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Akihiko Takahashi
University of Tokyo
Yoshihiko Uchida
Osaka University

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Akihiko Takahashi[#]

Yoshihiko Uchida[*]

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Abstract

In the present paper, we propose a new computational technique with the Asymptotic Expansion (AE) approach to achieve variance reduction of the Monte-Carlo integration appearing especially in finance. We extend the algorithm developed by Takahashi and Yoshida (2003) to the second order asymptotics. Moreover, we apply the AE to approximate time dependent differentials of the target value in Newton (1994)'s scheme.

Our numerical examples include pricing of average and basket options when the underlying state variables follow Constant Elasticity of Variance (CEV) processes.

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[#] Graduate School of Economics, the University of Tokyo.

[*] Graduate School of Economics, Osaka University (uchida@econ.osaka-u.ac.jp).

1. Introduction

In the present paper, we propose a new computational technique with the Asymptotic Expansion (AE) approach to achieve variance reduction of the Monte-Carlo integration appearing especially in finance. In order to compute control variables, we utilize the analytic approximation based on the AE in Takahashi (1999) and Kunitomo and Takahashi (2003a). We extend the algorithm developed by Takahashi and Yoshida (2003) to the second order asymptotics. This scheme gives us more precise estimate efficiently than the second order AE when the precision of the AE is not satisfactory for practical purpose. Moreover, we apply the AE to approximate time dependent differentials of the target value in Newton (1994)'s scheme.

Through numerical experiments, we observe remarkable acceleration of convergence, which implies broad applications of our techniques. Our numerical examples include pricing of average and basket options when the underlying state variables follow Constant Elasticity of Variance (CEV) processes.

The organization of this paper is as follows. In the next section, we review the result of Takahashi and Yoshida (2005) and extend it to the higher order asymptotics. Moreover, in order to demonstrate the broad usage of the AE, we propose applying the AE to Newton (1994)'s scheme. In section 3, we review the outline of the AE approach followed by deriving the pricing formulas of average and basket options of which the underlying state variables are described by CEV processes. In section 4, we present the result of numerical experiments.

2. Variance reduction technique

2.1 The Extension of Takahashi and Yoshida (2005)

Suppose that the $\mathbf{R}^d$ valued processes $X_u(s,y)$ ($s \le u \le T$, $y \in \mathbf{R}^d$) follow the stochastic integral equation[1]:

$$X_u^{(\varepsilon)}(t,y) = y + \int_t^u V_0\left(X_{s-}^{(\varepsilon)}(t,y),\varepsilon\right)ds + \int_t^u V\left(X_{s-}^{(\varepsilon)}(t,y),\varepsilon\right)dB_s\ , \tag{1}$$

where ε $(0<\varepsilon\le 1)$ is a parameter.

For a stochastic approximation of $u(0,y)\equiv E\left[f\left(X_T^{(\varepsilon)}(0,y)\right)\right]$, an estimator by *crude Mote Carlo* simulations is expressed as

$$G(n,N) = \frac{1}{N}\sum_{j=1}^{N}\left[f\left(\bar{X}_T^{(\varepsilon)}\right)\right]_j . \tag{2}$$

Here, $[Z]_j$ ($j=1,\ldots,N$) denote the realized value of the *i*th independent trial of any random variable *Z*, and the discretized approximation of $\bar{X}^{(\varepsilon)}$ based on Euler-Maruyama scheme is given by[2]

$$\bar{X}_u^{(\varepsilon)} = y + \int_0^u V_0\left(\bar{X}_{\eta(s)}^{(\varepsilon)},\varepsilon\right)ds + \int_0^u V\left(\bar{X}_{\eta(s)}^{(\varepsilon)},\varepsilon\right)dB_s\ , \tag{3}$$

$$\text{where } \eta(s) = \left[\frac{ns}{T}\right]\frac{T}{n}.$$

From the mathematical point of view, we should note that it is not a trivial thing to justify this type of approximation based on the Monte Carlo method, which has been often used in practice. In particular when $f(\cdot)$ is not a smooth function such as cash flow functions for options, we need a careful discussion on its mathematical foundation. While Takahashi and Yoshida (2005) have investigated this problem in some details, we shall focus on the practical aspects of our method for financial applications throughout this paper.

Definition 1

A modified new estimator of $u(0,y)$ is defined by

$$G^*(\varepsilon,n,N) = E\left[\hat{f}\left(X_T^{(0)}(0,y)\right)\right] + \frac{1}{N}\sum_{j=1}^{N}\left[f\left(\bar{X}_T^{(\varepsilon)}\right) - \hat{f}\left(\bar{X}_T^{(0)}\right)\right]_j . \tag{4}$$

Here, we assume that $E\left[\hat{f}\left(X_T^{(0)}(0,y)\right)\right]$ is calculated analytically. We call the

[1] While we omit the description of some mathematical setting, we assume that all the necessary mathematical conditions to modify the formula are satisfied.

[2] [*x*] indicates the largest integer that is not greater than *x*.

Monte Carlo simulation which uses (4) *hybrid* (*Monte Carlo*) *method* hereafter.

This estimate can be explained intuitively. When the difference between $\left[f\left(\bar{X}_T^{(\varepsilon)}\right)\right]_j - u(0,y)$ and $\left[\hat{f}\left(\bar{X}_T^{(0)}\right)\right]_j - E\left[\hat{f}\left(X_T^{(0)}(0,y)\right)\right]$ is small for each independent copy *j*, then we can expect that the error of $G^*(\varepsilon,n,N)$ minus true value $u(0,y)$ can be small because errors of $\left[f\left(\bar{X}_T^{(\varepsilon)}\right)\right]_j$ and $\left[\hat{f}\left(\bar{X}_T^{(0)}\right)\right]_j$ can be cancelled out. Then we have

$$G^*(\varepsilon,n,N) - u(0,y)$$
$$= \frac{1}{N}\sum_{j=1}^{N}\left[\left\{ f\left(\bar{X}_T^{(\varepsilon)}\right) - E\left[f\left(X_T^{(\varepsilon)}(0,T)\right)\right]\right\} - \left\{ \hat{f}\left(\bar{X}_T^{(0)}\right) - E\left[\hat{f}\left(X_T^{(0)}(0,T)\right)\right]\right\}\right]_j, \tag{5}$$

and we denote $\bar{X}_t^{(0)}$ as $\bar{X}_t^{(\varepsilon)}$ by setting $\varepsilon = 0$, then we expect that the correlation between $\bar{X}_t^{(0)}$ and $\bar{X}_t^{(\varepsilon)}$ is positively high. Hence the correlation between $\left[f\left(\bar{X}_T^{(\varepsilon)}\right)\right]_j$ and $\left[\hat{f}\left(\bar{X}_T^{(0)}\right)\right]_j$ becomes positively high. This type of estimate in (4) could be similar to the control variate technique, which has been known in the Monte Calro simulation.

The advantage of the technique is due to the AE approach, because it is a unified method in a sense that it is applicable to the broad class of processes. We notice that it is difficult to find control variables whose expectation can not be derived analytically. Furthermore, usual variance reduction techniques may use control variables that could be applied to very narrow class of processes.

For example, the pricing algorithm for an European average call option which extends Example two in Takahashi and Yoshida (2005) to the second order asymptotics is as follows:

Suppose that the reference asset price process follows

$$dS_t^{(\varepsilon)} = rS_t^{(\varepsilon)}dt + \varepsilon\sigma\left(S_t^{(\varepsilon)},t\right)dB_t,\quad S_0^{(\varepsilon)} = S_0. \tag{6}$$

The underlying asset of an average option at time zero with maturity *T* is defined by

$$A_T^{(\varepsilon)} \equiv \frac{1}{T}\int_0^T S_t^{(\varepsilon)} dt \,. \tag{7}$$

The price at t=0 of an option with maturity T and strike price K (>0) is represented by

$$C_A^{(\varepsilon)} = e^{-rT}\left(A_T^{(\varepsilon)} - K\right)^+. \tag{8}$$

It is rewritten as

$$C_A^{(\varepsilon)} = e^{-rT}\varepsilon\left(\frac{1}{T}X_{2,T}^{(\varepsilon)} + k_T\right)^+, \tag{9}$$

where $X_{1,t}^{(\varepsilon)} \equiv \dfrac{S_t^{(\varepsilon)} - S_t^{(0)}}{\varepsilon}$, $X_{2,t}^{(\varepsilon)} \equiv \dfrac{t\left(A_t^{(\varepsilon)} - A_t^{(0)}\right)}{\varepsilon}$ and $k_T \equiv \dfrac{A_T^{(0)} - K}{\varepsilon}$.

Considering the discussion of Takahashi (1999), we introduce

$$\hat{C}_A(x) = e^{-rT}\varepsilon\left[\frac{x}{T} + k_T + \varepsilon\left(c_T\left(\frac{x}{T}\right)^2 + f_T\right)\right]1_{\left\{-k_T < \frac{x}{T}\right\}}. \tag{10}$$

Here, c_T, and f_T are obtained by the AE. We shall discuss about these terms in the next section briefly.

By definition, we have

$$S_t^{(0)} = e^{rt}S_0, \quad A_t^{(0)} = \frac{1}{t}\frac{S_0}{r}\left(e^{rt} - 1\right). \tag{11}$$

$X_{1,t}^{(\varepsilon)}$ and $X_{2,t}^{(\varepsilon)}$ follow

$$dX_{1,t}^{(\varepsilon)} = rX_{1,t}^{(\varepsilon)}dt + \sigma\left(\varepsilon X_{1,t}^{(\varepsilon)} + S_t^{(0)}, t\right)dB_t, \quad X_{1,0}^{(\varepsilon)} = 0, \tag{12}$$

$$dX_{2,t}^{(\varepsilon)} = X_{1,t}^{(\varepsilon)}dt, \;\; X_{2,0}^{(\varepsilon)} = 0. \tag{13}$$

And also $X_{1,t}^{(0)}$ and $X_{2,t}^{(0)}$ follow

$$dX_{1,t}^{(0)} = rX_{1,t}^{(0)}dt + \sigma\left(S_t^{(0)}, t\right)dB_t, \quad X_{1,0}^{(0)} = 0, \tag{14}$$

$$dX_{2,t}^{(0)} = X_{2,t}^{(0)}dt, \;\; X_{2,0}^{(0)} = 0. \tag{15}$$

Then, we can introduce the modified estimator of $C_A^{(\varepsilon)}$ as

$$E\left[\hat{C}_A\left(X_{2,T}^{(0)}\right)\right]+\frac{1}{N}\sum_{j=1}^{N}\left[C_A^{(\varepsilon)}\left(\overline{X}_{2,T}^{(\varepsilon)}\right)-\hat{C}_A\left(\overline{X}_{2,T}^{(0)}\right)\right]_j . \tag{16}$$

We notice that we can obtain $E\left[\hat{C}_A\left(X_{2,T}^{(0)}\right)\right]$ by the AE analytically. Here, $\overline{X}_{2,t}^{(\varepsilon)}$ and $\overline{X}_{2,t}^{(0)}$ denote the calculated value by the Euler-Maruyama scheme.

Notice that calculating (14) does not require the evaluation of $\sigma\left(S_t^{(0)},t\right)$ path by path while computing (12) requires. Therefore, the amount of calculation of the present technique is as large as the crude Monte Carlo method.

The algorithm shown above is based on the second order asymptotic value. For other concrete applications using the first order asymptotic value, see Takahashi and Yoshida (2005).

2.2 Newton (1994)'s estimator with AE

Newton (1994) derives the following formula

$$f\left(X_T(0,y)\right)=E\left[f\left(X_T(0,y)\right)\right]+\int_0^T\frac{\partial}{\partial x}u\left(t,X_t(0,y)\right)\sigma\left(t,X_t(0,y)\right)dB_t \quad a.s., \tag{17}$$

$$\text{where } E\left[f\left(X_T(0,y)\right)\middle|X_t(0,y)\right]=u\left(t,X_t(0,y)\right) \ a.s. \ .$$

and show that we can obtain the ideal estimator of $E\left[f\left(X_T(0,y)\right)\right]$ calculating

$$f\left(X_T(0,y)\right)-\int_0^T\frac{\partial}{\partial x}u\left(t,X_t(0,y)\right)\sigma\left(t,X_t(0,y)\right)dB_t \ , \tag{18}$$

path by path.

Since neither $u\left(t,X_t(0,y)\right)$ nor $\partial/\partial x\, u\left(t,X_t(0,y)\right)$ is known, we approximate $u\left(t,X_t(0,y)\right)$ by the AE and differentiate it with respect to *x*. That is to replace $\partial/\partial x\, u\left(t,X_t(0,y)\right)$ with the AE formula. We call the Monte Carlo simulation which uses (18) *Control Variate* (*Monte Carlo*) *method* hereafter.

Unfortunately, since we need compute $u\left(t,X_t(0,y)\right)$ not only path by path but also at each time step, this technique consumes the larger computation resources.

3. Examples[3]

3.1 Average call option

Let $S_t^{(\varepsilon)}$ be the reference asset. Then the underlying asset of an average option at time zero with maturity T is defined by

$$A_T^{(\varepsilon)} \equiv \frac{1}{T}\int_0^T S_t^{(\varepsilon)} dt . \tag{19}$$

We obtain

$$A_T^{(\varepsilon)} = A_T^{(0)} + \varepsilon \frac{1}{T}\int_0^T M_t^{\langle 1\rangle} dt + \frac{\varepsilon^2}{2}\frac{1}{T}\int_0^T M_t^{\langle 2\rangle} dt + o\left(\varepsilon^2\right), \tag{20}$$

where

$$M_t^{\langle 1\rangle} = \int_0^t e^{r(t-s)}\sigma\left(S_s^{(0)}, s\right) dB_s ,$$

$$M_t^{\langle 2\rangle} = 2\int_0^t e^{r(t-s)}\partial\sigma\left(S_s^{(0)}, s\right) M_s^{\langle 1\rangle} dB_s .$$

The first term of the right hand of (20) is a deterministic function.

We introduce $X_T^{(\varepsilon)}$ as

$$X_T^{(\varepsilon)} \equiv \frac{A_T^{(\varepsilon)} - A_T^{(0)}}{\varepsilon} \equiv g_T^{\langle 1\rangle} + \varepsilon g_T^{\langle 2\rangle} + o(\varepsilon) . \tag{21}$$

$g_T^{\langle 1\rangle}$ can be expressed as

$$\begin{aligned} g_T^{\langle 1\rangle} &= \frac{1}{T}\int_0^T M_s^{\langle 1\rangle} ds = \frac{1}{T}\int_0^T\int_0^u e^{r(u-s)}\sigma\left(S_s^{(0)}, s\right) dB_s du \\ &= \frac{1}{T}\int_0^T\int_s^T e^{r(u-s)}\sigma\left(S_s^{(0)}, s\right) du dB_s \\ &= \frac{1}{T}\int_0^T\left[\frac{e^{r(T-s)}-1}{r}\right]\sigma\left(S_s^{(0)}, s\right) dB_s. \end{aligned} \tag{22}$$

Since $g_T^{\langle 1\rangle}$ is an integral with respect to Brownian motion whose integrand is a deterministic function, the distribution of $X_T^{(\varepsilon)}$ approaches normal as $\varepsilon \to 0$. Then, we obtain the following proposition.

[3] For the general discussion of the application including basket and average options, see Takahashi (1999). For the mathematical validity of AE, see Kunitomo and Takahashi (2003a),

Proposition 2

The asymptotic expansion of $f_{X_t^{(\varepsilon)}}(x)$, the distribution function of $X_t^{(\varepsilon)}$, is as follows:

$$f_{X_t^{(\varepsilon)}}(x) = n[x;0,\Sigma_t] + \varepsilon \frac{-\partial\left\{\left(c_t x^2 + f_t\right) n[x;0,\Sigma_t]\right\}}{\partial x} + o(\varepsilon), \tag{23}$$

where

$$\Sigma_t = \frac{1}{t^2}\int_0^t \left[\frac{e^{r(t-s)}-1}{r}\right]^2 \sigma\left(S_s^{(0)}, s\right)^2 ds, \tag{24}$$

$$c_t = \frac{1}{\Sigma_t^2}\frac{1}{t^3}\int_0^t\int_0^v e^{r(v-s)}\left[\frac{e^{r(t-s)}-1}{r}\right]\sigma\left(S_s^{(0)}, s\right)\partial\sigma\left(S_s^{(0)}, s\right) \int_0^s e^{r(s-u)}\left[\frac{e^{r(t-u)}-1}{r}\right]\sigma\left(S_u^{(0)}, u\right)^2 du ds dv, \tag{25}$$

$$f_t = -c_t\Sigma_t. \tag{26}$$

See section 2.1 in Takahashi (1999) for the proof.

We shall consider the situation that the underlying values of an average option at time zero depends on the reference assets price process before zero[4]. We introduce $A_{\delta,t}$ with a positive constant, δ,

$$A_{\delta,t} = \frac{1}{t+\delta}\int_{-\delta}^t S_u du. \tag{27}$$

Here, δ represents elapsed time since contract date until time zero. And we assume that r is constant.

We can modify the payoff of the option at time zero with maturity T as

[4]We can use this equation for the evaluation of the current value of the standing contracts.

$$
\begin{aligned}
C_{A,\delta}(T) &= E\left[e^{-rT}\left(A_{\delta,T}-K\right)^+\right] \\
&= E\left[e^{-rT}\left(\frac{1}{T+\delta}\int_{-\delta}^{T} S_u du - K\right)^+\right] \\
&= \frac{1}{T+\delta}E\left[e^{-rT}\left(\int_0^T S_u du - \left(K(T+\delta)-\int_{-\delta}^{0} S_u du\right)\right)^+\right] \\
&= \frac{T}{T+\delta}E\left[e^{-rT}\left(\frac{1}{T}\int_0^T S_u du - \frac{1}{T}\left(K(T+\delta)-\int_{-\delta}^{0} s_u du\right)\right)^+\right].
\end{aligned}
\tag{28}
$$

We rewrite the above formula with

$$
\kappa_{\delta,T,K} = \frac{1}{T}\left(K(T+\delta)-\int_{-\delta}^{0} S_u du\right). \tag{29}
$$

Assuming that r is constant, we obtain

$$
\begin{aligned}
&E\left[e^{-rT}\left(\frac{1}{T}\int_0^T S_u du - \frac{1}{T}\left(K(T+\delta)-\int_{-\delta}^{0} s_u du\right)\right)^+\right] \\
&= e^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_u du - \kappa_{\delta,T,K}\right)^+\right].
\end{aligned}
\tag{30}
$$

We only need to replace strike price with $\kappa_{\delta,T,K}$ so that we can forget the dependency of the underlying value on the reference asset before time zero.

Let $C_{A,\delta}^{(\varepsilon)}(T,x)$ be the price of an European average option at time zero with maturity T, elapsed time δ and $S_0^{(\varepsilon)} = x$. $C_{A,\delta}^{(\varepsilon)}(T,x)$ is expressed with the strike price K as follows:

$$
\begin{aligned}
C_{A,\delta}^{(\varepsilon)}(T,x) &= \frac{T}{T+\delta}E\left[e^{-rT}\left(A_T^{(\varepsilon)}-\kappa_{\delta,T,K}\right)^+\right] = \frac{T}{T+\delta}e^{-rT}\varepsilon E\left[\left(\frac{A_T^{(\varepsilon)}-\kappa_{\delta,T,K}}{\varepsilon}\right)^+\right] \\
&= \frac{T}{T+\delta}e^{-rT}\varepsilon E\left[\left(X_T^{(\varepsilon)}+k_T\right)^+\right],
\end{aligned}
\tag{31}
$$

where

$$
k_T = \frac{A_T^{(0)}-\kappa_{\delta,T,K}}{\varepsilon}.
$$

Then, we obtain the following theorem.

Theorem 3

When $\kappa_{\delta,T,K} > 0$, the asymptotic expansion of the price of the average option with consideration of elapsed time, $C^{(\varepsilon)}_{A,\delta}(T)$, is as follows:

$$\begin{aligned} C^{(\varepsilon)}_{A,\delta}(T) &= \frac{T}{T+\delta} e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_u du - \kappa_{\delta,T,K}\right)^+\right] \\ &= \frac{T}{T+\delta} e^{-rT}\left\{\varepsilon\left(\Sigma_T n\left[k_T;0,\Sigma_T\right] + k_T N\left(\frac{k_T}{\sqrt{\Sigma_T}}\right)\right) + \varepsilon^2 f_T k_T n\left[k_T;0,\Sigma_T\right]\right\} + o\left(\varepsilon^2\right). \end{aligned} \tag{32}$$

When $\kappa_{\delta,T,K} \leq 0$, the price is trivial, which is as follows:

$$\begin{aligned} C_{A,\delta}(T) &= \frac{T}{T+\delta} e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_u du - \kappa_{\delta,T,K}\right)^+\right] \\ &= \frac{T}{T+\delta} e^{-rT}\left(\frac{1}{T} E\left[\int_0^T S_u du\right] - \kappa_{\delta,T,K}\right) \\ &= \frac{1}{T+\delta} e^{-rT}\left(S_0\left(\frac{e^{rT}-1}{r}\right) - T\kappa_{\delta,T,K}\right). \end{aligned} \tag{33}$$

Some of the derivation has been already discussed above. For the option pricing formula by the AE, see section 2.1 and 3.2 in Takahashi (1999).

3.2 Basket call option

The price of a European basket call option can be derived similarly. First, we define a basket by

$$I(t) \equiv \sum_{i=1}^n \alpha_i S_{i,t}, \qquad I(0) = \sum_{i=1}^n \alpha_i S_{i,0} = x. \tag{34}$$

Here, $S_{i,t}$ denotes the ith asset price at time t and α_i is the amount of the ith asset in the basket. The payoff of a basket call option can be expressed as

$$C^{(\varepsilon)}_B(T,x) = E\left[e^{-rT}\left(I(T)-K\right)^+\right]. \tag{35}$$

We assume that the price processes of risky assets can be described with d ($n \geq d$) independent Brownian Motions as

$$dS^{(\varepsilon)}_{i,t} = rS^{(\varepsilon)}_{i,t} + \varepsilon\sum_{j=1}^d \sigma_{ij}\left(t, S^{(\varepsilon)}_{i,t}\right) dB_{j,t}, \qquad S^{(\varepsilon)}_{i,0} = x_i, \qquad \sum_{i=1}^n \alpha_i x_i = x. \tag{36}$$

Since the value of the basket is the linear combination of risky assets, we obtain the AE of the basket.

$$I_t^{(\varepsilon)} = \sum_{i=1}^{n} \alpha_i S_{i,t}^{(0)} + \varepsilon \sum_{i=1}^{n} \alpha_i M_{i,t}^{\langle 1 \rangle} + \frac{\varepsilon^2}{2} \varepsilon \sum_{i=1}^{n} \alpha_i M_{i,t}^{\langle 2 \rangle} + o\left(\varepsilon^2\right). \tag{37}$$

To avoid lousy formula, we define that[5]

$$\sigma_I^{(0)}(t) \equiv \left(\sum_{i=1}^{n} \alpha_i \sigma_{i1}\left(S_{i,t}^{(0)}, t\right), \sum_{i=1}^{n} \alpha_i \sigma_{i2}\left(S_{i,t}^{(0)}, t\right), \ldots, \sum_{i=1}^{n} \alpha_i \sigma_{id}\left(S_{i,t}^{(0)}, t\right) \right)', \tag{38}$$

$$\sigma_i^{(0)}(t) \equiv \left(\sigma_{i1}\left(S_{i,t}^{(0)}, t\right), \sigma_{i2}\left(S_{i,t}^{(0)}, t\right), \ldots, \sigma_{id}\left(S_{i,t}^{(0)}, t\right) \right)', \tag{39}$$

and $$\partial\sigma_i^{(0)}(t) \equiv \left(\partial\sigma_{i1}\left(S_{i,t}^{(0)}, t\right), \partial\sigma_{i2}\left(S_{i,t}^{(0)}, t\right), \ldots, \partial\sigma_{id}\left(S_{i,t}^{(0)}, t\right) \right)'. \tag{40}$$

We introduce $X_t^{(\varepsilon)}$ as follows:

$$X_t^{(\varepsilon)} = \frac{I_t^{(\varepsilon)} - I_t^{(0)}}{\varepsilon} = g_t^{\langle 1 \rangle} + \varepsilon g_t^{\langle 2 \rangle} + o(\varepsilon). \tag{41}$$

Then, we obtain the next proposition.

Proposition 4

The asymptotic expansion of $f_{X_t^{(\varepsilon)}}(x)$, the distribution function of $X_t^{(\varepsilon)}$, is as follows:

$$f_{X_t^{(\varepsilon)}}(x) = n[x; 0, \Sigma_t] + \varepsilon \frac{-\partial\left\{\left(c_t x^2 + f_t\right) n[x; 0, \Sigma_t]\right\}}{\partial x} + o(\varepsilon), \tag{42}$$

where

$$\Sigma_t = \int_0^t e^{2r(t-s)} \sigma_I^{(0)}(s)' \sigma_I^{(0)}(s)\, ds, \tag{43}$$

$$c_t = \sum_{i=1}^{n} \alpha_i c_{i,t}, \tag{44}$$

$$f_t = \sum_{i=1}^{n} \alpha_i f_{i,t}, \tag{45}$$

$$f_{i,t} = -c_{i,t} \Sigma_t, \tag{46}$$

$$c_{i,t} = \frac{1}{\Sigma_t^2} e^{3rt} \int_0^t \left[\int_0^s e^{-2ru} \sigma_I^{(0)}(u)' \sigma_i^{(0)}(u)\, du \right] e^{-rs} \sigma_I^{(0)}(s)' \partial\sigma_i^{(0)}(s)\, ds. \tag{47}$$

[5] Dash represents transpose.

See section 3.1 in Takahashi (1999) for the proof.

Replacing $A_T^{(0)}$ with $I_T^{(0)}$ in (5), we have

$$k_T \equiv \frac{I_T^{(0)} - K}{\varepsilon}. \tag{48}$$

Then, calculating $E\left[e^{-rT}\varepsilon\left(X_T^{(\varepsilon)} + k_T \right)^{+} \right]$ with proposition 5, we obtain the next theorem. This is similar to the derivation of theorem 3.

Theorem 5

The asymptotic expansion of the price of a European basket call option is as follows:

$$C_B^{(\varepsilon)}(T,x) = e^{-rT}\left\{ \varepsilon\left(\Sigma_T n[k_T;0,\Sigma_T] + k_T N\left(\frac{k_T}{\sqrt{\Sigma_T}} \right) \right) + \varepsilon^2 f_T k_T n[k_T;0,\Sigma_T] \right\} + o\left(\varepsilon^2\right). \tag{49}$$

4. Numerical Experiments

We shall show the result of numerical experiments. In this section, we assume all the necessary coefficients which define the underlying process.

4.1 European average call option under CEV process

Suppose that S_t follows

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dB_t, \quad S_0 = x. \tag{50}$$

We shall represent this by the AE. $S_t^{(\varepsilon)}$ is expressed with some positive constant b as

$$dS_t^{(\varepsilon)} = rS_t^{(\varepsilon)} dt + \varepsilon b\left(S_t^{(\varepsilon)} \right)^{\gamma} dB_t, \quad S_0^{(\varepsilon)} = x. \tag{51}$$

Then, we obtain

$$\begin{aligned} \Sigma_T &= \frac{1}{T^2}\int_0^T \left[\frac{e^{r(T-s)} - 1}{r} \right]^2 b^2 \left(xe^{rs} \right)^{2\gamma} ds \\ &= \frac{b^2\left(\left(e^{rT}x\right)^{2\gamma} - x^{2\gamma}\left(\left(e^{rT}-1\right)\gamma\left(e^{rT}(2\gamma-1) - 2\gamma + 3 \right) + 1 \right) \right)}{2r^3T^2\gamma(\gamma-1)(2\gamma-1)}, \end{aligned} \tag{52}$$

$$
\begin{aligned}
c_T = & \frac{1}{2x(r-2r\gamma)^2\left(\left(e^{rT}x\right)^{2\gamma}-x^{2\gamma}\left(\left(-1+e^{rT}\right)\gamma\left(3-2\gamma+e^{rT}\left(-1+2\gamma\right)\right)+1\right)\right)^2}\times \\
& Tr^3(1-2\gamma)^2\gamma^3\left(\frac{x\left(e^{rT}x\right)^{-1+4\gamma}(-7+10\gamma)}{3-16\gamma+16\gamma^2}+e^{rT}x^{4\gamma}\left(2-2\gamma+e^{rT}\left(-1+2\gamma\right)\right)^2\right. \\
& -\frac{2x^{2\gamma}\left(e^{rT}x\right)^{2\gamma}\left(2-2\gamma+e^{rT}\left(-1+2\gamma\right)\right)}{\gamma} \\
& \left.-\frac{2x^{4\gamma}(-1+\gamma)^2\left(-\left(e^{rT}(1-4\gamma)^2(-3+4\gamma)\right)+2(1-2\gamma)^2(-3+4\gamma)+4e^{2rT}\gamma\left(1-6\gamma+8\gamma^2\right)\right)}{\gamma\left(3-16\gamma+16\gamma^2\right)}\right).
\end{aligned}
\tag{53}
$$

We set the simulation conditions as $x=100$, $r=0.05$, $\sigma=0.3S_0^{(1-\gamma)}$. For the Monte Carlo simulation, we divide one year into 250 time steps. Table 1 expresses the price of the option with $\gamma=0.6$. Column A is the true value that is computed by the crude Monte Carlo method with 25 million trials. Column B and column C are the prices by the AE with the first order[6] and the second order approximation respectively. Column D and column E are the rate of errors of the column B and C respectively, which are defined by ((the price by the AE)-(true value))/(true value). The price of the AE with the second order approximation is nearer to the true value than that with the first order. As for the second order approximation, the rate of errors are sufficiently small except 40% OTM cases. In these cases, the prices are very small and the differences between the prices by the AE and true values are 0.01559029 (1 year maturity) and 0.01221904 (2 years maturity), which are also very small. Overall, except far out of the money cases, utilizing the second order AE method, we can satisfy almost all the practical requirements of calculation speed and accuracy.

Table 2 expresses the price of the option with $\gamma=0.9$. We can derive the similar implication to Table 1.

Two tables of Table 3 express the performance of the simulation algorithms with $\gamma=0.6$. For each parameter and simulation algorithm, (1) we generate 5,000 paths (trials), compute the prices and take the average, (2) we repeat

[6] See Takahashi (1999) for the pricing formulae of the first order approximation.

taking the average 100 times (cases), (3) we extract result of the simulation. For the upper table, each column (column A, B and C) has two sub-columns: rmse and worst stand for relative mean square error and the biggest error among 100 cases respectively. Column A, B and C express the results of the crude Monte Carlo, the hybrid Monte Carlo and the control variate Monte Carlo method respectively. The standard variance is shown in the lower table. The true value is the same as table 1. The ratio of the upper table is that between rmses. While both the hybrid and the control variate method Monte Carlo reduce “rmse”, “worst” and “std variance,” the effect of the former is remarkabe. Using the ratio of the standard variances, we can say that the convergence speed of the hybrid Monte Carlo method is 40~2800 times faster than the crude Monte Carlo method. We notice that comparing column E of Table 1 with column B of Table 3 the hybrid Monte Carlo method gives us more precise estimate than the second order AE.

Table 4 expresses the performance of the simulation algorithms with $\gamma = 0.9$. We can derive the similar implication to Table 3. We notice that comparing column E of Table 2 with column B of Table 4 the hybrid Monte Carlo method improves accuracy especially in the cases of out of the money.

Table 5 gives one of the most interesting features. This expresses the actual calculation time (seconds). We stress the importance of the fact that (B)/(A) is nearly equal to one. This implies that the hybrid Monte Carlo method does require only 10% of extra computation resources. Considering the results of Table 3 and 4, this means that we can reduce calculation time dramatically utilizing the hybrid Monte Carlo method. Of course, even in the control variate Monte Carlo method’s case, overall performance of the calculation is improved. This shows that the AE can be utilized broadly.

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	12.49935416	12.71707769	12.51212379	1.7419%	0.1022%
ATM	1yr	6.73428441	6.74813443	6.74813443	0.2057%	0.2057%
20%OTM	1yr	3.15072673	2.96296259	3.16791649	-5.9594%	0.5456%
30%OTM	1yr	1.28206878	1.04351219	1.29243429	-18.6072%	0.8085%
40%OTM	1yr	0.14403973	0.06074728	0.12844944	-57.8260%	-10.8236%
20%ITM	2yrs	14.50204118	14.84184228	14.53528053	2.3431%	0.2292%
ATM	2yrs	9.29054199	9.32109049	9.32109049	0.3288%	0.3288%
20%OTM	2yrs	5.59585027	5.32558408	5.63214584	-4.8298%	0.6486%
30%OTM	2yrs	3.17763067	2.73707722	3.21513521	-13.8642%	1.1803%
40%OTM	2yrs	0.87235868	0.50676043	0.86013964	-41.9092%	-1.4007%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 1

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	12.40891668	12.73864061	12.42928304	2.6572%	0.1641%
ATM	1yr	6.74971348	6.77358186	6.77358186	0.3536%	0.3536%
20%OTM	1yr	3.26788639	2.98452551	3.29388308	-8.6711%	0.7955%
30%OTM	1yr	1.42747034	1.05663114	1.43376246	-25.9788%	0.4408%
40%OTM	1yr	0.21342712	0.06254478	0.16666380	-70.6950%	-21.9107%
20%ITM	2yrs	14.39446223	14.90694606	14.44190901	3.5603%	0.3296%
ATM	2yrs	9.34353313	9.39178041	9.39178041	0.5164%	0.5164%
20%OTM	2yrs	5.80130055	5.39068786	5.85572491	-7.0779%	0.9381%
30%OTM	2yrs	3.47136478	2.78793444	3.51583295	-19.6877%	1.2810%
40%OTM	2yrs	1.14296844	0.52569883	1.07185137	-54.0058%	-6.2221%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 2

		(A) crude Monte		(B) Hybrid		(C) CntlVar			
K	T	rmse	worst	rmse	worst	rmse	worst	(A)/(B)	(A)/(C)
20%ITM	1yr	1.69317%	4.31945%	0.02724%	0.07183%	0.20104%	0.31511%	62.155	8.422
ATM	1yr	1.97610%	5.01059%	0.07054%	0.17683%	0.44260%	0.74814%	28.013	4.465
20%OTM	1yr	3.58571%	10.12856%	0.12399%	0.32688%	0.67818%	1.21087%	28.918	5.287
30%OTM	1yr	4.46133%	11.47246%	0.23399%	0.67431%	1.16765%	2.34854%	19.067	3.821
40%OTM	1yr	15.15842%	48.81630%	0.21522%	0.52056%	2.45432%	6.82826%	70.433	6.176
20%ITM	2yrs	1.81773%	4.90331%	0.05200%	0.16108%	0.13640%	0.31729%	34.957	13.327
ATM	2yrs	2.63279%	7.54903%	0.07350%	0.20970%	0.21642%	0.54405%	35.823	12.165
20%OTM	2yrs	3.19567%	8.32458%	0.11394%	0.31197%	0.34159%	0.78243%	28.047	9.355
30%OTM	2yrs	4.19172%	14.36365%	0.16266%	0.45048%	0.50975%	1.30455%	25.770	8.223
40%OTM	2yrs	7.52163%	19.34298%	0.50170%	1.24380%	1.21223%	3.49212%	14.992	6.205

5,000 trials, 100 cases for each parameter.

K	T	(A) crude Monte std variance	(B) Hybrid std variance	(C) CntlVar std variance	ratio bet. std vars (A)/(B)	(A)/(C)
20%ITM	1yr	1.1884	0.0004	0.0014	2,818.632	835.841
ATM	1yr	2.4616	0.0013	0.0059	1,908.292	416.409
20%OTM	1yr	5.4863	0.0040	0.0248	1,363.974	221.532
30%OTM	1yr	13.1978	0.0158	0.1066	832.926	123.853
40%OTM	1yr	103.5923	2.3962	1.7743	43.231	58.384
20%ITM	2yrs	1.5489	0.0011	0.0027	1,423.393	579.373
ATM	2yrs	2.6239	0.0025	0.0061	1,051.780	433.182
20%OTM	2yrs	4.5195	0.0053	0.0161	855.146	281.369
30%OTM	2yrs	8.0590	0.0115	0.0460	702.940	175.262
40%OTM	2yrs	27.9386	0.1267	0.3684	220.455	75.838

Table 3

K	T	(A) crude Monte rmse	worst	(B) Hybrid rmse	worst	(C) CntlVar rmse	worst	(A)/(B)	(A)/(C)
20%ITM	1yr	1.44463%	4.41825%	0.10057%	0.21532%	0.24850%	0.37131%	14.364	5.813
ATM	1yr	2.31410%	7.34999%	0.17781%	0.34180%	0.52911%	0.81597%	13.015	4.374
20%OTM	1yr	3.70666%	11.24126%	0.32092%	0.72156%	0.89547%	1.54219%	11.550	4.139
30%OTM	1yr	5.70582%	16.27347%	0.56203%	1.22544%	1.38500%	2.62893%	10.152	4.120
40%OTM	1yr	14.04384%	37.37138%	3.26302%	8.25889%	3.34409%	8.52676%	4.304	4.200
20%ITM	2yrs	1.88159%	6.46842%	0.11671%	0.29809%	0.21346%	0.40181%	16.123	8.815
ATM	2yrs	2.55599%	7.27398%	0.19272%	0.43598%	0.32756%	0.64067%	13.263	7.803
20%OTM	2yrs	2.91609%	7.14232%	0.24243%	0.59779%	0.47983%	1.03366%	12.029	6.077
30%OTM	2yrs	4.37348%	11.22706%	0.39518%	1.09299%	0.73480%	1.74208%	11.067	5.952
40%OTM	2yrs	6.86312%	19.99852%	0.30195%	1.09889%	1.32158%	3.00296%	22.729	5.193

5,000 trials, 100 cases for each parameter.

K	T	(A) crude Monte std variance	(B) Hybrid std variance	(C) CntlVar std variance	ratio bet. std vars (A)/(B)	(A)/(C)
20%ITM	1yr	1.2790	0.0014	0.0019	932.294	677.968
ATM	1yr	2.6676	0.0044	0.0075	610.397	357.986
20%OTM	1yr	5.8316	0.0135	0.0327	430.860	178.120
30%OTM	1yr	13.3960	0.0504	0.1475	265.913	90.851
40%OTM	1yr	85.7051	4.5613	2.1226	18.790	40.377
20%ITM	2yrs	1.7507	0.0046	0.0040	381.312	432.585
ATM	2yrs	2.9456	0.0108	0.0093	272.050	316.286
20%OTM	2yrs	5.0198	0.0228	0.0255	219.750	196.848
30%OTM	2yrs	8.6865	0.0512	0.0746	169.710	116.421
40%OTM	2yrs	26.8724	0.4035	0.5448	66.594	49.328

Table 4

# of trials	(A) Crude Monte	(B) Hybrid	(C) CtrlVar	(B)/(A)	(C)/(A)
1,000	0.2813	0.3125	1.6875	1.111	6.000
10,000	2.4844	2.6719	14.8786	1.075	5.989
20,000	4.9063	5.4531	29.8923	1.111	6.093
100,000	24.5781	27.3750	154.6323	1.114	6.291

Table 5

4.2 European basket call option with n assets under CEV process[7]

Let S_t and B_t be $\mathbf{R}^n$ valued stochastic process and independent n dimensional Brownian Motion respectively. Suppose the price of the asset, S_t follows

$$dS_t = rS_t dt + I\left(\left(S_{i,t}\right)^{\gamma_i}\right)\sigma dB_t, \quad S_0 = x \in \mathbf{R}^n, , \tag{54}$$

where

$r \in \mathbf{R}$ and $\sigma \in \mathbf{R}$ are constant,

and $I\left(y_i\right) = \begin{pmatrix} y_1 & & O \\ & \ddots & \\ O & & y_n \end{pmatrix}$.

Then, we represent $S_t^{(\varepsilon)}$ using some positive constant b$\in \mathbf{R}^n$ as

[7] The concrete representations of Σ_t, $c_{i,t}$ are available upon the request.

$$dS_t^{(\varepsilon)} = rS_t^{(\varepsilon)}dt + \varepsilon I\left(b_i\left(S_{i,t}^{(\varepsilon)}\right)^{\gamma_i}\right)\hat{\lambda}dB_t, \quad S_0^{(\varepsilon)} = x. \tag{55}$$

Assuming that the $n\times n$ matrix $\hat{\lambda}$ has Cholesky decomposition, let the lower triangular matrix be λ, and we can modify (55) to

$$dS_t^{(\varepsilon)} = rS_t^{(\varepsilon)}dt + \varepsilon I\left(b_i\left(S_{i,t}^{(\varepsilon)}\right)^{\gamma_i}\right)\lambda dB_t, \quad S_0^{(\varepsilon)} = x. \tag{56}$$

Then, we have

$$\Sigma_t = \int_0^t e^{2r(t-s)}\sigma_I^{(0)}(s)'\sigma_I^{(0)}(s)ds, \tag{57}$$

$$c_{i,t} = \frac{1}{\Sigma_t^2}e^{3rt}\int_0^t\left[\int_0^s e^{-2ru}\sigma_I^{(0)}(u)'\sigma_i^{(0)}(u)du\right]e^{-rs}\sigma_I^{(0)}(s)'\partial\sigma_i^{(0)}(s)ds. \tag{58}$$

where

$$\sigma_I^{(0)}(s) \equiv \sum_{i=1}^n \alpha_i\sigma_i^{(0)},$$

$$\sigma_i^{(0)}(s) \equiv \text{transpose of } i\text{th row of}\left(I\left(b_i\left(S_{i,t}\right)^{\gamma_i}\right)\lambda\right),$$

$$\partial\sigma_i^{(0)}(s) \equiv \left.\frac{\partial\sigma_i^{(\varepsilon)}(s)}{\partial S_i^{(\varepsilon)}}\right|_{S_i^{(\varepsilon)}=S_i^{(0)}}.$$

4.2.1 European basket call option with two assets

We suppose S_t follows

$$dS_t^i = rS_t^i dt + \sum_{j=1}^2 \sigma^{ij}\left(S_t^i\right)^{\gamma^i}dB_t^j, \quad S_0^i = x^i \quad (i=1,2). \tag{59}$$

We set the simulation conditions as $S_0^i = 100$, $r = 0.05$, $\sigma^{ij} = 0.3\left(S_0^i\right)^{(1-\gamma^i)}$, ($i$=1,2). We also suppose each asset takes 50% of the basket. And for the Monte Carlo simulation, we divide one year into 250 time steps. Table 6 expresses the price of the option with $\gamma = 0.6$ and $\hat{\lambda} = I_d$. Comparing this result with that of the average option case, the accuracy of the approximation is slightly better, but we observe that the characteristics are very similar to each other. Overall, except far out of the money cases, utilizing the second order AE method, we can satisfy almost all the practical requirements of calculation speed and accuracy.

Table 7 expresses the price of the option with $\gamma = 0.9$ and $\hat{\lambda} = I_d$. We can derive the similar implication to Table 6.

Table 8 expresses the price of the option with $\gamma = 0.6$ and $\hat{\lambda} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. We can derive the similar implication to Table 6 and Table 7.

Table 9, 10, 11 express the performances of the simulation algorithms with $\gamma = 0.6$ and $\hat{\lambda} = I_d$, $\gamma = 0.9$ and $\hat{\lambda} = I_d$, and $\gamma = 0.6$ and $\hat{\lambda} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$ respectively. The improvement by utilizing the hyprid Monte Carlo is remarkably. From the ratio of the standard variances, we can say that the convergence speed of the hybrid Monte Carlo method is 10~650 times faster than the crude one. We also notice that the hybrid Monte Carlo method improves the accuracy when the precision of the second order AE is not satisfactory.

Table 12 expresses the actual calculation time (seconds). As in the case of the average option, we can find the fact that (B)/(A) is nearly equal to one.

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	21.58449592	21.91484983	21.59537753	1.5305%	0.0504%
ATM	1yr	8.37264248	8.37891636	8.37891636	0.0749%	0.0749%
20%OTM	1yr	2.22354692	1.91484982	2.23432211	-13.8831%	0.4846%
30%OTM	1yr	0.99673734	0.72325016	0.99513944	-27.4382%	-0.1603%
40%OTM	1yr	0.40980386	0.22946927	0.39343213	-44.0051%	-3.9950%
20%ITM	2yrs	23.75393902	24.34591519	23.78727415	2.4921%	0.1403%
ATM	2yrs	11.72064834	11.73284501	11.73284501	0.1041%	0.1041%
20%OTM	2yrs	4.87965526	4.34591515	4.90455619	-10.9381%	0.5103%
30%OTM	2yrs	2.96608739	2.35809561	2.98571581	-20.4981%	0.6618%
40%OTM	2yrs	1.73796428	1.17677381	1.73511233	-32.2901%	-0.1641%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 6

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	21.49460357	21.95490132	21.46884396	2.1415%	-0.1198%
ATM	1yr	8.48940761	8.44173067	8.44173067	-0.5616%	-0.5616%
20%OTM	1yr	2.48084201	1.95490131	2.44095867	-21.2001%	-1.6077%
30%OTM	1yr	1.21974229	0.74607045	1.16322375	-38.8338%	-4.6336%
40%OTM	1yr	0.56959162	0.23985193	0.49439313	-57.8905%	-13.2022%
20%ITM	2yrs	23.69764633	24.48592433	23.62962071	3.3264%	-0.2871%
ATM	2yrs	12.03942987	11.90867567	11.90867567	-1.0860%	-1.0860%
20%OTM	2yrs	5.45615965	4.48592429	5.34222790	-17.7824%	-2.0881%
30%OTM	2yrs	3.55487043	2.46340958	3.43363195	-30.7033%	-3.4105%
40%OTM	2yrs	2.27687312	1.24746280	2.12088287	-45.2116%	-6.8511%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 7

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	22.71735009	23.21655438	22.76144673	2.1975%	0.1941%
ATM	1yr	10.22349407	10.26203483	10.26203483	0.3770%	0.3770%
20%OTM	1yr	3.63176073	3.21655437	3.67166201	-11.4326%	1.0987%
30%OTM	1yr	1.98872558	1.54585901	2.01372033	-22.2689%	1.2568%
40%OTM	1yr	1.03346054	0.66421283	1.03177629	-35.7293%	-0.1630%
20%ITM	2yrs	25.68231230	26.52967739	25.79066387	3.2994%	0.4219%
ATM	2yrs	14.28003136	14.36974176	14.36974176	0.6282%	0.6282%
20%OTM	2yrs	7.16647093	6.52967735	7.26869087	-8.8857%	1.4264%
30%OTM	2yrs	4.90100198	4.08450289	4.99874001	-16.6598%	1.9942%
40%OTM	2yrs	3.28003520	2.42073748	3.35150000	-26.1978%	2.1788%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 8

K	T	(A) crude Monte rmse	worst	(B) Hybrid rmse	worst	(A)/(B)	ratio of std vars
20%ITM	1yr	0.57247%	2.11432%	0.05469%	0.18375%	10.468	116.217
ATM	1yr	1.06549%	2.89570%	0.10752%	0.30802%	9.910	89.762
20%OTM	1yr	2.47095%	6.18395%	0.28494%	0.85038%	8.672	63.805
30%OTM	1yr	3.42828%	9.48989%	0.41288%	1.31510%	8.303	52.091
40%OTM	1yr	5.81575%	18.25744%	0.77309%	2.29925%	7.523	37.912
20%ITM	2yrs	0.80623%	2.47425%	0.11267%	0.31136%	7.156	56.552
ATM	2yrs	1.11634%	2.48094%	0.18546%	0.46768%	6.019	46.446
20%OTM	2yrs	1.75587%	4.45897%	0.27797%	0.61263%	6.317	37.584
30%OTM	2yrs	2.42987%	6.81598%	0.37658%	1.10531%	6.452	32.875
40%OTM	2yrs	3.11685%	14.66067%	0.56497%	1.34763%	5.517	28.813

20,000 trials, 100 cases for each parameter.

Table 9

K	T	(A) crude Monte rmse	worst	(B) Hybrid rmse	worst	(A)/(B)	ratio of std vars
20%ITM	1yr	0.69977%	2.03948%	0.08510%	0.22680%	8.223	49.380
ATM	1yr	1.06798%	3.33770%	0.19102%	0.45967%	5.591	35.852
20%OTM	1yr	2.49303%	6.12488%	0.42960%	1.27127%	5.803	23.736
30%OTM	1yr	3.40935%	8.18267%	0.75750%	2.22337%	4.501	18.693
40%OTM	1yr	4.42942%	12.17258%	1.24398%	3.07976%	3.561	13.328
20%ITM	2yrs	0.73714%	1.93463%	0.16848%	0.44136%	4.375	23.556
ATM	2yrs	0.99079%	2.61396%	0.27867%	0.71735%	3.555	18.028
20%OTM	2yrs	2.07158%	5.05733%	0.23390%	0.59402%	8.857	13.645
30%OTM	2yrs	2.27417%	7.10532%	0.66713%	2.31033%	3.409	11.725
40%OTM	2yrs	2.96570%	8.25342%	0.92505%	2.32947%	3.206	10.116

20,000 trials, 100 cases for each parameter.

Table 10

K	T	(A) crude Monte rmse	worst	(B) Hybrid rmse	worst	(A)/(B)	ratio of std vars
20%ITM	1yr	0.60834%	1.47800%	0.05008%	0.10602%	12.148	650.503
ATM	1yr	1.17193%	2.95817%	0.08413%	0.23467%	13.930	502.013
20%OTM	1yr	1.91457%	4.50384%	0.11110%	0.34603%	17.233	418.228
30%OTM	1yr	2.42859%	7.58760%	0.14537%	0.38379%	16.706	369.648
40%OTM	1yr	3.74073%	11.40472%	0.22285%	0.52894%	16.786	256.348
20%ITM	2yrs	0.80736%	2.35756%	0.05331%	0.14069%	15.146	317.170
ATM	2yrs	1.12164%	2.90605%	0.07760%	0.18374%	14.455	255.993
20%OTM	2yrs	1.70795%	4.38880%	0.10273%	0.25005%	16.626	231.451
30%OTM	2yrs	2.11188%	6.75970%	0.13440%	0.35935%	15.713	222.585
40%OTM	2yrs	2.31228%	5.88210%	0.17853%	0.42656%	12.952	203.666

20,000 trials, 100 cases for each parameter.

Table 11

# of trials	(A) Crude Monte	(B) Hybrid	(B)/(A)
1,000	0.0469	0.0625	1.333
10,000	0.4844	0.5938	1.226
20,000	4.6875	5.0469	1.077
100,000	9.5000	10.3125	1.086

Table 12

4.2.2 European basket call option with five assets

We suppose S_t follows

$$dS_t^i = rS_t^i dt + \sum_{j=1}^{5} \sigma^{ij} \left(S_t^i\right)^{\gamma^i} dB_t^j, \quad S_0^i = x^i \quad (i = 1, 2, ..., 5). \tag{60}$$

We set the simulation conditions as $S_0^i = 100$, $r = 0.05$, $\sigma^{ij} = 0.3\left(S_0^i\right)^{(1-\gamma^i)}$, ($i$=1,2, ...5). We also suppose each asset takes 20% of the basket. And for the Monte Carlo simulation, we divide one year into 250 time steps. Table 13 and Table 14 express the prices of the option with $\gamma = 0.6$ and $\hat{\lambda} = I_d$, and $\gamma = 0.9$

and $\hat{\lambda} = I_d$, respectively. Comparing this result with that of the two assets case, the accuracy of the approximation is slightly worse, but we observe that the characteristics are very similar to each other. Overall, except out of the money cases, utilizing the second order AE method, we can satisfy almost all the practical requirements of calculation speed and accuracy. Moreover, we notice that it is very difficult that calculating the price of options of which underlying asset has complicated structure. We obtain it in a twinkling of an eye.

Table 15 and 16 express the performances of the simulation algorithms with $\gamma = 0.6$ and $\hat{\lambda} = I_d$, and $\gamma = 0.9$ and $\hat{\lambda} = I_d$ respectively. The effect of utilizing the hyprid Monte Carlo is very well. From the ratio of the standard variances, we can say that the convergence speed of the hybrid Monte Carlo method is 4~76 times faster than the crude Monte Carlo method. We also notice that the hybrid Monte Carlo method improves the accuracy when the precision of the second order AE is not satisfactory

Table 17 expresses the actual calculation time (seconds). As in the case of the average option, we can find the fact that (B)/(A) is nearly equal to one. In the case of 1,000 trials, (B)/(A) is not near to one, because of the complexity of the option, we need so much pre-processing calculation especially for the five assets case. But the pre-processing is done only once for one simulation so that the greater number of the trials the smaller number of the (B)/(A). From computational point of view, this feature is very important. The fact that (B)/(A) is nearly equal to one for the large number trials even in the cases of very complicated options enables us to parallelize the computation without serious problems[8].

We remain the consideration under parallel processing for the next research.

[8] As in the case of both the Intel processors and the fastest super computing processors, the trend of the improvement of the computation power is due to parallerization. Therefore, from practical point of view, we need to pay attention to the scalability of the parallerization of our algorithms.

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	20.28929581	20.38431153	20.28195784	0.4683%	-0.0362%
ATM	1yr	5.30743951	5.29929201	5.29929201	-0.1535%	-0.1535%
20%OTM	1yr	0.49060990	0.38431151	0.48666520	-21.6666%	-0.8040%
30%OTM	1yr	0.09816019	0.05488990	0.09211739	-44.0813%	-6.1561%
40%OTM	1yr	0.01502470	0.00487760	0.01170632	-67.5361%	-22.0862%
20%ITM	2yrs	21.10433339	21.34007865	21.09031468	1.1170%	-0.0664%
ATM	2yrs	7.44457320	7.42050274	7.42050274	-0.3233%	-0.3233%
20%OTM	2yrs	1.60036248	1.34007861	1.58984258	-16.2641%	-0.6573%
30%OTM	2yrs	0.61576331	0.41936737	0.60125496	-31.8947%	-2.3562%
40%OTM	2yrs	0.21084724	0.10454405	0.19272944	-50.4172%	-8.5929%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 13

K	T	(A) crude Monte	(B) AE 1st Order	(C) AE 2nd Order	(D) rate of diff	(E) rate of diff
20%ITM	1yr	20.26498684	20.39720862	20.23990541	0.6525%	-0.1238%
ATM	1yr	5.40146269	5.33901927	5.33901927	-1.1560%	-1.1560%
20%OTM	1yr	0.59238501	0.39720861	0.55451182	-32.9476%	-6.3933%
30%OTM	1yr	0.14435017	0.05805043	0.11647844	-59.7850%	-19.3084%
40%OTM	1yr	0.02971169	0.00531893	0.01635639	-82.0982%	-44.9496%
20%ITM	2yrs	21.09423163	21.40299771	21.01623877	1.4637%	-0.3697%
ATM	2yrs	7.70372804	7.53170781	7.53170781	-2.2329%	-2.2329%
20%OTM	2yrs	1.92213506	1.40299767	1.78975661	-27.0084%	-6.8871%
30%OTM	2yrs	0.85089405	0.45024321	0.73792492	-47.0859%	-13.2765%
40%OTM	2yrs	0.35294523	0.11594175	0.25961763	-67.1502%	-26.4425%

(D)={(B)-(A)}/(A), (E)={(C)-(A)}/(A),
crude Monte : 25,000,000 trials

Table 14

		(A) crude Monte		(B) Hybrid			
K	T	rmse	worst	rmse	worst	(A)/(B)	ratio of std vars
20%ITM	1yr	0.42014%	1.08677%	0.05475%	0.16829%	7.673	76.837
ATM	1yr	1.13766%	3.99027%	0.13812%	0.47988%	8.237	56.511
20%OTM	1yr	3.66004%	8.54030%	0.61195%	2.05561%	5.981	29.999
30%OTM	1yr	7.17961%	17.86614%	1.83090%	5.59845%	3.921	19.404
40%OTM	1yr	17.55049%	46.53202%	6.32235%	16.14387%	2.776	9.159
20%ITM	2yrs	0.55322%	1.40204%	0.09114%	0.26925%	6.070	37.690
ATM	2yrs	1.16860%	3.19490%	0.20595%	0.63206%	5.674	29.246
20%OTM	2yrs	2.52678%	10.35230%	0.51600%	1.47408%	4.897	19.600
30%OTM	2yrs	3.66453%	9.02592%	0.94991%	2.66898%	3.858	15.481
40%OTM	2yrs	5.88319%	13.95806%	1.85550%	5.40032%	3.171	11.378

20,000 trials, 100 cases for each parameter.

Table 15

K	T	(A) crude Monte rmse	worst	(B) Hybrid rmse	worst	(A)/(B)	ratio of std vars
20%ITM	1yr	0.50538%	1.28845%	0.08247%	0.25874%	6.128	33.661
ATM	1yr	1.22147%	3.34248%	0.22857%	0.62241%	5.344	23.414
20%OTM	1yr	3.49785%	8.86552%	0.92710%	2.42557%	3.773	11.832
30%OTM	1yr	6.22005%	17.09113%	2.20000%	7.85495%	2.827	7.476
40%OTM	1yr	14.51955%	40.32392%	6.98003%	17.55973%	2.080	3.728
20%ITM	2yrs	0.59881%	1.55055%	0.14306%	0.37638%	4.186	16.301
ATM	2yrs	1.11286%	3.34611%	0.31742%	0.78337%	3.506	11.846
20%OTM	2yrs	2.22898%	6.49894%	0.22784%	0.56676%	9.783	7.603
30%OTM	2yrs	3.60794%	11.19228%	1.48275%	4.12166%	2.433	5.927
40%OTM	2yrs	5.06255%	14.67636%	2.33434%	6.03359%	2.169	4.438

20,000 trials, 100 cases for each parameter.

Table 16

# of trials	(A) Crude Monte	(B) Hybrid	(B)/(A)
1,000	0.1250	0.8594	6.875
10,000	1.3125	1.8750	1.429
20,000	12.0938	17.2344	1.425
100,000	24.5469	35.0469	1.428

Table 17

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