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Akihiko Takahashi
University of Tokyo
Kohta Takehara
University of Tokyo

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A Hybrid Asymptotic Expansion Scheme:
an Application to Long-term Currency Options *

Akihiko Takahashi and Kohta Takehara [†]

Graduate School of Economics, The University of Tokyo,

Bunkyo-ku, Hongo 7-3-1, Tokyo 113-0033, Japan

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Abstract

This paper develops a general approximation scheme, henceforth called a hybrid asymptotic expansion scheme for valuation of multi-factor European path-independent derivatives. Specifically, we apply it to pricing long-term currency options under a market model of interest rates and a general diffusion stochastic volatility model with jumps of spot exchange rates.

Our scheme is very effective for a type of models in which there exist correlations among all the factors whose dynamics are not necessarily affine nor even Markovian so long as the randomness is generated by Brownian motions. It can also handle models that include jump components under an assumption of their independence of the other random variables when the characteristic functions for the jump parts can be analytically obtained.

An asymptotic expansion approach provides a closed-form approximation formula for their values, which can be calculated in a moment and thus can be used for calibration or for an explicit approximation of Greeks of options. Moreover, this scheme develops Fourier transform method with an asymptotic expansion as well as with closed-form characteristic functions obtainable in parts of a model, extending the method proposed by Takehara and Takahashi[2008] to be applicable to a general class of models.

It also introduces a characteristic-function-based Monte Carlo simulation method with the asymptotic expansion as a control variable in order to make full use of analytical approximations by the asymptotic expansion and of the closed-form characteristic functions.

Finally, a series of numerical examples shows the effectiveness of our scheme.

Keywords: Currency option, libor market model, stochastic volatility, asymptotic expansion, Monte Carlo simulation

1 Introduction

This paper develops a hybrid asymptotic expansion scheme, a general approximation scheme for valuation of multi-factor European path-independent derivatives. As its application, we evaluate long-term currency options under a market model of interest rates and a general diffusion stochastic volatility model with jumps of spot exchange rates. This scheme provides a closed-form approximation formula for option values, which can be calculated in a moment.

Our scheme is effective even for a type of models where there exist correlations among all the factors whose dynamics are not necessarily affine nor even Markovian so long as the source of the uncertainty is generated by Brownian motions. It can also handle models that include jump processes when their characteristic functions are analytically available under an assumption that the jump parts are independent of the other

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[†]E-mail: fin.tk.house@gmail.com. Research Fellow of the Japan Society for the Promotion of Science(JSPS).

random variables. It is well-known that analytical approximations of the option values under those models are very hard to be obtained although it is important in practice. It is stressed that the proposed hybrid asymptotic expansion scheme can be applied to these types of models and hence is very useful for practical purposes such as pricing, hedging and calibration because the analytically-derived formula needs few seconds for calculation and because an effective Monte Carlo scheme with an asymptotic expansion attains further accuracy.

We first note that the limiting distribution of the asymptotic expansion in our scheme differs from a normal distribution as in an asymptotic expansion method previously developed by Yoshida[1992b], Takahashi[1995,1999] and Kunitomo and Takahashi[2001,2003]. In particular, the scheme transforms a target random variable to a martingale with a certain measure change as well as with a transformation of variable. Consequently, an approximation of the drift term in the target stochastic process is not necessary, and the asymptotic expansion for the transformed variable provides more precise approximations of the option prices at the same order of expansions compared with the results by Takahashi and Takehara[2007] that relies on the previous method.

Moreover, our hybrid scheme develops Fourier transform method with an asymptotic expansion as well as with closed-form characteristic functions obtainable in parts of a model. Under an assumption that random variables whose characteristic functions are analytically available are independent of the other random components, we can concentrate on asymptotic expansions for random variables of which characteristic functions are unknown in closed form. For instance, the analytical results such as in Heston[1993] ord Carr and Wu[2005] can be combined with libor market models of interest rates to evaluate European long-term options. This can be considered as an extension of Takahashi and Takehara[2008] to general models; The preceding paper assumed that stochastic interest rates are independent of the spot foreign exchange rate and its stochastic volatility, and that the stochastic volatility model is Heston[1993]'s model or more generally the models in which the characteristic function of the spot exchange rate at the option's maturity is available, such as the affine model. These assumptions sometimes prevent us from stable and reasonable calibration to the markets with an extreme skew as seen in recent JPY-USD market. In this paper we can apply our hybrid scheme to general classes of models without such strong assumptions.

In addition, our scheme also introduces a characteristic-functions-based Monte Carlo simulation and a variance reduction method with the asymptotic expansion as a control variable in order to make full use of both analytical approximations by the asymptotic expansion and closed-form characteristic functions assumed available. The scheme is regarded as an extension of Takahashi and Yoshida[2005] that developed a variance reduction method with an asymptotic expansion. We also note that our method may be used together with other acceleration methods such as *antithetic variables technique* and an extrapolation method of Talay and Tubaro[1990] to pursue further variance reduction of Monte Carlo simulation.

Finally, a series of numerical examples confirms the effectiveness of our scheme. In particular, our method is very effective for complicated but practically important cases in a sense that it can provide closed-form approximation formulas for models where there exist correlations among all the diffusion factors in addition to jump components: For these models, few alternatives for an analytical evaluation, which is required for fast pricing, hedging or calibration of model parameters, are available. Specifically, we test our scheme for models including the one that does not belong to the affine class in terms of pricing and calibration. Moreover, it is also shown that a combination of our characteristic-function-based Monte Carlo simulations with an asymptotic expansion is quite effective to achieve further accuracy; in particular, under the model such as an affine one whose characteristic function is known in closed form, the simulation is dramatically accelerated.

The asymptotic expansion approach related with this paper originated from Yoshida[1992b] and Kunitomo and Takahashi[1992] that applied it to pricing average options. Then, Takahashi[1995,1999] and Kunitomo and Takahashi[2001] presented the second or third order schemes for pricing various types of options in a general Markovian setting and interest rates options in term structure models of Heath, Jarrow and Morton[1992] which is not necessarily Markovian. Moreover, Takahashi and Yoshida[2004] extended the method to dynamic portfolio problems.

For mathematical justification of the method based on Watanabe[1987] in the Malliavin calculus, see Chapter 7 of Malliavin and Thalmaier[2006], Yoshida[1992a], Kunitomo and Takahashi[2003] and Takahashi and Yoshida[2004,2005].

Other applications and extensions of asymptotic expansions to numerical problems in finance are found as follows: Kawai[2003], Kobayashi, Takahashi and Tokioka[2003], Takahashi and Saito[2003], Kunitomo and Takahashi[2004], Muroi[2005], Takahashi[2005], Matsuoka, Takahshi and Uchida[2006], Takahashi and Uchida[2006], Osajima[2006,2007], Kunitomo and Kim[2007], Kusuoka and Osajima[2007] and Takahashi and Takehara[2007,2008].

Many other papers have considered related topics to our concerns: For analytical approximations in option pricing, see for instance, Duffie, Pan and Singleton[1999], Fouque, Papanicolaou and Sircar[1999, 2000], Siopacha and Teichmann[2007], Faulhaber[2002], Henry-Labordere[2005a,b,2006], Piterbarg[2003,2005a,b,2006], Antonov, Misirpashaev and Piterbarg[2007], Davydov and Linetsky[2003], Gorovoi and Linetsky[2004] and Linetsky[2004a,b,c,d].

For variance reduction in Monte Carlo simulations by using *control variate technique* (refer to chapter 3 of Robert and Casella[2000] or section 4.1 of Glasserman[2003] on basics of *control variate technique*), see Kemna and Vorst[1990], Newton[1994], Fournié, Lasry and Touzi[1997], Milshtein and Schoenmakers[2002], for instance. For a general survey of researches on Monte Carlo methods in finance, refer to Glasserman[2003].

The organization of the paper is as follows: After the next section describes a basic structure of our model and a certain transformation of a target random variable, Section 3 derives an approximation formula based on an asymptotic expansion of the target variable. Section 4 introduces a characteristic-function-based Monte Carlo simulation and also proposes a variance reduction method with the asymptotic expansion. Both of Section 3 and 4 present numerical examples. The final section states concluding remarks. Some details omitted in the main body of this paper are explained in Appendices.

2 European Currency Options under a Market Model of Interest Rates and a General Diffusion Stochastic Volatility Model with Jump of Spot Exchange Rates

2.1 The underlying model for European currency options

Let $(\Omega, \mathcal{F}, \tilde{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T^* < \infty})$ be a complete probability space with a filtration satisfying the usual conditions. First we briefly state basics of European currency options. The payoffs of call and put options with maturity $T \in (0, T^*]$ and strike rate $K > 0$ are expressed as $(S(T) - K)^+$ and $(K - S(T))^+$ respectively where $S(t)$ denotes the spot exchange rate at time $t \geq 0$ and x^+ denotes $\max(x, 0)$. In this paper we will concentrate on valuation of a call option since a value of a put option can be obtained through the put-call parity or similar method. We note that the spot exchange rate $S(T)$ can be expressed in terms of a foreign exchange forward(forex forward) rate with maturity T. That is, $S(T) = F_T(T)$ where $F_T(t)$, $t \in [0, T]$ denotes the time t value of the forex forward rate with maturity T. It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t, T)$ and $P_f(t, T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively.

Hence, our objective is to obtain the present value of the payoff $(F_T(T) - K)^+$. In particular, we need to evaluate:

$$V(0; T, K) = P_d(0, T) \times \mathbf{E}^P \left[(F_T(T) - K)^+ \right] \tag{1}$$

where $V(0; T, K)$ denotes the value of an European call option at time 0 with maturity T and strike rate K, and $\mathbf{E}^P[\cdot]$ denotes an expectation operator under EMM(Equivalent Martingale Measure) P with its associated numeraire of the domestic zero coupon bond maturing at T (we use a term of *the domestic terminal measure* in what follows).

Next, with a log-price of the forex forward $f_T(t) := ln(\frac{F_T(t)}{F_T(0)})$, (1) can be rewritten as:

$$V(0; T, K) = P_d(0, T) \times F_T(0) \, \mathbf{E}^P \left[(e^{f_T(T)} - e^k)^+ \right]$$

where $k := ln(\frac{K}{F_T(0)})$ denotes a log-strike rate. Here we note that $e^{f_T(T)} = F_T(T)$ is a martingale under the domestic terminal measure.

The following proposition is well known (e.g. Heston[1993]).

Proposition 1 *Let $\Phi_T^P(u)$ denote a characteristic function of $f_T(T)$ under P. Then, $V(0;T,K)$ is given by:*

$$V(0;T,K) \;=\; P_d(0,T) \times \left[F_T(0) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk}\Phi_T^P(u-i)}{iu}\} du \right\} \right.$$
$$\left. - K \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk}\Phi_T^P(u)}{iu}\} du \right\} \right] \tag{2}$$

where $i := \sqrt{-1}$ and $Re(x)$ denotes a real part of x.

Then, we need to know the characteristic function of $f_T(T)$ under the domestic terminal measure P for pricing the option. For this objective, a market model and a stochastic volatility model with a jump are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})} - 1 \right) \frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})} - 1 \right) \frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$, and $P_d(t,T_j)$ and $P_f(t,T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time $t(\leq T_j)$ respectively; $n(t) = \min\{i : t \leq T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})} - 1 \right) \frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})} - 1 \right) \frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

Under the standard settings of the cross-currency libor market model, we have to consider the following system of stochastic differential equations(henceforth called S.D.E.s) under the domestic terminal measure P to price options. Detailed arguments on the framework of these S.D.E.s are given in A.

As for the domestic and foreign interest rates we assume forward market models; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}(t) \;=\; f_{dj}(0) + \int_0^t \left\{ -f_{dj}(s)\tilde{\gamma}_{dj}'(s) \sum_{i=j+1}^N \frac{\tau_i f_{di}(s)\tilde{\gamma}_{di}(s)}{1+\tau_i f_{di}(s)} \right\} ds + \int_0^t f_{dj}(s)\tilde{\gamma}_{dj}'(s) dW_s \tag{3}$$

$$f_{fj}(t) \;=\; f_{fj}(0) + \int_0^t f_{fj}(s)\tilde{\gamma}_{fj}'(s) \left\{ -\sum_{i\in\hat{J}_{j+1}} \frac{-\tau_i f_{fi}(s)\tilde{\gamma}_{fi}(s)}{1+\tau_i f_{fi}(s)} + \sum_{i\in\hat{J}_{N+1}} \frac{-\tau_i f_{di}(s)\tilde{\gamma}_{di}(s)}{1+\tau_i f_{di}(s)} \right\} ds$$
$$- \int_0^t f_{fj}(s)\tilde{\gamma}_{fj}'(s)\tilde{\sigma}(s)\bar{\sigma} ds + \int_0^t f_{fj}(s)\tilde{\gamma}_{fj}'(s) dW_s \tag{4}$$

where x' denotes the transpose of x, $\hat{J}_{j+1} := \{0, 1, \cdots, j\}$, and W is a D dimensional Brownian motion under the domestic terminal measure; $\tilde{\gamma}_{dj}(s)$, $\tilde{\gamma}_{fj}(s)$ are D dimensional vector-valued functions of time-parameter s; $\bar{\sigma}$ denotes a D dimensional constant vector satisfying $||\bar{\sigma}|| = 1$ and $\tilde{\sigma}(t)$, the volatility of the spot exchange rate, is specified to follow a $\mathbf{R}_{++}$-valued general time-inhomogeneous Markovian process as follows:

$$\tilde{\sigma}(t) = \tilde{\sigma}(0) + \int_0^t \mu(\{f_{dj}(s)\}, \tilde{\sigma}(s), s) ds + \int_0^t \tilde{\omega}'(\tilde{\sigma}(s), s) dW_s \tag{5}$$

where $\mu(\cdots, s)$ is concretely defined in A and $\tilde{\omega}(x, s)$ is a function of x and s. We stress that our scheme introduced in this paper can be applied not only in this setting but also in the model where all of interest rates and the spot exchange rate have (not necessarily Markovian)stochastic volatilities as long as they are driven by Brownian motions.

Finally, we consider the process of the forex forward $F_{N+1}(t)$. Since $F_{N+1}(t) \equiv F_{T_{N+1}}(t)$ can be expressed as $F_{N+1}(t) = S(t)\frac{P_f(t,T_{N+1})}{P_d(t,T_{N+1})}$, it is a martingale under the domestic terminal measure as we mentioned. Then,

we assume that the log-price $f_{N+1}(t) = ln(\frac{F_{N+1}(t)}{F_{N+1}(0)})$ follows

$$f_{N+1}(t) \quad = \quad ln(\frac{F_{N+1}(t)}{F_{N+1}(0)}) = Z(t) + A(t) \tag{6}$$

where $Z(t)$ is an exponential-martingale continuous process given by

$$Z(t) \quad = \quad -\frac{1}{2}\int_0^t \|\tilde{\sigma}_Z(s)\|^2 \, ds + \int_0^t \tilde{\sigma}_Z^{'}(s) dW_s \tag{7}$$

where

$$\tilde{\sigma}_Z(t) \quad := \quad \sum_{j \in \hat{J}_{N+1}} \left(\frac{-\tau_j f_{fj}(t)\tilde{\gamma}_{fj}(t)}{1 + \tau_j f_{fj}(t)} - \frac{-\tau_j f_{dj}(t)\tilde{\gamma}_{dj}(t)}{1 + \tau_i f_{dj}(t)} \right) + \tilde{\sigma}(t)\bar{\sigma}$$

and $A(t)$ denotes a continuous or jump process that is an exponential-martingale independent from $Z(t)$. Further, we assume that the characteristic function of $A(t)$ is known in closed-form. e.g. $A(t)$ is a compound Poisson process, a variance gamma process, an inverse Gaussian process, a CGMY model or the Stochastic Skew Model(Carr and Wu[2005]).

We here emphasize the generality of our framework investigated in this paper. For the stochastic volatility, a general time-inhomogeneous Markovian process is assumed, which is not necessarily classified in the affine model such as in Heston[1993]: In addition, we can incorporate a jump process in our model. These settings are flexible enough to capture the complexity of movements of the underlying asset and to calibrate our model to the market with ease even in the severely skewed environment as in a recent JPY-USD market.

2.2 A transformation of the underlying stochastic differential equations

Let $\Phi_{N+1}^P(t, u)$ denote the characteristic function of $f_{N+1}(t)$ under P. Then, $\Phi_{N+1}^P(t, u)$ can be decomposed as;

$$\Phi_{N+1}^P(t, u) = \Phi_Z^P(t, u)\Phi_A^P(t, u) \tag{8}$$

where $\Phi_Z^P(t, u)$ and $\Phi_A^P(t, u)$ denote the characteristic functions of $Z(t)$ and $A(t)$ under P, respectively.

For evaluation of European currency options, an explicit expression of $\Phi_{N+1}^P(T_{N+1}, u)$ is necessary. However, the process $Z(t)$ is too complicated to obtain the analytical expression of $\Phi_Z^P(T_{N+1}, u)$ (see Section 6.3.2 in Brigo and Mercurio[2006] or Section 25.5 in Björk[2004]) while that of $\Phi_A^P(T_{N+1}, u)$ is assumed to be known. Then, later we will suggest to utilize the asymptotic expansion for the approximation of $\Phi_Z^P(T_{N+1}, u)$.

In (7), $Z(t)$, the key process for evaluation of options, has a nonzero drift. Thus, unless we provide the approximation which has not any error in the drift term, even the first moment(i.e. the expectation value) of that approximation will not match the target's. Contrarily, if we can eliminate its drift term by some means, that is the objective process will be a martingale, its first moment can be much easily kept by using a martingale process as an approximation. In this light, here we consider a certain change of measures so that the main objective process of our expansion will be martingale.

For a fixed u(an argument of $\Phi_Z^P(T_{N+1}, u)$) we define a new probability measure Q_u on $(\Omega, \mathcal{F}_{T_{N+1}})$ with a Radon-Nikodym derivative of

$$\frac{dQ_u}{dP} = \exp\left(-\frac{1}{2}\int_0^{T_{N+1}} \|\lambda_u(s)\|^2 \, ds - \int_0^{T_{N+1}} \lambda_u^{'}(s) dW_s \right) \tag{9}$$

where

$$\lambda_u(t) := \left((-iu) + i\sqrt{u^2 + iu} \right) \tilde{\sigma}_Z(t) = \tilde{h}(u)\tilde{\sigma}_Z(t)$$

and $\tilde{h}(u) := (-iu) + i\sqrt{u^2 + iu}$.

Then $\Phi_Z^P(T_{N+1}, u)$, the characteristic function of $Z(T_{N+1})$ under the measure P, is expressed as that of another random variable $\hat{Z}(T_{N+1})$ under Q_u with a transformation of variable $h(\cdot)$:

$$
\begin{aligned}
\Phi_Z^P(T_{N+1}, u) &= \mathbf{E}^P\left[\exp\left(iuZ(T_{N+1})\right)\right] \\
&= \mathbf{E}^{Q_u}\left[\exp\left(ih(u)\int_0^{T_{N+1}} \tilde{\sigma}_Z'(s)dW_s^{Q_u}\right)\right] \\
&=: \Phi_{\hat{Z}}^{Q_u}(T_{N+1}, h(u))
\end{aligned}
\tag{10}
$$

where $\mathbf{E}^{Q_u}[\cdot]$ is an expectation operator under Q_u; $W_t^{Q_u} := W_t + \int_0^t \lambda_u'(s)ds$ is now a Brownian motion under that measure; $\Phi_{\hat{Z}}^{Q_u}(t, v)$ denotes the characteristic function of $\hat{Z}(t) := \int_0^t \tilde{\sigma}_Z'(s)dW_s^{Q_u}$ and $h(u) := \sqrt{u^2 + iu}$.

Now, we have the martingale objective process for the approximation. Then, in the following section, we will apply the asymptotic expansion method to the underlying system of S.D.E.s under Q_u.

3 An Approximation Scheme based on the Asymptotic Expansion Approach

3.1 The asymptotic expansion approach

The asymptotic expansion approach describes the processes of forward rates as $f_{dj}^{(\epsilon)}(t)$ and $f_{fj}^{(\epsilon)}(t)$, and of the volatility of the spot forex as $\sigma^{(\epsilon)}(t)$, which explicitly depend upon a parameter $\epsilon \in (0, 1]$, and expands the processes around $\epsilon = 0$, that is asymptotic expansions are made around deterministic processes.

First, to fit the framework of the asymptotic expansion, the processes of $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$ and $\sigma^{(\epsilon)}(t)$ in (3), (4) and (5) are redefined under the measure Q_u with a parameter ϵ as follows; for $j = n(t) - 1, n(t), n(t) + 1, \cdots, N$,

$$
\begin{aligned}
f_{dj}^{(\epsilon)}(t) &= f_{dj}(0) + \epsilon^2 \int_0^t \left\{ -f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s) \sum_{i=j+1}^N \frac{\tau_i f_{di}^{(\epsilon)}(s)\gamma_{di}(s)}{1 + \tau_i f_{di}^{(\epsilon)}(s)} \right\} ds \\
&\quad - \epsilon^2 \tilde{h}(u) \int_0^t f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s)\sigma_Z^{(\epsilon)}(s)ds + \epsilon \int_0^t f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s)dW_s^{Q_u}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
f_{fj}^{(\epsilon)}(t) &= f_{fj}^{(\epsilon)}(0) + \epsilon^2 \int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s) \left\{ \sum_{i \in \hat{J}_{j+1}} \frac{\tau_i f_{fi}^{(\epsilon)}(s)\gamma_{fi}(s)}{1 + \tau_i f_{fi}^{(\epsilon)}(s)} - \sum_{i \in \hat{J}_{N+1}} \frac{\tau_i f_{di}^{(\epsilon)}(s)\gamma_{di}(s)}{1 + \tau_i f_{di}^{(\epsilon)}(s)} \right\} ds \\
&\quad - \epsilon^2 \int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s)\sigma^{(\epsilon)}(s)\bar{\sigma}ds - \epsilon^2 \tilde{h}(u) \int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s)\sigma_Z^{(\epsilon)}(s)ds \\
&\quad + \epsilon \int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s)dW_s^{Q_u}
\end{aligned}
\tag{12}
$$

and

$$
\sigma^{(\epsilon)}(t) = \sigma(0) + \int_0^t \mu^{(\epsilon)}(s)ds - \epsilon^2 \tilde{h}(u)\int_0^t \omega'(\sigma^{(\epsilon)}(s), s)\sigma_Z^{(\epsilon)}(s)ds + \epsilon \int_0^t \omega'(\sigma^{(\epsilon)}(s), s)dW_s^{Q_u}.
\tag{13}
$$

Note that they are redefined with replacement of $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\tilde{\sigma}(t)$ and $\tilde{\omega}(x, t)$ in the previous section by $\epsilon\gamma_{dj}(t)$, $\epsilon\gamma_{fj}(t)$, $\epsilon\sigma^{(\epsilon)}(t)$ and $\epsilon\omega(x, t)$ respectively, and with applying the change of measure.

Then $\hat{Z}^{(\epsilon)}(t)$, the analogy of $\hat{Z}(t)$, is given by

$$
\hat{Z}^{(\epsilon)}(t) = \epsilon \int_0^t \sigma_Z^{(\epsilon)'}(s)dW_s^{Q_u}
\tag{14}
$$

where

$$
\sigma_Z^{(\epsilon)}(t) := \sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(t)\gamma_{fj}(t)}{1 + \tau_j f_{fj}^{(\epsilon)}(t)} - \sum_{j \in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(t)\gamma_{dj}(t)}{1 + \tau_i f_{dj}^{(\epsilon)}(t)} + \sigma^{(\epsilon)}(t)\bar{\sigma}.
$$

Of course, in this framework, variables and functions such as $F_{N+1}^{(\epsilon)}(t)$, $f_{N+1}^{(\epsilon)}(t)$, $\Phi_{N+1}^{P,(\epsilon)}(t,u)$, $\Phi_Z^{P,(\epsilon)}(t,u)$ or $\Phi_{\hat{Z}}^{Q_u,(\epsilon)}(t,v)$ explicitly depend on ϵ.

Next, we expand the processes of forward interest rates and of the volatility of the spot forex up to the second order of $\epsilon(\epsilon^2$-order) around $\epsilon = 0$ to obtain the third order asymptotic expansion of $\hat{Z}^{(\epsilon)}(t)$. These expansions can be obtained by differentiating the right hand side of the equations (11), (12), (13) and (14) with respect to ϵ and setting $\epsilon = 0$. Here only the results are stated as the following proposition. Justification of these and the consequent results is mainly found in Kunitomo and Takahashi[2003](see also our Appendix).

Proposition 2 *The asymptotic expansion of $\hat{Z}^{(\epsilon)}(t)$ up to the third order is expressed as follows:*

$$\hat{Z}^{(\epsilon)}(t) = \epsilon \hat{G}_t^{Q_u,(1)} + \epsilon^2 \hat{G}_t^{Q_u,(2)} + \epsilon^3 \hat{G}_t^{Q_u,(3)} + o(\epsilon^3) \tag{15}$$

where $\hat{G}_t^{Q_u,(1)} := \int_0^t \sigma_Z^{(0)'}(s)dW_s^{Q_u}$ and each of $\hat{G}_t^{Q_u,(2)}$ and $\hat{G}_t^{Q_u,(3)}$ is expressed as a sum of (iterated) Itô integrals which is given in B concretely.

Remark 1 *Since (iterated) Itô integrals always have zero means, the martingale property of $\hat{Z}^{(\epsilon)}(t)$ is kept at any order of this expansion.*

3.2 A pricing formula

Next, we define a random variable $X^{(\epsilon)} := \frac{\hat{Z}^{(\epsilon)}(T)}{\epsilon}$ with $T \equiv T_{N+1}$. First, we note that $\Phi^{P,(\epsilon)}(u) := \Phi_{N+1}^{P,(\epsilon)}(T,u)$ which is necessary for pricing options is now expressed as;

$$\begin{aligned} \Phi^{P,(\epsilon)}(u) &= \Phi_Z^{P,(\epsilon)}(u)\Phi_A^P(u) \\ &= \Phi_{\hat{Z}}^{Q_u,(\epsilon)}(h(u))\Phi_A^P(u) = \Phi_X^{Q_u,(\epsilon)}(\epsilon h(u))\Phi_A^P(u) \end{aligned} \tag{16}$$

where $\Phi_{\hat{Z}}^{Q_u,(\epsilon)}(v) := \Phi_{\hat{Z}}^{Q_u,(\epsilon)}(T,v)$, which is an analogy of $\Phi_{\hat{Z}}^{Q_u}(v)$; $\Phi_A^P(u) := \Phi_A^P(T,u)$; $\Phi_X^{Q_u,(\epsilon)}(v)$ denotes the characteristic function of $X^{(\epsilon)}$ under Q_u.

Second, we also note that $X^{(\epsilon)}$ is expanded up to the second order as follows:

$$X^{(\epsilon)} = g_1 + \epsilon g_2 + \epsilon^2 g_3 + o(\epsilon^2), \tag{17}$$

where $g_1 := \hat{G}_T^{Q_u,(1)}$, $g_2 := \hat{G}_T^{Q_u,(2)}$ and $g_3 := \hat{G}_T^{Q_u,(3)}$.

Finally, note that the first order term g_1 follows a normal distribution with mean 0 and variance Σ:

$$\Sigma := \int_0^T ||\sigma_Z^{(0)}(s)||^2 ds. \tag{18}$$

Using the following theorem, we will obtain an approximation of $\Phi^{P,(\epsilon)}(u)$, the characteristic function of the terminal log-price $f_{N+1}^{(\epsilon)}(T)$.

Theorem 1 *With the assumption of $\Sigma > 0$, an asymptotic expansion of $\Phi_X^{Q_u,(\epsilon)}(v)$, the characteristic function of $X^{(\epsilon)}$ under Q_u, is given by*

$$\Phi_X^{Q_u,(\epsilon)}(v) = \left[1 + D_2^{Q_u,(\epsilon)}(iv)^2 + D_3^{Q_u,(\epsilon)}(iv)^3 + D_4^{Q_u,(\epsilon)}(iv)^4 + D_5^{Q_u,(\epsilon)}(iv)^5 + D_6^{Q_u,(\epsilon)}(iv)^6\right]\Phi_{0,\Sigma}(v) + o(\epsilon^2) \tag{19}$$

where $\Phi_{\mu,\Sigma}(v) := e^{i\mu v - \frac{\Sigma}{2}v^2}$.

$D_2^{Q_u,(\epsilon)}, D_3^{Q_u,(\epsilon)}, D_4^{Q_u,(\epsilon)}, D_5^{Q_u,(\epsilon)}$ and $D_6^{Q_u,(\epsilon)}$ are constants for pre-specified ϵ and u, whose derivations are explained in C. Each subscript corresponds to the order of (iv) in the equation (19).

Finally, we provide an approximation formula for valuation of European call options written on $F_{N+1}^{(\epsilon)}(T)$ by direct application of Theorem 1 to Proposition 1.

Theorem 2 *Assume $\Sigma > 0$. Let $\hat{V}(0; T_{N+1}, K)$ be an approximated value of $V(0; T_{N+1}, K)$ which denotes the exact value of the option with maturity T_{N+1} and strike rate K. Then, $\hat{V}(0; T_{N+1}, K)$ is given by:*

$$
\hat{V}(0; T_{N+1}, K) \quad := \quad P_d(0, T_{N+1}) \left[F_{N+1}(0) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk} \hat{\Phi}^{(\epsilon)}(u-i)}{iu} \} du \right\} \right.
$$
$$
\left. - K \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk} \hat{\Phi}^{(\epsilon)}(u)}{iu} \} du \right\} \right].
$$

(20)

Here, $k := ln(\frac{K}{F_{N+1}(0)})$, $\hat{\Phi}^{(\epsilon)}(u) := \hat{\Phi}_X^{Q_u,(\epsilon)}(\epsilon h(u)) \times \Phi_A^P(u)$, and $\hat{\Phi}_X^{Q_u,(\epsilon)}(v)$ is defined as;

$$
\hat{\Phi}_X^{Q_u,(\epsilon)}(v) = \left[1 + D_2^{Q_u,(\epsilon)}(iv)^2 + D_3^{Q_u,(\epsilon)}(iv)^3 + D_4^{Q_u,(\epsilon)}(iv)^4 + D_5^{Q_u,(\epsilon)}(iv)^5 + D_6^{Q_u,(\epsilon)}(iv)^6 \right] \times \Phi_{0,\Sigma}(v).
$$

Remark 2 *Note that since $h(-i) = 0$ and A is assumed to be an exponential martingale, $\mathbf{E}^P[e^{f_{N+1}^{(\epsilon)}(T)}] = \Phi^{P,(\epsilon)}(u)$ is approximated by $\hat{\Phi}^{(\epsilon)}(-i) = \hat{\Phi}_X^{Q_{-i},(\epsilon)}(\epsilon h(-i)) \times \Phi_A^P(-i) = 1$, which means that in our approximation the exponential-martingale property of $f_{N+1}^{(\epsilon)}$ is kept.*

Especially, when $A \equiv 0$ the first-order approximation of the option price coincides $BS(\Sigma^{\frac{1}{2}}; F_{N+1}(0), K, T_{N+1})$ which is the Black-Scholes price under the case where the stochastic interest rates and the stochastic volatility would be replaced by (their limiting-)deterministic processes:

$$
BS(\sigma; F, K, T) := P_d(0, T) \left[FN(d_+) - KN(d_-) \right]
$$

(21)

where

$$
d_\pm := \frac{ln(F/K) \pm \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}, \quad N(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.
$$

Remark 3 *Using this approximation formula, we can also provide analytical approximations of Greeks of the option, sensitivities of the option price to the factors. Note that our approximation for the underlying characteristic function does not depend upon the initial value of the spot forex. Thus in particular, Δ and Γ, the first and second derivatives of the option value with respect to $S(0)$ respectively, can be explicitly approximated with ease. For example $\hat{\Delta}$, the approximation of Δ, is given by*

$$
\hat{\Delta} \quad := \quad \frac{\partial \hat{V}(0; T_{N+1}, K)}{\partial S(0)}
$$
$$
= \quad \frac{P_d(0, T_{N+1})}{S(0)} \left[F_{N+1}(0) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk} \hat{\Phi}^{(\epsilon)}(u-i)}{iu} \} du \right\} \right.
$$
$$
\left. + F_{N+1}(0) \left\{ \frac{1}{\pi} \int_0^\infty Re\left(e^{-iuk} \hat{\Phi}^{(\epsilon)}(u-i) \right) du \right\} - K \left\{ \frac{1}{\pi} \int_0^\infty Re\left(e^{-iuk} \hat{\Phi}^{(\epsilon)}(u) \right) du \right\} \right].
$$

For other risk parameters such as Θ or Vega, sensitivities of the option price with respect to t and $\sigma(0)$ respectively, their approximations are given in the easy way with our formula such as the difference quotient method, which needs few seconds for calculation and has satisfactory accuracies.

Remark 4 *In practice, we are often interested in the accuracy of our formulas for the option prices whose underlying variables follow the SDEs (3), (4), (5) and (7) with a particular set of parameters such as $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\tilde{\sigma}(0)$ and $\tilde{\omega}(x, t)$. From this point of view, given some particular value of ϵ, $\gamma_{dj}(t)$, $\gamma_{fj}(t)$, $\sigma(0)$ and $\omega(x, t)$ in (11), (12), (13) and (14) should be scaled so that $\epsilon\gamma_{dj}(t) = \tilde{\gamma}_{dj}(t)$, $\epsilon\gamma_{fj}(t) = \tilde{\gamma}_{fj}(t)$, $\epsilon\sigma(0) = \tilde{\sigma}(0)$*

and $\epsilon\omega(x,t) = \tilde{\omega}(x,t)$ *for an arbitrary* $t \in [0,T]$. *For instance,* $\gamma(t)$ *is scaled to be* $\gamma(t) := \frac{\tilde{\gamma}(t)}{\epsilon}$ *where* ϵ *is fixed at a pre-specified constant through our procedure of expansions.*

Moreover, it can be shown that the approximated prices are unchanged whatever $\epsilon \in (0,1]$ *is taken in evaluation, as long as above conditions are met. We will see this briefly in D.*

3.3 Numerical examples(1)

In this subsection, we examine the effectiveness of our method through a series of numerical examples under the model of the spot forex not classified in the affine one. First, the approximate option prices by our closed-form formula are compared with their estimates by Monte Carlo simulations. Second, the formula is also utilized for calibration of the model parameters to the recent JPY-USD market with extreme skews.

3.3.1 Model specification

First of all, the processes of the domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose $D = 4$, that is the dimension of a Brownian motion is set to be four; it represents the uncertainty in the domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in our framework correlations among all factors are allowed.

Next, we specify a volatility process, not a variance process as in affine-type models, of the spot exchange rate under the domestic risk-neutral measure as follows;

$$\sigma^{(\epsilon)}(t) = \sigma(0) + \kappa \int_0^t (\theta - \sigma^{(\epsilon)}(s))ds + \epsilon\omega' \int_0^t \sqrt{\sigma^{(\epsilon)}(s)}d\hat{W}_s \tag{22}$$

where θ and κ represent the level and speed of its mean-reversion respectively, and ω denotes a volatility vector on the volatility. In this section the parameters are set as follows; $\epsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* \bar{v}$ where $\omega^* = 0.1$ and $\bar{v}$ denotes a four dimensional constant vector given below.

We further suppose that initial term structures of the domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all j, $f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) = \gamma_d^* \bar{\gamma}_d 1_{\{t<T_j\}}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \bar{\gamma}_f 1_{\{t<T_j\}}(t)$. Here, γ_d^* and γ_f^* are constant scalars, and $\bar{\gamma}_d$ and $\bar{\gamma}_f$ denote four dimensional constant vectors. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors $\bar{\gamma}_d$, $\bar{\gamma}_f$, $\bar{\sigma}$ and $\bar{v}$ can be determined to satisfy $||\bar{\gamma}_d|| = ||\bar{\gamma}_f|| = ||\bar{\sigma}|| = ||\bar{v}|| = 1$ and $V'V = \underline{C}$ where $V := (\bar{\gamma}_d, \bar{\gamma}_f, \bar{\sigma}, \bar{v})$.

In the following subsection, we consider three different cases for f_d, γ_d^*, f_f and γ_f^* as in Table 1. For correlations, four sets of parameters are considered: In the case "Corr.1", all the factors are independent: In "Corr.2", there exists only the correlation of -0.5 between the spot exchange rate and its volatility (i.e. $\bar{\sigma}'\bar{v} = -0.5$) while there are no correlations among the others: In "Corr.3", the correlation between the interest rates and the spot exchange rate are allowed while there are no correlations among the others: the correlation between the domestic ones and the spot forex is $0.5(\bar{\gamma}_d'\bar{\sigma} = 0.5)$ and the correlation between the foreign ones and the spot forex is $-0.5(\bar{\gamma}_f'\bar{\sigma} = -0.5)$. In these three cases, $A(t) \equiv 0$ for simplicity, that is there is assumed no component such as a jump whose characteristic function is available in closed form.

Finally in "Corr.4", correlations among most factors are considered; $\bar{\gamma}_d'\bar{\gamma}_f = 0.3$ between the domestic and foreign interest rates; $\bar{\gamma}_d'\bar{\sigma} = 0.5$, $\bar{\gamma}_f'\bar{\sigma} = -0.5$ between the interest rates and the spot forex; and $\bar{\sigma}'\bar{v} = -0.5$ between the spot forex and its volatility. In this case also $A(t)$, a jump component, will be taken into account: $A(t)$ is assumed to be a compound Poisson process with its intensity λ and with random jumps following $N(m, s^2)$; $\lambda = 1$, $m = -0.05$ and $s = 0.05$. In this case, the characteristic function of $A(t)$ is given by

$$\Phi_A^P(t,u) = \exp\left(\lambda t \left(e^{imu - \frac{1}{2}s^2 u^2} - 1\right) - iu\lambda t \left(e^{m + \frac{1}{2}s^2} - 1\right)\right).$$

It is well known that (both of exact and approximate)evaluation of the long-term options is a hard task in the case with a complex structure of correlations and/or with a jump component, such as "Corr.3" or "Corr.4". Because the method with an asymptotic expansion introduced in Takahashi and Takehara[2007](TT[2007])

Table 1: Initial domestic/foreign forward interest rates and their volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.2	0.05	0.2
case (ii)	0.02	0.5	0.05	0.2
case (iii)	0.05	0.2	0.02	0.5

can deal with that evaluation in all these cases but for "Corr.4", their estimator will be compared to our estimator in the following examinations.

Lastly, we make another assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $t \in [t, T_{n(t)}]$.

3.3.2 Examinations of our closed-form approximation formula

In this subsection the accuracy of our pricing formula is examined. We show numerical examples for evaluation of call options calculated by Monte Carlo (henceforth called M.C.) simulations and by our approximation formula of the third order with maturities of five or ten years and with different parameters for the interest rates and correlations set in the previous subsection.

Each estimator based on the M.C. simulations is obtained by 1,000,000 trials with *antithetic variables method*. The estimators by the pricing formula via the third-order asymptotic expansion given in TT[2007] are added for comparison, except for the case "Corr.4" where the approximated prices cannot be directly given by their method.

We set $K \sim F_{N+1}(0) \pm \sqrt{\Sigma}$ for OTM and ITM where Σ was defined in (18), and then moneyness for those strikes, $\frac{K}{F_{N+1}(0)}$ are approximately given by $1 \pm \frac{\sqrt{\Sigma}}{F_{N+1}(0)}$ respectively. Thus, in the longer maturity the prices of deeper OTM/ITM options are given in the figures, and also deeper OTM and ITM options are considered for the cases "Corr.3" and "Corr.4" because the underlying forex forward is more volatile in those cases than in the other two cases.

Figures 1.-4. show results in our numerical investigations. Figure 1. reports differences of the estimators by formulas in a five-year maturity and Figure 3. does in a ten-year maturity. In Figure 2. and Figure 4., they are shown in terms of implied volatilities respectively. The differences shown in those figures between the approximations by the formula given in this paper or in TT[2007] and those by M.C. simulations are defined as (the approximate value by asymptotic expansions)-(the estimate by M.C. simulations).

Absolute levels of the differences of our approximation are on average 0.025/0.06% in prices/in implied volatilities for a five-year maturity and 0.100/0.18% for a ten-year maturity. Most of the differences in our approximations are less than 0.1/0.2% for five years and 0.25/0.4% for ten years. Moreover, in "Corr.4" in which most of existing methods for analytical evaluation including TT[2007] are difficult to be applied, our formula also works well. This stability of performances of our method, even in the complicated settings with many correlated processes and/or with a jump process in addition, can be advantageous in practice.

3.3.3 Calibration to the market

In this subsection, our model parameters are calibrated with the third order asymptotic expansion formula to observed volatilities with maturities of five and ten years in the JPY/USD currency option market. Market makers in OTC currency option markets usually provide quotes on Black-Scholes implied volatilities and moneyness of an option is expressed in terms of Black-Scholes delta, rather than its strike price. We use the data of volatility surfaces on Sep 27, Oct 30 and Dec 07, 2007, after the beginning of the *subprime-loan* crash, which consist of 25 delta put, 10 delta put, at-the-money, 10 delta call, and 25 delta call with their maturities of seven and ten years (These data are provided by Forex Division of Mizuho Corporate Bank, Ltd.). We also construct domestic/foreign forward interest rates' term structures and volatilities using the

data downloaded from Bloomberg on swap rates and cap volatilities in each market.

Tables 3.-4. and Figures 5.-10. show the data on volatility surfaces and our calibrated parameters. In Table 3. and Figures 5.-7., calibration results to the observed volatility smile for five or ten year, separately. Additionally, Table 4. and Figures 8.-10 show results in joint calibration to the volatilities for five and ten years.

Most of absolute errors in separate calibration are less than 0.01%, in joint calibration less than 0.3%., which seems small enough for a practical purpose.

Consequently, we conclude that our formula is flexible enough for calibration of the observed surfaces, which is a hard task with other time-consuming methods such as numerical ones. We may use the calibrated parameters for valuation of illiquid options and more complicated currency derivatives.

4   A Characteristic-function-based Monte Carlo Simulation with an Asymptotic Expansion

In this section, we will introduce a Monte Carlo simulation scheme which incorporates the analytically obtained characteristic function. Further, with the asymptotic expansion as a control variable, variance of this characteristic-function-based(ch.f.-based) M.C. is reduced.

4.1   A characteristic-function-based Monte Carlo simulation

In a usual M.C. procedure, we discretize the S.D.E.s (3), (4), (5) and (6), and generate $\{f^j\}_{j=1}^M$, M samples of $f_{N+1}^{(\epsilon)}(T)$. Then the approximation for the option value, a discounted average of terminal payoffs, is obtained by;

$$\hat{V}_{MC}^{payoff}(0, M; T, K) := P_d(0, T) \frac{1}{M} \sum_{j=1}^M (e^{f^j} - K)^+. \tag{23}$$

On the other hand, by the pricing formula (2) in Proposition 1, the option price can be expressed as a certain functional of the characteristic function of the underlying log-process.

$$V(0; T, K) = \mathbf{\Psi}(\Phi^{P,(\epsilon)}; T, K)$$

where

$$
\begin{aligned}
\mathbf{\Psi}(\Phi; T, K) \quad := \quad & P_d(0, T) \times \left[F_T(0) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk}\Phi(u - i)}{iu} \} du \right\} \right. \\
& \left. -K \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{ \frac{e^{-iuk}\Phi(u)}{iu} \} du \right\} \right].
\end{aligned}
\tag{24}
$$

Since $\Phi^{P,(\epsilon)}(u)$ is defined by $\mathbf{E}^P \left[e^{iuf_{N+1}^{(\epsilon)}(T)} \right] = \mathbf{E}^P \left[e^{iuZ^{(\epsilon)}(T)} \right] \times \mathbf{E}^P \left[e^{iuA(T)} \right]$, an alternative approximation with M.C. can be constructed;

$$\hat{V}_{MC}^{chf}(0, M; T, K) \quad := \quad \mathbf{\Psi}(\hat{\Phi}_{MC}^P(\,\cdot\,; M); T, K) \tag{25}$$

$$\hat{\Phi}_{MC}^P(u; M) \quad = \quad \hat{\Phi}_{Z,MC}^P(u; M) \times \Phi_A^P(u) := \left(\frac{1}{M} \sum_{j=1}^M e^{iuZ^j} \right) \Phi_A^P(u) \tag{26}$$

where $\{Z^j\}_{j=1}^M$ are samples of $Z^{(\epsilon)}(T)$. Here it is stressed that in this approximation there does not exist any error caused by M.C. for the (jump or continuous) part A.

4.2 The asymptotic expansion as a control variable

Further, this ch.f.-based scheme can be much refined through a better estimation for $\Phi_Z^{P,(\epsilon)}(u)$ by M.C., achieved with our asymptotic expansion of the first order. Since $\Phi_Z^{P,(\epsilon)}(u)$ is expressed as $\Phi_X^{Q_u,(\epsilon)}(\epsilon h(u))$, it is done by the approximation of $\Phi_X^{Q_u,(\epsilon)}(\epsilon h(u))$ with M.C.. In what follows in this section, we abbreviate ϵ(or set $\epsilon = 1$) for simplicity.

Here, in order to avoid the influence appearing in this variance reduction procedure caused by the variable transformation $h(\cdot)$, we use the following relationship

$$\mathbf{E}^{Q_u}\left[e^{ih(u)g_1}\right] = \exp\left(-\frac{1}{2}iu\Sigma\right)\mathbf{E}^{Q_u}\left[e^{iug_1}\right], \tag{27}$$

i.e. $\Phi_{g_1}^{Q_u}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right) \times \Phi_{g_1}^{Q_u}(u)$. $\Phi_{g_1}^{Q_u}(v)$ is the characteristic function of g_1, which is equivalent to $\hat{\Phi}_X^{Q_u,(\epsilon)}(v)$ in Theorem 1 if the expansion is made only up to the first order. This equation can be easily checked with recalling $\Phi_{g_1}^{Q_u}(v) = \Phi_{0,\Sigma}(v) = \exp(-\frac{\Sigma}{2}v^2)$.

Thus on the one hand, the closed-form characteristic function of g_1 evaluated at $v = h(u)$ is given by

$$\Phi_{g_1}^{Q_u}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right)\Phi_{0,\Sigma}(u). \tag{28}$$

But on the other hand, generating samples of g_1 following $N(0,\Sigma)$, $\{g^j\}_{j=1}^M$, we can further approximate the right hand side of (27) by

$$\hat{\Phi}_{g_1,MC}^{Q_u}(u;M) := \exp\left(-\frac{1}{2}iu\Sigma\right)\frac{1}{M}\sum_{j=1}^M\left(e^{iug^j}\right). \tag{29}$$

Note that because only the distribution of g_1 matters here, we can simulate samples of $\tilde{g}_1 := \int_0^T \sigma_Z^{(0)\prime}(s)dW_s$ following $N(0,\Sigma)$instead of those of g_1, not under the measure Q_u but under P as well as other random variables simulated for (26).

Using two functions in (28) and (29), which both are the first-order approximations for $\Phi_Z^{Q_u,(\epsilon)}(h(u))$, define two following estimators for the option price.

$$\hat{V}_{ana}^{AE}(0;T,K) := \mathbf{\Psi}\left(\Phi_{g_1}^{Q_u}(h(\cdot)) \times \Phi_A^P; T, K\right) \tag{30}$$

$$\hat{V}_{MC}^{AE}(0,M;T,K) := \mathbf{\Psi}\left(\hat{\Phi}_{g_1,MC}^{Q_u}(\,\cdot\,;M) \times \Phi_A^P; T, K\right) \tag{31}$$

Finally, using $\Phi_{g_1}^{Q_u}(h(u))$ as a control variable, we can construct the more sophisticated estimator $\hat{V}^{CV}(0,M;T,K)$ for the option price $V(0;T,K)$ as

$$
\begin{aligned}
\hat{V}^{CV}(0,M;T,K) &:= \hat{V}_{MC}^{chf}(0,M;T,K) + \left(\hat{V}_{ana}^{AE}(0;T,K) - \hat{V}_{MC}^{AE}(0,M;T,K)\right) \tag{32}\\
&= \mathbf{\Psi}\left(\left\{\hat{\Phi}_{Z,MC}^P(\,\cdot\,;M) + \left(\Phi_{g_1}^{Q_u}(h(\cdot)) - \hat{\Phi}_{g_1,MC}^{Q_u}(\,\cdot\,;M)\right)\right\} \times \Phi_A^P; T, K\right)
\end{aligned}
$$

where $T = T_{N+1}$ and

$$\hat{\Phi}_{Z,MC}^P(u;M) = \frac{1}{M}\sum_{j=1}^M e^{iuZ^j}$$

$$\Phi_{g_1}^{Q_u}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right) \times \Phi_{0,\Sigma}(u)$$

$$\hat{\Phi}_{g_1,MC}^{Q_u}(u;M) = \exp\left(-\frac{1}{2}iu\Sigma\right) \times \frac{1}{M}\sum_{j=1}^M\left(e^{iug^j}\right).$$

Remark 5 *Here we note the following fact.*

$$V(0; T, K) - \hat{V}^{CV}(0, M; T_K)$$

$$= \left(V(0; T, K) - \hat{V}^{chf}_{MC}(0, M; T, K) \right) - \left(\hat{V}^{AE}_{ana}(0; T, K) - \hat{V}^{AE}_{MC}(0, M; T, K) \right)$$

$$= \mathbf{\Psi} \left(\left\{ \left(\Phi^{P,(\epsilon)}_Z - \hat{\Phi}^P_{Z,MC}(\,\cdot\,; M) \right) - \left(\Phi^{Q_u}_{g_1}(h(\cdot)) - \hat{\Phi}^{Q_u}_{g_1,MC}(h(\cdot); M) \right) \right\} \times \Phi^P_A; T, K \right)$$

where $\Phi^{P,(\epsilon)}_Z$ is the exact characteristic function of $Z^{(\epsilon)}(T)$. The former in the first parentheses is the exact characteristic function of $Z^{(\epsilon)}(t)$ and the latter is its approximation by Monte Carlo simulations. Similarly, the former in the second parentheses is the exact one of g_1, the first-order expansion for $Z^{(\epsilon)}(t)$, and the latter is its approximation. Thus, in the case where the first and second term in the braces cancel each other out, an error of our hybrid estimator is expected to be small.

Remark 6 *We here also summarize procedures introduced in this section.*

1. *Discretize the processes of $f^{(\epsilon)}_{dj}(t)$, $f^{(\epsilon)}_{fj}(t)$, $\sigma^{(\epsilon)}(t)$ and of $Z^{(\epsilon)}(t)$ under P and generate $\{Z^j\}^M_{j=1}$, M samples of $Z^{(\epsilon)}(T)$.*

2. *Also generate $\{\tilde{g}^j\}^M_{j=1}$, samples of $\tilde{g}_1 = \int_0^T \sigma^{(0)'}_Z(s) dW_s$ instead of g_1, under P with the same sequence of random numbers used in 1.*

3. *Calculate $\hat{\Phi}^P_{Z,MC}(u; M)$ with $\{Z^j\}$ for each u, which is the characteristic function of $Z^{(\epsilon)}(T)$ approximated by M.C..*

4. *Similarly calculate $\hat{\Phi}^{Q_u}_{g_1,MC}(u; M)$ with $\{\tilde{g}^j\}$ for each u, the approximation for $\Phi^{Q_u}_{g_1}(h(u))$ by M.C..*

5. *Using the estimators calculated in 3. and 4., approximate $\Phi^{P,(\epsilon)}_Z(u)$ by*

$$\hat{\Phi}^P_{Z,MC}(u; M) + \left(\Phi^{Q_u}_{g_1}(h(u)) - \hat{\Phi}^{Q_u}_{g_1,MC}(u; M) \right)$$

   where $\Phi^{Q_u}_{g_1}(u)$ is the exact characteristic function of g_1 given in closed-form.

6. *Inverting the estimated characteristic function in 5. via the pricing functional $\mathbf{\Psi}(\,\cdot\,; T, K)$ given in (24), we finally obtain the estimator for the option price with the first-order asymptotic expansion as a control variable.*

4.3 Numerical examples(2)

Here the convergence of our hybrid estimator with the asymptotic expansion as a control variable is compared to that of a "crude M.C."(only with the antithetic variables method). In this subsection, the following three estimators are examined; $\hat{V}^{payoff}_{MC}(0, M; T, K)$ in (23) is a standard M.C. estimator which averages the discounted terminal payoffs; $\hat{V}^{chf}_{MC}(0, M; T, K)$ in (25) is obtained via the Fourier inversion of the characteristic function approximated by M.C.; and $\hat{V}^{CV}(0, M; T, K)$ in (32) is the estimator with a use of the first-order asymptotic expansion as a control variable. We apply the antithetic variables method to all estimators.

First, in Table 2., comparisons of their convergences in the same model(indicated by "CIR-type vol.") and the same parameters of "Corr.2" assumed in examples in the previous section are shown. It lists up ratios of variances of $\hat{V}^{chf}_{MC}(0, M; T, K)$ and $\hat{V}^{CV}(0, M; T, K)$ to that of $\hat{V}^{payoff}_{MC}(0, M; T, K)$ with the same 1,000,000 trials: Strictly speaking, we show the variances of a series of these estimators calculated with each 1,000 paths.

The ch.f.-based M.C. seems to have almost the same variance with a crude M.C. in this setting. Contrarily to this, usage of our asymptotic expansion as a control variable for the ch.f.-based M.C. reduces its variances; reducing more for OTM than for ATM in these investigations. They are reduced around to 15% in five years and to 20% in ten years of a crude M.C.'s.

Table 2: Comparisons of variances of our estimators, given in terms of ratios to that of a crude M.C.

| Moneyness | 5y | Variance ratio | |
		CIR-type vol.	SSM
1	Chf/Crude	1.013	0.045
	CV/Crude	0.161	0.007
1.3	Chf/Crude	0.980	0.060
	CV/Crude	0.112	0.011
Moneyness	10y	Variance ratio	
		CIR-type vol.	SSM
1	Chf/Crude	1.014	0.191
	CV/Crude	0.207	0.035
1.4	Chf/Crude	1.009	0.171
	CV/Crude	0.158	0.029

Second, we investigate the convergence of our hybrid estimators in the case where the analytically-obtained component $A(t)$ exists. In such cases, their convergences are expected to be faster than those in the previous settings are, since we need not approximate a whole part of the characteristic function of the underlying asset but do by M.C. only parts without known analytical expressions of their characteristic functions.

In particular, for $A(t)$ we assume the following Stochastic Skew Model(SSM) in Carr and Wu[2005], which well captures a time-varying behavior of smiles or skews observed in currency option markets:

$$
\begin{aligned}
A(t) &= \xi_L(t) + \xi_R(t) \\
&:= \left(-\frac{1}{2} \int_0^t V_L(s) ds + \int_0^t \sqrt{V_L(s)} \bar{\xi}'_L d\hat{W}_s \right) + \left(-\frac{1}{2} \int_0^t V_R(s) ds + \int_0^t \sqrt{V_R(s)} \bar{\xi}'_R d\hat{W}_s \right) \\
V_k(t) &= V_k(0) + \kappa_k \int_0^t (\theta_k - V_k(s)) \, ds + \omega_k \int_0^t \sqrt{V_k(s)} \bar{v}'_k d\hat{W}_s, \quad for \ k = L, R
\end{aligned}
$$

under the domestic risk-neutral measure, where ξ_L and ξ_R are assumed to be independent (hence $\bar{\xi}'_L \bar{\xi}_R = 0$). Further, for the correlations among those components we assume $\bar{\xi}'_L \bar{v}_L < 0$, $\bar{\xi}'_R \bar{v}_R > 0$; $\bar{\xi}'_L \bar{v}_R = \bar{\xi}'_R \bar{v}_L = \bar{v}'_L \bar{v}_R = 0$. These conditions mean that $V_L(V_R)$ correlates to the spot forex negatively(positively) and is independent of the other processes.

This can be regarded as a double Heston-type model which consists of two independent stochastic variance processes correlating to the spot forex in opposite directions. For simplicity, the jump components appearing in the original paper of SSM are omitted here with little loss of generality.

Then the characteristic function of $A(t)$ is given by

$$
\begin{aligned}
\Phi^P_A(t; u) &= \Phi^P_{\xi_L}(t, u) \times \Phi^P_{\xi_R}(t, u), \\
\Phi^P_{\xi_k}(t, u) &= \left(\cosh \frac{\eta_k t}{2} + \frac{\kappa_k - i\rho_k \omega_k u}{\eta_k} \sinh \frac{\eta_k t}{2} \right)^{-\frac{2\kappa_k \theta_k}{\omega_k^2}} \\
&\quad \times \exp \left(\frac{\kappa_k \theta_k (\kappa_k - i\rho_k \omega_k u)t}{\omega_k^2} - \frac{(u^2 + iu)V_k(0)}{\eta_k \coth \frac{\eta_k t}{2} + \kappa_k - i\rho_k \omega_k u} \right),
\end{aligned}
$$

where $\eta_k := \sqrt{\omega_k^2(u^2 + iu) + (\kappa_k - i\rho_k \omega_k u)^2}$ and $\rho_k := \bar{\xi}'_k \bar{v}_k$ (See Daffie, Pan and Singleton[1999] or Carr and Wu[2005] for details).

In the investigations made here, the parameters are set as follows. For interest rates, the parameters of case (ii) in Table 1. are used except for $\bar{\gamma}'_d \bar{\gamma}_f = 0.5$; for the stochastic volatility in $Z(t)$, $\sigma(0) = \theta = \omega^* = 0$ are assumed so as to ensure $\sigma^{(\epsilon)}(t) \equiv 0$, that is the objective $\hat{Z}(t)$ of our asymptotic expansion consists only of the domestic and foreign interest rates. Finally, for SSM, $V_k(0) = \theta_k = 0.0075$, $\kappa_k = 0.5$, $\omega_k = 0.1$ for $k = L, R$; the correlation $\rho_L(\rho_R)$ between $V_L(V_R)$ and the spot forex is assumed to -0.5(0.5). Other

correlation parameters among factors are all assumed to be zero. These settings can be interpreted as SSM under a market model of interest rates.

In Table 2., as well as in the examination with the CIR-type volatility model, comparisons of the variances in this setting(indicated by "SSM") are made. Contrary to results in the previous case, in this case the ch.f-based M.C. estimator has the variances of around 5% in five years and 20% in ten years of a crude M.C's, reduced by usage of the analytically solved characteristic function as we expected. The scheme proposed in Section 4.2 further cuts down those variances to around 1% in five years and 3% in ten years of the originals. Thus it can be said that, in such cases where the closed characteristic function of a part of the underlying is available, incorporation of this knowledge and our analytical approximation for another part's via the ch.f-based M.C. scheme dramatically accelerates the convergence of M.C. simulations.

In Figures 11-12.,we present these variance-reduction effects. It is stressed that at most 5,000 paths for five years and 50,000 paths for ten years are enough to obtain the accuracy within 0.01.

5   Concluding Remarks

This paper presented the hybrid asymptotic expansion scheme that provides extensions, refinement and integration of the methods previously developed by Yoshida[1992b], Takahashi[1995,1999], Kunitomo and Takahashi[2001,2003], and Takahashi and Yoshida[2005]. It provided the accurate closed-form approximation formula for option pricing which needs few seconds for calculation and enables us to obtain an analytical approximation for Greeks. It also developed the new variance reduction technique in the characteristic-function-based Monte Carlo simulations with the asymptotic expansion we had derived as the control variable.

The proposed scheme is valid for models even in which analytical approximations of the option values are very difficult to be obtained although it is important in practice: Specifically, our scheme is applied to models that allow correlations among all the factors whose dynamics are not necessarily affine nor even Markovian as long as a source of uncertainty is generated by Brownian motions. It is also applied to models that include jump processes under the assumption that they are independent of the other random variables when the characteristic functions of the jump components can be analytically obtained. Moreover, the numerical examples in the cross-currency settings confirmed the effectiveness of our method especially for the practically important cases.

Our next research plans are as follows: We will develop a general but computationally feasible scheme in practice for higher order expansions in order to achieve more precise approximations. We will also pursue an efficient method for the evaluation of multi-factor path-dependent or/and American derivatives. In fact, our proposed scheme can be applied to average options under a general setting of the underlying factors as in this paper.

A   Details for the Underlying Model

In this section we will consider in detail our underlying framework, which is assumed to consist of a market model of interest rates, of a general stochastic volatility model of the spot forex, and of a possibly jumping component with its analytically available characteristic function.

We first model dynamics of underlying forward interest rates' processes under the terminal measure of the each market. $\mathbf{R}_{++}$-valued processes of domestic forward interest rates under the domestic terminal measure can be specified as; for $j = n(t) - 1, n(t), n(t) + 1, \cdots, N$,

$$f_{dj}(t) = f_{dj}(0) + \int_0^t \left\{ -f_{dj}(s)\tilde{\gamma}'_{dj}(s) \sum_{i=j+1}^N \frac{\tau_i f_{di}(s)\tilde{\gamma}_{di}(s)}{1 + \tau_i f_{di}(s)} \right\} ds + \int_0^t f_{dj}(s)\tilde{\gamma}'_{dj}(s)dW_s$$

Similarly, $\mathbf{R}_{++}$-valued processes of foreign ones under the foreign terminal measure are specified as;

$$f_{fj}(t) \;=\; f_{fj}(0) + \int_0^t \left\{ -f_{fj}(s)\tilde{\gamma}'_{fj}(s) \sum_{i=j+1}^{N} \frac{\tau_i f_{fi}(s)\tilde{\gamma}_{fi}(s)}{1+\tau_i f_{fi}(s)} \right\} ds + \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s) dW_s^f$$

where W^f is a D dimensional Brownian motion under this measure.

Finally, we assume that a spot exchange rate $S(t)$ and its volatility $\tilde{\sigma}(t)$ follow $\mathbf{R}_{++}$-valued stochastic processes under the domestic risk-neutral measure (not under the domestic terminal measure) :

$$S(t) \;=\; S(0) + \int_0^t S(s)(r_d(s) - r_f(s))ds + \int_0^t S(s)\tilde{\sigma}(s)\bar{\sigma}' d\hat{W}_s + \int_0^t S(s)d\tilde{A}(s)$$

$$\tilde{\sigma}(t) \;=\; \tilde{\sigma}(0) + \int_0^t \tilde{\mu}(\tilde{\sigma}(s), s)ds + \int_0^t \tilde{\omega}'(\tilde{\sigma}(s), s)d\hat{W}_s$$

where $\hat{W}$ is a D dimensional Brownian motion under the domestic risk-neutral measure and $\tilde{A}$ is some (possibly jump) martingale independent of $\hat{W}$; $r_d(t)$ and $r_f(t)$ denote domestic and foreign instantaneous spot interest rates respectively.

We next note the following well known relations among Brownian motions under different measures;

$$W_t \;=\; \hat{W}_t - \int_0^t \tilde{\sigma}_{dN+1}(s)ds$$

$$\;=\; W_t^f + \int_0^t \{ \tilde{\sigma}_{fN+1}(s) - \tilde{\sigma}_{dN+1}(s) + \tilde{\sigma}(s)\bar{\sigma} \}ds$$

where $\tilde{\sigma}_{dN+1}(t)$ and $\tilde{\sigma}_{fN+1}(t)$ are volatilities of domestic and foreign zero coupon bonds with the maturity T_{N+1}, that is,

$$\tilde{\sigma}_{dN+1}(t) := \sum_{i \in J_{N+1}(t)} \frac{-\tau_i f_{di}(t)\tilde{\gamma}_{di}(t)}{1+\tau_i f_{di}(t)}, \quad \tilde{\sigma}_{fN+1}(t) := \sum_{i \in J_{N+1}(t)} \frac{-\tau_i f_{fi}(t)\tilde{\gamma}_{fi}(t)}{1+\tau_i f_{fi}(t)}$$

and $J_{j+1}(t) = \{n(t)-1, n(t), n(t)+1, \cdots, j\}$. Since $\gamma_{fi}(t) = 0$ and $\gamma_{di}(t) = 0$ for all i such that $T_i \le t$, the set of indices $J_{j+1}(t)$ can be changed into $\hat{J}_{j+1} := \{0, 1, \cdots, j\}$, which does not depend on t.

By the above equations, expressions of those processes under different measures are unified into those under the same measure, the domestic terminal one:

$$f_{fj}(t) \;=\; f_{fj}(0) + \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s) \left\{ -\sum_{i \in \hat{J}_{j+1}} \frac{-\tau_i f_{fi}(s)\tilde{\gamma}_{fi}(s)}{1+\tau_i f_{fi}(s)} + \sum_{i \in \hat{J}_{N+1}} \frac{-\tau_i f_{di}(s)\tilde{\gamma}_{di}(s)}{1+\tau_i f_{di}(s)} \right\} ds$$

$$\qquad - \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s)\tilde{\sigma}(s)\bar{\sigma}ds + \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s)dW_s$$

$$\tilde{\sigma}(t) \;=\; \tilde{\sigma}(0) + \int_0^t \mu(s)ds + \int_0^t \tilde{\omega}'(\tilde{\sigma}(s), s)dW_s,$$

where

$$\mu(t) \;=\; \mu(\{f_{dj}(t)\}, \tilde{\sigma}(t), t)$$

$$\;:=\; \tilde{\mu}(\tilde{\sigma}(t), t) + \tilde{\omega}'(\tilde{\sigma}(t), t) \sum_{i \in \hat{J}_{N+1}} \frac{-\tau_i f_{di}(t)\tilde{\gamma}_{di}(t)}{1+\tau_i f_{di}(t)}.$$

Finally, we consider the process of the forex forward $F_{N+1}(t)$. Since it can be expressed as $F_{N+1}(t) := S(t)\frac{P_f(t, T_{N+1})}{P_d(t, T_{N+1})}$, we easily notice that it is a martingale under the domestic terminal measure, and obtain its log-process, $f_{N+1}(t)$ under that measure by Itô's formula:

$$f_{N+1}(t) \;=\; ln(\frac{F_{N+1}(t)}{F_{N+1}(0)}) = Z(t) + A(t)$$

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where

$$\begin{aligned}
Z(t) &= -\frac{1}{2}\int_0^t \|\tilde{\sigma}_Z(s)\|^2\,ds + \int_0^t \tilde{\sigma}_Z'(s)dW_s \\
\tilde{\sigma}_Z(t) &:= \tilde{\sigma}_{fN+1}(t) - \tilde{\sigma}_{dN+1}(t) + \tilde{\sigma}(t)\bar{\sigma} \\
&= \sum_{j\in\hat{J}_{N+1}}\left(\frac{-\tau_j f_{fj}(t)\tilde{\gamma}_{fj}(t)}{1+\tau_j f_{fj}(t)} - \frac{-\tau_j f_{dj}(t)\tilde{\gamma}_{dj}(t)}{1+\tau_i f_{dj}(t)}\right) + \tilde{\sigma}(t)\bar{\sigma}
\end{aligned}$$

and $A(t)$ denotes an exponential-martingale process, which may be a jump process or a affine-structured one, obtained through Itô's formula. Because of the independence of $\tilde{A}$, A is again independent of other factors driven by Brownian motions. Additionally, we assume that the characteristic function of $A(t)$ is available in closed form.

B   Concrete Expansions in Proposition 2

First we state the following lemma. While its derivation is straightforward done by formal Taylor expansions of the underlying S.D.E.s with respect to ϵ, expression of each term in the lemma is somewhat complicated, and hence omitted.

Lemma 1 *The asymptotic expansions of the domestic, foreign forward rates and the volatility of the spot forex are given as follows:*

$$\begin{aligned}
f_{dj}^{(\epsilon)}(t) &= f_{dj}(0) + \epsilon G_{dj}^{Q_u,(1)}(t) + \epsilon^2 G_{dj}^{Q_u,(2)}(t) + o(\epsilon^2) & (33)\\
f_{fj}^{(\epsilon)}(t) &= f_{fj}(0) + \epsilon G_{fj}^{Q_u,(1)}(t) + \epsilon^2 G_{fj}^{Q_u,(2)}(t) + o(\epsilon^2) & (34)\\
\sigma^{(\epsilon)}(t) &= \sigma^{(0)}(t) + \epsilon G_{\sigma}^{Q_u,(1)}(t) + \epsilon^2 G_{\sigma}^{Q_u,(2)}(t) + o(\epsilon^2). & (35)
\end{aligned}$$

Here $\sigma^{(0)}(t)$ satisfies the equation $\sigma^{(0)}(t) = \sigma(0) + \int_0^t \mu^{(0)}(s)ds$.

Then, the asymptotic expansion of $\hat{Z}^{(\epsilon)}(t)$ up to the third order of $\epsilon(\epsilon^3$-order) can be derived again by formal Taylor expansions. Proposition 2 is below restated in details.

Proposition 3 *The asymptotic expansion of $\hat{Z}^{(\epsilon)}(t)$ up to the third order is expressed as follows:*

$$\hat{Z}^{(\epsilon)}(t) = \epsilon\hat{G}_t^{Q_u,(1)} + \epsilon^2\hat{G}_t^{Q_u,(2)} + \epsilon^3\hat{G}_t^{Q_u,(3)} + o(\epsilon^3) \qquad (36)$$

where

$$\hat{G}_t^{Q_u,(1)} := \int_0^t \sigma_Z^{(0)}(s)'dW_s^{Q_u}, \qquad (37)$$

$$\hat{G}_t^{Q_u,(2)} := \int_0^t\left[\sum_{i\in\hat{J}_{N+1}}\left(g_{fi}^{(1)}(f_{fi}(0),s)G_{fi}^{Q_u,(1)}(s) - g_{di}^{(1)}(f_{di}(0),s)G_{di}^{Q_u,(1)}(s)\right) + G_{\sigma}^{Q_u,(1)}(s)\bar{\sigma}\right]'dW_s^{Q_u},$$

$$(38)$$

$$\begin{aligned}
\hat{G}_t^{Q_u,(3)} :=& \sum_{i\in\hat{J}_{N+1}}\int_0^t G_{fi}^{Q_u,(2)}(s)(g_{fi}^{(1)}(f_{fi}(0),s))'dW_s^{Q_u} + \frac{1}{2}\sum_{i\in\hat{J}_{N+1}}\int_0^t (G_{fi}^{Q_u,(1)}(s))^2(g_{fi}^{(2)}(f_{fi}(0),s))'dW_s^{Q_u} \\
&- \sum_{i\in\hat{J}_{N+1}}\int_0^t G_{di}^{Q_u,(2)}(s)(g_{di}^{(1)}(f_{di}(0),s))'dW_s^{Q_u} - \frac{1}{2}\sum_{i\in\hat{J}_{N+1}}\int_0^t (G_{di}^{Q_u,(1)}(s))^2(g_{di}^{(2)}(f_{di}(0),s))'dW_s^{Q_u} \\
&+ \int_0^t G_{\sigma}^{Q_u,(2)}(s)\bar{\sigma}'dW_s^{Q_u} & (39)
\end{aligned}$$

and

$$
\begin{cases}
g_{di}^{(1)}(x,t) := \left(\frac{-\tau_i}{(1+\tau_i x)^2}\right)\gamma_{di}(t), \quad g_{fi}^{(1)}(x,t) := \left(\frac{-\tau_i}{(1+\tau_i x)^2}\right)\gamma_{fi}(t), \\[2mm]
g_{di}^{(2)}(x,t) := \left(\frac{2\tau_i^2}{(1+\tau_i x)^3}\right)\gamma_{di}(t), \quad g_{fi}^{(2)}(x,t) := \left(\frac{2\tau_i^2}{(1+\tau_i x)^3}\right)\gamma_{fi}(t).
\end{cases}
$$

(proof) Recall that

$$
\hat{Z}^{(\epsilon)}(t) \;=\; \epsilon \int_0^t \sigma_Z^{(\epsilon)'}(s)\,dW_s^{Q_u} \tag{40}
$$

where

$$
\sigma_Z^{(\epsilon)}(t) \;=\; \sum_{j\in \hat{J}_{N+1}} \left(g_{fj}^{(0)}(f_{fj}^{(\epsilon)}(t),t) - g_{dj}^{(0)}(f_{dj}^{(\epsilon)}(t),t) \right) + \sigma^{(\epsilon)}(t)\bar{\sigma},
$$

$$
g_{dj}^{(0)}(x,t) \;:=\; \frac{-\tau_j x}{1+\tau_j x}\gamma_{dj}(t), \quad g_{fj}^{(0)}(x,t) := \frac{-\tau_j x}{1+\tau_j x}\gamma_{fj}(t)
$$

and note that

$$
\frac{\partial^k g_{dj}^{(0)}(x,t)}{\partial x^k} = g_{dj}^{(k)}(x,t), \quad \frac{\partial^k g_{fj}^{(0)}(x,t)}{\partial x^k} = g_{fj}^{(k)}(x,t), \; k=1,2.
$$

With formal Taylor expansion of (40) around $\epsilon = 0$, we have

$$
\hat{Z}^{(\epsilon)}(t) \;=\; \epsilon \frac{\partial \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon}\Big|_{\epsilon=0} + \frac{\epsilon^2}{2}\frac{\partial^2 \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon^2}\Big|_{\epsilon=0} + \frac{\epsilon^3}{6}\frac{\partial^3 \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon^3}\Big|_{\epsilon=0} + o(\epsilon^3),
$$

and set $\hat{G}_t^{Q_u,(1)} := \frac{\partial \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon}\Big|_{\epsilon=0}$, $\hat{G}_t^{Q_u,(2)} := \frac{1}{2}\frac{\partial^2 \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon^2}\Big|_{\epsilon=0}$ and $\hat{G}_t^{Q_u,(3)} := \frac{1}{6}\frac{\partial^3 \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon^3}\Big|_{\epsilon=0}$. As for $\hat{G}_t^{Q_u,(1)}$, differentiating the equation (40) with respect to ϵ once, we have

$$
\frac{\partial \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon} \;=\; \int_0^t \sigma_Z^{(\epsilon)'}(s)\,dW_s^{Q_u} + \epsilon \int_0^t \frac{\partial}{\partial \epsilon}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u}.
$$

Then, setting $\epsilon = 0$, we obtain the expression of $\hat{G}_t^{Q_u,(1)}$, that is (37).
Similarly, differentiating the equation (40) with respect to ϵ twice and three times, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2 \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon^2} \;=\;& \frac{1}{2}\left(2\int_0^t \frac{\partial}{\partial \epsilon}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u} + \epsilon \int_0^t \frac{\partial^2}{\partial \epsilon^2}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u}\right) \\
=\;& \int_0^t \left[\sum_{i\in \hat{J}_{N+1}}\left(g_{fi}^{(1)}(f_{fi}^{(\epsilon)}(s),s)\frac{\partial f_{fi}^{(\epsilon)}(s)}{\partial \epsilon} - g_{di}^{(1)}(f_{di}^{(\epsilon)}(s),s)\frac{\partial f_{di}^{(\epsilon)}(s)}{\partial \epsilon}\right) + \frac{\partial \sigma^{(\epsilon)}(s)}{\partial \epsilon}\bar{\sigma}\right]' dW_s^{Q_u} \\
& + \frac{\epsilon}{2}\int_0^t \frac{\partial^2}{\partial \epsilon^2}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u},
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{6}\frac{\partial^3 \hat{Z}^{(\epsilon)}(t)}{\partial \epsilon^3} \;=\;& \frac{1}{6}\left(3\int_0^t \frac{\partial^2}{\partial \epsilon^2}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u} + \epsilon \int_0^t \frac{\partial^3}{\partial \epsilon^3}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u}\right) \\
=\;& \frac{1}{2}\left[\int_0^t \sum_{i\in \hat{J}_{N+1}}\left(g_{fi}^{(1)}(f_{fi}^{(\epsilon)}(s),s)\frac{\partial^2 f_{fi}^{(\epsilon)}(s)}{\partial \epsilon^2} - g_{di}^{(1)}(f_{di}^{(\epsilon)}(s),s)\frac{\partial^2 f_{di}^{(\epsilon)}(s)}{\partial \epsilon^2}\right)' dW_s^{Q_u}\right. \\
& + \int_0^t \sum_{i\in \hat{J}_{N+1}}\left(g_{fi}^{(2)}(f_{fi}^{(\epsilon)}(s),s)\left(\frac{\partial f_{fi}^{(\epsilon)}(s)}{\partial \epsilon}\right)^2 - g_{di}^{(2)}(f_{di}^{(\epsilon)}(s),s)\left(\frac{\partial f_{di}^{(\epsilon)}(s)}{\partial \epsilon}\right)^2\right)' dW_s^{Q_u} \\
& \left. + \frac{\partial^2 \sigma^{(\epsilon)}(s)}{\partial \epsilon^2}\bar{\sigma}' dW_s^{Q_u}\right] + \frac{\epsilon}{6}\int_0^t \frac{\partial^3}{\partial \epsilon^3}\left(\sigma_Z^{(\epsilon)'}(s)\right)dW_s^{Q_u}.
\end{aligned}
$$

Letting $\epsilon = 0$ and substituting

$$
\begin{cases}
f_{dj}^{(\epsilon)}(t)|_{\epsilon=0} = f_{dj}(0), \quad f_{fj}^{(\epsilon)}(t)|_{\epsilon=0} = f_{fj}(0), \\[2mm]
\dfrac{\partial^k f_{dj}^{(\epsilon)}(t)}{\partial \epsilon^k}\Big|_{\epsilon=0} = k! G_{dj}^{Q_u,(k)}(t), \quad \dfrac{\partial^k f_{fj}^{(\epsilon)}(t)}{\partial \epsilon^k}\Big|_{\epsilon=0} = k! G_{fj}^{Q_u,(k)}(t), \\[2mm]
\dfrac{\partial^k \sigma^{(\epsilon)}(t)}{\partial \epsilon^k}\Big|_{\epsilon=0} = k! G_{\sigma}^{Q_u,(k)}(t); \quad k = 1, 2
\end{cases}
$$

into these, (38) and (39) are obtained. $\square$

The formal calculation used in this section is justified by Theorem 3.1 of Kunitomo and Takahashi[2003].

C   Coefficients in Theorem 1

In this section, the derivations of Theorem 1 and of its coefficients are briefly explained. Differentiating the distribution function of $X^{(\epsilon)}$ obtained by substituting $d = 1$, $\phi^{(\epsilon)}(x) \equiv 1$ and $B = (-\infty, x]$ in Theorem 3.4 of Kunitomo and Takahashi[2003], we get the asymptotic expansion of the density function of $X^{(\epsilon)}$ as

$$
\begin{aligned}
\phi_X^{Q_u,(\epsilon)}(x) &= \phi_{0,\Sigma}(x) - \epsilon \frac{\partial}{\partial x}\left\{ \mathbf{E}^{Q_u}[g_2 | g_1 = x]\phi_{0,\Sigma}(x)\right\} \\
&\quad -\epsilon^2 \frac{\partial}{\partial x}\left\{ \mathbf{E}^{Q_u}[g_3 | g_1 = x]\phi_{0,\Sigma}(x)\right\} + \frac{1}{2}\epsilon^2 \frac{\partial^2}{\partial x^2}\left\{ \mathbf{E}^{Q_u}[g_2^2 | g_1 = x]\phi_{0,\Sigma}(x)\right\} + o(\epsilon^2),
\end{aligned}
$$

where $\phi_{\mu,\Sigma}(x)$ is the density function of $N(\mu, \Sigma)$. Then, by Fourier transformation of this density function, the characteristic function of $X^{(\epsilon)}$ is obtained in the form with conditional expectations.

Since the above expectations conditional on $g_1 = x$ are given as polynomials of x by

$$
\begin{aligned}
\mathbf{E}^{Q_u}[g_2 | g_1 = x] &= C_{2,1}^{Q_u}\frac{x}{\Sigma} + C_{2,2}^{Q_u}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\
\mathbf{E}^{Q_u}[g_3 | g_1 = x] &= C_{3,1}^{Q_u}\frac{x}{\Sigma} + C_{3,2}^{Q_u}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) + C_{3,3}^{Q_u}\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) \\
\mathbf{E}^{Q_u}[g_2^2 | g_1 = x] &= C_{4,0}^{Q_u} + C_{4,1}^{Q_u}\frac{x}{\Sigma} + C_{4,2}^{Q_u}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\
&\quad + C_{4,3}^{Q_u}\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) + C_{4,4}^{Q_u}\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right),
\end{aligned}
$$

with certain constants $\{C_{i,j}^{Q_u}\}$, the coefficients $D_2^{Q_u,(\epsilon)}$, $D_3^{Q_u,(\epsilon)}$, $D_4^{Q_u,(\epsilon)}$, $D_5^{Q_u,(\epsilon)}$ and $D_6^{Q_u,(\epsilon)}$ in Theorem 1 are reduced to sums of $\{C_{i,j}^{Q_u}\}$ as

$$
\begin{cases}
D_2^{Q_u,(\epsilon)} := \epsilon C_{2,1}^{Q_u} + \epsilon^2 C_{3,1}^{Q_u} + \frac{1}{2}\epsilon^2 C_{4,0}^{Q_u} \\
D_3^{Q_u,(\epsilon)} := \epsilon C_{2,2}^{Q_u} + \epsilon^2 C_{3,2}^{Q_u} + \frac{1}{2}\epsilon^2 C_{4,1}^{Q_u} \\
D_4^{Q_u,(\epsilon)} := \epsilon^2 C_{3,3}^{Q_u} + \frac{1}{2}\epsilon^2 C_{4,2}^{Q_u} \\
D_5^{Q_u,(\epsilon)} := \frac{1}{2}\epsilon^2 C_{4,3}^{Q_u} \\
D_6^{Q_u,(\epsilon)} := \frac{1}{2}\epsilon^2 C_{4,4}^{Q_u}.
\end{cases}
$$

Here the derivation only of $C_{2,1}^{Q_u}$ and $C_{2,2}^{Q_u}$ are shown while the others done in a similar manner are omitted due to limitation of spaces. Formulas necessary for those calculations are found in Takahashi and Takehara[2007].

For evaluation of $\mathbf{E}^{Q_u}[g_2 | g_1 = x]$, we note that g_1 and g_2 are expressed as

$$
g_1 = \hat{G}_T^{Q_u,(1)} = \int_0^T \sigma_Z^{(0)}(s)' dW_s^{Q_u}
$$

$$
g_2 = \hat{G}_T^{Q_u,(2)} = \int_0^T \Big[\sum_{i \in \hat{J}_{N+1}} g_{fi}^{(1)}(f_{fi}(0), s)G_{fi}^{Q_u,(1)}(s) - \sum_{i \in \hat{J}_{N+1}} g_{di}^{(1)}(f_{di}(0), s)G_{di}^{Q_u,(1)}(s) + G_{\sigma}^{Q_u,(1)}(s)\bar{\sigma}\Big]' dW_s^{Q_u}
$$

where $T \equiv T_{N+1}$. Then,

$$
\begin{aligned}
\mathbf{E}^{Q_u}[g_2|g_1 = x] &= \mathbf{E}^{Q_u}\left[\int_0^T \sum_{i \in \hat{J}_{N+1}} G_{fi}^{Q_u,(1)}(s)(g_{fi}^{(1)}(f_{fi}(0),s))' dW_s^{Q_u}|g_1 = x\right] \\
&\quad - \mathbf{E}^{Q_u}\left[\int_0^T \sum_{i \in \hat{J}_{N+1}} G_{di}^{Q_u(1)}(s)(g_{di}^{(1)}(f_{di}(0),s))' dW_s^{Q_u}|g_1 = x\right] \\
&\quad + \mathbf{E}^{Q_u}\left[\int_0^T G_{\sigma}^{Q_u,(1)}(s)\bar{\sigma}' dW_s^{Q_u}|g_1 = x\right].
\end{aligned}
$$

Each term in the right hand side of this equation is evaluated as

1.

$$
\begin{aligned}
&\mathbf{E}^{Q_u}\left[\int_0^T G_{di}^{Q_u,(1)}(s)(g_{di}^{(1)}(f_{di}(0),s))' dW_s^{Q_u}|g_1 = x\right] \\
&= \left\{f_{di}(0)\int_0^T \left(\int_0^s \gamma_{di}'(r)\sigma_Z^{(0)}(r)dr\right)(g_{di}^{(1)}(f_{di}(0),s))'\sigma_Z^{(0)}(s)ds\right\}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\
&=: a_{2i}^{Q_u,d}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
&\mathbf{E}^{Q_u}\left[\int_0^T G_{fi}^{Q_u,(1)}(s)(g_{fi}^{(1)}(f_{fi}(0),s))' dW_s^{Q_u}|g_1 = x\right] \\
&= \left\{f_{fi}(0)\int_0^T \left(\int_0^s \gamma_{fi}'(r)\sigma_Z^{(0)}(r)dr\right)(g_{fi}^{(1)}(f_{fi}(0),s))'\sigma_Z^{(0)}(s)ds\right\}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\
&=: a_{2i}^{Q_u,f}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
\end{aligned}
$$

3.

$$
\begin{aligned}
&\mathbf{E}^{Q_u}\left[\int_0^T G_{\sigma}^{Q_u,(1)}(s)\bar{\sigma}' dW_s^{Q_u}|g_1 = x\right] \\
&= \left\{\int_0^T \left(\int_0^s Y_r^{-1}\partial_\epsilon \mu(r)dr\right) Y_s\bar{\sigma}'\sigma_Z^{(0)}(s)ds\right\}\frac{x}{\Sigma} \\
&\quad + \left\{\int_0^T \left(\int_0^s Y_r^{-1}\omega'(r)\sigma_Z^{(0)}(r)dr\right) Y_s\bar{\sigma}'\sigma_Z^{(0)}(s)ds\right\}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\
&=: b_2^{Q_u}\frac{x}{\Sigma} + c_2^{Q_u}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)
\end{aligned}
$$

where $Y_s := e^{\int_0^s \partial_\sigma \mu(r)dr}$, $\partial_\epsilon \mu(s) := \partial_\epsilon \mu^{(\epsilon)}(s)|_{\epsilon=0}$, $\partial_\sigma \mu(s) := \partial_\sigma \mu^{(\epsilon)}(s)|_{\epsilon=0}$ and $\omega(s) := \omega(\sigma^{(0)}(s),s)$.
Then, $C_{2,1}^{Q_u}$ and $C_{2,2}^{Q_u}$ are defined by

$$
\begin{aligned}
C_{2,1}^{Q_u} &:= b_2^{Q_u} \\
C_{2,2}^{Q_u} &:= \sum_{i \in \hat{J}_{N+1}} (a_{2i}^{Q_u,f} - a_{2i}^{Q_u,d}) + c_2^{Q_u}.
\end{aligned}
$$

In fact, the coefficients $C_{2,1}^{Q_u}$ and $C_{2,2}^{Q_u}$ do not depend on u. However, those which correspond to the expansion up to the higher order than the second order sometimes do.

D Supplement for Remark 4

In remark 4, it was mentioned that as long as some conditions are met, which are natural for practical purposes, how you choose a value of the parameter ϵ does not matter to option prices by our approximation. More precisely, we can obtain the same approximated prices regardless of a value of ϵ under the constraints $\epsilon\gamma_{dj}(t) = \tilde{\gamma}_{dj}(t)$, $\epsilon\gamma_{fj}(t) = \tilde{\gamma}_{fj}(t)$, $\epsilon\sigma(0) = \tilde{\sigma}(0)$ and $\epsilon\omega(x,t) = \tilde{\omega}(x,t)$ for an arbitrary $t \in [0,T]$ in (11), (12), (13) and (14). We here see this briefly.

Suppose we rewrite the system of SDEs (11), (12), (13) and (14) not with ϵ but with another constant parameter $\delta \in (0,1]$ and define $c = \delta/\epsilon$. In order to guarantee that the above conditions are satisfied, $\gamma_{dj}(t)$, $\gamma_{fj}(t)$, $\sigma(0)$ and $\omega(x,t)$ must be replaced by $\frac{\gamma_{dj}(t)}{c}$, $\frac{\gamma_{fj}(t)}{c}$, $\frac{\sigma(0)}{c}$ and $\frac{\omega(x,t)}{c}$, respectively in the underlying SDEs.

In (20) in Theorem 2, all that depends on ϵ is only $\hat{\Phi}^{(\epsilon)}(\cdot)$ or equivalently $\hat{\Phi}_X^{Q_u,(\epsilon)}(\epsilon h(\cdot))$. Thus, it is sufficient to show

$$\hat{\Phi}_X^{Q_u,(\delta)}(\delta v') = \hat{\Phi}_X^{Q_u,(\epsilon)}(\epsilon v'). \tag{41}$$

Moreover, due to definition of $\hat{\Phi}_X^{Q_u,(\epsilon)}(\cdot)$ in (19), it is equivalent to see

$$\Phi_{0,\Sigma^{(\delta)}}(cv) = \Phi_{0,\Sigma^{(\epsilon)}}(v), \tag{42}$$

$$\text{and} \quad D_l^{Q_u,(\delta)} = \frac{1}{c^l}D_l^{Q_u,(\epsilon)}; \ for \ l = 2,\cdots,6 \tag{43}$$

where $\Sigma^{(\delta)}$ and $\Sigma^{(\epsilon)}$ are defined in the same way as Σ in (18) under the SDEs with the re-scaled parameters;

$$(18) \qquad \Sigma := \int_0^T ||\sigma_Z^{(0)}(s)||^2 ds.$$

Since under our re-specification of (11), (12), (13) and (14) with $\delta = c\epsilon$ we have $\Sigma^{(\delta)} = \frac{1}{c^2}\Sigma^{(\epsilon)}$, the former part can be shown as

$$\begin{aligned}\Phi_{0,\Sigma^{(\delta)}}(cv) &= \exp\left\{-\frac{\Sigma^{(\delta)}}{2}(cv)^2\right\} \\ &= \exp\left\{-\frac{\Sigma^{(\epsilon)}}{2}v^2\right\} = \Phi_{0,\Sigma^{(\epsilon)}}(v).\end{aligned}$$

Also, the latter can be shown with easy calculations while they are omitted due to their length.

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Table 3: A separate calibration to the observed implied volatilities for a five-year and ten-year maturity.

Date	07/09/27	Type	Separate

5y	10put	25put	ATM	25call	10call
Market I.V.	16.66%	10.92%	7.90%	6.07%	6.96%
Model I.V.	16.66%	10.92%	7.90%	6.07%	6.95%
differences	0.00%	-0.01%	0.00%	0.00%	-0.01%

10y	10put	25put	ATM	25call	10call
Market I.V.	20.00%	13.59%	10.50%	8.34%	9.50%
Model I.V.	20.01%	13.59%	10.50%	8.35%	9.49%
differences	0.01%	-0.00%	0.00%	0.01%	-0.02%

Date	07/10/30	Type	Separate

5y	10put	25put	ATM	25call	10call
Market I.V.	15.58%	11.20%	8.10%	6.60%	6.38%
Model I.V.	15.59%	11.20%	8.10%	6.60%	6.38%
differences	0.01%	0.00%	0.00%	-0.00%	0.00%

10y	10put	25put	ATM	25call	10call
Market I.V.	19.29%	13.49%	10.35%	8.19%	8.69%
Model I.V.	19.29%	13.48%	10.34%	8.19%	8.69%
differences	-0.00%	-0.01%	-0.01%	0.00%	0.00%

Date	07/12/07	Type	Separate

5y	10put	25put	ATM	25call	10call
Market I.V.	17.20%	11.72%	8.45%	6.62%	7.00%
Model I.V.	17.19%	11.71%	8.45%	6.62%	7.00%
differences	-0.01%	-0.01%	0.00%	0.00%	0.00%

10y	10put	25put	ATM	25call	10call
Market I.V.	20.46%	13.98%	10.65%	8.38%	9.26%
Model I.V.	20.46%	13.98%	10.65%	8.38%	9.26%
differences	-0.00%	0.00%	0.00%	-0.00%	0.00%

Figure 1: A comparison of the accuracy of each estimator in prices for a five-year maturity.
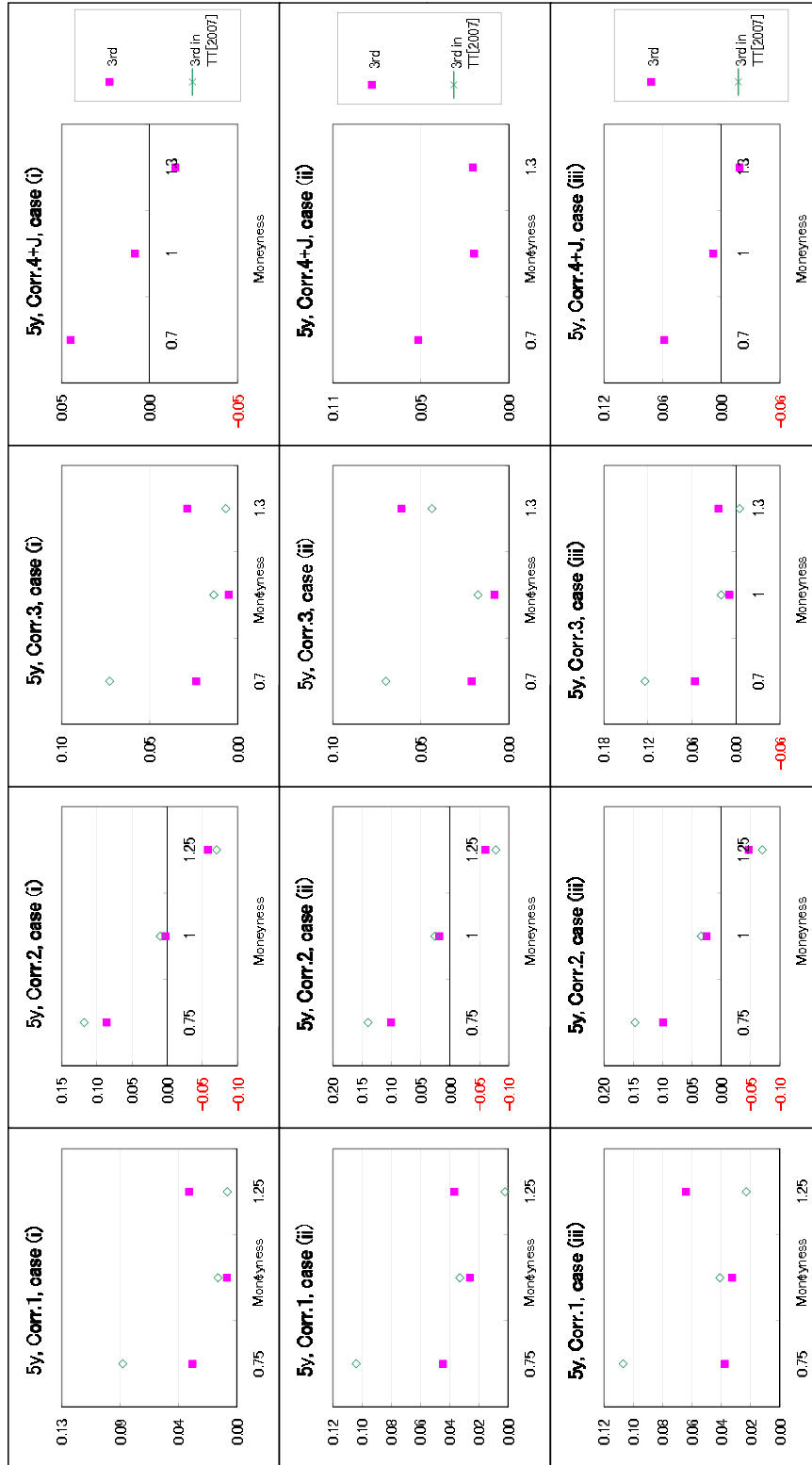
Figure 2: A comparison of the accuracy of each estimator in implied volatilities for a five-year maturity.
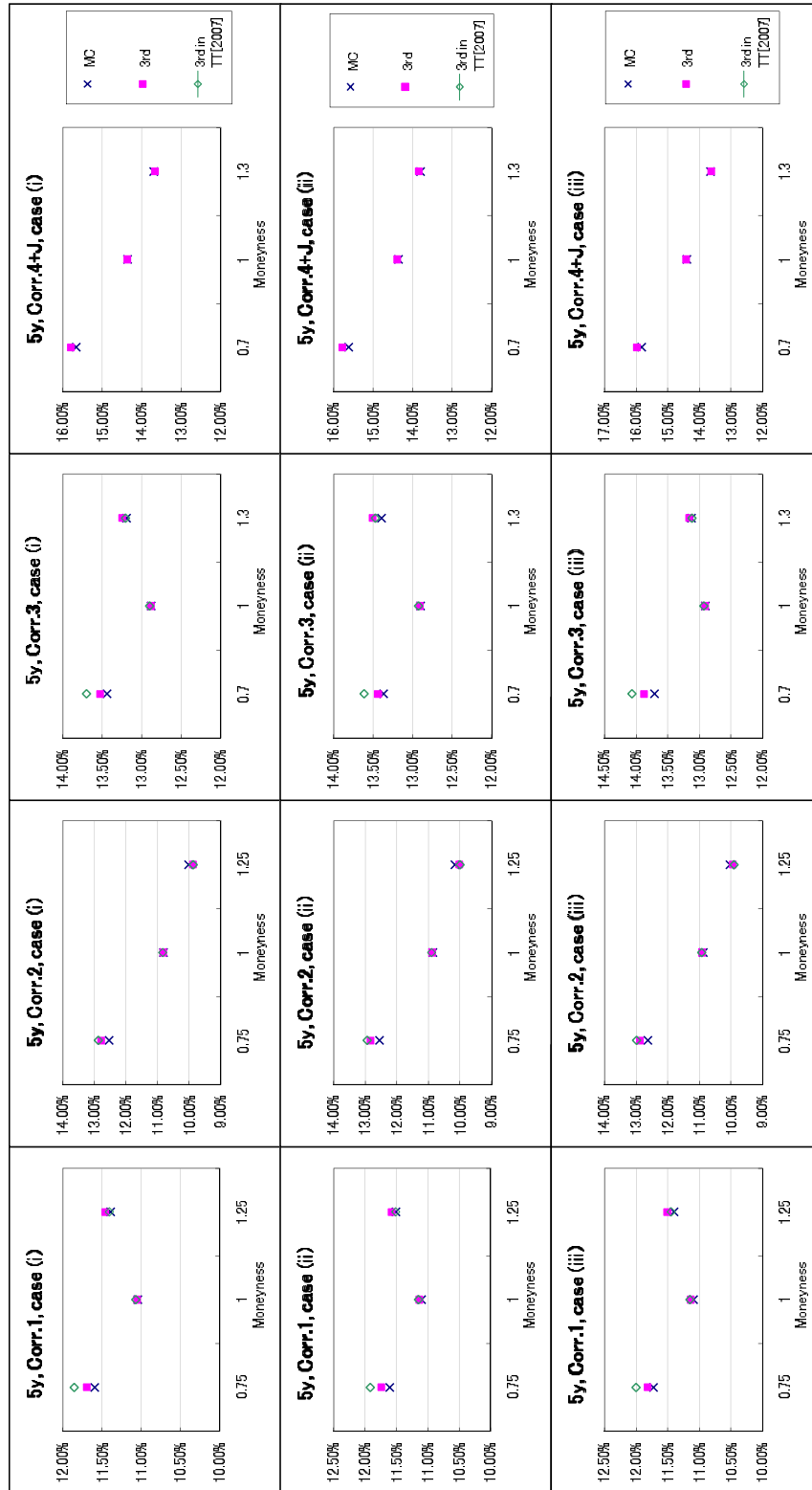
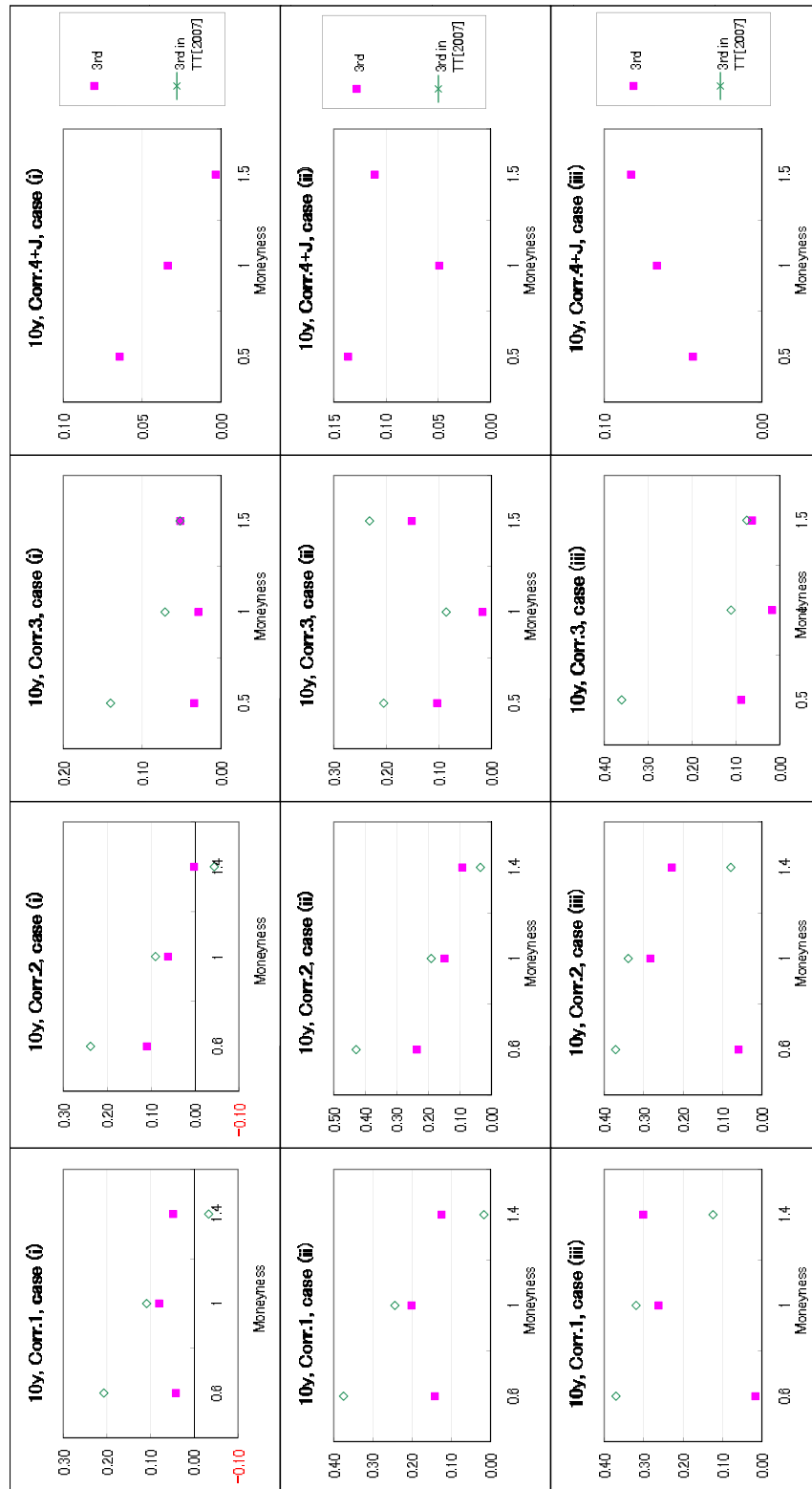Figure 3: A comparison of the accuracy of each estimator in prices for a ten-year maturity.

Figure 4: A comparison of the accuracy of each estimator in implied volatilities for a ten-year maturity.
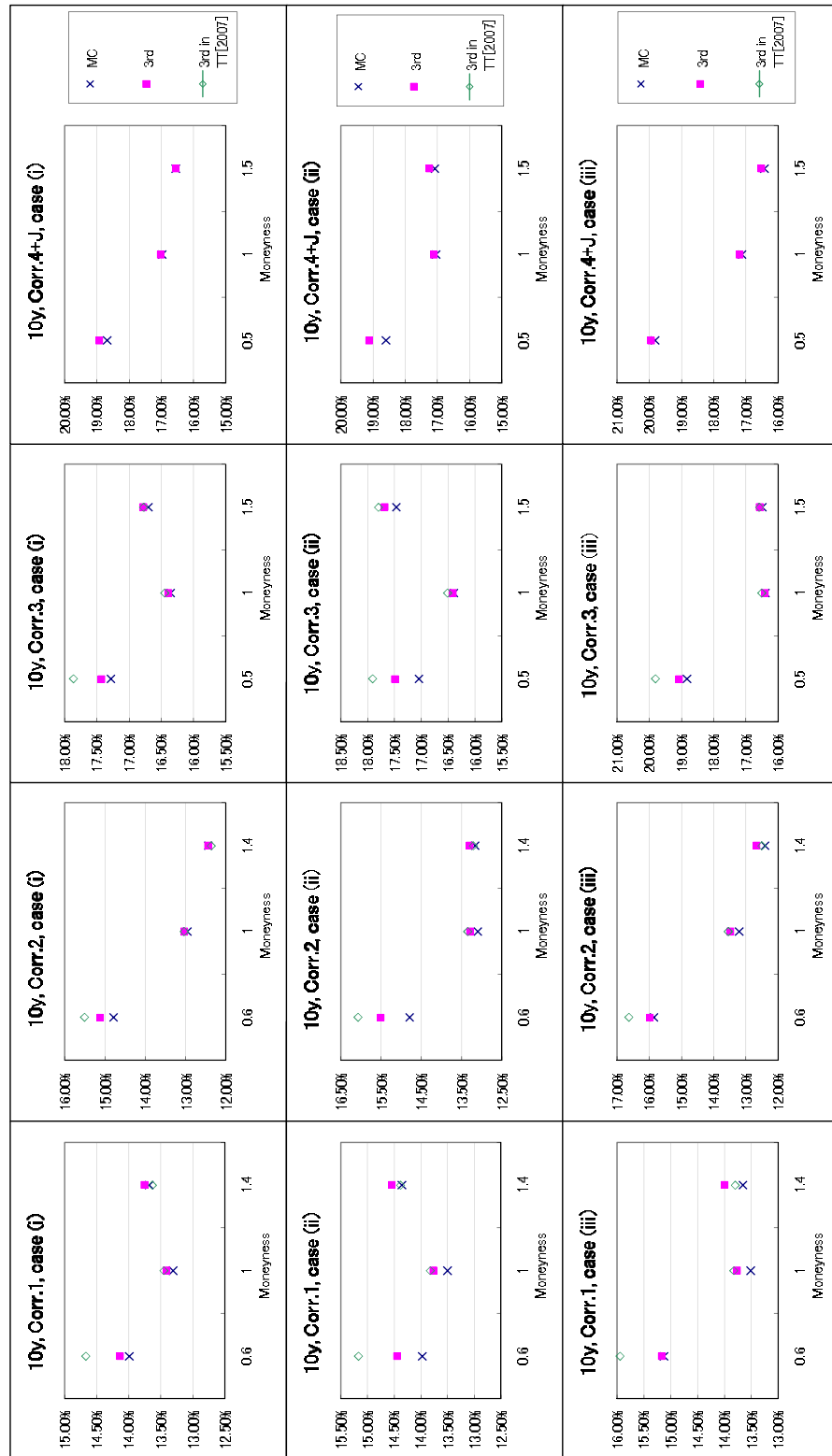
Table 4: A joint calibration to the observed implied volatilities for a five-year and ten-year maturity.

Date	07/09/27	Type	Joint

5y	10put	25put	ATM	25call	10call
Market I.V.	16.66%	10.92%	7.90%	6.07%	6.96%
Model I.V.	16.88%	11.14%	7.88%	6.07%	6.87%
differences	0.22%	0.22%	-0.02%	0.00%	-0.09%
10y	10put	25put	ATM	25call	10call
Market I.V.	20.00%	13.59%	10.50%	8.34%	9.50%
Model I.V.	19.76%	13.63%	10.51%	8.37%	9.13%
differences	-0.24%	0.04%	0.01%	0.03%	-0.37%

Date	07/10/30	Type	Joint

5y	10put	25put	ATM	25call	10call
Market I.V.	15.58%	11.20%	8.10%	6.60%	6.38%
Model I.V.	15.90%	11.11%	8.09%	6.58%	6.50%
differences	0.32%	-0.09%	-0.01%	-0.01%	0.12%
10y	10put	25put	ATM	25call	10call
Market I.V.	19.29%	13.49%	10.35%	8.19%	8.69%
Model I.V.	18.99%	13.57%	10.35%	8.19%	8.69%
differences	-0.30%	0.08%	0.00%	0.01%	-0.11%

Date	07/12/07	Type	Joint

5y	10put	25put	ATM	25call	10call
Market I.V.	17.20%	11.72%	8.45%	6.62%	7.00%
Model I.V.	17.40%	11.77%	8.45%	6.68%	6.94%
differences	0.20%	0.05%	-0.03%	0.06%	-0.05%
10y	10put	25put	ATM	25call	10call
Market I.V.	20.46%	13.98%	10.65%	8.38%	8.87%
Model I.V.	20.25%	13.87%	10.68%	8.38%	8.87%
differences	-0.21%	-0.11%	0.03%	0.00%	-0.39%

Figure 5: A separate calibration to the observed implied volatilities for a five-year or ten-year maturity on Sep 27, 2007.
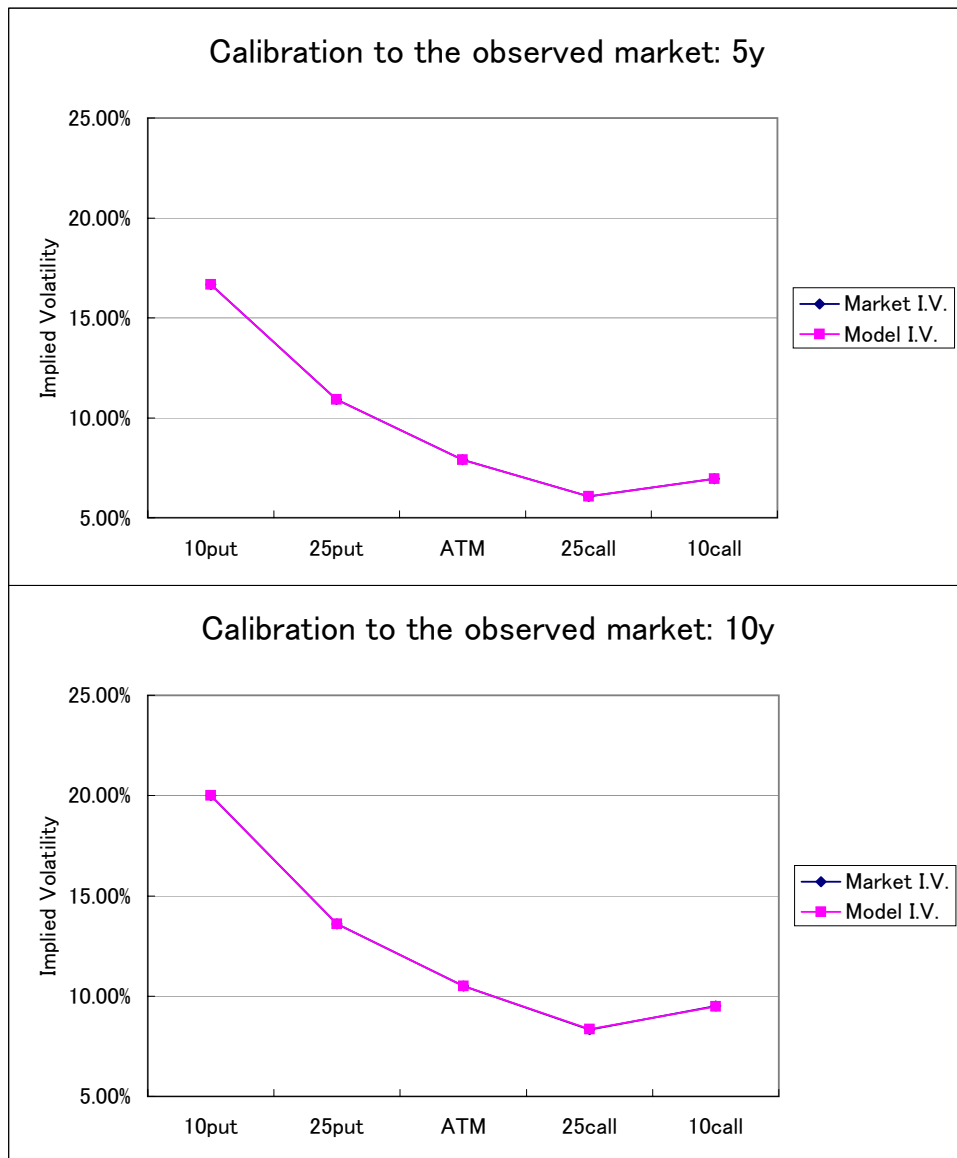
Figure 6: A separate calibration to the observed implied volatilities for a five-year or ten-year maturity on Oct 30, 2007.
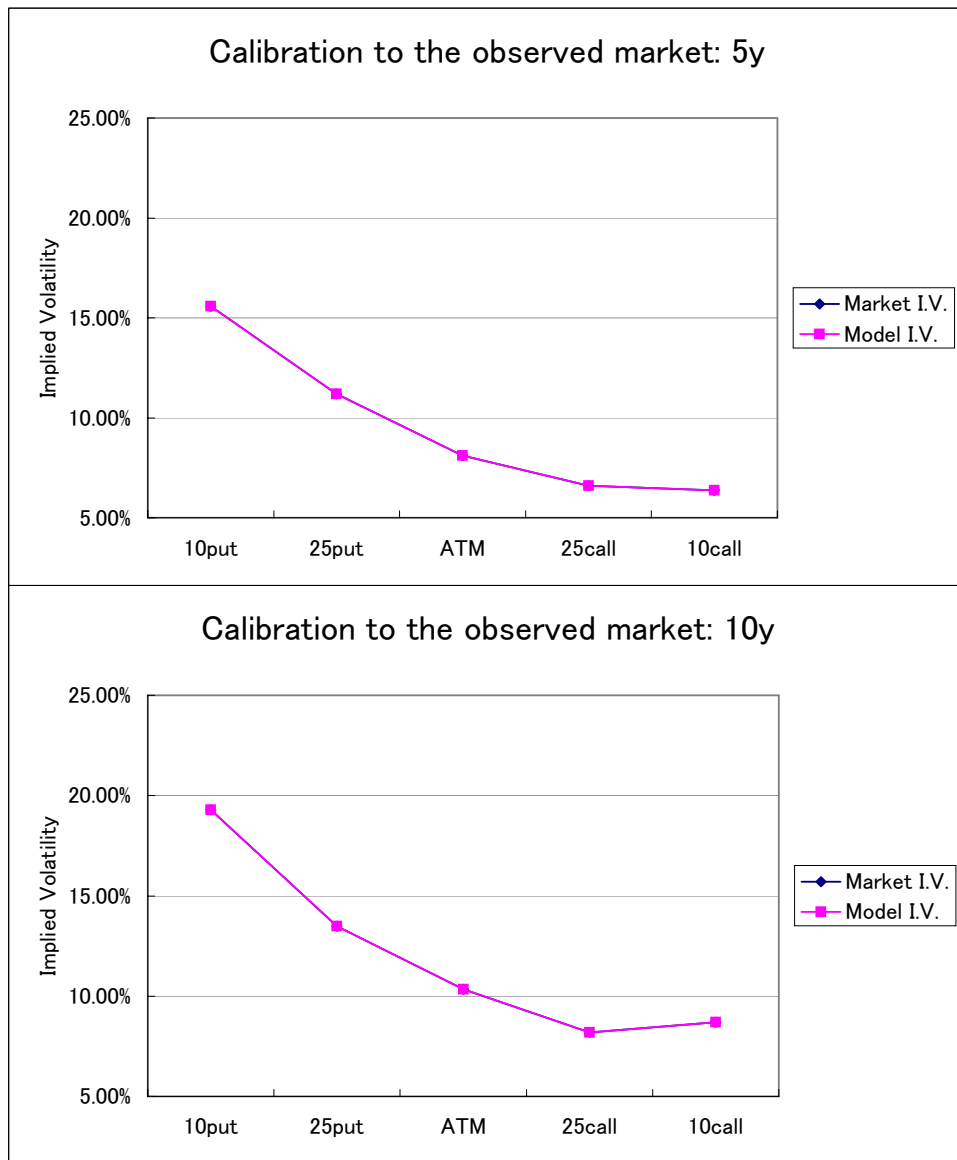
Figure 7: A separate calibration to the observed implied volatilities for a five-year or ten-year maturity on Dec 07, 2007.
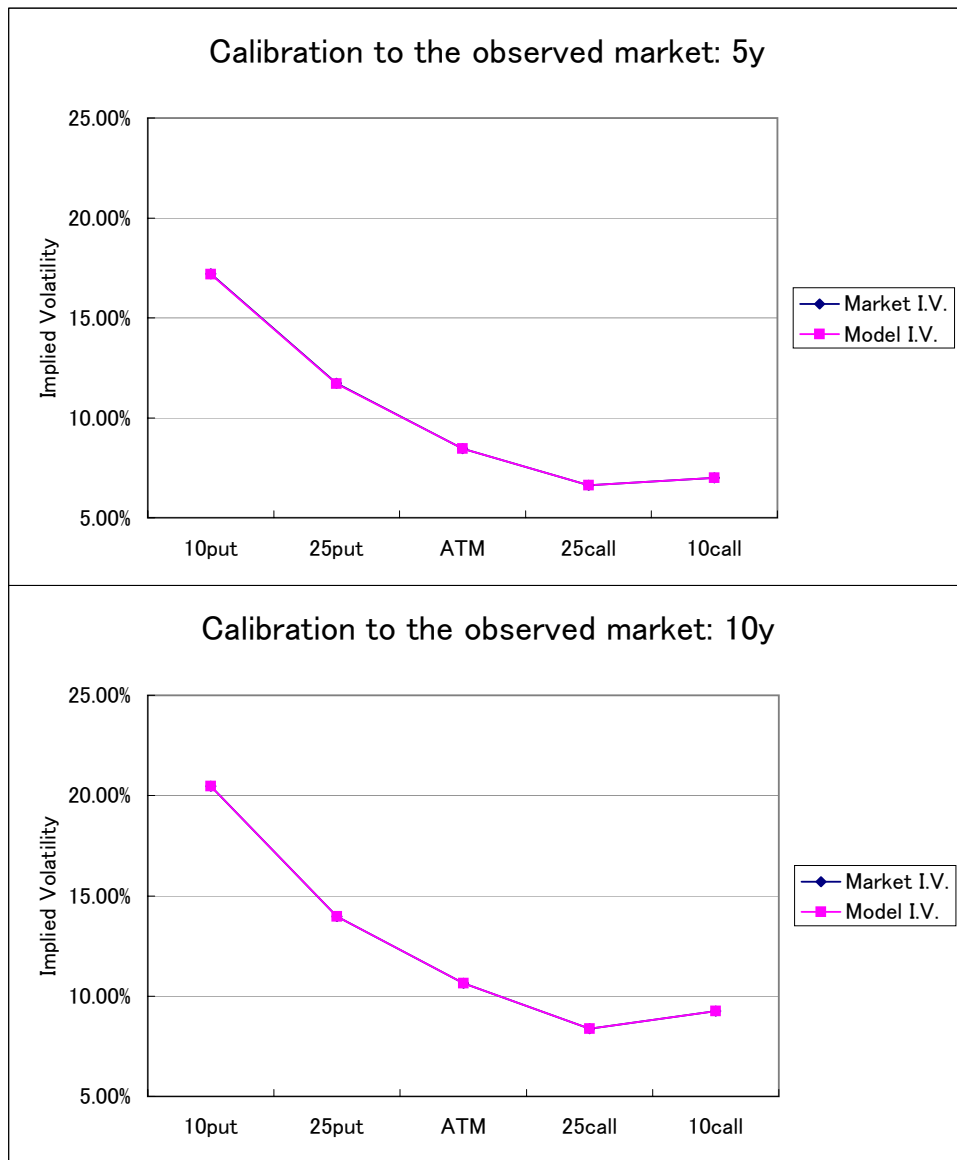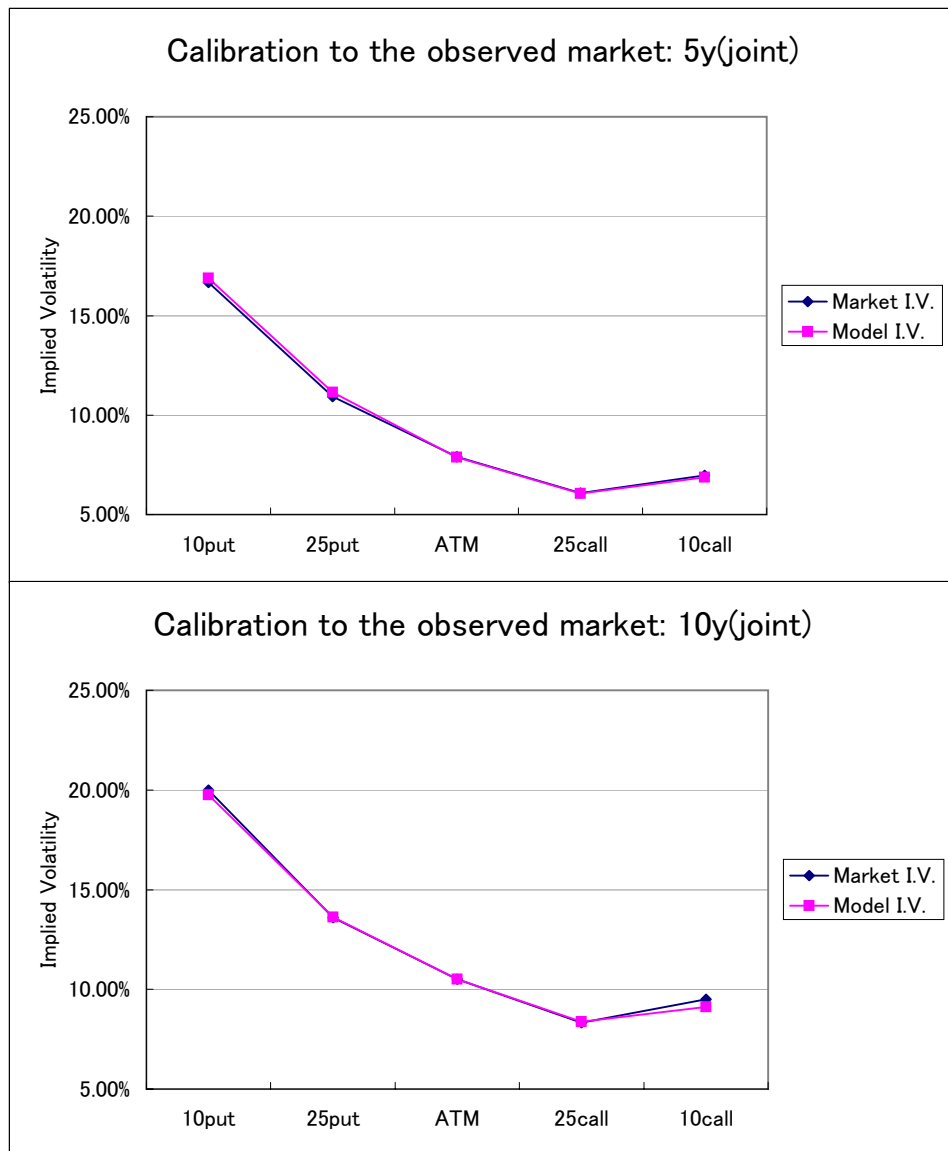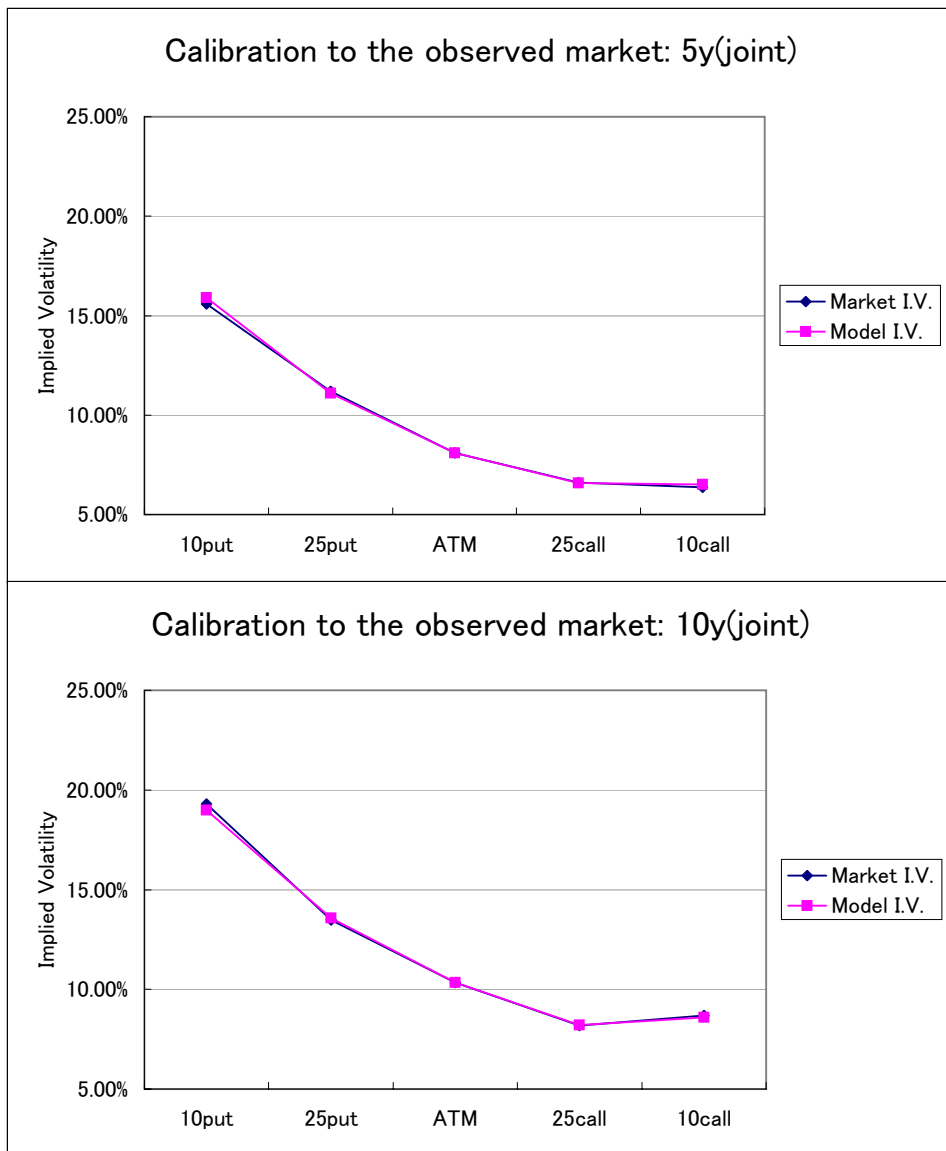
Figure 8: A joint calibration to the observed implied volatilities for a five-year and ten-year maturity on Sep 27, 2007.

Figure 9: A joint calibration to the observed implied volatilities for a five-year and ten-year maturity on Oct 30, 2007.

Figure 10: A joint calibration to the observed implied volatilities for a five-year and ten-year maturity on Dec 07, 2007.
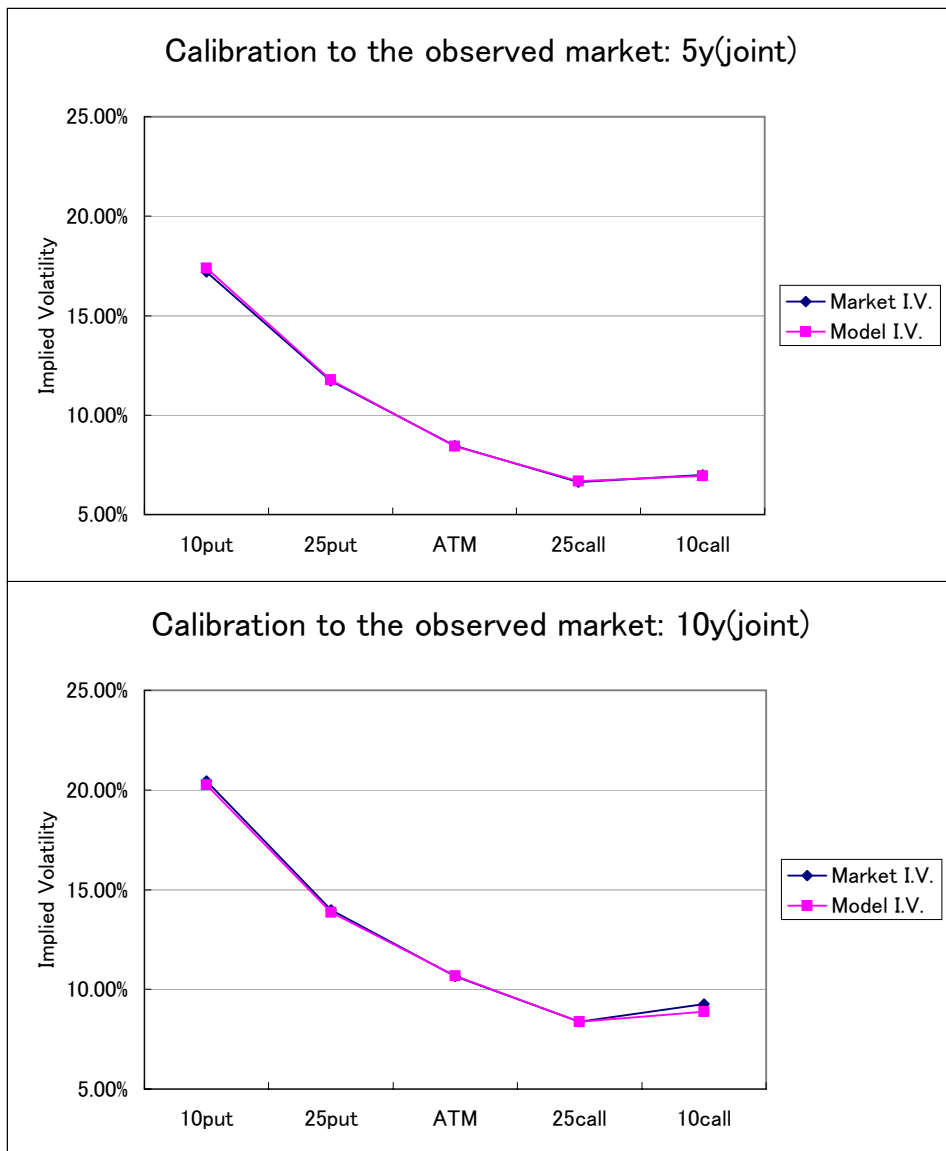
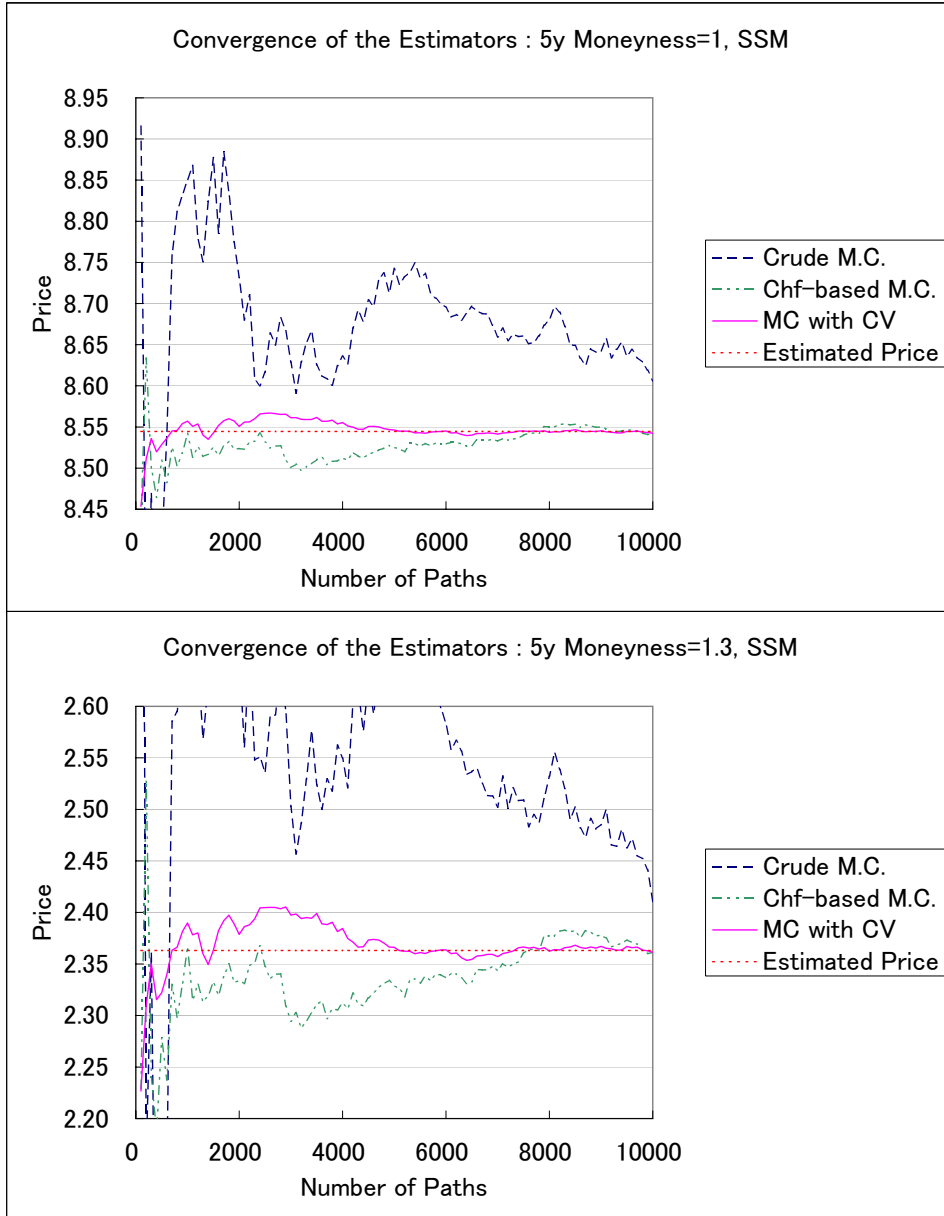Figure 11: Convergences of the estimators for a five-year maturity in SSM.

Figure 12: Convergences of the estimators for a ten-year maturity in SSM.

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