the recent years, including those of the recent financial crisis, does not alter significantly the results. A separate analysis shows even greater statistical significance due to the longer data set available.

### *3.2 Student-t Log-Returns of the Marginals*

Platen and Rendek (2008) have investigated the large class of *symmetric generalized hyperbolic* distributions, aiming to identify an appropriate distribution within this class, which leads to the best fit of the log-returns of the EWI104s denominated in different currencies. They have shown that the Student-t distribution with about four degrees of freedom is clearly the typical estimated log-return distribution of such a diversified equity index in all considered currency denominations. Figure 3 represents the logarithm of the histogram for the normalized pooled data of log-returns taken for all EWI104s currency denominations for the time span from 02 January, 1973 to 10 March, 2006 comprising 173,200 observations. The solid line in the left panel of Figure 3 represents the normal density, and the solid line in the right panel characterizes the standardized Student-t density given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\sigma\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\sigma^2\nu}\right)^{-\frac{\nu+1}{2}}$$

with $\nu = 4$ degrees of freedom and standard deviation $\sigma_X$ given by $\sigma_X = \sigma\sqrt{\frac{\nu}{\nu-2}}$ with $\sigma = 0.7$. One observes an excellent fit of the log-returns of the EWI104s to the Student-t density compared to a poor fit to the normal density. In addition, Figure 4 represents the QQ plot for the pooled data of log-returns. The QQ plot contrasts the quantiles of the data with the quantiles from the normal (left panel) and from the Student-t (right panel) distribution. A deviation of the log-returns from the 45 degrees line in the left panel indicates strong non-normality of the data. The performance improves considerably in the right panel for the Student-t case. Parameter estimates and standard errors obtained from fitting the univariate Student-t distribution to log-returns of index denominations are given in Platen and Rendek (2008).

### *3.3 Fitting Static and Time-varying Copulae*

The following subsection aims to fit a parametric copula, that is, to estimate the copula dependence parameter, assuming that the unknown marginal distributions are nuisance parameters. [2] Following Platen and Rendek (2008), assume that the marginals have a Student-t distribution. The estimation of the dependence parameter of the Student-t copula proceeds as follows: First, transform the original data to the "copula scale" by applying a probability integral transform to obtain uniformly $[0, 1]$-distributed values. Then, apply the IFM method to estimate different copulae assuming that the marginals are Student-t distributed. The estimated number of degrees of freedom for the marginals is taken from Platen and Rendek (2008) for each of the currency denominations of the EWI104s index.

A *static copula* is assumed to estimate the average dependence parameter using log-return data from the time interval covering the whole time span from 02 January, 1973 to 10 March, 2006. The paper fixes different one parametric copula models, as well as mixture copulae models summarized in Table 1, and uses the 20 different currency denominations listed above. In total it obtains 190 estimated copulae of currency pairs for each of the following copula models: Clayton, Frank, Gaussian, Gumbel, Plakett, Student-t, and the mixture models of Clayton & survival Clayton, Clayton & Gumbel, survival Clayton & survival Gumbel, Gumbel & survival Gumbel type. Figure 5 shows the estimated dependence parameter for each pair of currency denominations and one-parametric copula models. Figures 6 and 7 show the estimated copula dependence parameters for 190 pairs of currency denominations and different mixture copula models. The parameters $\theta_1$ and $\theta_2$ plotted in the first and the second panels are the dependence parameters for the first and second terms of the mixture, respectively. The mixture parameter $\theta_3$ plotted in the third panel gives the proportion of the first term. Box-plots exhibited on the bottom of each plot show the extreme of the lower whisker (2.5%-quantile), low quartile, median, upper quartile and the extreme of the upper whisker (97.5%-quantile) of the estimated parameters. Not surprisingly, one observes high correlations between European countries and lower correlations between countries on different continents. The average estimated copula dependence parameters taken over 190 currency pairs together with their standard errors, minimum and maximum values are given

[2] For the EML and IFM method one needs to make an assumption about the parametric form of the marginal distributions, whereas the CML method uses empirical marginal distributions.

in the first three columns of Table 1. In the case of mixture models, the parameters $\theta_1$ and $\theta_2$ are the dependence parameters for the first and second terms of the mixture, respectively, and $\theta_3$ is the mixture parameter which gives the proportion of the first term.

To judge the performance of each fitted model, the *Akaike information criterion* (AIC) is provided, see Akaike (1974):

$$AIC = -2L(\alpha; x_1, \dots, x_T) + 2q,$$

where $q$ is the number of parameters of the family of distributions fitted. The AIC combines two components: the log-likelihood $L$, which measures the goodness-of-fit, and a penalty term $2q$, accounting for model complexity. The better the model fits the data, the larger is the likelihood, that is, smaller values of the AIC indicate a better fit. In the last column of Table 1 all models are ranked by their AIC (model ranking is given in parentheses). One observes that the Student-t copula provides the best fit across different copulae models, followed by the mixture Gumbel & survival Gumbel copula.

The *time-varying* dependence parameter is estimated using log-returns from a moving window with a fixed size of n=250, corresponding to one year of observations. A subsample of log-returns $\{\widehat{X_t}\}_{t=s-n+1}^{s}$ is scrolling in time for $s \in \{n, ..., T\}$ generating a time-series for the dependence parameter $\{\widehat{\theta}_t\}_{t=n}^{T}$. The static case, on the contrary, estimates the dependence parameter based on the whole series of observations. Figure 8 represents the estimated time-varying dependence parameter between the denominations of the EWI104s in Euro and US Dollar, estimated using the IFM method for a Student-t copula with Student-t marginals. Note that the dependence parameter dramatically declined before the 1987 crash. Potentially, such a decline could serve as an indicator for a likely major market correction. The dashed line in Figure 8 corresponds to the static case. One observes that the dependence structure between denominations of the EWI104s in different currencies is not constant but varies over time. In addition, Figure 9 plots the 250 observations used for the estimation from the moving window corresponding to maximal dependence on 11 October, 1977 (left panel) and minimal dependence on 30 September, 1985 (right panel).

## 4 Risk Management Applications

The following applies the estimation techniques described above to the inference of the dependence parameter, the *Value-at-Risk* (VaR) and the *expected shortfall* (ES) of a benchmarked portfolio, that is, a portfolio expressed in units the EWI104s. This is different from the usual set up, when calculating VaR or ES for a portfolio denominated in units of a currency. Below the log-returns of the benchmarked portfolio are examined. They do not use a certain currency as denominator but the EWI104s as reference unit, which reflects the movements of a market as a whole. This means, the resulting VaR and ES numbers measure the risk, or uncertainty of the portfolio against this index. In some sense they measure its idiosyncratic or specific market risk, as proposed in Platen and Stahl (2003) under the benchmark approach.

The study includes the following currencies: US dollar(USD), Euro(EUR), Canadian dollar(CAD), British pound(GBP), Australian dollar(AUD) and Singapore dollar(SGD). Three-month T-Bill rates covering the time period from 01 January, 1990 to 10 March, 2006 were used to calculate the savings account prices $B^{USD}$, $B^{EUR}$, $B^{CAD}$, $B^{GBP}$, $B^{AUD}$ and $B^{SGD}$. The total number of observations corresponds to 4225. This is the longest time span for which the T-Bill data for EUR were available in *Datastream Thomson Financial.* The first example considers a two-dimensional case, fitting bivariate copulae to the benchmarked savings accounts for USD and EUR. The second example examines a six-dimensional portfolio which uses the entire set of currencies.

Three different copulae models are specified to estimate the copula dependence parameter, the VaR and the ES of the portfolio. These include the Student-t copula with Student-t marginals [3], the Gaussian copula with Student-t marginals, and the Gaussian copula with normal marginals. For the Gaussian copula with Gaussian marginals the conditional Gaussian *RiskMetrics* framework is recovered, see Morgan/Reuters (1996). For all three models the dependence parameter $\{\widehat{\theta}_t\}_{t=n}^{T}$ is estimated using the IFM method, see Section 2.3. Marginal parameters are estimated time-varying prior to the estimation of $\{\widehat{\theta}_t\}$. A moving window of size $n = 250$ days is applied, which corresponds to about one year of observations.

### *4.1 VaR for a Benchmarked Portfolio*

Denote by $B_t^j$ the saving account of the $j^{\text{th}}$ currency, $j \in \{1, ..., d\}$, on day $t$, and by $S_t^{j,EWI}$ the value of the EWI104s denominated in the $j^{\text{th}}$ currency on day $t$. Consider a portfolio with strategy $w = (w_1, ..., w_d)^\top$, composed of savings accounts of different currencies, which invests in $w^j$ units of $j^{\text{th}}$ currency's savings account $B_t^j$. The benchmarked value on day $t$ of a portfolio with strategy $w = (w_1, ..., w_d)^\top$, where $w_j$ also represents the number of units invested in the $j^{\text{th}}$ benchmarked savings account $\widehat{B}_t^j$ on day $t$, is given by

$$\widehat{V}_t = \sum_{j=1}^{d} w_j \widehat{B}_t^j \quad \text{with} \quad \widehat{B}_t^j = \frac{B_t^j}{S_t^{j,EWI}}. \tag{33}$$

The benchmarked daily profit and loss (P&L), representing daily changes in the benchmarked portfolio, is of the form:

$$\widehat{L}_{t+1} = \widehat{V}_{t+1} - \widehat{V}_t. \tag{34}$$

Denoting the daily benchmarked $j^{\text{th}}$ savings account's log-return on day $t$ by $\widehat{X}_{t+1}^j = \ln(\widehat{B}_{t+1}^j) - \ln(\widehat{B}_t^j)$, one can rewrite (34) as

$$\widehat{L}_{t+1} = \sum_{j=1}^{d} w_j \widehat{B}_t^j \left\{ \exp(\widehat{X}_{t+1}^j) - 1 \right\}. \tag{35}$$

[3] Log-returns of the benchmarked savings account $\widehat{B}^j$ are Student-t distributed because they are the negative values of the log-returns of the discounted EWI104s, which have a Student-t distribution as shown previously.

The one-day VaR on day $t$ for significance level $\alpha$ is given by the $\alpha$-quantile of the distribution of the benchmarked P&L, that is,

$$VaR_t(\alpha) = F^{-1}_{\widehat{L}_{t+1}}(\alpha). \tag{36}$$

Furthermore, the one day expected shortfall (ES) at the $t^{\text{th}}$ day can be computed as

$$ES_t(\alpha) = \frac{1}{N_{t+1}} \sum_{i=1}^{N_{t+1}} \widehat{L}_{t+1,i} \mathbf{1}_{\{\widehat{L}_{t+1,i} \leq VaR_t(\alpha)\}}, \tag{37}$$

where $N_{t+1}$ denotes the number of simulated portfolio returns with value less than or equal to $VaR_t(\alpha)$, and $\widehat{L}_{t+1,i}$ is the $i^{\text{th}}$ outcome of the $N_{t+1}$ samples. Obviously, $ES_t(\alpha) \geq VaR_t(\alpha)$. Note that the ES has been proposed as a remedy for the deficiencies of VaR, which is, in general, not a coherent risk measure, see Föllmer and Schied (2004) and McNeil et al. (2005).

The *RiskMetrics* VaR procedure assumes that the conditional distribution of log-returns is multivariate normal $\widehat{X}_t^j \sim N(0, \Sigma_t^j)$ with a covariance matrix $\Sigma_t^j$, estimated using exponential smoothing:

$$(\widehat{\sigma}_t^j)^2 = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} (\widehat{X}_s^j)^2.$$

The parameter $\lambda \in (0, 1)$ is the, so-called, exponential moving average decay factor, which is set to the value of 0.05 since this level provides the best backtesting results, see Morgan/Reuters (1996).

Under the copulae-based approach one can first correct the contemporaneous volatility in the log-return process using e.g. a GARCH(1,1) process:

$$(\sigma_t^j)^2 = \alpha_0 + \alpha_1 (\widehat{X}_{t-1}^j)^2 + \beta_1 (\sigma_{t-1}^j)^2 \text{ and } \widehat{X}_t^j = \sigma_t^j \varepsilon_t^j. \tag{38}$$

Here $\varepsilon_t = (\varepsilon_t^1, \ldots, \varepsilon_t^d)^\top$ are standardized innovations for benchmarked savings accounts in the respective currency with a joint distribution $F_\varepsilon$ and continuous marginal distributions $F^j$, $j \in \{1, \ldots, d\}$, and parameters $\alpha_0 > 0$, $\alpha_1 \geq 0$, $0 \leq \beta_1 < 1$. The innovations $\varepsilon$ have a joint distribution function given by

$$F_\varepsilon(\varepsilon^1, \ldots, \varepsilon^d) = C_\theta\{F^1(\varepsilon^1), \ldots, F^d(\varepsilon^d)\}. \tag{39}$$

For details on the above model specification, see McNeil et al. (2005), Chen and Fan (2006a), Chen and Fan (2006b), Chen et al. (2006) and Dias and Embrechts (2010).

In order to estimate the VaR and the ES in the given set-up, one first has to estimate the copula dependence parameter from a sample of pairs of log-returns of the marginals. The estimated parameter is then used to generate Monte-Carlo samples of the P&L using the specified copula. Their quantiles at different confidence levels give estimates for VaR. Finally, averaging over the worst $100\alpha\%$ cases leads to the estimated ES.

The VaR and the ES estimation procedure described above has been applied to different sub-samples of log-returns $\{\widehat{X}_t\}_{t=1}^T$. Time-varying estimation uses subsets of size $n$, that is, a moving window of size $n$, $\{\widehat{X}_t\}_{t=s-n+1}^s$ scrolling in time for $s \in \{n, ..., T\}$. It generates a time-series for the dependence parameter $\{\widehat{\theta}_t\}_{t=n}^T$ and a time-series for the VaR: $\{\widehat{VaR}_t\}_{t=n}^T$. The static case, on the contrary, estimates at once the dependence parameter and the VaR number based on the entire series of observations.

Backtesting is applied to evaluate the performance of the procedure. It uses part of the information available to estimate the one period ahead risk measures and afterwards compares these with the values actually observed. More precisely, it compares the estimated values of VaR with the true realizations $\{\widehat{L}_t\}_{t=n}^T$ of the benchmarked P&L by computing the exceedance ratio:

$$\widehat{\alpha} = \frac{1}{T-n} \sum_{t=n}^{T} \mathbf{1}_{\{\widehat{L}_t < \widehat{VaR}_t(\alpha)\}}. \tag{40}$$

Whenever $\widehat{L}_t < \widehat{VaR}_t$, one can say that a violation of the VaR has occurred. By definition, VaR is a number such that the loss should occur with probability larger than $\alpha$. Therefore, the ratio of the number of such violations to the total number of observations in the backtesting period should be close to the target level $\alpha$.

### *4.1.1 A Two-Dimensional Portfolio*

This subsection considers a two-dimensional benchmarked portfolio of investments in the benchmarked savings account for USD: $\widehat{B}_t^{USD} = B_t^{USD}/S_t^{USD,EWI}$ and that for EUR: $\widehat{B}_t^{EUR} = B_t^{EUR}/S_t^{EUR,EWI}$. This portfolio simply invests $w^{USD}$ units in the USD savings account and $w^{EUR}$ units in the EUR savings account. For the Student-t copula with Student-t marginals, Figure 10 presents the global parameter estimate (dashed line) corresponding to the static case, as well as the time-varying parameter estimate $\{\widehat{\theta}_t\}_{t=n}^T$ (solid line) as it evolves when using a moving window of size $n = 250$ days.

Based on $N = 10000$ simulations one estimates the VaR and the ES for confidence levels $\alpha \in \{10.0\%, 5.0\%, 1.0\%, 0.05\%, 0.01\%\}$. Average one-day estimates for the portfolio $w = (1, 1)^\top$ are reported in Table 2, with standard errors reported in parentheses. It can be seen that together with an assumption on the marginals, also the dependence structure given by the copula function has a great influence on the VaR and the ES. For the lower significance levels, e.g. 10.0%, 5.0%, one observes similar VaR numbers in the case of the Gaussian copula (with either normal, or Student-t marginals) as for the Student-t copula. When increasing the significance levels to e.g. 0.5% or 0.1%, the leptokurtic effect becomes much stronger, and VaR will, therefore, be underestimated if normality is assumed. This leads to the smaller VaR and ES numbers for the Gaussian copula with normal marginals compared to those of the Student-t copula. Furthermore, one observes that the assumption on the distributional form of the marginals has greater effect on the VaR and ES estimates, compared to the effect of the choice of the copula. Not surprisingly, the errors increase as the estimated risk measures go further into the tails of the distribution.

To confirm the statistical outperformance of the Student-t copula with Student-t marginals over the two other models, the exceedance ratios for several confidence levels are reported in Table 3. The data set has 4225 observations and uses 3975 of these in the backtesting procedure, since the moving window has length 250. Table 3 compares the performance of the methods by checking how close the percentages of VaR violations are to the targets of 10.0%, 5.0%, 1.0%, 0.5%, 0.1%. For all copula models one observes that the estimated exceedance ratios $\hat{\alpha}$ are consistent with a specified target confidence level $\alpha$. For the Gaussian copula with normal marginals the estimated 1.0%, 0.5% and 0.1% VaR is violated, respectively, by 3.09% , 2.21% and 1.28% of observations, that is, the model underestimates the VaR. Changing distributional assumption for the marginals from normal to Student-t leads to a significant improvement. The Student-t assumption on the marginals allows a better capturing of outliers in the tails, leading to percentages of VaR violations close to the above targets. VaR violations amount to 1.48%, 0.75%, 0.17% for the Gaussian copula with Student-t marginals, and 1.33%, 0.65%, 0.15% for the Student-t copula with Student-t marginals. The overall performance is summarized by computing the relative squared and absolute errors $\sum_\alpha((\alpha-\widehat{\alpha})/\alpha)^2$ and $\sum_\alpha |\alpha-\widehat{\alpha}|/\alpha$, respectively, reported in the last two columns of Table 3. One observes that the Student-t copula with Student-t marginals leads to the smallest errors and, thus, to the best backtesting results.

Finally, Figure 12 shows P&L outcomes and the VaR estimated at different confidence levels of 10.0%, 5.0%, 1.0% and 0.5% together with exceedances computed at the 0.5% significance level. The upper panel corresponds to the Gaussian copula with normal marginals (*RiskMetrics*). The lower panel refers to the Student-t copula with Student-t marginals. As in Table 3, one observes that the number of violations at the level 0.5% for the Gaussian copula model with normal marginals corresponds to 88, which considerably exceeds the target number of 19.9. Employing the Student-t copula with Student-t marginals shows that the fit improves significantly and leads to the more realistic number of violations of 26.

#### *4.1.2 A Six-Dimensional Portfolio*

This subsection considers a benchmarked portfolio with six constituents and invests in $w^j$ units of the $j^{\text{th}}$ currency's benchmarked savings account. The currencies include the USD, EUR, CAD, GBP, AUD and SGD. Figure 11 presents the global parameter estimate (dashed line) corresponding to the static case, as well as, the time-varying parameter estimate $\{\widehat{\theta}_t\}_{t=n}^T$ (solid line) for the Student-t copula with Student-t marginals, as it evolves when using a moving window of size $n = 250$ days.

Table 4 reports average one-day estimates for the VaR and the ES of the portfolio $w = (1,1,1,1,1,1)^\top$, estimated for confidence levels $\alpha \in \{10.0\%, 5.0\%, 1.0\%, 0.5\%, 0.1\%\}$, together with standard errors reported in parentheses. One makes similar observations as in the two-dimensional case: increasing significance levels lead to increased VaR and ES. For the high significance levels the VaR and ES estimates are smaller in the case of the normal marginals compared to the case of the Student-t marginals, that is, the normality assumption leads to critical underestimation of VaR.

Exceedance ratios for all confidence levels are reported in Table 5. Similar to the two-dimensional case, changing distributional assumption for the marginals from normal to

Student-t leads to a significant improvement in terms of the exceedance ratios. Furthermore, comparing both copula models with Student-t marginals, one observes that the Gaussian copula with Student-t marginals performs slightly better for the confidence levels of 10.0% and 5.0%, and is outperformed by the Student-t copula with Student-t marginals for the significance levels of 1.0%, 0.5% and 0.1%. Overall, the Student-t copula with Student-t marginals outperforms the two other models, as can be observed by computing relative squared and absolute errors given in the last column of Table 5.

Figure 13 shows P&L realizations and estimated VaR numbers at different confidence levels for the Gaussian copula with normal marginals and the Student-t copula with Student-t marginals in the upper and the lower panel, respectively. Exceedances are plotted for the confidence level $\alpha = 0.5\%$ (i.e., the target number of VaR violations is 19.9). One observes that the Gaussian copula with normal marginals leads to a number of exceedances of 97 (2.44% of VaR violations), that is, the model underestimates the VaR. The Student-t copula with Student-t marginals shows a significant improvement leading to a more realistic number of VaR violations of 32.

### *4.2 VaR for a Portfolio Expressed in Units of Currencies*

The previous subsection has examined the log-returns of benchmarked savings accounts. These findings have been used to calculate the VaR and the ES for a benchmarked portfolio with a value $\widehat{V}_t$ expressed in units of the EWI104s:

$$\widehat{V}_t = \sum_{j=1}^{d} w_j \widehat{B}_t^j = \sum_{j=1}^{d} w_j \frac{B_t^j}{S_t^{j,EWI}}. \tag{41}$$

In order to calculate the VaR for the portfolio when it is expressed in units of the $j*^{\text{th}}$ currency, one has to consider the benchmarked portfolio value multiplied by the EWI104s denominated in the $j*^{\text{th}}$ currency:

$$V_t^{j*} = \widehat{V}_t S_t^{j*,EWI} = \sum_{j=1}^{d} w_j \frac{B_t^j}{S_t^{j,EWI}} S_t^{j*,EWI} = w_{j*} B_t^{j*} + \sum_{j=1, j\neq j*}^{d} w_j FX^{j*,j} B_t^j, \tag{42}$$

where $S_t^{j*,EWI}$ denotes the EWI104s denominated in currency $j*$ and $FX^{j*,j}$ is a foreign exchange rate between currency $j*$ and currency $j$. Equations (41) and (42) imply that one can use the non-benchmarked savings account of currency $j*$ and the exchange rates $FX^{j*,j}$, $j \in \{1, ..., d\}$, to estimate a copula dependence parameter.

Figure 14 displays in the upper panel the estimated $\widehat{VaR}(V_t^{j*})$ of the portfolio expressed in units of different currencies $j* \in \{USD, EUR, CAD, GBP, AUD, SGD\}$ for a confidence level of 1.0%. The estimated VaR and ES, computed at different significance levels, are summarized in Table 6 together with exceedances given in Table 7. One observes that the estimated values for the VaR differ significantly for the same portfolio in different currency denominations. For example, the average one-day VaR is higher for the portfolio expressed in AUD when compared to e.g. its denomination in SGD. To understand the causes for these differences, one needs to consider the changes in the benchmarked portfolio value, as well as, those in the currency denominations of the benchmark. The latter can significantly vary.

The dynamics of the $j^{\text{th}}$ savings account in the $j^{\text{th}}$ currency are given by the equation:

$$dB_t^j = r_t^j B_t^j dt, \tag{43}$$

for $t \geq 0$ and $B_0^j = 1$, where $r_t^j$ is the short rate for the $j^{\text{th}}$ currency. The stochastic differential equation for the benchmark, the GOP (approximated by the EWI104s) when denominated in the $j^{\text{th}}$ currency, has according to Platen and Heath (2006) the form:

$$dS_t^{j,EWI} = S_t^{j,EWI} \left\{ r_t^j dt + \sum_{k=1}^{d} \theta_t^{j,k} (\theta_t^{j,k} dt + dW_t^k) \right\}. \tag{44}$$

Here $k \in \{1, ..., d\}$ refers to the $k^{\text{th}}$ risk factor, which is modeled by the $k^{\text{th}}$ independent Wiener process $W^k$. For the $j^{\text{th}}$ currency denomination one may call $\theta_t^{j,k}$ the *general market volatility* under the $j^{\text{th}}$ denomination with respect to the $k^{\text{th}}$ source of uncertainty $W^k$. When denominating the benchmark in the $j^{\text{th}}$ currency, it measures the movements of the market as a whole and $\theta_t^{j,k}$ measures its general movements that are caused by the $k^{\text{th}}$ Wiener process. As in Platen and Stahl (2003), one can take the view that the benchmark (the GOP) captures the *general market risk*. Note that the benchmarked GOP is trivially constant and, thus, has zero drift and zero volatility.

The stochastic differential equation of the benchmarked portfolio value $\widehat{V}_t$ at time $t$ is obtained by a straightforward application of the *Itô* formula to (41), leading to the SDE:

$$d\widehat{V}_t = \sum_{j=1}^{d} w_j d\widehat{B}_t^j = -\sum_{j=1}^{d} w_j B_t^j \sum_{k=1}^{d} \theta_t^{j,k} dW_t^k = \widehat{V}_t \sum_{k=1}^{d} b_t^k dW_t^k. \tag{45}$$

One notes that the benchmarked portfolio value is driftless in the given continuous financial market and its volatility with respect to $W^k$ is given by the expression

$$b_t^k = -\sum_{j=1}^{d} \frac{w_j \widehat{B}_t^j}{\widehat{V}_t} \theta_t^{j,k}. \tag{46}$$

The above representation turns out to be convenient in the discussion on general and specific market risk as follows: One may call $b_t^k$, defined in (46), the *specific volatility* of the portfolio with respect to the $k^{\text{th}}$ risk factor, since it is the volatility of the portfolio when expressed in units of the benchmark. It measures the specific fluctuations of the benchmarked portfolio value with respect to the $k^{\text{th}}$ uncertainty in so far as these have not been already captured by the *general market risk* of the benchmark. Note that the *specific volatility* of the benchmark equals zero.

By applying the *Itô* formula to $V_t^{j*} = \widehat{V}_t \cdot S_t^{j*,EWI}$, one obtains the SDE for the portfolio value when expressed in units of $j*^{\text{th}}$ currency:

$$\begin{aligned} dV_t^{j*} &= d\widehat{V}_t \cdot S_t^{j*,EWI} \\ &= \widehat{V}_t \cdot dS_t^{j*,EWI} + S_t^{j*,EWI} \cdot d\widehat{V}_t + d\left[\widehat{V}, S^{j*,EWI}\right]_t \\ &= V_t^{j*} \left\{ r_t^{j*} dt + \sum_{k=1}^{d} \left(\theta_t^{j*,k} + b_t^k\right)\left(\theta_t^{j*,k} dt + dW_t^k\right) \right\}. \end{aligned} \tag{47}$$

From (47) the volatility with respect to the $k^{\text{th}}$ Wiener process of the portfolio, when expressed in units of the $j*^{\text{th}}$ currency, follows in the form $(\theta_t^{j*,k} + b_t^k)$, where

$$b_t^k = -\sum_{j=1}^{d} \frac{w_j \widehat{B}_t^j}{V_t^j} S_t^{j,EWI} \theta_t^{j,k} = -\sum_{j=1}^{d} \frac{w_j B_t^j}{V_t^j} \theta_t^{j,k}, \tag{48}$$

see (46). If $\theta_t^{j*,k} \neq 0$, then the $k^{\text{th}}$ Wiener process drives in the $j*^{\text{th}}$ currency denomination the benchmark with volatility $\theta_t^{j*,k}$, and as before one may call $\theta_t^{j*,k}$ the general market volatility with respect to the $k^{\text{th}}$ Wiener process $W^k$. As previously discussed, since it is the $k^{\text{th}}$ volatility of the benchmark, when expressed in units of the $j*^{\text{th}}$ currency, one can interpret it as a measure of the $k^{\text{th}}$ general market risk, that is, the fluctuations of the market as a whole in the $j*^{\text{th}}$ currency denomination, caused by $W^k$. Then, one may call the expression $(\theta_t^{j*,k} + b_t^k)$ the *total market volatility* of the portfolio in the $j*^{\text{th}}$ currency denomination with respect to $W^k$. It measures the general volatility (captured by $\theta_t^{j*,k}$) plus the specific volatility (captured by $b_t^k$) of the portfolio, when expressed in units of the $j*^{\text{th}}$ currency caused by the $k^{\text{th}}$ Wiener process.

By referring to the upper panel of Figure 14, one observes that the estimated VaR of the portfolio, when denominated in different currencies, is higher (in absolute terms) if the volatility of the portfolio expressed in the respective currency is lower. For example, the total market volatility $\theta_t^{SGD,k} - b_t^k$ of the portfolio, expressed in units of SGD, is lower than the total market volatility $\theta_t^{AUD,k} - b_t^k$ of the portfolio, expressed in units of AUD, which indicates that the SGD denomination of the portfolio appears to be less risky from the SGD perspective than its AUD denomination is from the AUD perspective. This in turn leads to higher VaR numbers for the portfolio denominated in AUD compared to the portfolio denominated in SGD. If regulatory capital has to be provided under similar rules in each economic environment, then the portfolio would be better located as a business in Singapore than in Australia.

Different riskiness of portfolios, when expressed in units of different currencies, is strongly related to the fluctuations of the exchange rates. From the investor's perspective, these fluctuations make it more difficult to decide where to invest. Clearly, taking into account merely the interest rates across countries is not sufficient, since the impact of the volatility of the particular exchange rate might lead to substantial losses. Therefore, when measuring portfolio risk by a risk measure like VaR with respect to a given currency, an investor may have to hedge the risks caused by the floating exchange rate involved via purchasing respective forward or futures contracts.

The collateral of major pension funds and international banks consists mostly of a diversified portfolio in equities. If the equity market moves significantly downward in a reference currency, then this collateral will be severely diminished from the perspective of this currency. As a consequence, if the liabilities of this fund or bank are mostly in international equities, then it is not so much the risk that is measured by some risk measure, such as VaR, with respect to a given currency that is important, it is more the risk measured with respect to a broadly diversified index (some benchmark) what matters. For the risk management of some companies, which are primarily exposed to the world equity risk, using the above benchmark perspective seems to be more appropriate than the choice of a particular currency perspective. A major downward move in the world equity market should not trigger extra capital requirements if the liabilities of the firm are mostly moving as the benchmark (a diversified world equity index). Clearly, for companies which have only liabilities in fixed income of a particular given currency, the risk measure has to be chosen with respect to that currency. The paper suggests to allow for both views in risk measurement and management.

## 5 Conclusions

This paper studies the dependency among log-returns of different currency denominations of a benchmark, which is chosen to be a well diversified global equity portfolio. Motivated by earlier findings, which showed that log-returns of a diversified world equity stock index in different currency denominations follow closely the Student-t distribution with about four degrees of freedom, the paper considers different copulae for modeling the dependence structure between log-returns of different currency denominations.

The Student-t property allows to capture in an effective way the dependent extreme movements of log-returns. This property can be well observed in diversified equity index log-returns and is important in portfolio risk management, where extreme losses have a tendency to occur simultaneously. The paper applies copulae to the estimation of the Value-at-Risk and the expected shortfall of a portfolio constructed from savings accounts of different currencies. The proposed methodology allows to split market risk into its general and specific part.

The study considers three different copulae models: the Student-t copula with Student-t marginals, the Gaussian copula with Student-t marginals, and the Gaussian copula with normal marginals, the latter known as the conditional Gaussian *RiskMetrics* framework. It shows that merely changing the distributional assumption for the marginals leads to a significantly better performance in terms of backtesting for the case of the Gaussian copula with Student-t marginals, compared to the Gaussian copula with normal marginals. Furthermore, for a portfolio of benchmarked savings accounts, changing the dependence structure from that of the Gaussian to the Student-t copula, leads to even better results in terms of the exceedance ratios. The Student-t copula with Student-t marginals leads to the smallest absolute and relative squared errors, and thus, yields the best backtesting results.

The Student-t copula with Student-t marginals turns out to be the most reliable model in the study. It provides both, a flexible and realistic fit for the joint distribution and the dependence structure. The backtesting results confirm that the Student-t copula with Student-t marginals captures also the joint extreme values in different currency denominations most successfully. These findings could allow investors to improve the allocation of required regulatory capital.

| Copula model | mean($\hat{\theta_1}$)($\hat{s.e.}$) | med($\hat{\theta_1}$) | min($\hat{\theta_1}$) | max($\hat{\theta_1}$) | mean($\hat{\theta_2}$)($\hat{s.e.}$) | med($\hat{\theta_2}$) | min($\hat{\theta_2}$) | max($\hat{\theta_2}$) | mean($\hat{\theta_3}$)($\hat{s.e.}$) | med($\hat{\theta_3}$) | min($\hat{\theta_3}$) | max($\hat{\theta_3}$) | AIC |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Clayton | 3.278(2.437) | 2.196 | 1.265 | 19.283 | - | - | - | - | - | - | - | - | -7767.55(10) |
| Frank | 10.96(6.824) | 8.007 | 5.040 | 56.636 | - | - | - | - | - | - | - | - | -8925.03(7) |
| Gaussian | 0.810(0.082) | 0.784 | 0.647 | 0.9899 | - | - | - | - | - | - | - | - | -7980.49(9) |
| Gumbel | 3.243(1.743) | 2.477 | 1.833 | 15.340 | - | - | - | - | - | - | - | - | -9063.73(6) |
| Plakett | 814.9(1541) | 23.61 | 9.984 | 4376.4 | - | - | - | - | - | - | - | - | -8054.06(8) |
| Student-t | 0.831(0.092) | 0.809 | 0.631 | 0.9930 | - | - | - | - | - | - | - | - | **-10713.2(1)** |
| Clayton & surv.Clayton | 8.630(19.59) | 3.296 | 0.698 | 182.5 | 6.720(14.46) | 3.560 | 0.634 | 102.43 | 0.534(0.075) | 0.529 | 0.253 | 0.732 | -10130.44(5) |
| Clayton & Gumbel | 1.306(1.191) | 0.955 | 0.264 | 8.263 | 7.085(9.693) | 4.118 | 1.010 | 94.970 | 0.320(0.100) | 0.314 | 0.046 | 0.728 | -10384.80(4) |
| surv. Clayton & surv. Gumbel | 1.168(2.003) | 0.608 | 0.010 | 15.36 | 6.344(6.278) | 3.803 | 2.640 | 50.000 | 0.277(0.108) | 0.279 | 0.021 | 0.990 | -10406.78(3) |
| Gumbel & surv. Gumbel | 9.204(11.31) | 4.689 | 1.415 | 50.00 | 6.720(14.46) | 3.560 | 0.634 | 102.43 | 0.534(0.075) | 0.529 | 0.253 | 0.732 | **-10673.96(2)** |

Table 1. An average (over 190 pairs of currency denominations) for the copula dependence parameter with standard errors (in parentheses), the minimum and the maximum values for different one-parametric copulae, as well as, for the mixture copula models. In case of mixture models, parameters $\theta_1$ and $\theta_2$ are the dependence parameters for the first and the second terms of the mixture, respectively, and $\theta_3$ is a mixture parameter which gives the proportion of the first term. For each model fitted, the last column provides the AIC and model ranking (in parentheses).

| Copula | Marginals | $\alpha$ | $VaR_t(s.e.)$ | $ES_t(s.e)$ |
|---|---|---|---|---|
| Student-t | Student-t | 0.1 | 0.009280 (0.004086) | 0.014887 (0.006526) |
| | | 0.05 | 0.012847 (0.005637) | 0.018927 (0.008268) |
| | | 0.01 | 0.022144 (0.009755) | 0.030036 (0.013200) |
| | | 0.005 | 0.026866 (0.011839) | 0.035819 (0.015851) |
| | | 0.001 | 0.040222 (0.017860) | 0.052380 (0.023546) |
| Gaussian | Student-t | 0.1 | 0.009594 (0.004226) | 0.009594 (0.004226) |
| | | 0.05 | 0.013127 (0.005774) | 0.013127 (0.005774) |
| | | 0.01 | 0.021986 (0.009598) | 0.021986 (0.009598) |
| | | 0.005 | 0.026336 (0.011493) | 0.026336 (0.011493) |
| | | 0.001 | 0.038097 (0.016505) | 0.038097 (0.016505) |
| Gaussian | Normal | 0.1 | 0.009289 (0.004231) | 0.012804 (0.005846) |
| | | 0.05 | 0.011958 (0.005456) | 0.015096 (0.006905) |
| | | 0.01 | 0.017060 (0.007815) | 0.019645 (0.009019) |
| | | 0.005 | 0.018942 (0.008685) | 0.021384 (0.009847) |
| | | 0.001 | 0.022803 (0.010538) | 0.024938 (0.011527) |

Table 2

For a benchmarked portfolio, $w = (1, 1)^\top$, of USD and EUR savings accounts an average one-day VaR and ES (standard errors in parentheses) is shown, estimated using different copula models and different confidence levels $\alpha$.

| Copula | Marginals | 0.1 | 0.05 | 0.01 | 0.005 | 0.001 | $\sum_\alpha ((\alpha - \widehat{\alpha})/\alpha)^2$ | $\sum_\alpha \lvert\alpha - \widehat{\alpha}\rvert/\alpha$ |
|---|---|---|---|---|---|---|---|---|
| Student-t | Student-t | 0.117248 | 0.059754 | 0.013307 | 0.006528 | 0.001506 | 0.006694 | 1.50986 |
| Gaussian | Student-t | 0.112729 | 0.057745 | 0.014813 | 0.007532 | 0.001757 | 0.006992 | 2.02689 |
| Gaussian | Normal | 0.126287 | 0.072809 | 0.030881 | 0.022094 | 0.012804 | 0.258692 | 18.0299 |

Table 3

Exceedance ratio $\hat{\alpha}$ for one-day VaR estimated for different confidence levels $\alpha$ for a benchmarked portfolio, $w = (1, 1)^\top$, of USD and EUR savings accounts.

| Copula | Marginals | $\alpha$ | $VaR_t(s.e.)$ | $ES_t(s.e)$ |
|---|---|---|---|---|
| Student-t | Student-t | 0.1 | 0.023172 (0.010894) | 0.037205 (0.017369) |
| | | 0.05 | 0.032097 (0.015048) | 0.047354 (0.022090) |
| | | 0.01 | 0.055371 (0.025897) | 0.075130 (0.034960) |
| | | 0.005 | 0.067222 (0.031297) | 0.089900 (0.041905) |
| | | 0.001 | 0.100544 (0.047141) | 0.131090 (0.062125) |
| Gaussian | Student-t | 0.1 | 0.024125 (0.011292) | 0.024125 (0.011292) |
| | | 0.05 | 0.032705 (0.015303) | 0.032705 (0.015303) |
| | | 0.01 | 0.053443 (0.024978) | 0.053443 (0.024978) |
| | | 0.005 | 0.063269 (0.029559) | 0.063269 (0.029559) |
| | | 0.001 | 0.089072 (0.042433) | 0.089072 (0.042433) |
| Gaussian | Normal | 0.1 | 0.024262 (0.012162) | 0.033361 (0.016645) |
| | | 0.05 | 0.031200 (0.015583) | 0.039314 (0.019561) |
| | | 0.01 | 0.044368 (0.022051) | 0.051047 (0.025351) |
| | | 0.005 | 0.049241 (0.024450) | 0.055534 (0.027530) |
| | | 0.001 | 0.059202 (0.029338) | 0.064735 (0.031998) |

Table 4

For a benchmarked portfolio, $w = (1, 1, 1, 1, 1, 1)^\top$, of USD, EUR, CAD, GBP, AUD and SGD savings accounts an average one-day VaR and ES (standard errors in parentheses) is shown, estimated using different copula models and different confidence levels $\alpha$.

| Copula | Marginals | 0.1 | 0.05 | 0.01 | 0.005 | 0.001 | $\sum_\alpha ((\alpha - \widehat{\alpha})/\alpha)^2$ | $\sum_\alpha \lvert\alpha - \widehat{\alpha}\rvert/\alpha$ |
|---|---|---|---|---|---|---|---|---|
| Student-t | Student-t | 0.122019 | 0.060507 | 0.015566 | 0.008034 | 0.001757 | 0.012568 | 2.35073 |
| Gaussian | Student-t | 0.114487 | 0.057745 | 0.018077 | 0.009289 | 0.002762 | 0.016606 | 3.72727 |
| Gaussian | Normal | 0.122521 | 0.074818 | 0.031383 | 0.024354 | 0.013809 | 0.302100 | 19.5397 |

Table 5

Exceedance ratio $\hat{\alpha}$ for one-day VaR estimated for different confidence levels $\alpha$ for a benchmarked portfolio, $w = (1, 1, 1, 1, 1, 1)^\top$, of USD, EUR, CAD, GBP, AUD and SGD savings accounts.

| Currency | $\alpha$ | $VaR_t(s.e.)$ | $ES_t(s.e)$ |
|---|---|---|---|
| USD | 0.1 | 0.019222 (0.005730) | 0.030935 (0.009097) |
| | 0.05 | 0.026613 (0.007894) | 0.039366 (0.011591) |
| | 0.01 | 0.046040 (0.013443) | 0.062676 (0.018236) |
| | 0.005 | 0.056049 (0.016476) | 0.075246 (0.022303) |
| | 0.001 | 0.084177 (0.025105) | 0.110340 (0.034743) |
| EUR | 0.1 | 0.032620 (0.015248) | 0.052373 (0.024336) |
| | 0.05 | 0.045141 (0.021067) | 0.066581 (0.030880) |
| | 0.01 | 0.077861 (0.036093) | 0.105726 (0.048705) |
| | 0.005 | 0.094510 (0.043846) | 0.126104 (0.058193) |
| | 0.001 | 0.142203 (0.065949) | 0.185699 (0.086454) |
| CAD | 0.1 | 0.029146 (0.013508) | 0.046904 (0.021888) |
| | 0.05 | 0.040401 (0.018728) | 0.059691 (0.027748) |
| | 0.01 | 0.069779 (0.032784) | 0.094791 (0.044379) |
| | 0.005 | 0.084887 (0.039804) | 0.113355 (0.053192) |
| | 0.001 | 0.127329 (0.059502) | 0.166198 (0.078934) |
| GBP | 0.1 | 0.029227 (0.007581) | 0.047062 (0.012252) |
| | 0.05 | 0.040499 (0.010528) | 0.059886 (0.015603) |
| | 0.01 | 0.070055 (0.018433) | 0.095428 (0.025283) |
| | 0.005 | 0.085147 (0.022432) | 0.114009 (0.030286) |
| | 0.001 | 0.127819 (0.034758) | 0.166912 (0.048126) |
| AUD | 0.1 | 0.042367 (0.007986) | 0.068012 (0.013013) |
| | 0.05 | 0.058637 (0.011173) | 0.086469 (0.016678) |
| | 0.01 | 0.101203 (0.019654) | 0.137564 (0.027189) |
| | 0.005 | 0.122775 (0.024168) | 0.164125 (0.032760) |
| | 0.001 | 0.183788 (0.037994) | 0.240048 (0.054777) |
| SGD | 0.1 | 0.017119 (0.005088) | 0.027598 (0.008157) |
| | 0.05 | 0.023733 (0.007030) | 0.035135 (0.010343) |
| | 0.01 | 0.041078 (0.012144) | 0.056082 (0.016477) |
| | 0.005 | 0.049987 (0.014789) | 0.067156 (0.019996) |
| | 0.001 | 0.075456 (0.022950) | 0.098904 (0.031244) |

Table 6
Average one-day VaR and ES (standard errors in parentheses), estimated using Student-t copula with Student-t marginals for different confidence levels; computed for a portfolio $w = (1,1,1,1,1,1)^\top$ constructed of USD, EUR, CAD, GBP, AUD and SGD savings accounts, expressed in units of these currencies.

| Currency | 0.1 | 0.05 | 0.01 | 0.005 | 0.001 | $\sum_\alpha((\alpha-\widehat{\alpha})/\alpha)^2$ | $\sum_\alpha\lvert\alpha-\widehat{\alpha}\rvert/\alpha$ |
|---|---|---|---|---|---|---|---|
| USD | 0.08561 | 0.04745 | 0.01230 | 0.00703 | 0.00025 | 0.004110 | 1.58090 |
| EUR | 0.10193 | 0.05523 | 0.01054 | 0.00527 | 0.00075 | 0.000691 | 0.48190 |
| CAD | 0.12830 | 0.07055 | 0.01607 | 0.00854 | 0.00075 | 0.022710 | 2.25900 |
| GBP | 0.12453 | 0.06503 | 0.01180 | 0.00402 | 0.00126 | 0.011120 | 1.18190 |
| AUD | 0.14010 | 0.07909 | 0.01506 | 0.00552 | 0.00075 | 0.035680 | 1.84280 |
| SGD | 0.10193 | 0.05298 | 0.01080 | 0.00628 | 0.00100 | 0.000607 | 0.41490 |

Table 7
Exceedance ratio $\hat{\alpha}$ for one-day VaR, estimated using Student-t copula with Student-t marginals for different confidence levels, for a portfolio $w = (1,1,1,1,1,1)^\top$; constructed from USD, EUR, CAD, GBP, AUD and SGD savings accounts, expressed in units of respective currencies.

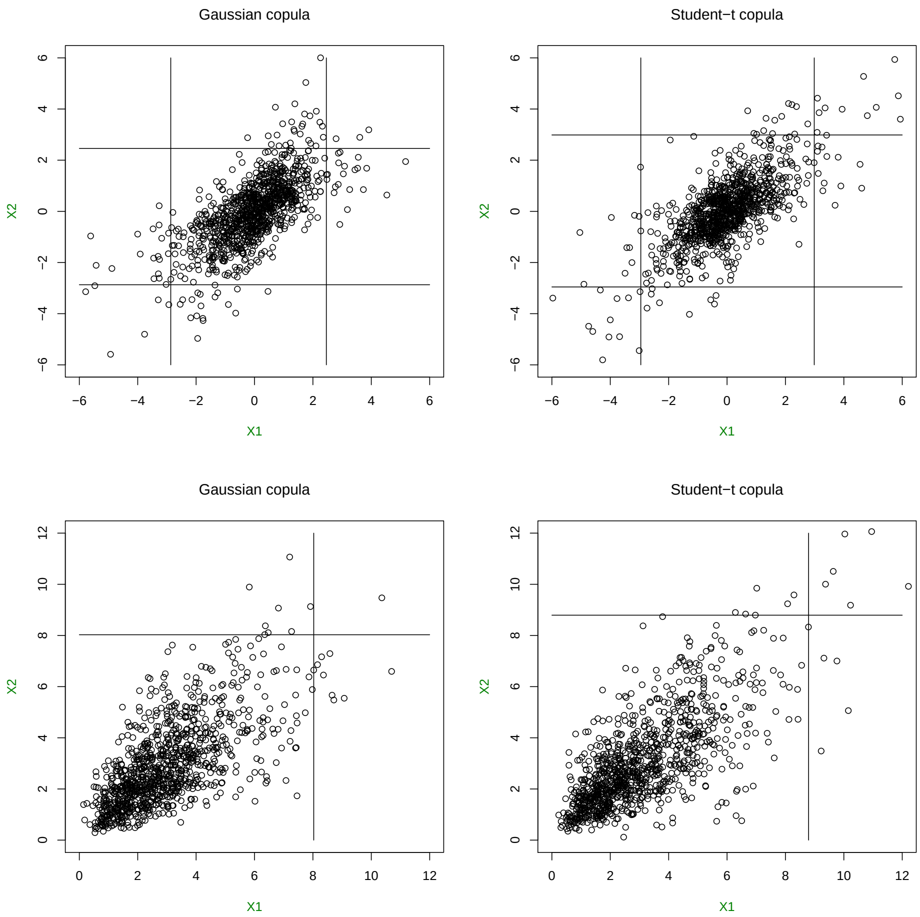



Fig. 1. 1000 bivariate realizations simulated from the Gaussian copula (left panel) and the Student-t copula (right panel), with identical Student-t marginal distributions with four degrees of freedom (upper panel), and Gamma(3,1) marginal distributions (lower panel), for given identical correlation $\theta = 0.7$.

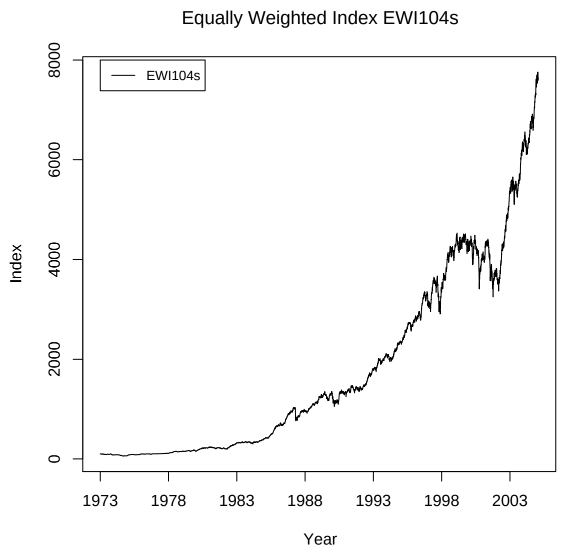



Fig. 2. Equi-weighted index EWI104s in USD constructed from 104 world industry sector indices, plotted from 02 January, 1973 to 10 March, 2006.

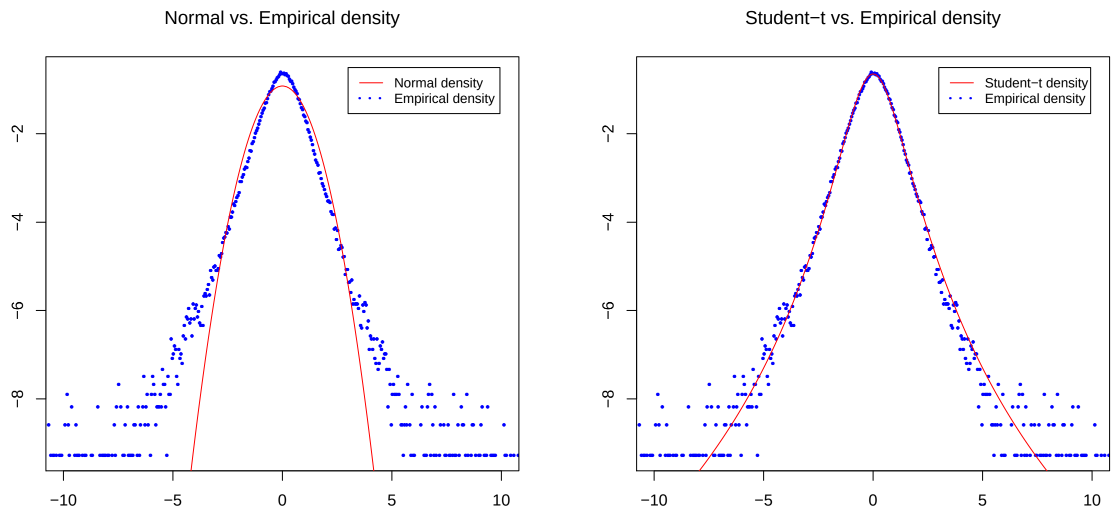



Fig. 3. Logarithm of the histogram for the pooled data of the EWI104s log-returns vs. normal density (left panel) and vs. Student-t density with four degrees of freedom (right panel). Pooled normalized data are used for 20 currency denominations for the time span from 02 January, 1973 to 10 March, 2006.

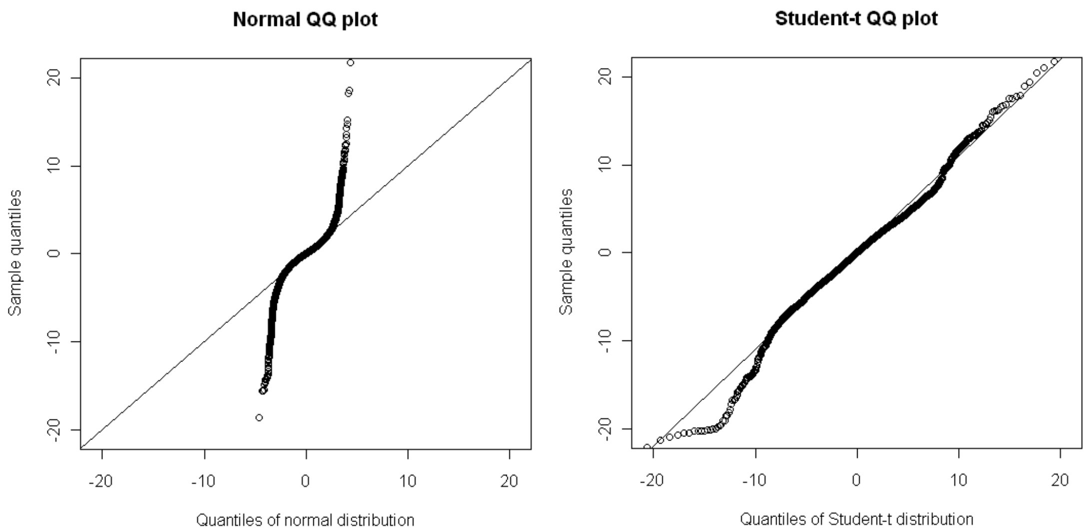



Fig. 4. QQ plots of the data against the quantiles of the fitted normal distribution (left panel) and Student-t distribution with four degrees of freedom (right panel). Pooled data are taken for 20 currency denominations from 02 January, 1973 to 10 March, 2006.

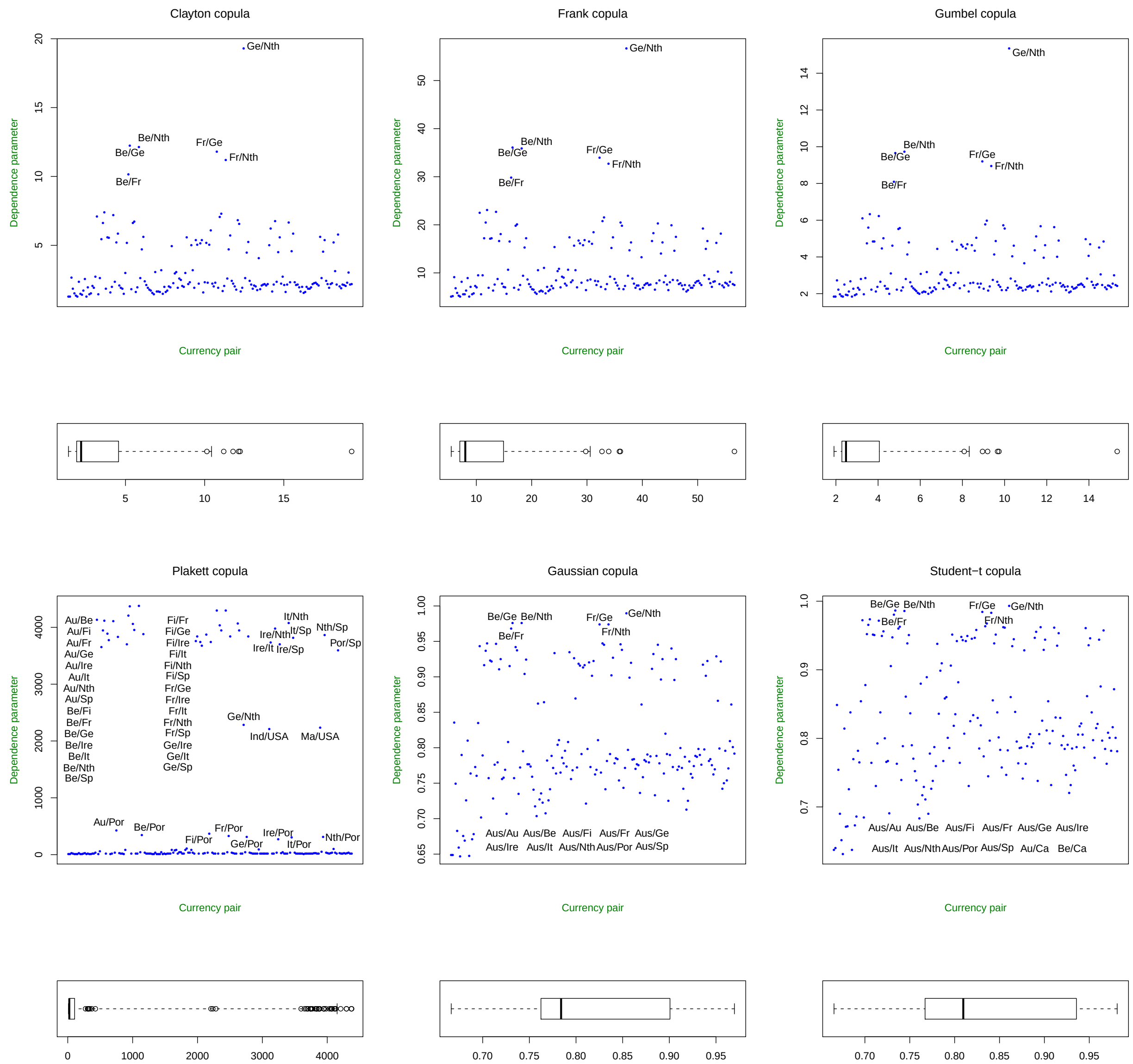



Fig. 5. Copula dependence parameters estimated for 190 pairs of currency denominations and different copula models, together with the box-plots: The extreme of the lower whisker (2.5%-quantile), low quartile, median, upper quartile and the extreme of the upper whisker (97.5%-quantile) are plotted. Estimated using data of EWI104s log-returns for the time span from 02 January, 1973 to 10 March, 2006.

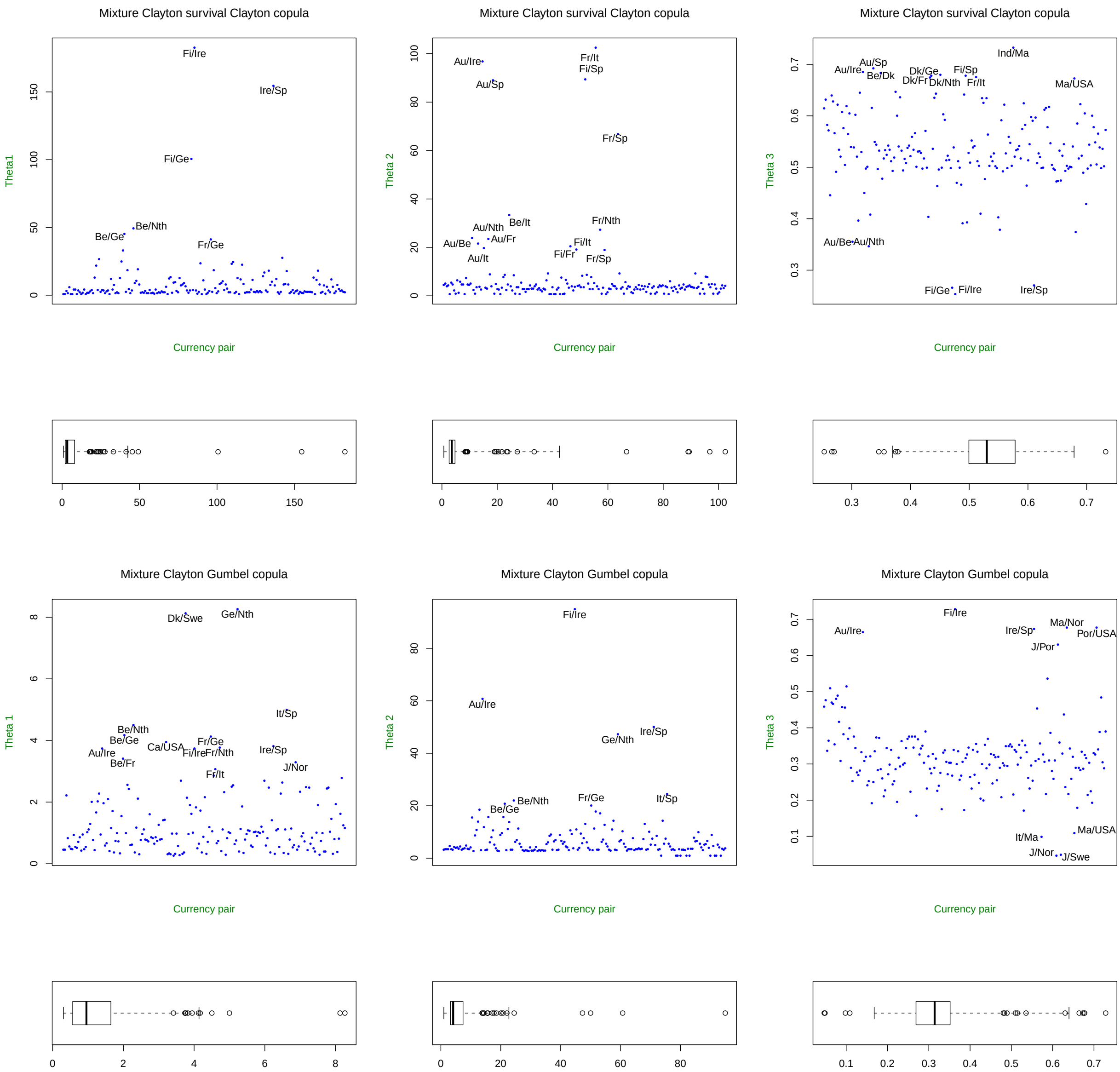



Fig. 6. Copula dependence parameters estimated for 190 pairs of currency denominations and two mixture copula models: mixture Clayton & survival Clayton and mixture Clayton & Gumbel. Parameters $\theta_1$ and $\theta_2$ are the dependence parameters for the first and the second term of the mixture, respectively, and parameter $\theta_3$ gives the proportion of the first term in the mixture model. Box-plots show the extreme of the lower whisker (2.5%-quantile), low quartile, median, upper quartile and the extreme of the upper whisker (97.5%-quantile). Estimated using data of EWI104s log-returns for the time span from 02 January, 1973 to 10 March, 2006.

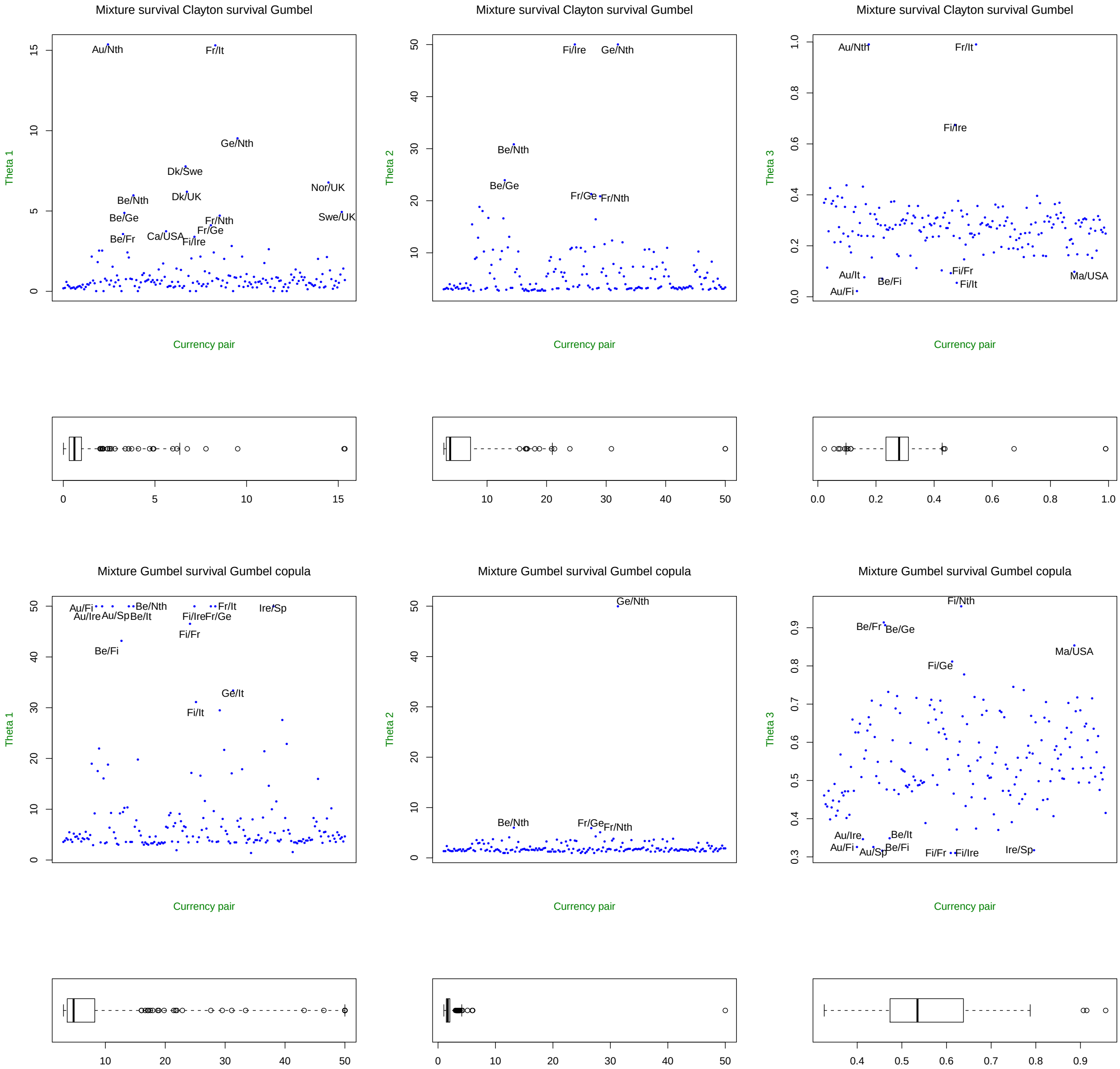



Fig. 7. Copula dependence parameters estimated for 190 pairs of currency denominations and two mixture copula models: mixture survival Clayton & survival Gumbel and mixture Gumbel & survival Gumbel. Parameters $\theta_1$ and $\theta_2$ are the dependence parameters for the first and the second terms of the mixture, respectively, and parameter $\theta_3$ gives the proportion of the first term in the mixture model. Box-plots show the extreme of the lower whisker (2.5%-quantile), low quartile, median, upper quartile and the extreme of the upper whisker (97.5%-quantile). Estimated using data of EWI104s log-returns for the time span from 02 January, 1973 to 10 March, 2006.

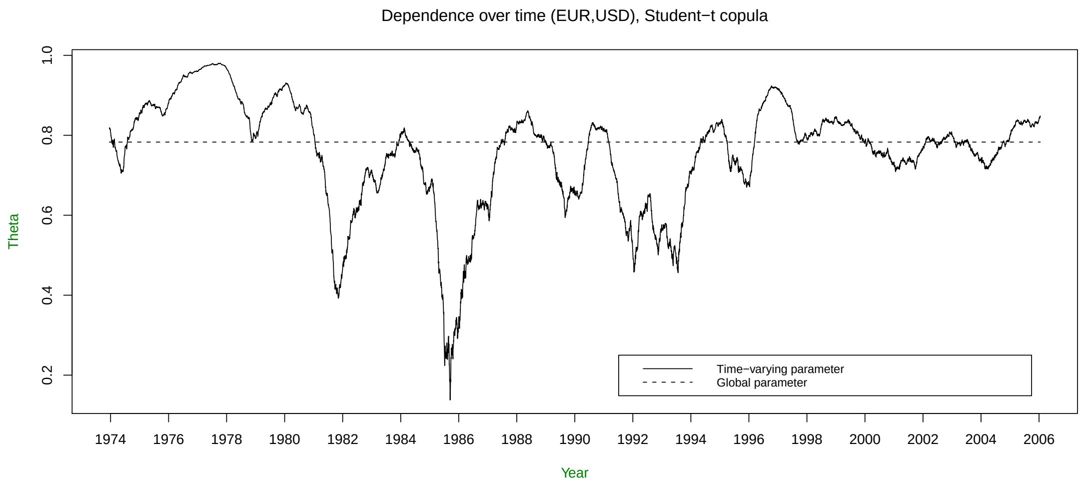



Fig. 8. Copula dependence parameter for portfolio of denominations of the EWI104 in EUR and USD. Estimated using a Student-t copula with Student-t marginals for the log-return data from 02 January, 1973 to 10 March, 2006.

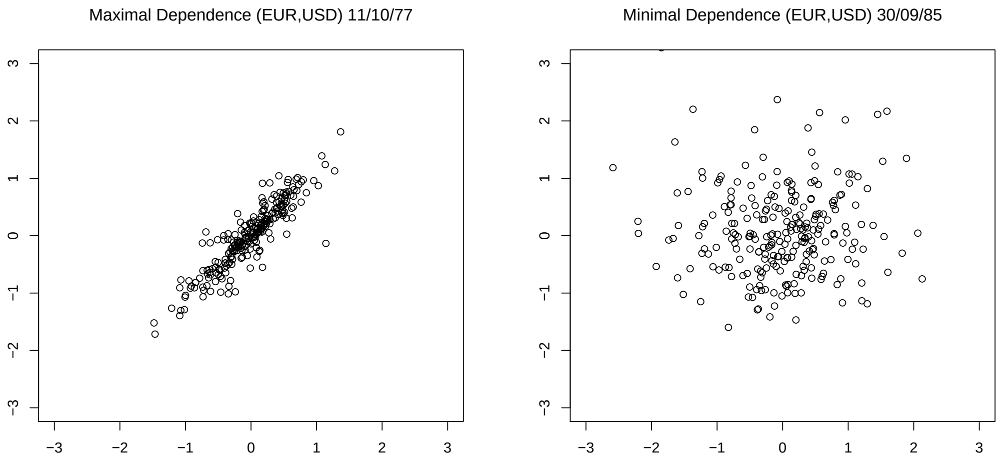



Fig. 9. Log-returns of the EWI104s in EUR and USD at maximal dependence on 11 October, 1977 (left panel) and minimal dependence on 30 September, 1985 (right panel).

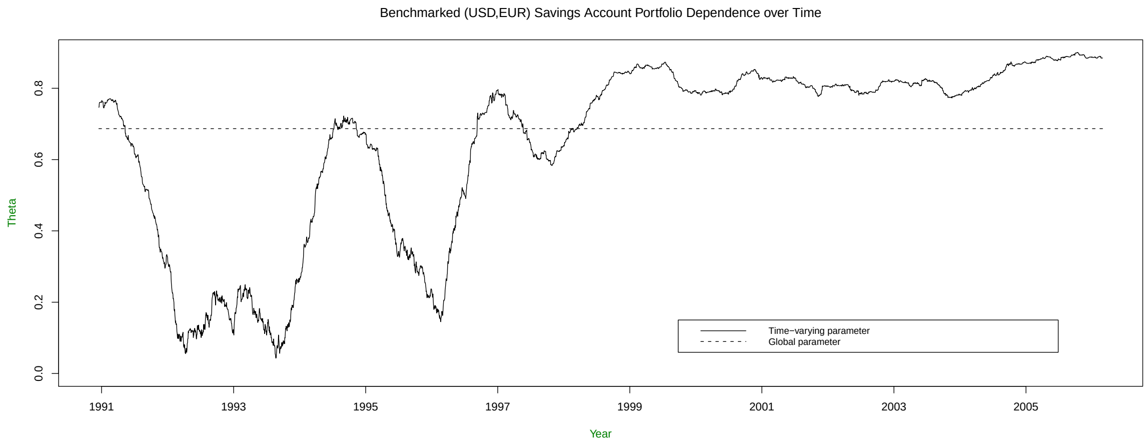



Fig. 10. Copula dependence parameter for portfolio of benchmarked savings accounts in USD and in EUR. Estimated using the Student-t copula with Student-t marginals for the log-return data covering time period from 02 January, 1991 to 10 March, 2006.

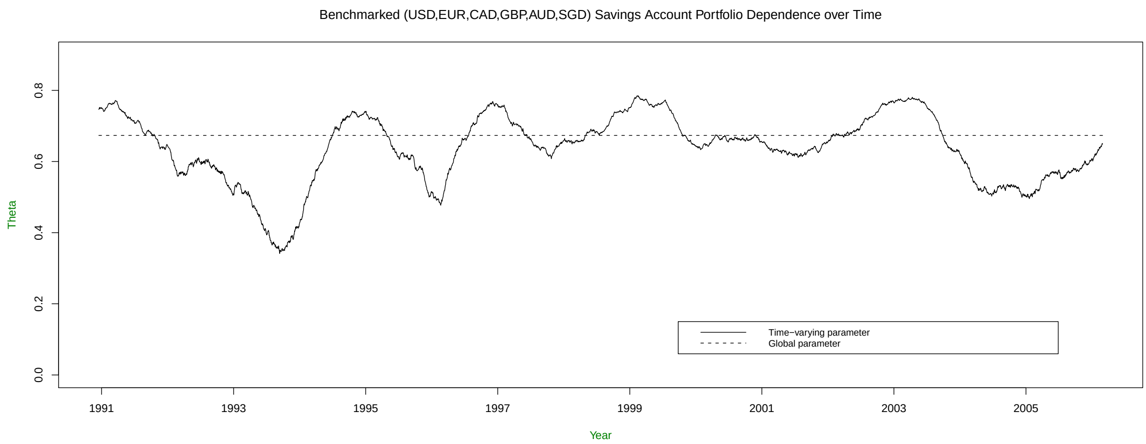
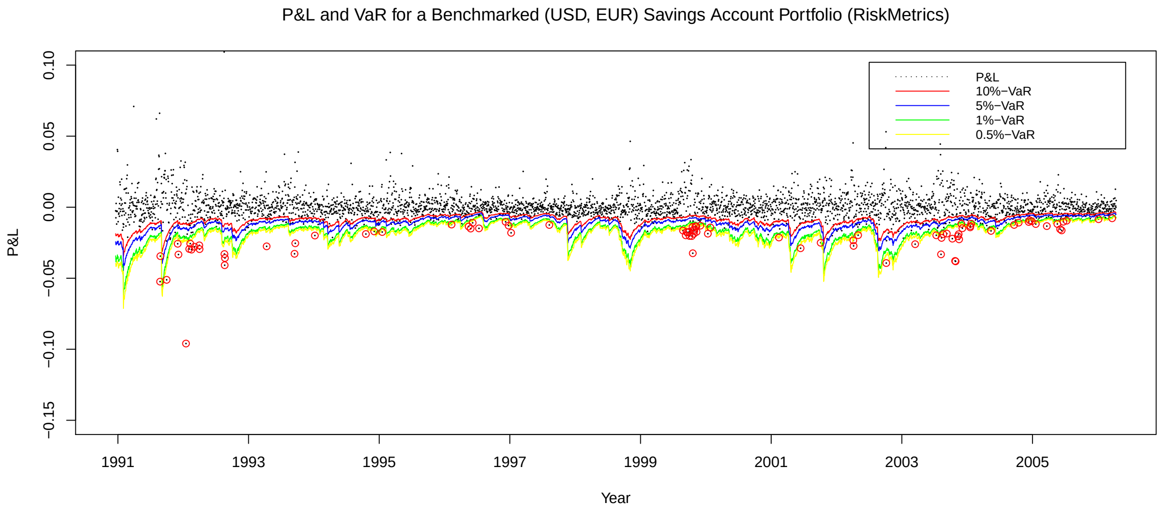



Fig. 11. Copula dependence parameter for portfolio of benchmarked savings accounts in USD, EUR, CAD, GBP, AUD, SGD. Estimated using the Student-t copula with Student-t marginals for the log-return data covering time period from 02 January, 1991 to 10 March, 2006.



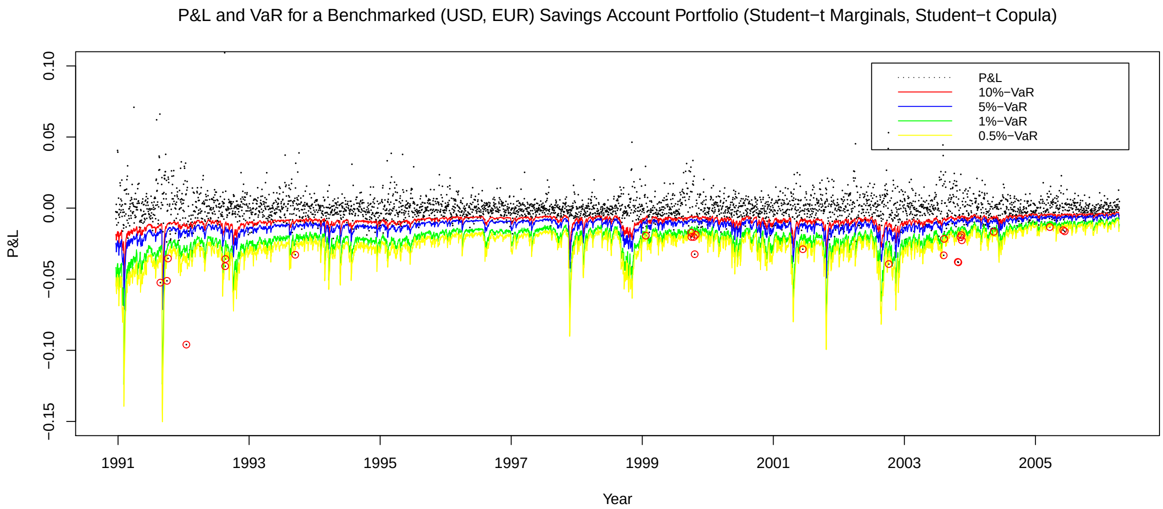
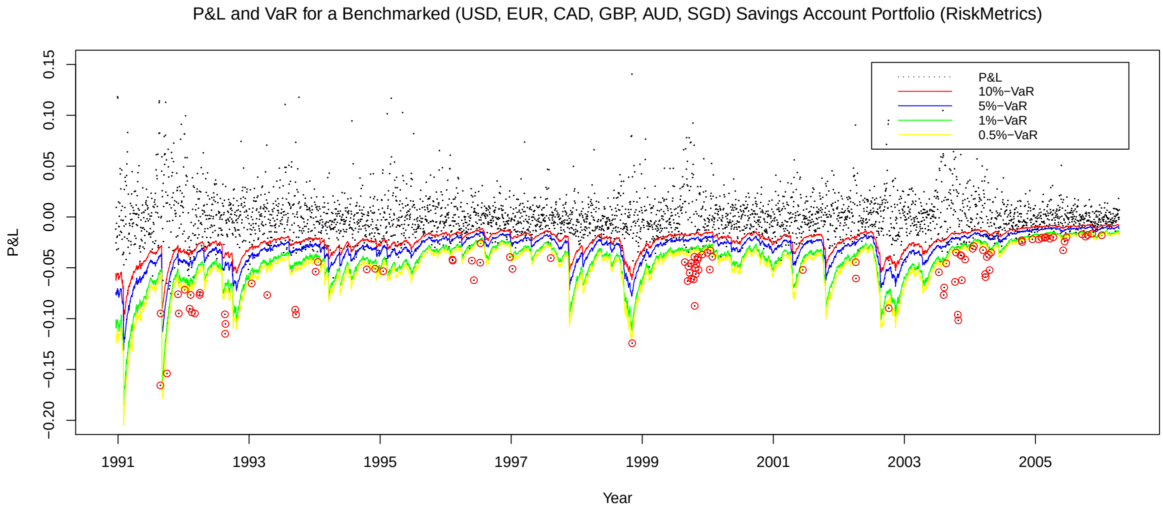




Fig. 12. P&L and estimated VaR for portfolio $w = (1, 1)^\top$ of benchmarked USD and EUR savings accounts at different confidence levels $\alpha \in \{10.0\%, 5.0\%, 1.0\%, 0.5\%\}$. Estimated using RiskMetrics (upper panel) and Student-t copula with Student-t marginals (lower panel). Exceedances are computed at level $\alpha = 0.5\%$.



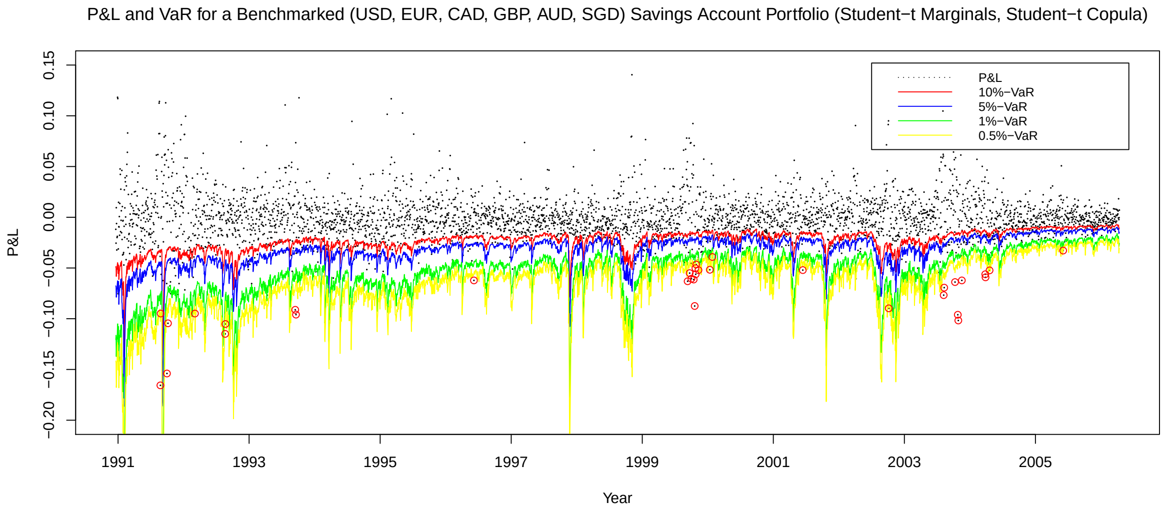




Fig. 13. P&L and estimated VaR for portfolio $w = (1, 1, 1, 1, 1, 1)^\top$ of benchmarked savings accounts in USD, EUR, CAD, GBP, AUD, SGD at different confidence levels $\alpha \in \{10.0\%, 5.0\%, 1.0\%, 0.5\%\}$. Estimated using RiskMetrics (upper panel) and Student-t copula with Student-t marginals (lower panel). Exceedances are computed at level $\alpha = 0.5\%$.

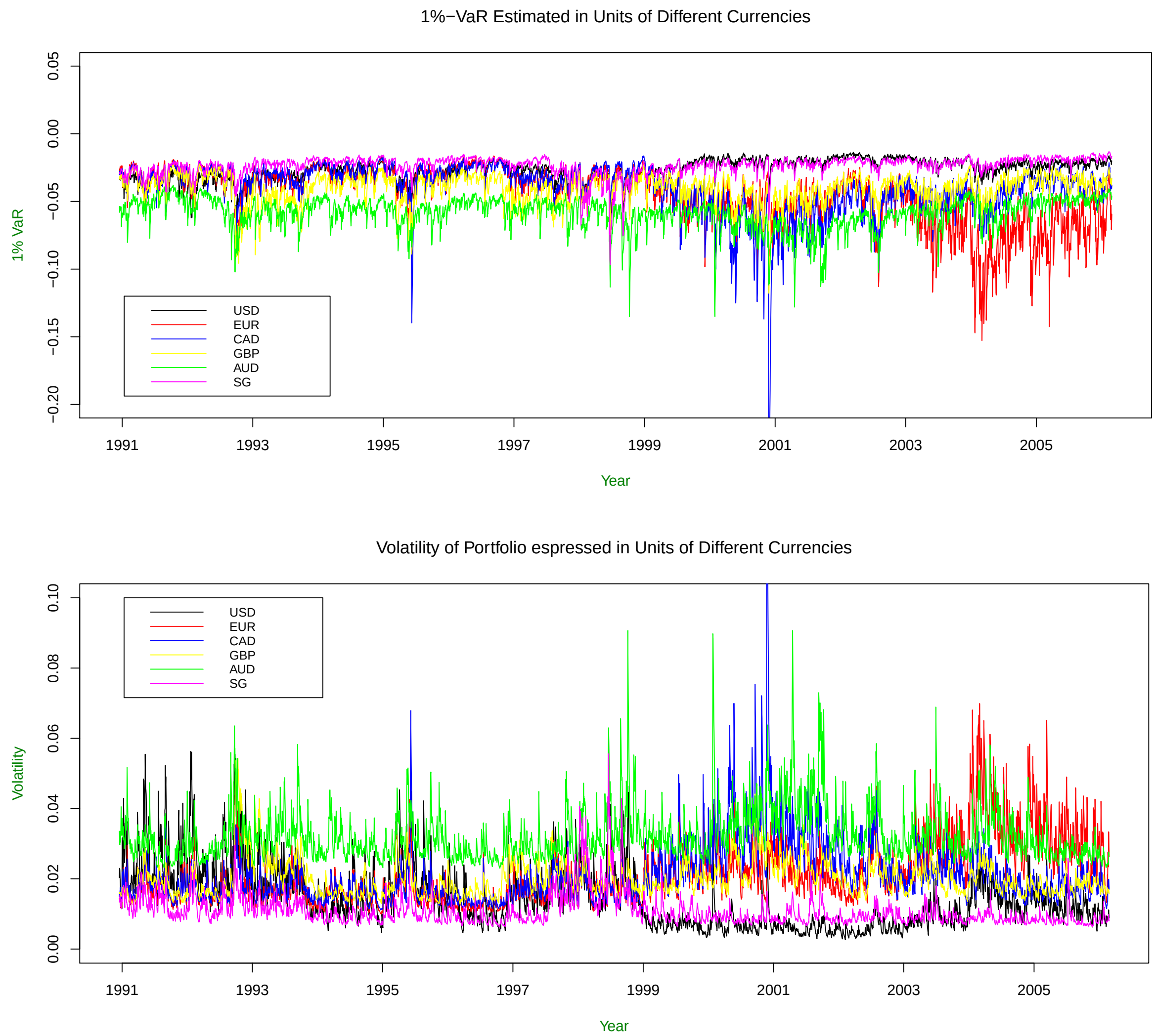



Fig. 14. Estimated VaR at level $\alpha = 1.0\%$ (upper panel) and estimated daily volatility (lower panel) for portfolio $w = (1, 1, 1, 1, 1, 1)^\top$ constructed of USD, EUR, CAD, GBP, AUD and SGD savings accounts, expressed in units of respective currency. Estimated using Student-t copula with Student-t marginals.

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