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Collateral Posting and Choice of Collateral Currency -Implications for Derivative Pricing and Risk Management-

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Collateral Posting and Choice of Collateral Currency *

–*Implications for Derivative Pricing and Risk Management*–

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Abstract

In recent years, we have observed the dramatic increase of the use of collateral as an important credit risk mitigation tool. It has become even rare to make a contract without collateral agreement among the major financial institutions. In addition to the significant reduction of the counterparty exposure, collateralization has important implications for the pricing of derivatives through the change of effective funding cost. This paper has demonstrated the impact of collateralization on the derivative pricing by constructing the term structure of swap rates based on the actual market data. It has also shown the importance of the "choice" of collateral currency. Especially, when the contract allows multiple currencies as eligible collateral and free replacement among them, the paper has found that the embedded "cheapest-to-deliver" option can be quite valuable and significantly change the fair value of a trade. The implications of these findings for market risk management have been also discussed.

Keywords : swap, collateral, derivatives, Libor, currency, OIS, EONIA, Fed-Fund, CCS, basis, risk management, HJM, FX option, CSA, CVA, term structure

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1 Introduction

Collateralization in the privately negotiated derivatives market has continued to grow at a rapid pace over the past decade. According to the ISDA (International Swaps and Derivatives Association), about 70% of the trade volumes for all the OTC (over-the-counter) derivatives were collateralized at the end of 2009, which was merely 30% in 2003.(See [5] for the detail.) Among the major financial institutions, it has become quite rare to make a contract without a collateral agreement in these days, and coverage is up to 78% and 84% for all the OTC and fixed income derivatives, respectively. One can expect that the use of collateral continues to grow at a faster pace due to the recent financial crisis and the heightened attention to the counterparty credit risk. A stringent collateral management will be also crucial for the operations of central clearing houses, which is now under the active debate. The importance of the collateralization is mainly twofold: 1) reduction of the counterparty credit risk, and 2) change of the funding cost of the trade. The first one is well recognized, and there exist a large number of studies in the context of credit value adjustments. Although it is not as obvious as the first one, the second effect is also quite important, which is the main topic of this work.

Let us consider the situation where the firm A has a positive present value (PV) in the contract with the firm B with high credit quality. From the view point of the firm A, it is equivalent to providing a loan to the counterparty B with the principal equal to its PV. Since the firm A has to wait for the payment from the firm B until the maturity of the contract, it is clear that A has to finance its loan and hence the funding cost should be reflected in the pricing of the contract. If the firm A has (and continues to be) Libor credit quality, the funding cost is given by the Libor of its funding currency since it makes the present value of "funding" zero. This is the main reason why Libor (London inter-bank offered rate) has been widely used as a proxy of the discounting rate in the derivative pricing. [1] The situation is depicted in Fig. 1.

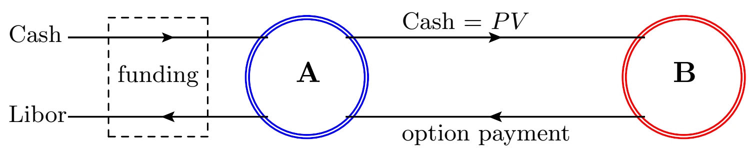

Figure 1: Unsecured trade with external funding

The above situation drastically changes when the trade is collateralized, which is depicted in Fig. 2. Let us assume that the trade has been made with a CSA (credit support annex, a legal document regulating credit support for derivative transactions), requiring cash collateral with zero minimum transfer amount as well as threshold. In this case, there is no external funding need for the firm A, since the cash amount equal to the PV is posted from the firm B. Now, the firm A has to pay the counterparty B the margin whose interest rate is called "collateral rate" applied to the posted collateral amount. This effectively makes the funding cost of the contract equal to the collateral rate. According to [5], in the current financial market the most popular collateral is the cash of the developed

[1]In reality, we need appropriate CVA to handle the credit risk, since there is no guarantee that the relevant firms continue to be Libor-credit quality.

countries, more than 80% of the collateral used, and the typical choice of collateral rate is the over-night (ON) rate of the corresponding currency [2].

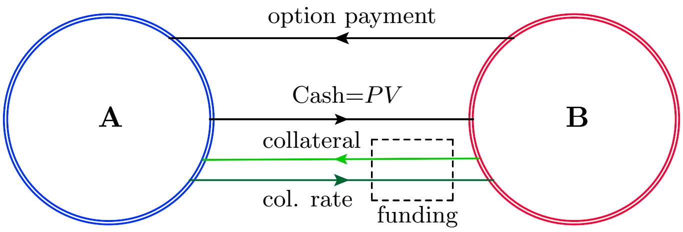

Figure 2: Secured trade with external funding

In recent years, the latter effect has gained significant attention among market participants, since they have experienced through this financial crisis under which the difference between collateral rate and Libor can be neither negligible nor stable [3]. In our previous works, "Note on Construction of Multiple Swap Curves with and without Collateral" [1] and "A Market Model of Interest Rates with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies" [2], we have presented a systematic procedures for the construction of multiple swap curves in the presence of collateral, and their no-arbitrage dynamics in an HJM (Heath-Jarrow-Morton)-type framework.

In this brief note, we have constructed the swap curves based on the method given in the previous works using the actual market data, and demonstrated the importance of collateralization in pricing of derivatives [4]. Furthermore, considering the fact that the collateral currency is not necessarily the same as the payment currency, which is in fact unavoidable in multi-currency trades, and also the fact that quite large basis spreads have existed in the cross currency swap (CCS) market for many years, it is a natural question to ask what is the impact on derivative pricing from different choice of the collateral currency. In fact, by making use of the information in CCS markets, we have found that the selection of the collateral currency has non-negligible impact on derivative prices. For example, in the recent financial crisis, the cost to post USD cash as the collateral was found to be quite expensive compared to other currencies, probably due to the so-called "safe-haven" demand. These differences among collateral currencies give rise to another interesting twist. When the relevant CSA allows multiple choices of collateral currencies and free replacement among them, the payer of the collateral has the "cheapest-to-deliver" options. Although the available length of historical data is quite limited due to the recent birth of OIS (overnight index swap), we have found that the right to change the collateral currency can be quite valuable especially when the cross currency market is volatile.

[2]Cash collateral is becoming increasingly popular, which is partly because it is free from the issues associated with rehypothecation, and the required time for settlement is shorter than other assets, such as government bonds.

[3]See Fig. 4 in a later section to see the historical behavior.

[4]All the market data used in this note were taken from Bloomberg.

2 Pricing under the collateralization

This section reviews [1], our results on pricing derivatives under the collateralization. Let us make the following simplifying assumptions about the collateral contract.

1. *Full collateralization (zero threshold) by cash.*
2. *The collateral is adjusted continuously with zero minimum transfer amount.*

In fact, the daily margin call is now quite popular in the market, which makes the above assumptions a reasonable proxy of the reality. Since the assumptions allow us to neglect the loss given default of the counterparty, we can treat each trade/payment separately without worrying about the non-linearity arising from the netting effects and the asymmetric handling of exposure.

We consider a derivative whose payoff at time T is given by $h^{(i)}(T)$ in terms of currency "i". We suppose that the currency "j" is used as the collateral for the contract. Note that the instantaneous return (or cost when it is negative) by holding the cash collateral at time t is given by

$$y^{(j)}(t) = r^{(j)}(t) - c^{(j)}(t) \ , \tag{2.1}$$

where $r^{(j)}$ and $c^{(j)}$ denote the risk-free interest rate and collateral rate of the currency j, respectively. If we denote the present value of the derivative at time t by $h^{(i)}(t)$ (in terms of currency i), the collateral amount posted from the counterparty is given by $\left(h^{(i)}(t)/f_x^{(i,j)}(t)\right)$, where $f_x^{(i,j)}(t)$ is the foreign exchange rate at time t representing the price of the unit amount of currency j in terms of currency i. These considerations leads to the following calculation for the collateralized derivative price,

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T)\right] \\ &+ f_x^{(i,j)}(t) E_t^{Q_j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s)\left(\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}\right) ds\right] \ , \end{aligned} \tag{2.2}$$

where $E_t^{Q_i}[\cdot]$ is the time t conditional expectation under the risk-neutral measure of currency i, where the money-market account of currency i is used as the numeraire. Here, the second term takes into account the return/cost from holding the collateral. By aligning the measure to Q_i, we obtain

$$h^{(i)}(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds\right] \ . \tag{2.3}$$

Now, it is easy to see that

$$X(t) := e^{-\int_0^t r^{(i)}(s)ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds \tag{2.4}$$

is a Q_i-martingale under appropriate integrability conditions. This tells us that the process of the option value can be written as

$$dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right) h^{(i)}(t) dt + dM(t) \tag{2.5}$$

with some Q_i-martingale M.

As a result, we have the following theorem:

Theorem 1 *Suppose that $h^{(i)}(T)$ is a derivative's payoff at time T in terms of currency "i" and that the currency "j" is used as the collateral for the contract. Then, the value of the derivative at time t, $h^{(i)}(t)$ is given by*

$$
\begin{aligned}
h^{(i)}(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right)h^{(i)}(T)\right] && (2.6)\\
&= D^{(i)}(t,T)E_t^{\mathcal{T}^c_{(i)}}\left[e^{-\int_t^T y^{(i,j)}(s)ds}h^{(i)}(T)\right] \ , && (2.7)
\end{aligned}
$$

where

$$y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s) \tag{2.8}$$

with $y^{(i)}(s) = r^{(i)}(s) - c^{(i)}(s)$ and $y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s)$. Here, we have also defined the collateralized zero-coupon bond of currency i as

$$D^{(i)}(t,T) = E_t^{Q_i}\left[e^{-\int_t^T c^{(i)}(s)ds}\right] \tag{2.9}$$

and the collateralized forward measure $\mathcal{T}^c_{(i)}$, where the collateralized zero-coupon bond is used as the numeraire; thus, $E_t^{\mathcal{T}^c_{(i)}}[\cdot]$ denotes the time t conditional expectation under the measure $\mathcal{T}^c_{(i)}$.

As a corollary of the theorem, we have

$$
\begin{aligned}
h(t) &= E_t^{Q}\left[e^{-\int_t^T c(s)ds}h(T)\right]\\
&= D(t,T)E_t^{\mathcal{T}^c}[h(T)] && (2.10)
\end{aligned}
$$

when the payment and collateral currencies are the same. This result is clearly showing the fact that the effective funding cost is given by the collateral rate, as in Fig. 2, regardless of the risk-free rate of the corresponding currency.

A different derivation using the process of the collateral account is available in [1]. As an early related research, there is work of Johannes and Sundaresan [6], where the authors introduced an "convenience yield" to reflect the cost/benefit of the collateral, and put more emphasis on the empirical analysis of its dynamics based on the US treasury and swap markets.

3 Curve Construction in Single Currency

In this section, we will construct the relevant yield curves in a single currency market. For the details of the procedures, see [1, 3]. Here, we briefly summarize the set of formulas needed to strip the relevant discounting factors and forward Libors;

$$\mathrm{OIS}_N \sum_{n=1}^{N} \Delta_n D(0,T_n) = D(0,T_0) - D(0,T_N) \ , \tag{3.1}$$

$$\mathrm{IRS}_M \sum_{m=1}^{M} \Delta_m D(0,T_m) = \sum_{m=1}^{M} \delta_m D(0,T_m) E^{\mathcal{T}^c_m}[L(T_{m-1},T_m;\tau)] \ , \tag{3.2}$$

$$\sum_{n=1}^{N} \delta_n D(0,T_n)\left(E^{\mathcal{T}^c_n}[L(T_{n-1},T_n;\tau_S)] + TS_N\right) = \sum_{m=1}^{M} \delta_m D(0,T_m) E^{\mathcal{T}^c_m}[L(T_{m-1},T_m;\tau_L)] \ . \tag{3.3}$$

These are the consistency conditions to give the market quotes of various swaps [5]. We have denoted the market observed OIS (Overnight Index Swap) rate, IRS (Interest Rate Swap) rate and TS (Tenor Swap) spread respectively as OIS_N, IRS_M and TS_N, where the subscripts represent the lengths of swaps. $\{T_i\}_{i\geq 0}$ are the reset/payment times of each instrument. We distinguish the day-count fraction of fixed and floating legs by Δ and δ, which are not necessarily the same among different instruments. $L(T_{m-1}, T_m; \tau)$ is the Libor with tenor τ whose reset and payment times are T_{m-1} and T_m, respectively. In the third formula, we have distinguished the two different tenors by τ_S and τ_L $(> \tau_S)$. Although we have used the same payment frequencies in fixed and floating legs of IRS, there is no difficulty if it is not the case as in the US market.

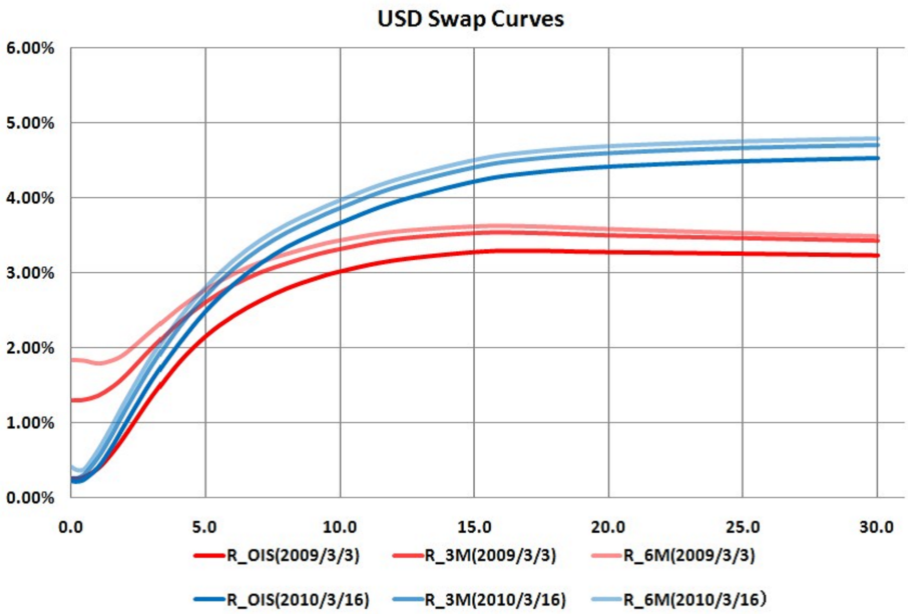

Figure 3: USD zero rate curves of Fed-Fund rate, $3m$ and $6m$ Libors.

In Fig. 3, we have given examples of calibrated yield curves for USD market on 2009/3/3 and 2010/3/16, where R_{OIS}, R_{3m} and R_{6m} denote the zero rates for OIS (Fed-Fund rate), 3m and 6m forward Libor, respectively. $R_{\mathrm{OIS}}(\cdot)$ is defined as

$$R_{\mathrm{OIS}}(T) = -\ln(D(0,T))/T \ . \tag{3.4}$$

For the forward Libor, the zero-rate curve $R_\tau(\cdot)$ is determined recursively through the relation

$$E^{\mathcal{T}^c_m}[L(T_{m-1}, T_m; \tau)] = \frac{1}{\delta_m}\left(\frac{e^{-R_\tau(T_{m-1})T_{m-1}}}{e^{-R_\tau(T_m)T_m}} - 1\right) \ . \tag{3.5}$$

In the actual calculation of $D(0,\cdot)$, we have used the Fed-Fund vs 3m-Libor basis swap, where the two parties exchange 3m Libor and the compounded Fed-Fund rate with spread,

[5] If the payments are compounded in TS, the corresponding formula turns out slightly more complicated. However, the effect from compounding is negligibly small and does not cause any meaningful change to the result.

which seems more liquid and a larger number of quotes available than the usual OIS. In the end of 2008, the Libor-OIS spread was quite significant and more than one percentage point in the short end of the curve. Although it tightened rapidly, there still exists spread around 20bps in early 2010. In Fig. 4, one can see the historical behavior of the spread between 1yr IRS and OIS for USD, JPY and EUR, where the underlying floating rates of IRS are 3m-Libor for USD and EUR and 6m-Libor for JPY. In Fig. 5, we have also given an example of JPY and EUR swap curves on 2010/3/16.

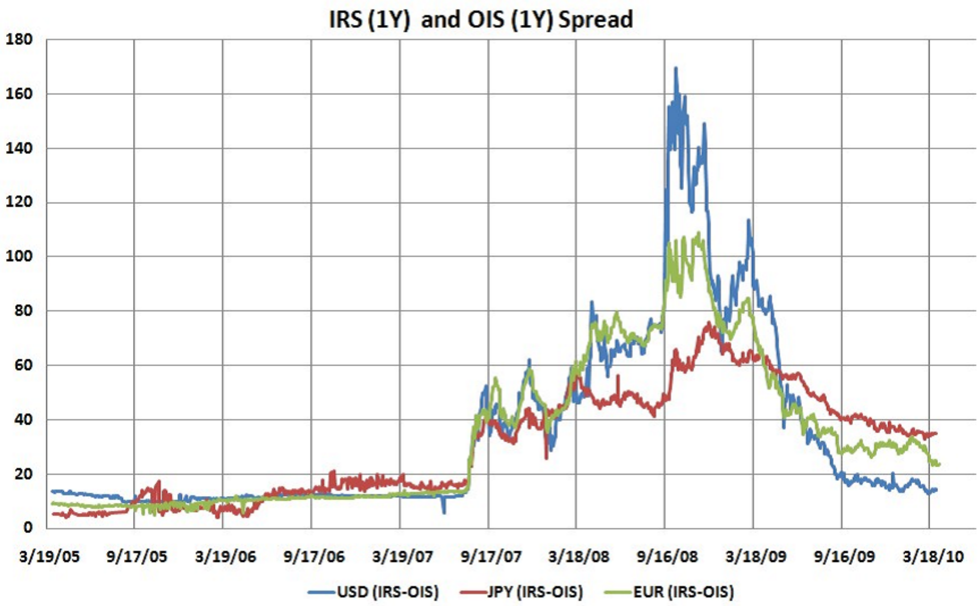

Figure 4: Difference between 1yr IRS and OIS. Underlying floating rates are 3m-Libor for USD and EUR, and 6m-Libor for JPY.

Remarks: In the above calculations, we have assumed that the conditions given in the previous section are satisfied, and also that all the instruments are collateralized by the cash of domestic currency which is the same as the payment currency. Cautious readers may worry about the possibility that the market quotes contain significant contributions from market participants who use a foreign currency as collateral. In fact, some of the major financial firms prefer USD cash collateral regardless of the payment currency of the contracts. It gives rise to additional factors in discounting as in Eq.(2.6), and changes the present value of the relevant cash flow. However, the induced changes in IRS/TS quotes are very small and impossible to distinguish from the bid/offer spreads in normal circumstances, because the correction appears both in the fixed and floating legs which keeps the market quotes almost unchanged [6]. However, as we will see in the later sections, the present values of off-the-market swaps can be significantly affected when the collateral currency is different.

[6] As for cross currency swaps, the change can be a few basis point, and hence comparable to the market bid/offer spreads.

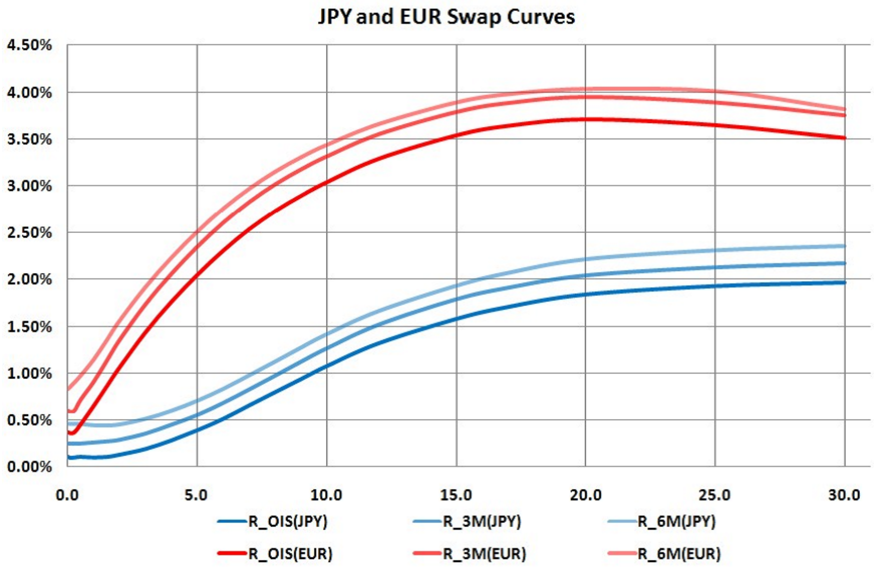

Figure 5: Zero rate curves of OIS rate, $3m$ and $6m$ Libor for JPY and EUR as of 2010/3/16.

4 Curve Construction in Multiple Currencies

4.1 Calibration Procedures

In this section, we will discuss how to make the term structure consistent with CCS (cross currency swap) markets. The current market is dominated by USD crosses where 3m USD Libor flat is exchanged with 3m Libor of a different currency with additional basis spread. There are two types of CCS, one is CNCCS (Constant Notional CCS), and the other is MtMCCS (Mark-to-Market CCS). In a CNCCS, the notionals of both legs are fixed at the inception of the trade and kept constant until its maturity. On the other hand, in a MtMCCS, the notional of USD leg is reset at the start of every calculation period of the Libor while the notional of the other leg is kept constant throughout the contract period. Although the required calculation becomes a bit more complicated, we will use MtMCCS for calibration due to its better liquidity [7]. We consider a MtMCCS of (i, j) currency pair, where the leg of currency i (intended to be USD) needs notional refreshments. We assume that the collateral is posted in currency i, which seems common in the market.

The value of j-leg of a T_0-start T_N-maturing MtMCCS is calculated as

$$\begin{aligned} PV_j &= -D^{(j)}(0,T_0)E^{\mathcal{T}^c_{0,(j)}}\left[e^{-\int_0^{T_0} y^{(j,i)}(s)ds}\right] + D^{(j)}(0,T_N)E^{\mathcal{T}^c_{n,(j)}}\left[e^{-\int_0^{T_N} y^{(j,i)}(s)ds}\right] \\ &+ \sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(0,T_n)E^{\mathcal{T}^c_{n,(j)}}\left[e^{-\int_0^{T_n} y^{(j,i)}(s)ds}\Big(L^{(j)}(T_{n-1},T_n;\tau)+B_N\Big)\right], \qquad (4.1) \end{aligned}$$

where the basis spread B_N is available as a market quote. In generic situations, we need to solve a set of SDEs (stochastic differential equations) to evaluate the expectations involved

[7]As for the details of MtMCCS and CNCCS, see [2, 3].

in this equation. For interested readers, we have given a complete set of SDEs in HJM-type framework with fully stochastic basis spreads in Appendix A.

In [2], we have assumed that all of the $\{y^{(k)}(\cdot)\}$ and hence $\{y^{(i,j)}(\cdot)\}$ are deterministic functions of time to make the curve construction tractable. Here, we slightly relax the assumption allowing randomness of $\{y^{(i,j)}(\cdot)\}$. As long as we assume that $\{y^{(i,j)}(\cdot)\}$ is independent from the dynamics of Libors and collateral rates, the procedures of bootstrapping given in [2] can be applied in the same way [8]. Under this assumption, we obtain

$$\begin{aligned} PV_j &= -D^{(j)}(0,T_0)e^{-\int_0^{T_0} y^{(j,i)}(0,s)ds} + D^{(j)}(0,T_N)e^{-\int_0^{T_N} y^{(j,i)}(0,s)ds} \\ &+ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(0,T_n)e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds}\left(E^{\mathcal{T}^c_{n,(j)}}[L^{(j)}(T_{n-1},T_n;\tau)] + B_N\right) . \end{aligned} \quad (4.2)$$

Here, we have defined, $y^{(j,i)}(t,s)$, the forward rate of $y^{(j,i)}(s)$ at time t as [9]

$$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{Q_j}\left[e^{-\int_t^T y^{(j,i)}(s)ds}\right] . \quad (4.3)$$

Note that non-zero correlations among $\{y^{(i,k)}\}_{i,k}$ themselves do not pose any difficulty on curve construction.

On the other hand, the present value of i-leg in terms of currency j is given by

$$\begin{aligned} PV_i &= -\sum_{n=1}^{N} E^{Q_i}\left[e^{-\int_0^{T_{n-1}} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\right] / f_x^{(i,j)}(0) \\ &\quad + \sum_{n=1}^{N} E^{Q_i}\left[e^{-\int_0^{T_n} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\left(1 + \delta_n^{(i)} L^{(i)}(T_{n-1},T_n;\tau)\right)\right] / f_x^{(i,j)}(0) \\ &= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0,T_n) E^{\mathcal{T}^c_{n,(i)}}\left[\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} B^{(i)}(T_{n-1},T_n;\tau)\right] , \end{aligned} \quad (4.4)$$

where

$$B^{(i)}(t,T_k;\tau) = E_t^{\mathcal{T}^c_{k,(i)}}\left[L^{(i)}(T_{k-1},T_k;\tau)\right] - \frac{1}{\delta_k^{(i)}}\left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)} - 1\right) , \quad (4.5)$$

which represents a Libor-OIS spread. Since we found no persistent correlation between FX and Libor-OIS spread in historical data, we have treated them as independent variables. Even if a non-zero correlation exists in a certain period, the expected correction seems not numerically important due to the typical size of bid/offer spreads for MtMCCS (about a few bps at the time of writing). Since 3-month timing adjustment of FX is safely negligible, an approximate value of i-leg is obtained as

$$PV_i \simeq \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0,T_n)\frac{D^{(j)}(0,T_{n-1})}{D^{(i)}(0,T_{n-1})} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} B^{(i)}(0,T_n;\tau). \quad (4.6)$$

[8] In practice, it would not be a problem even if there is a non-zero correlation as long as it does not meaningfully change the model implied quotes compared to the market bid/offer spreads.

[9] Since we are assuming the independence from the collateral rate, the measure change within the same currency gives no difference.

Here, we have used the following expression of the forward FX collateralized with currency i:

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{D^{(j)}(t,T)}{D^{(i)}(t,T)}e^{-\int_t^T y^{(j,i)}(t,s)ds} \ . \tag{4.7}$$

Using Eqs. (4.2) and (4.6), the term structure of $\{y^{(j,i)}(0,\cdot)\}$ can be extracted by the relation $PV_i = PV_j$, a consistency condition for the market quotes of MtMCCS.

Under the above approximation, (i,j)-MtMCCS par spread is expressed as

$$\begin{aligned} B_N \ &= \ \Bigg[\Bigg\{\sum_{n=1}^N \delta_n^{(i)} D_{T_n}^{(i)}\left(\frac{D_{T_{n-1}}^{(j)}}{D_{T_{n-1}}^{(i)}}\right) e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} B_{T_n}^{(i)} - \sum_{n=1}^N \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} B_{T_n}^{(j)}\Bigg\} \\ &- \sum_{n=1}^N D_{T_n}^{(j)} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds}\left(e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(0,s)ds} - 1\right)\Bigg] \Big/ \sum_{n=1}^N \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} \ , \end{aligned} \tag{4.8}$$

where we have shortened the notations as $D^{(k)}(0,T) = D_T^{(k)}$ and $B^{(k)}(0,T;\tau) = B_T^{(k)}$.

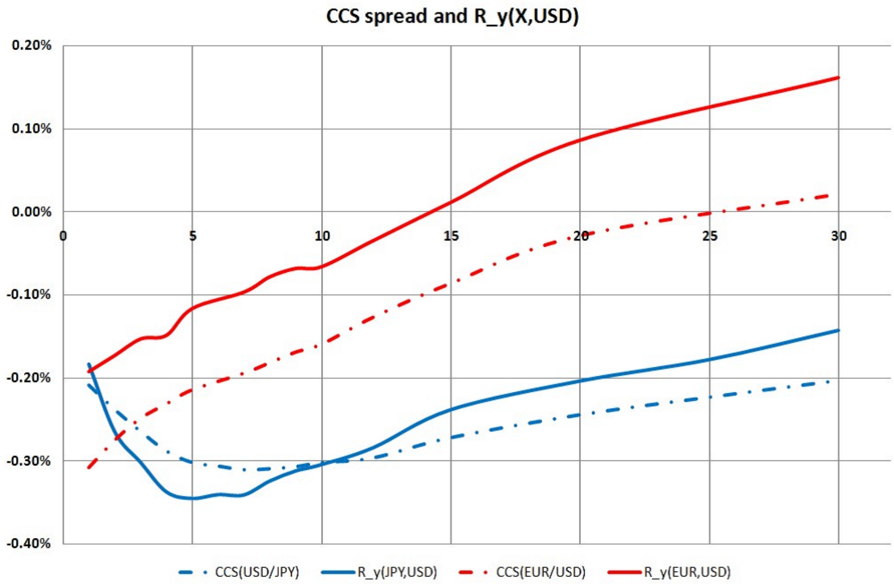

Figure 6: MtMCCS par spreads, $R_{y(JPY,USD)}$ and $R_{y(EUR,USD)}$ as of 2010/3/16.

In Fig. 6, we have given examples of calibration for EUR/USD and USD/JPY MtM-CCS as of 2010/3/16. We have plotted the zero rates of $y^{(j,i)}$ defined as

$$R_{y(j,i)}(T) = -\frac{\ln\left(E^{Q_j}\left[e^{-\int_0^T y^{(j,i)}(s)ds}\right]\right)}{T} = \frac{1}{T}\int_0^T y^{(j,i)}(0,s)ds \tag{4.9}$$

together with the term structure of MtMCCS basis spreads. It is easy to expect that there are significant contributions from the second line of Eq. (4.8) to the CCS basis spreads from the similarities between $R_{y(X,USD)}$ and CCS quotes. The implied forward FXs derived from Eq. (4.7) were well within the bid/offer spreads [10].

4.2 Historical Behavior

Before going to discuss implications for the pricing of derivatives, let us first check the historical behavior of $R_{y(EUR,USD)}$ and $R_{y(JPY,USD)}$ given in Fig. 7 to 11 [11]. Although the available length of OIS data is quite limited, especially longer than a few years points, we were able to cover the volatile market after the collapse of Lehman Brothers, at least for $R_{y(EUR,USD)}$. For both cases, the term structures of R_y have quite similar shapes and levels to those of the corresponding CCS basis spreads. In Fig. 7, historical behavior of $R_{y(X,USD)}(T = 5y)$ (X = EUR, JPY) and corresponding 5y-MtMCCS spreads are given. One can see that a significant portion of CCS spreads movement stems from the change of $y^{(i,j)}$, rather than the difference of Libor-OIS spread between two currencies. The level (difference)-correlation between R_y and CCS spread is quite high, which is about 93% (75%) for EUR or about 70% (92%) for JPY for the historical series used in the figure, for example.

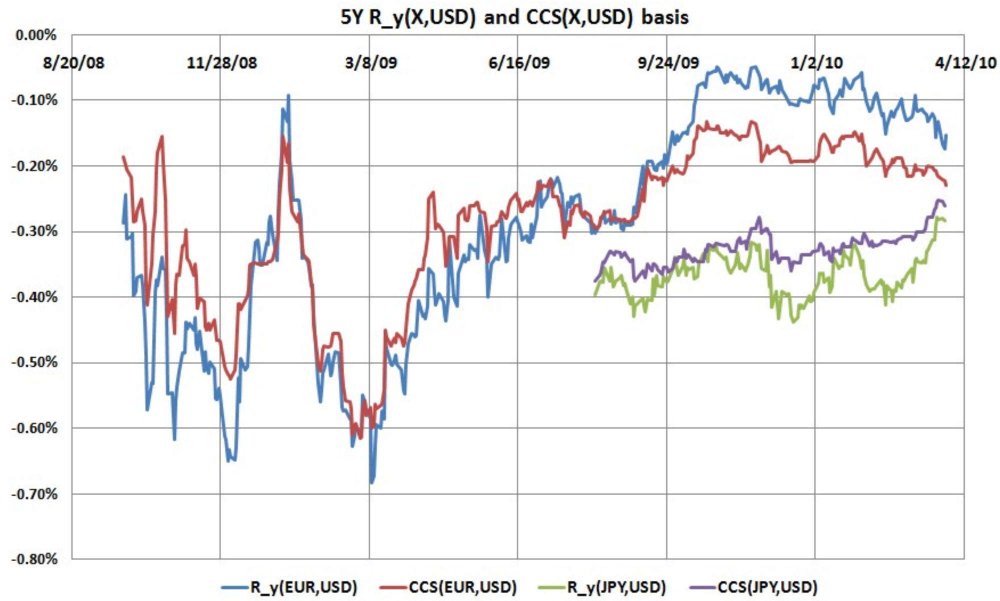

Figure 7: $R_{y(EUR,USD)}(5y)$, $R_{y(JPY,SD)}(5y)$ and corresponding quotes of 5y-MtMCCS.

The 3m-roll historical volatilities of $y^{(EUR,USD)}$ instantaneous forwards, which are annualized in absolute terms, are given in Fig. 12. In a calm market, they tend to be 50 bp or so, but they were more than a percentage point just after the market crisis, which is reflecting a significant widening of the CCS basis spread to seek USD cash in the

[10] In any case, it is quite wide for long maturities.

[11] Due to the lack of OIS data for JPY market, we have only a limited data for (JPY,USD) pair. We have used Cubic Monotone Spline for calibration although the figures are given in linear plots for ease. For spline technique, see [4], for example.

low liquidity market. Except the CCS basis spread, y does not seem to have persistent correlations with other variables such as OIS, IRS and FX forwards. At least, within our limited data, the 3m-roll historical correlations with these variables fluctuate mainly around $\pm 20\%$ or so.

5 Implications for Derivatives Pricing

Now, let us consider implications of collateralization for derivatives pricing. It is quite straightforward when the payment and collateral currencies are the same. As in Eq. (2.10), the discounting factor is given by the collateral rate which is under control of the relevant central bank. Traditionally, among financial firms, the Libor curve has been widely used to discount the future cash flows. However, this method would easily underestimate their values by several percentage points for long maturities, even with the current level of Libor-OIS spread, or $10 \sim 20$ bps. Considering the mechanism of collateralization, financial firms need to hedge the change of OIS in addition to the standard hedge against the movement of Libors. Especially, payers of floating rates need to be careful, since the overnight rate can move in the opposite direction to the Libor as was observed in this financial crisis. In Fig. 13, the present values of Libor floating legs with final principal ($= 1$) payment

$$PV = \sum_{n=1}^{N} \delta_n D(0,T_n) E^{\mathcal{T}_n^c}\left[L(T_{n-1},T_n;\tau)\right] + D(0,T_N) \tag{5.1}$$

are given for various maturities. If traditional Libor discounting is being used, the stream of Libor payments has the almost constant present value "1", which is obviously wrong from our results. This point is very important in risk-management, since financial firms may overlook the quite significant interest-rate risk exposure when they adopt the traditional interest rate model in their system.

If a trade with payment currency j is collateralized by foreign currency i, an additional "discounting" factor appears (See, Eq. (4.3).) [12]:

$$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{Q_j}\left[e^{-\int_t^T y^{(j,i)}(s)ds}\right] . \tag{5.2}$$

From Figs. 9 and 11, one can see that posting USD as collateral tends to be expensive from the view point of collateral payers, which is particularly the case when the market is illiquid. Note that, by the definition of collateral payers, they want to make $(-PV)$ (> 0) as small as possible.

In some cases, financial firms make contracts with CSA allowing several currencies as eligible collateral. Suppose that the payer of collateral has a right to replace a collateral currency whenever he wants. If this is the case, the collateral payer can adopt the optimal collateral strategy, which leads to the modification of the discounting factor as

$$E_t^{Q_j}\left[e^{-\int_t^T \max_{i\in C}\left\{y^{(j,i)}(s)\right\}ds}\right] , \tag{5.3}$$

where C is the set of eligible currencies. Although there is a tendency toward a CSA allowing only one collateral currency to reduce the operational burden, it does not seem

[12] Here, we are assuming independence of y from reference assets.

uncommon to accept the domestic currency and USD as eligible collateral, for example. In this case, the above formula turns out to be

$$E_t^{Q_j}\left[e^{-\int_t^T \max\left\{y^{(j,\mathrm{USD})}(s),0\right\}ds}\right] . \tag{5.4}$$

In Figs. 14 and 15, we have plotted the modifications of discounting factors given in Eq. (5.4), for j = EUR and JPY as of 2010/3/16. We have used the Hull-White model for the dynamics of $y^{(EUR,USD)}(\cdot)$ and $y^{(JPY,USD)}(\cdot)$, with a mean reversion parameter 1.5% per annum and the set of volatilities, $\sigma = 0, 25, 50$ and 75 bps [13], respectively. As can be seen from the historical volatilities given in Fig. 12, σ can be much higher under volatile environment. The curve labeled by USD (EUR, JPY) denotes the modification of the discount factor when only USD (EUR, JPY) is eligible as collateral. One can easily see that there is quite significant impact when the collateral currency is different, or chosen optimally. In the calculation, we have used daily-step Monte Carlo simulation. Although, in practice, we can expect that there are various obstacles to implement the optimal strategy, the fact that the payer can choose collateral will make the implementation simpler. Furthermore, considering various efforts being made to automate the margin call and collateral delivery electronically, proper collateral strategy will be an important issue for the major financial firms and central clearing houses in coming years.

6 Summary and Discussions

In this work, we have demonstrated the impact of collateralization on the derivative pricing by constructing the term structure of swap rates based on the actual data, and have also shown the importance of the "choice" of collateral currency. For a contract allowing multiple choice of collateral currencies and replacement among them, we have found that the embedded "cheapest-to-deliver" option can be quite valuable and significantly change the fair value of a trade.

As implications of these findings, let us emphasize a potential danger to use the traditional Libor-discounting model, which still seems quite common among financial firms. First of all, it can overlook the huge delta exposure to the Libor-OIS and MtMCCS (or closely related "y") spreads. Note that, even if a desk is dealing with only single currency products, it inevitably has exposure to the CCS spread through the modification of discounting factors if it accepts foreign currencies as collateral. Furthermore, if the firm adopts a CSA allowing free replacement of collateral currency, there may exist non-negligible exposure on CCS volatility through the embedded "cheapest-to-deliver" options in collateral contracts.

As a further research, it would be worthwhile to consider implications of stochastic nature of $\{y^{(i,j)}\}$ more thoroughly. For multi-currency options, such as cancelable cross currency swaps, the dynamics of $y^{(i,j)}$ term structure would be particularly important. Extension of the HJM framework given in [2] to allow a stochastic term structure of $\{y^{(i,j)}\}$ and studying implications for multi-currency and/or foreign-currency collateralized derivatives, and their risk-management will be one of our future research topics.

[13] These are annualized volatilities in absolute terms.

A HJM framework with fully stochastic basis spreads

In this appendix, we will give a generic HJM framework which allows to make all the basis spreads stochastic under the collateralization. Although most of the contents have been already covered in [2], we now make $y^{(i,k)}(t,s)$ stochastic, which seems crucial to handle any options on cross currency swap as well as the optimal collateral strategy.

Firstly, let us define the relevant forwards for each currency i, and currency pair (i,k):

$$c^{(i)}(t,T) = -\frac{\partial}{\partial T}\ln D^{(i)}(t,T) \left(= -\frac{\partial}{\partial T}\ln\left(E_t^{Q_i}\left[e^{-\int_t^T c^{(i)}(u)du}\right]\right)\right) , \quad (A.1)$$

$$B^{(i)}(t,T_k;\tau) = E_t^{\mathcal{T}^c_{k,(i)}}\left[L^{(i)}(T_{k-1},T_k;\tau)\right] - \frac{1}{\delta_k^{(i)}}\left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)} - 1\right) , \quad (A.2)$$

$$y^{(i,k)}(t,T) = -\frac{\partial}{\partial T}\ln\left(E_t^{Q_i}\left[e^{-\int_t^T y^{(i,k)}(s)ds}\right]\right) . \quad (A.3)$$

The SDE for $c^{(i)}(t,s)$ can be written as

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s)\cdot\left(\int_t^s \sigma_c^{(i)}(t,u)du\right)dt + \sigma_c^{(i)}(t,s)\cdot dW^{Q_i}(t) , \quad (A.4)$$

where $W^{Q_i}(t)$ is a d-dimensional Brownian motion under the Q_i-measure. $\sigma_c^{(i)}(t,s)$ is a d-dimensional vector and the following abbreviation has been used:

$$\sigma_c^{(i)}(t,s)\cdot dW^{Q_i}(t) = \sum_{n=1}^d \left[\sigma_c^{(i)}(t,s)\right]_n \times dW_n^{Q_i}(t) . \quad (A.5)$$

The consistent drift term of $D^{(i)}$ requires the drift of $c^{(i)}$ to have the form given in the equation (A.4). Although it is not explicitly written, $\sigma_c^{(i)}(t,s)$ can be a function of $c^{(i)}$ and/or other state variables. This point is true for other forwards and FX volatilities, too. In exactly the same way, the SDE for $y^{(i,k)}$ is given by

$$dy^{(i,k)}(t,s) = \sigma_y^{(i,k)}(t,s)\cdot\left(\int_t^s \sigma_y^{(i,k)}(t,u)du\right)dt + \sigma_y^{(i,k)}(t,s)\cdot dW^{Q_i}(t) . \quad (A.6)$$

The SDE for a Libor-OIS spread, $B^{(i)}(t,T)$, can be easily derived under the risk-neutral measure of currency i, Q_i by making use of Maruyama-Girsanov's theorem and the fact that it is a martingale under the forward measure $\mathcal{T}^c_{(i)}$, where $D^{(i)}(\cdot,T)$ is used as the numeraire:

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t,T;\tau)\cdot\left(\int_t^T \sigma_c^{(i)}(t,s)ds\right)dt + \sigma_B^{(i)}(t,T;\tau)\cdot dW^{Q_i}(t) . \quad (A.7)$$

Finally, as is well known, the SDE of the spot FX, $f_x^{(i,j)}(t)$, can be written as

$$\begin{aligned} df_x^{(i,j)}(t)/f_x^{(i,j)}(t) &= (r^{(i)}(t) - r^{(j)}(t))dt + \sigma_X^{(i,j)}(t)\cdot dW^{Q_i}(t) \\ &= \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW^{Q_i}(t) . \quad (A.8) \end{aligned}$$

When we simulate the forward rates of foreign currency j under the assumption that the currency i is used as a base currency, the following SDEs need to be used:

$$dc^{(j)}(t,s) = \sigma_c^{(j)}(t,s)\cdot\left[\left(\int_t^s \sigma_c^{(j)}(t,u)du\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_c^{(j)}(t,s)\cdot dW^{Q_i}(t) \ , \quad \text{(A.9)}$$

$$dy^{(j,k)}(t,s) = \sigma_y^{(j,k)}(t,s)\cdot\left[\left(\int_t^s \sigma_y^{(j,k)}(t,u)du\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_y^{(j,k)}(t,s)\cdot dW^{Q_i}(t) \ , \quad \text{(A.10)}$$

$$\frac{dB^{(j)}(t,T;\tau)}{B^{(j)}(t,T;\tau)} = \sigma_B^{(j)}(t,T;\tau)\cdot\left[\left(\int_t^T \sigma_c^{(j)}(t,s)ds\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_B^{(j)}(t,T;\tau)\cdot dW^{Q_i}(t) \ . \quad \text{(A.11)}$$

The equations $(A.4)-(A.11)$ given above completely determine the dynamics of interest rates and foreign exchange rates in the presence of collateral and basis spreads. Under the assumption of independence between $\{y^{(i,k)}\}_k$ and $\{c^{(i)},\ B^{(i)}\}$ for each "i", one can directly use the result of curve construction as the initial conditions for Monte Carlo simulation. (See the related discussion in [2].) In a single currency market, an example of implementation adopting OIS discounting with a specific choice of the volatility function is available in the recent work of Mercurio [7].

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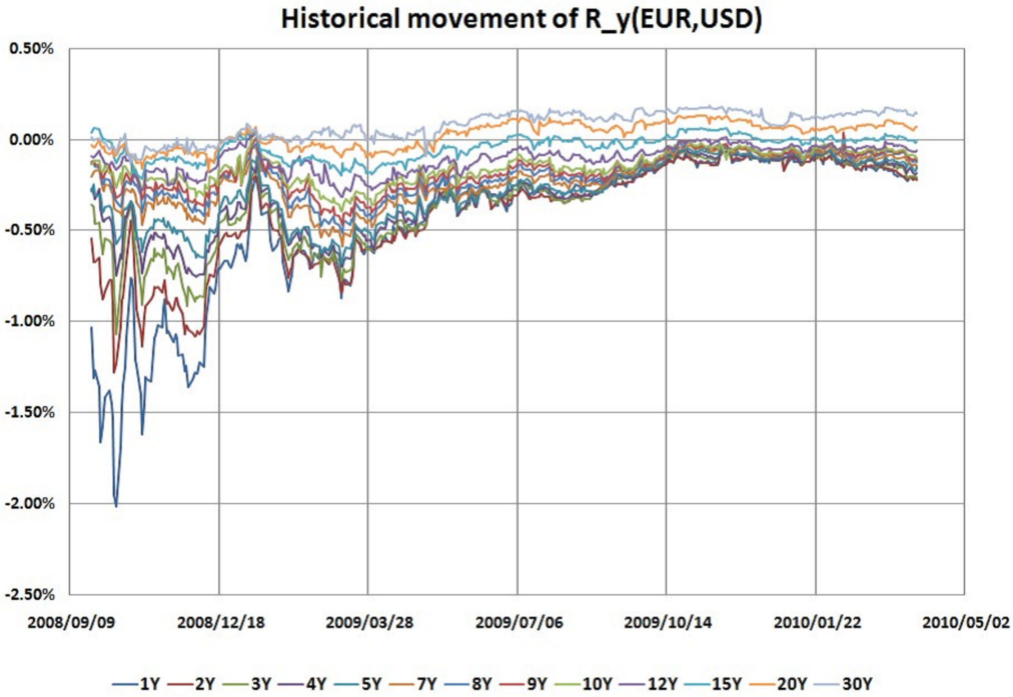

Figure 8: Historical movement of calibrated $R_{y(EUR,USD)}$.

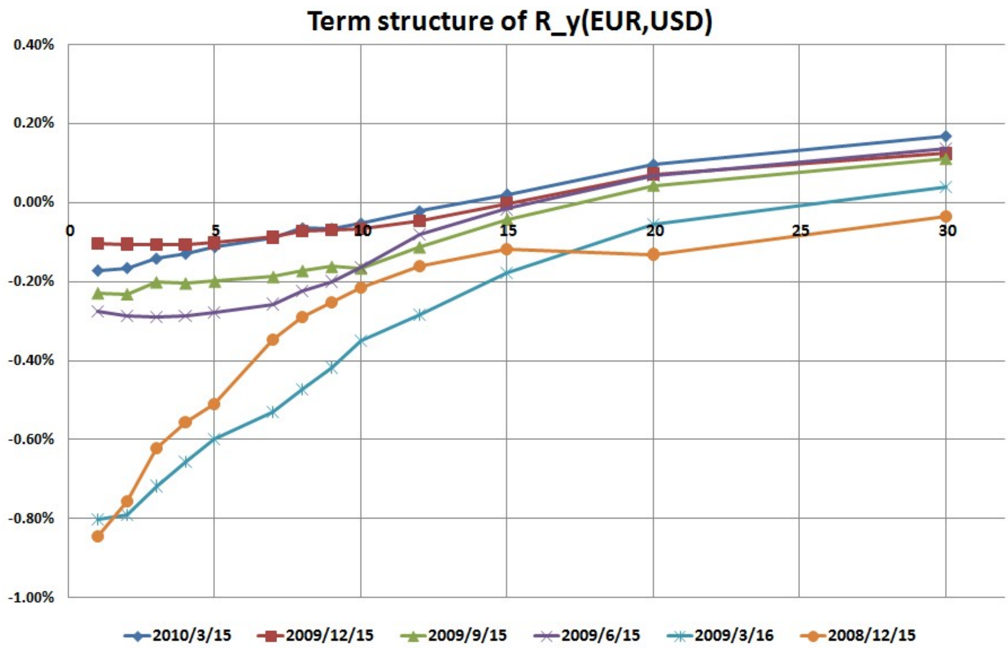

Figure 9: Examples of $R_{y(EUR,USD)}$ term structure.

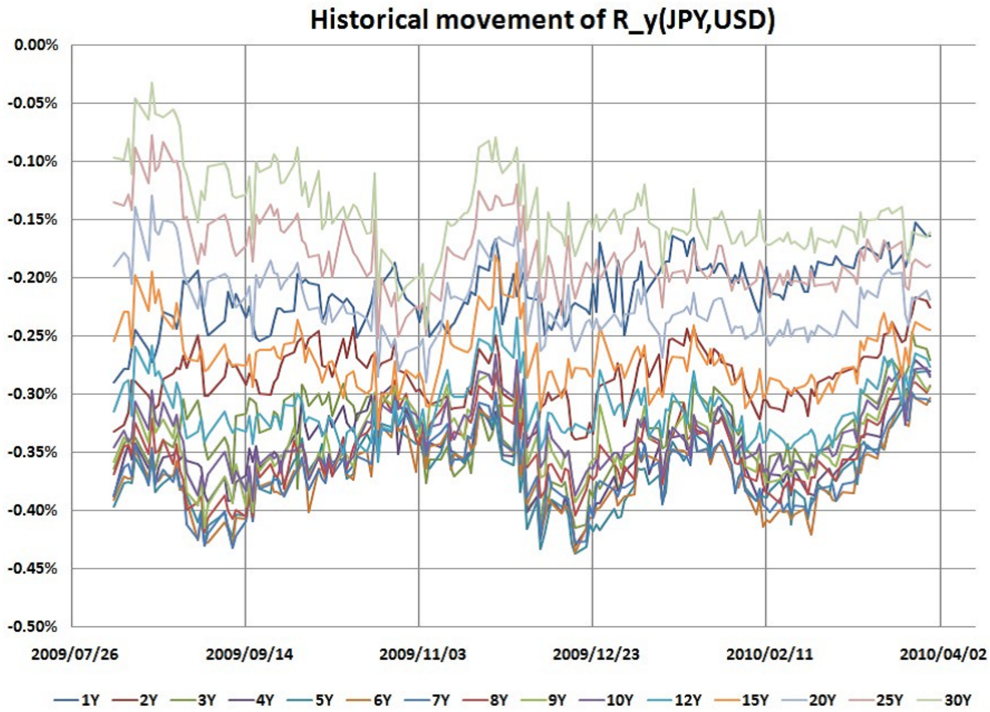

Figure 10: Historical movement of calibrated $R_{y(JPY,USD)}$.

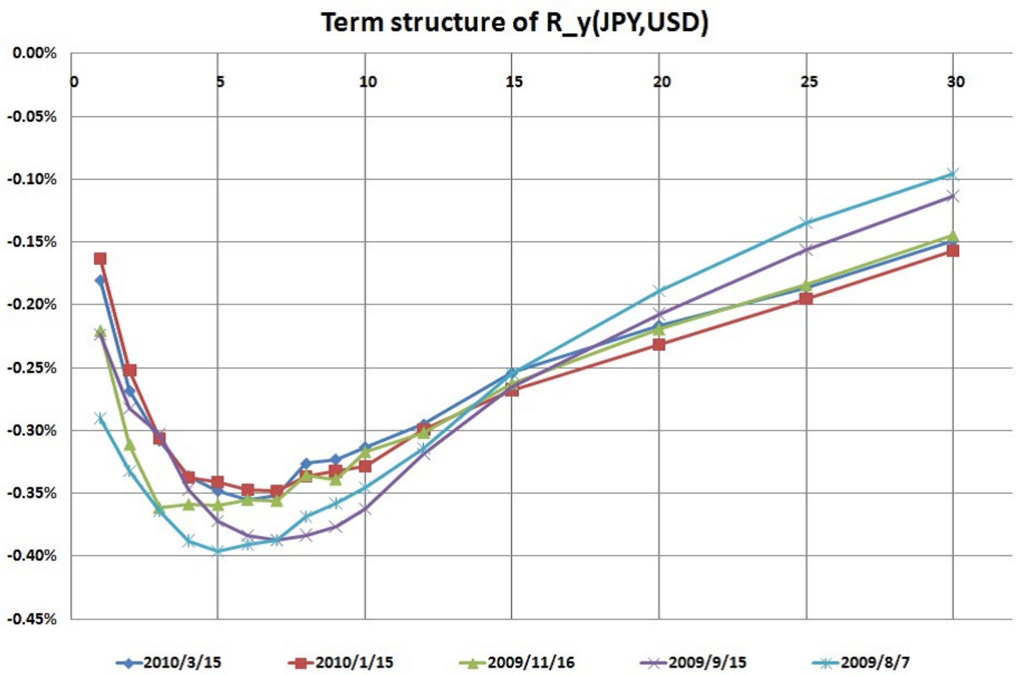

Figure 11: Examples of $R_{y(JPY,USD)}$ term structure.

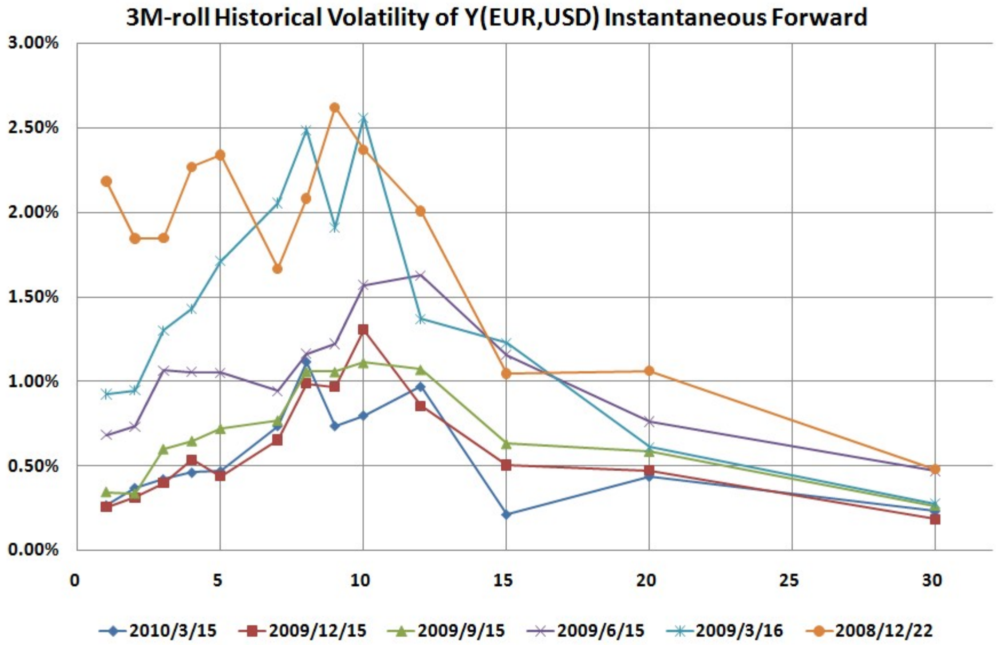

Figure 12: 3M-Roll historical volatility of $y^{(EUR,USD)}$ instantaneous forward. Annualized in absolute terms.

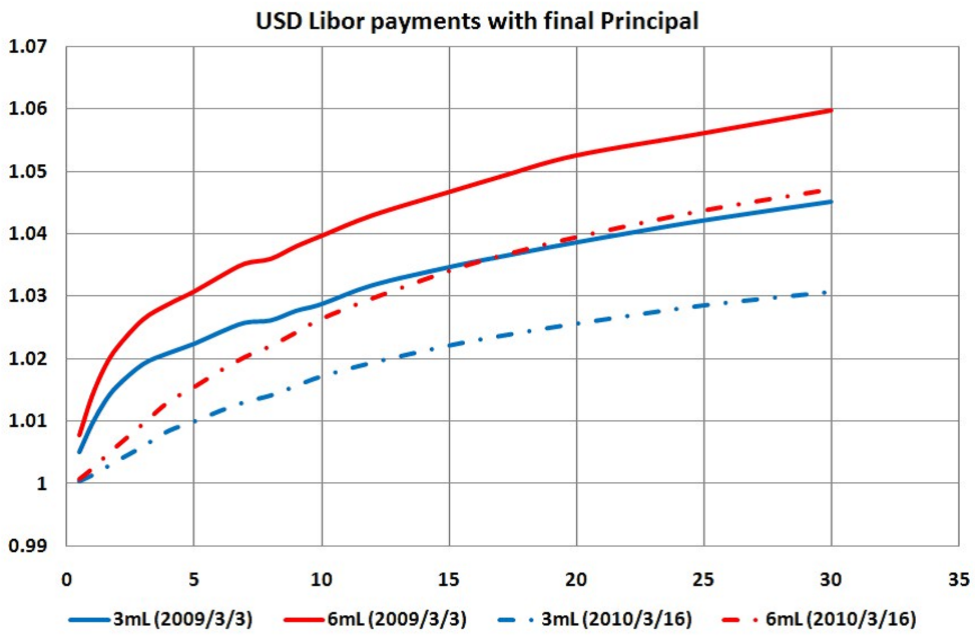

Figure 13: Present value of USD Libor stream with final principal (= 1) payment.

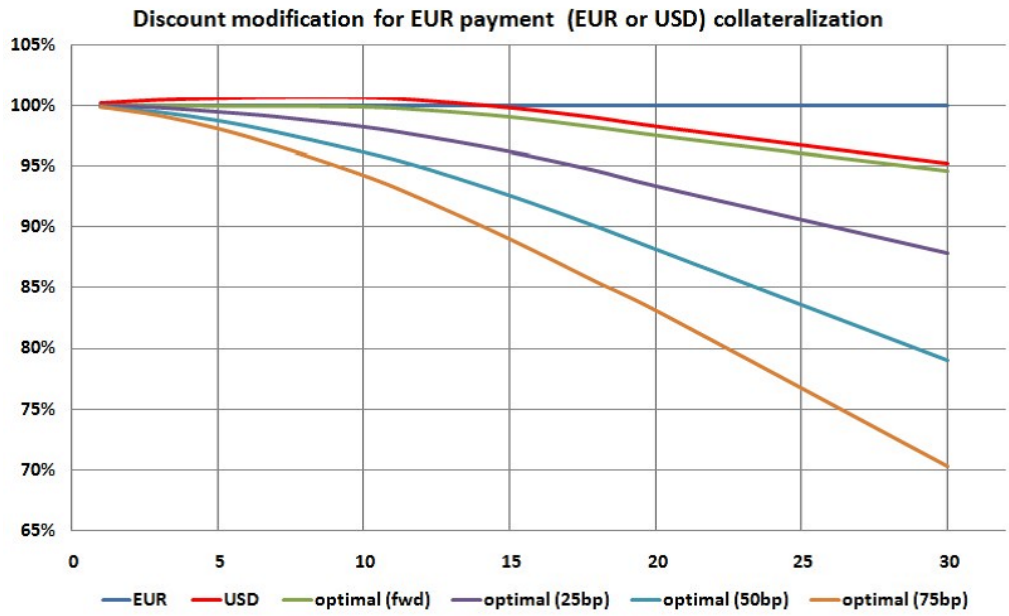

Figure 14: Modification of EUR discounting factors based on HW model for $y^{(EUR,USD)}$ as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.

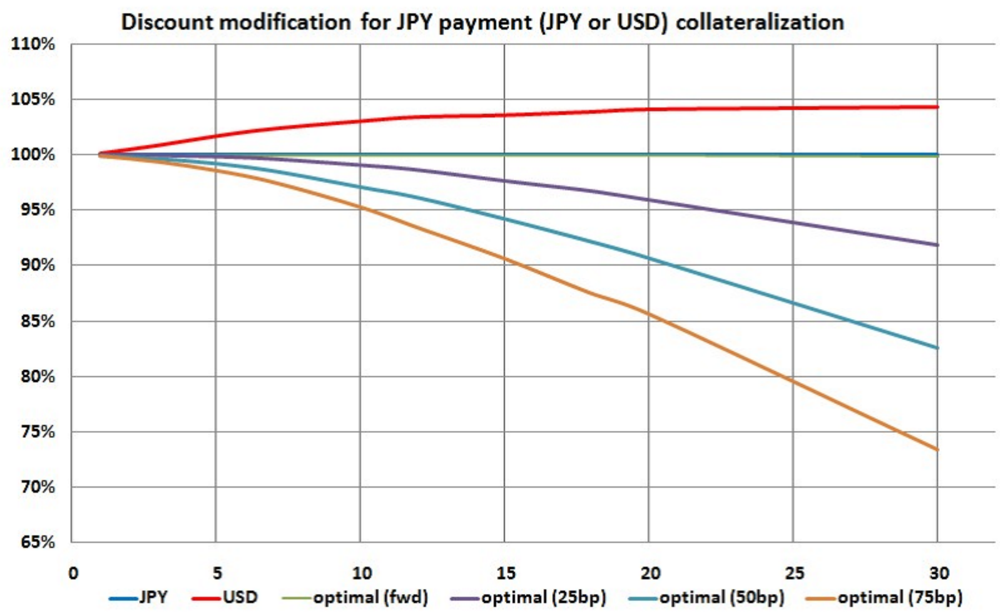

Figure 15: Modification of JPY discounting factors based on HW model for $y^{(JPY,USD)}$ as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.