View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Research Papers in Economics

CARF Working Paper

CARF-F-113

Pricing and Hedging of Long-term Futures and Forward Contracts by a Three-Factor Model

Kenichiro Shiraya
Akihiko Takahashi
University of Tokyo

First Version: December, 2007
Current Version: April, 2009

CARF is presently supported by AIG, Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Mizuho Financial Group, Inc., Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

CARF Working Papers can be downloaded without charge from:
http://www.carf.e.u-tokyo.ac.jp/workingpaper/index.cgi

Working Papers are a series of manuscripts in their draft form. They are not intended for circulation or distribution except as indicated by the author. For that reason Working Papers may not be reproduced or distributed without the written consent of the author.

Pricing and Hedging of Long-term Futures and Forward Contracts by a Three-Factor Model

Kenichiro Shiraya[+] Akihiko Takahashi[*]

Abstract

This paper shows pricing and hedging efficiency of a three factor stochastic mean reversion Gaussian model of commodity prices using oil and copper futures and forward contracts. The model is estimated using NYMEX WTI (light sweet crude oil) and LME Copper futures prices and is shown to fit the data well. Furthermore, it shows how to hedge based on a three-factor model and confirms that using three different futures contracts to hedge long-term contract outperforms the traditional parallel hedge based on a single futures position by time series data and simulation. It also finds that the three factor model outperforms its two-factor version in replication of actual term structures and that stochastic mean reversion models outperform constant mean reversion models in Out of Sample hedges.

1. Introduction

The term structure of commodities futures undergoes complex shape changes and a number of different models have been proposed for its estimation. In this paper, we propose a three-factor model to estimate the term structure of commodities futures, and then propose and verify effective hedging techniques for long-term futures and forwards estimated with the model, using as hedging instruments the short and medium-term futures that are tradable.

Black (1976) advocated the idea of trading commodities as "equities without dividends" and made use of geometric Brownian motion. However, given the complexity of the shapes associated with the term structure of commodities futures, the simple geometric Brownian motion model proposed by Black (1976) is not a good fit. To resolve this problem, mean reversion has been introduced. Unlike equities, when commodities prices rise, there is generally (albeit with a time lag) an increase in supply; conversely, when prices decline, supply decreases. The fact that prices are determined by the supply and demand balance means that the supply side adjusts supply volumes, which has the effect of constraining the potential for commodities prices to move in a single direction. That is why it is generally considered appropriate to employ mean reversion in commodities pricing models. Much empirical research has been done on this. For example, it is verified in Bessembinder et al (1995).

[+] Mizuho-DL Financial Technology Co., Ltd. The views expressed in this paper are those of the author and do not necessarily represent the views of Mizuho-DL Financial Technology Co., Ltd.

[*] Graduate School of Economics, the University of Tokyo

Nonetheless, even if mean reversion is used in a one-factor model, it is difficult to represent the complex term structure of commodities futures, leading Gibson - Schwartz (1990) to propose a model that supplements the fluctuation of spot prices with a convenience yield stochastic process, and Schwartz (1997) to propose a model that explicitly employs convenience yields and interest rates as the stochastic process.

On the other hand, different methods have been proposed that do not attempt to individually model commodities spot prices, convenience yields or interest rates but instead attempt direct modeling using state variables with a mean reversion of spot prices. Examples of direct modeling of spot prices include Schwartz - Smith (2000)'s two-factor mean reversion model, Casassus - Dufresne (2005)'s three-factor mean reversion model and Cortazar - Naranjo (2006)'s N-factor mean reversion model.

We use three-factor Gaussian models with constant mean reversion or without constant mean reversion. The models' parameters are estimated using a Kalman filter and have been confirmed to reproduce actual futures prices on the NYMEX WTI (light sweet crude oil) and LME Copper markets. Where our research differs from prior research is that we study cases both with and without a constant mean reversion level in the commodities price model and provide a detailed analysis not only of the model's ability to reproduce futures prices, but also its utility in hedging.

Commodities hedging is a long-debated topic. For example, Culp - Miller (1995), Mello - Parsons (1995) and many other papers have discussed it in terms of the Metallgesellschaft case. Culp - Miller (1995) explains that, like equities, etc., the forward prices for commodities are determined by the mechanism of "cost of carry" and argues that long-term forward contracts can be hedged by holding short-term futures and rolling over the contract months. On the other hand, Mello - Parsons (1995) acknowledges that it is possible to use short-term futures to hedge long-term forward contracts, but criticizes the hedging technique employed by Metallgesellschaft, which was to use the same number of units of short-term futures to hedge a unit of long-term forward contracts. They use the Gibson - Schwartz (1990) model to demonstrate that short-term prices are more sensitive to spot price changes than long-term prices and that the actual number of short-term futures required to hedge 1 unit of long-term forward contracts is approximately 0.3. Because of this, the trading of Metallgesellschaft, while having hedging elements, is deemed to be primarily futures speculation. Schwartz (1997) also comments on this point, using 1-3 factor models to calculate hedge positions and explaining that when one factor is used, the position is significantly less than 1, approximately 0.2-0.4, and even with two- and three-factors it is still, on a net basis, less than 1. Neuberger (1999) uses multiple contracts to hedge long-term exposure and shows the benefits of the simultaneous use of different hedging instruments.

Examples of research analyzing not only hedge positions but also hedging errors include Brenann - Crew (1997), Korn (2005) and Buhler - Korn - Schobel (2004). Brenann - Crew (1997) attempts to use a number of different expiring futures as hedging instruments for hedges under a two-factor model, but all of the futures it uses as hedges expire within 6 months, and the futures to be hedged are also extremely short at no more than 2 years. Buhler- Korn - Schobel (2004) uses several different models to compare

and analyze performance when hedging 10-year forward contracts. However, the futures used as hedging instruments are extremely short, expiring in no more than 2 months, and the data also only goes until 1996, so this analysis does not incorporate the rapid rises in commodities prices seen in recent years. Korn (2005) showed hedging error with one and two-factor models, but he didn't show it with a three factor model.

In this paper, we compare hedging error performed by Metallgesellschaft's parallel hedging and performed by multi-factor model based hedging. More specifically, we verify the stability of hedges based on two- and three-factor models that do and do not have a constant mean reversion level, and provide detailed analysis of the differences in hedge effectiveness due to differences in the way in which state variables are calculated and differences in the required futures units, and hedging error rate distribution (based on its simulations) due to differences in the contract months of the futures used as hedging instruments. We also use time series data to verify hedges for long-term forward contracts, for which interest rate factors have been taken into account. We find that the three-factor model without constant mean reversion level is possible to effectively hedge long-term futures against the complex changes in term structures of recent years.

In section 2 we propose a three-factor model including a two-factor model as a special case, which does not explicitly incorporate interest rates or convenience yields and use that model to derive an analytic solution for futures prices. section 3 makes use of Kalman filters to estimate the model's parameters. section 4 goes on to make use of short and medium term futures to create a hedging technique for long-term futures and to analyze performance when this hedging strategy is used. section 5 takes a more practical approach, analyzing hedges on "Out of Sample" and long-term forward contracts. section 6 uses a simulation to analyze how the form of distribution changes for the hedge error rate depending upon the selection of futures contract months. In the appendix we provide the expectation and covariance of the model expressed in futures prices and notes on the numbers of units of nearer maturity futures required to hedge long-term futures.

2. Model

We first describe a three-factor Gaussian model used for pricing and hedging futures and forward contracts. S_t represents spot prices of commodities at time t. The logarithm of spot prices at this time is expressed by the following equation.

(1) $$\log S_t = x_t^1 .$$

x^1 expresses a state variable corresponding to the spot price of the commodity and follows the stochastic differential equation shown below.

(2) $$dx_t^1 = \kappa\left(x_t^2 + x_t^3 - x_t^1\right)dt + \sigma_1 dW_t^1 ,$$

$$dx_t^2 = -\gamma x_t^2 dt + \sigma_2 dW_t^2,$$

$$dx_t^3 = \left(\alpha - \beta x_t^3\right)dt + \sigma_3 dW_t^3.$$

x^2 expresses a state variable corresponding to the difference between medium-term and long term commodity futures prices; x^3 is a state variable corresponding to the long-term portion of the term structure. W_t^i ($i = 1,2,3$) mutually have the following correlations in standard Browning motion under equivalent Martingale measures (EMM).

(3) $$dW_t^i \cdot dW_t^j = \rho_{ij} dt, \ i, j = 1,2,3.$$

Parameter κ expresses x^1's speed of reversion to $x^2 + x^3$; γ expresses x^2's speed of attenuation. If $\gamma > 0$, then x^2 is pulled back towards 0. β expresses the speed with which x^3 reverts to α/β when $\beta \neq 0$. Therefore, intuitively, if $\kappa > \gamma > \beta > 0$, over the course of time the spot price:

$$x^1 \text{ (spot price)} \rightarrow x^2 + x^3 \text{ (medium-term price)} \rightarrow x^3 \text{ (long-term price).}$$

is the trend expressed.

The stochastic differential equations of individual state variables can be analytically solved and expressed as follows:

(4)
$$x_t^1 = e^{-\kappa t} x_0^1 + \frac{\kappa}{\kappa - \gamma}\left(e^{-\gamma t} - e^{-\kappa t}\right)x_0^2 + \frac{\kappa}{\kappa - \beta}\left(e^{-\beta t} - e^{-\kappa t}\right)x_0^3$$
$$+ \frac{\alpha}{\beta}\left(1 - \frac{\kappa}{\kappa - \beta}e^{-\beta t} + \frac{\beta}{\kappa - \beta}e^{-\kappa t}\right) + \sigma_1 \int_0^t e^{-\kappa(t-s)} dW_s^1$$
$$+ \sigma_2 \frac{\kappa}{\kappa - \gamma}\int_0^t \left(e^{-\gamma(t-s)} - e^{-\kappa(t-s)}\right)dW_s^2 + \sigma_3 \frac{\kappa}{\kappa - \beta}\int_0^t \left(e^{-\beta(t-s)} - e^{-\kappa(t-s)}\right)dW_s^3,$$

$$x_t^2 = e^{-\gamma t} x_0^2 + \sigma_2 \int_0^t e^{-\gamma(t-s)} dW_s^2,$$

$$x_t^3 = e^{-\beta t} x_0^3 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma_3 \int_0^t e^{-\beta(t-s)} dW_s^3.$$

At this time, the futures price is expressed as shown below.

Theorem 2.1.

Using $G_T(t)$ to represent the price at time t of a future with expiration T, under EMM:

(5) $$G_T(t) = E_t[S_T] = \exp\left\{\mu_{11}(x_t^1, x_t^2, x_t^3, T-t) + \frac{\Sigma_{11}(T-t)}{2}\right\}.$$

In this equation, E_t expresses the conditional expectation at time t. For a discussion of μ_{11} and Σ_{11}, see Appendix 1.

Proof.

S_T is a log-normal distribution and the result can therefore be found by calculating the moment generating function of normal distribution. Q.E.D.

Next consider the market price of risk. $\theta(t) = (\theta_1(t), \theta_2(t), \theta_3(t))$ is the market price of risk for state variables x^1, x^2 and x^3. At this time, the following relationship holds true between observed measure P and equivalent Martingale measure Q.

(6) $$W_t^Q = W_t^P + \int_0^t \theta(u)du .$$

Therefore, under measure P, the stochastic differential equations that satisfy individual state variables are:

(7) $$dx_t^1 = \kappa\left(x_t^2 + x_t^3 - x_t^1\right)dt + \sigma_1\theta_1(t)dt + \sigma_1 dW_t^{1,P},$$

$$dx_t^2 = -\gamma x_t^2 dt + \sigma_2\theta_2(t)dt + \sigma_2 dW_t^{2,P},$$

$$dx_t^3 = \left(\alpha - \beta x_t^3\right)dt + \sigma_3\theta_3(t)dt + \sigma_3 dW_t^{3,P}.$$

In particular, rewriting $\theta(t)$ with the state variables and a time function $\theta\left(t, x^1, x^2, x^3\right)$:

(8) $$\theta_1\left(t, x^1, x^2, x^3\right) = -a\left(x_t^1 - x_t^2 - x_t^3\right),$$

$$\theta_2\left(t, x^1, x^2, x^3\right) = -bx_t^2,$$

$$\theta_3\left(t, x^1, x^2, x^3\right) = \begin{cases} c - dx_t^3 & (\beta \neq 0) \\ c & (\beta = 0) \end{cases}.$$

The stochastic differential equation described above can therefore be rewritten as:

(9) $$dx_t^1 = \hat{\kappa}\left(x_t^2 + x_t^3 - x_t^1\right)dt + \sigma_1 dW_t^{1,P},$$

$$dx_t^2 = -\hat{\gamma} x_t^2 dt + \sigma_2 dW_t^{2,P},$$

$$dx_t^3 = \left(\hat{\alpha} - \hat{\beta} x_t^3\right)dt + \sigma_3 dW_t^{3,P},$$

where

$$\hat{\kappa} = \kappa + \sigma_1 a\,,\ \hat{\gamma} = \gamma + \sigma_2 b\,,\ \hat{\alpha} = \alpha + \sigma_3 c\,,\ \hat{\beta} = \beta + \sigma_3 d\,.$$

Remark 2.1.

In the discussion above, when $\beta = 0$, solving for the limit will enable analytic expression. Also, when $\beta = 0$, x^3 doesn't have a constant mean reversion level and the model itself does not have an ultimate mean reversion level. Below, this paper refers to cases in which $\beta \neq 0$ as the "constant mean reversion model," and $\beta = 0$ as the "stochastic mean reversion model." Both types of models are essentially contained by Cortazar - Naranjo (2006) or Casassus - Dufresne (2005).

Remark 2.2.

A two-factor constant mean reversion or two-factor stochastic mean reversion model can be obtained by setting $x^2 \equiv 0$. These models are essentially the same as in Korn (2005) that used the two-factor models for analysis of hedging. For a two-factor model in the subsequent analysis, we put a restriction, $x^2 \equiv 0$ in our three-factor models.

3. Estimation of parameters

This section estimates the parameters in the model.

Using v_n and w_n as white noise with mean 0 and variance 1, the model described above can be expressed as the following system model and observation model.

[System model]

$$x_n = F_n x_{n-1} + C_n^x + Q_n v_n\,,$$

$$x_n = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ F_n = \begin{pmatrix} e^{-\hat{\kappa}\Delta t} & \dfrac{\hat{\kappa}}{\hat{\kappa}-\hat{\gamma}}\left\{e^{-\hat{\gamma}\Delta t} - e^{-\hat{\kappa}\Delta t}\right\} & \dfrac{\hat{\kappa}}{\hat{\kappa}-\hat{\beta}}\left\{e^{-\hat{\beta}\Delta t} - e^{-\hat{\kappa}\Delta t}\right\} \\ 0 & e^{-\hat{\gamma}\Delta t} & 0 \\ 0 & 0 & e^{-\hat{\beta}\Delta t} \end{pmatrix},$$

$$E[Q_n] = 0\,,\ G_n = Cov[Q_n] = \left(\Sigma_{ij}(\Delta t)\right),\ C_n^x = \begin{pmatrix} \dfrac{\hat{\alpha}}{\hat{\beta}}\left\{1 - \dfrac{\hat{\kappa}e^{-\hat{\beta}\Delta t} - \hat{\beta}e^{-\hat{\kappa}\Delta t}}{\hat{\kappa}-\hat{\beta}}\right\} \\ 0 \\ \dfrac{\hat{\alpha}}{\hat{\beta}}\left\{1 - e^{-\hat{\beta}\Delta t}\right\} \end{pmatrix}.$$

* $\Sigma_{ij}(\Delta t)$ expresses covariance. For specific formulas, see Appendix 1.

[Observation model]

$$y_n = H_n x_n + C_n^y + R_n w_n,$$

$$y_n = \begin{pmatrix} \log G_{T_{1_n}}(0) \\ \vdots \\ \log G_{T_{m_n}}(0) \end{pmatrix}, H_n = \begin{pmatrix} e^{-\kappa T_{1_n}} & \frac{\kappa}{\kappa-\gamma}\left\{e^{-\gamma T_{1_n}} - e^{-\kappa T_{1_n}}\right\} & \frac{\kappa}{\kappa-\beta}\left\{e^{-\beta T_{1_n}} - e^{-\kappa T_{1_n}}\right\} \\ \vdots & \vdots & \vdots \\ e^{-\kappa T_{m_n}} & \frac{\kappa}{\kappa-\gamma}\left\{e^{-\gamma T_{m_n}} - e^{-\kappa T_{m_n}}\right\} & \frac{\kappa}{\kappa-\beta}\left\{e^{-\beta T_{m_n}} - e^{-\kappa T_{m_n}}\right\} \end{pmatrix},$$

$$R_n = \begin{pmatrix} h_1^2 & & 0 \\ & \ddots & \\ 0 & & h_m^2 \end{pmatrix}, C_n^y = \begin{pmatrix} \frac{\Sigma_{11}(T_{1_n})}{2} + \frac{\alpha}{\beta}\left\{1 - e^{-\kappa T_{1_n}} - \frac{\kappa}{\kappa-\beta}\left\{e^{-\beta T_{1_n}} - e^{-\kappa T_{1_n}}\right\}\right\} \\ \vdots \\ \frac{\Sigma_{11}(T_{m_n})}{2} + \frac{\alpha}{\beta}\left\{1 - e^{-\kappa T_{m_n}} - \frac{\kappa}{\kappa-\beta}\left\{e^{-\beta T_{m_n}} - e^{-\kappa T_{m_n}}\right\}\right\} \end{pmatrix}.$$

R_n expresses observational errors where h_i, $i = (1, \cdots m)$ denote those standard deviations.

In light of the computational burden, the paper assumes that the observational error of futures at individual maturities is independent. Parameters are estimated using the Kalman filter of this state-space representation. More specifically, the following prediction and filtering are alternatingly repeated and a parameter set ϑ is obtained so as to maximize the log-likelihood.

[Prediction]

$$x_{n|n-1} = F_n x_{n-1|n-1} + C_n^x,$$

$$V_{n|n-1} = F_n V_{n-1|n-1} F_n^{\ t} + G_n.$$

[Filtering]

$$d_{n|n-1} = H_n V_{n|n-1} H_n^{\ t} + R_n,$$

$$K_n = V_{n|n-1} H_n^{\ t} d_{n|n-1}^{\ -1},$$

$$x_{n|n} = x_{n|n-1} + K_n\left(y_n - H_n x_{n|n-1} - C_n^y\right),$$

$$V_{n|n} = \left(I - K_n H_n\right) V_{n|n-1}.$$

[Log-likelihood]

$$l(\vartheta) = -\frac{1}{2}\left\{mN\log(2\pi) + \sum_{n=1}^{N}\log\left|\det\left(d_{n|n-1}\right)\right| + \sum_{n=1}^{N} u_n^{\ t} d_{n|n-1}^{\ -1} u_n\right\},$$

$$u_n = y_n - H_n x_{n|n-1} - C_n^y.$$

Even if optimal values are not set for the initial values of x and V, as calculation proceeds using the Kalman filter both approach optimal values. Therefore, the initial value problem can be avoided by discarding several steps of data when estimating parameters without using the likelihood calculation. Estimations of two-factor models are obtained similarly.

3.1 Estimation results

The constant mean reversion model and stochastic mean reversion model parameters were estimated using the procedure described above. The following data was used for the estimations.

NYMEX WTI (light sweet crude oil)

Data in 5-business day increments was used for the periods January 1997 - October 2002, January 1997 - October 2003 and January 1997 - November 2007; futures contract are, from the closest: Front Month, 1st DEC, 2nd DEC, 3rd DEC, 4th DEC, 5th DEC, 6th DEC and 7th DEC. Here, j-th DEC stands for the j-th contract expiring in December. If Front Month = 1st DEC, it was used as front month. Data until 10th DEC exists after April 2007. However, data from 8th DEC to 10th DEC are not used in estimation due to lack of reliability of the data.

LME Copper

Data in 5-business day increments was used for the periods September 2002 - November 2004 and September 2002 - December 2007; futures contract are, from the closest: Front Month, 1st DEC, 2nd DEC, 3rd DEC, 4th DEC, 5th DEC and 6th DEC. If Front Month = 1st DEC, it was used as front month.

These are liquid and typical assets of oil and metal futures. The choice of time period is the longest period for which the data has mid-term (7th DEC in WTI, 6th DEC in Copper) futures. Tables 1-4 show the parameters and observational errors R_n obtained using the data described above.

Table 1: *three-factor model (WTI)*

	Constant mean reversion model						Stochastic mean reversion model					
	-Nov 07	Std Err	-Oct 03	Std Err	-Oct 02	Std Err	-Nov 07	Std Err	-Oct 03	Std Err	-Oct 02	Std Err
κ	1.112	0.011	1.007	0.014	1.107	0.020	1.090	0.009	1.159	0.015	1.048	0.017
γ	0.275	0.007	0.159	0.012	0.293	0.018	0.262	0.007	0.284	0.009	0.253	0.011
α	0.006	0.010	0.052	0.023	0.311	0.024	-0.009	0.001	-0.007	0.001	-0.007	0.002
β	0.004	0.001	0.021	0.008	0.110	0.007	-	-	-	-	-	-
a	0.000	-	0.000	-	0.597	1.787	0.000	-	0.000	-	0.529	1.861
b	0.000	-	0.000	-	0.000	-	0.000	-	0.000	-	0.000	-
c	0.528	0.341	0.000	-	0.070	0.219	0.482	0.266	0.062	0.318	0.063	0.347
d	0.005	0.019	0.002	0.039	0.017	0.071	-	-	-	-	-	-
Sigma1	0.362	0.011	0.359	0.014	0.371	0.015	0.360	0.010	0.387	0.015	0.363	0.013
Sigma2	0.144	0.004	0.202	0.028	0.374	0.043	0.142	0.003	0.137	0.004	0.138	0.005
Sigma3	0.170	0.059	0.251	0.067	0.399	0.065	0.190	0.005	0.171	0.006	0.182	0.008
Rho12	0.155	0.047	0.048	0.077	-0.173	0.051	0.153	0.047	0.155	0.065	0.192	0.070
Rho23	-0.601	0.205	-0.851	0.199	-0.983	0.116	-0.539	0.033	-0.599	0.052	-0.619	0.062
Rho31	0.396	0.141	0.237	0.067	0.352	0.065	0.369	0.034	0.282	0.047	0.261	0.051
Front Month	0.041	0.001	0.049	0.002	0.047	0.002	0.041	0.001	0.047	0.002	0.045	0.002
1stDec	0.000	-	0.004	0.002	0.000	-	0.000	-	0.000	-	0.000	-
2ndDec	0.007	0.000	0.013	0.001	0.006	0.000	0.007	0.000	0.008	0.000	0.007	0.000
3rdDec	0.002	0.000	0.008	0.000	0.003	0.000	0.002	0.000	0.002	0.000	0.002	0.000
4thDec	0.004	0.000	0.001	0.000	0.004	0.000	0.004	0.000	0.005	0.000	0.005	0.000
5thDec	0.003	0.000	0.001	0.000	0.002	0.000	0.003	0.000	0.003	0.000	0.003	0.000
6thDec	0.000	-	0.001	0.000	0.002	0.000	0.000	-	0.000	-	0.000	-
7thDec	0.004	0.000	0.003	0.000	0.004	0.000	0.004	0.000	0.004	0.000	0.004	0.000
AIC	-27157.30		-16837.37		-14499.01		-27146.66		-16701.48		-14241.34	

Table 2: *three-factor model (Copper)*

	Constant mean reversion model				Stochastic mean reversion model			
	-Dec 07	Std Err	-Nov04	Std Err	-Dec 07	Std Err	-Nov04	Std Err
κ	0.766	0.046	0.918	0.173	0.740	0.039	0.930	0.180
γ	0.177	0.036	0.143	0.069	0.161	0.024	0.153	0.036
α	0.329	0.114	0.036	0.633	-0.059	0.010	-0.014	0.009
β	0.052	0.015	0.007	0.084	-	-	-	-
a	0.000	-	0.000	-	0.000	-	0.000	-
b	0.000	-	1.720	2.915	0.000	-	2.044	1.890
c	0.092	0.212	0.079	0.567	0.153	0.331	0.096	0.546
d	0.000	-	0.000	0.023	-	-	-	-
Sigma1	0.268	0.013	0.224	0.021	0.268	0.013	0.224	0.021
Sigma2	0.631	0.199	0.317	0.307	0.437	0.043	0.287	0.056
Sigma3	0.658	0.201	0.319	0.350	0.440	0.039	0.287	0.046
Rho12	0.074	0.080	0.368	0.339	0.256	0.069	0.402	0.112
Rho23	-0.929	0.057	-0.840	0.593	-0.845	0.033	-0.801	0.072
Rho31	0.237	0.082	0.130	0.250	0.200	0.070	0.155	0.139
Front Month	0.000	-	0.000	-	0.000	-	0.000	-
1stDec	0.007	0.000	0.006	0.001	0.007	0.000	0.006	0.001
2ndDec	0.003	0.000	0.001	0.001	0.003	0.000	0.001	0.001
3rdDec	0.005	0.000	0.004	0.000	0.005	0.000	0.003	0.000
4thDec	0.000	-	0.003	0.001	0.000	-	0.003	0.001
5thDec	0.010	0.001	0.002	0.002	0.010	0.001	0.002	0.002
6thDec	0.011	0.000	0.008	0.002	0.013	0.001	0.008	0.001
AIC	-9538.94		-4259.00		-9520.27		-4264.08	

Table 3: *two-factor model (WTI)*

	Constant mean reversion model						Stochastic mean reversion model					
	-Nov 07	Std Err	-Oct 03	Std Err	-Oct 02	Std Err	-Nov 07	Std Err	-Oct 03	Std Err	-Oct 02	Std Err
κ	0.218	0.004	0.042	0.003	0.511	0.007	0.334	0.006	0.379	0.008	0.395	0.007
α	-0.074	0.017	1.187	0.033	0.234	0.014	-0.009	0.001	-0.003	0.001	-0.005	0.001
β	0.012	0.001	0.390	0.010	0.084	0.002	-	-	-	-	-	-
a	0.000	-	0.000	-	0.000	-	0.000	-	0.000	-	0.000	-
c	0.072	0.434	2.093	1.300	0.080	0.862	0.218	0.293	0.035	0.389	0.000	-
d	0.001	0.002	1.617	0.698	0.000	0.262	-	-	-	-	-	-
Sigma1	0.426	0.027	0.171	0.009	0.225	0.012	0.193	0.006	0.175	0.008	0.190	0.010
Sigma3	0.520	0.036	0.865	0.129	0.222	0.076	0.180	0.005	0.160	0.005	0.167	0.006
Rho31	0.941	0.034	-0.748	0.145	0.114	0.061	0.540	0.029	0.419	0.041	0.375	0.047
Front Month	0.145	0.006	0.137	0.007	0.009	0.005	0.145	0.006	0.142	0.008	0.127	0.007
1stDec	0.101	0.009	0.087	0.009	0.002	0.003	0.075	0.005	0.075	0.006	0.066	0.005
2ndDec	0.029	0.002	0.016	0.001	0.000	0.001	0.016	0.001	0.016	0.001	0.015	0.001
3rdDec	0.009	0.000	0.002	0.000	0.000	0.000	0.002	0.000	0.003	0.000	0.002	0.001
4thDec	0.001	0.000	0.003	0.000	0.000	0.000	0.003	0.000	0.005	0.000	0.006	0.000
5thDec	0.002	0.000	0.000	-	0.000	-	0.000	-	0.004	0.000	0.004	0.000
6thDec	0.000	-	0.004	0.000	0.000	0.000	0.005	0.000	0.000	-	0.000	-
7thDec	0.004	0.000	0.007	0.000	0.000	0.001	0.009	0.000	0.005	0.000	0.005	0.000
AIC	-22211.90		-14422.80		-12770.44		-22632.97		-14387.29		-12253.04	

Table 4: *two-factor model (Copper)*

	Constant mean reversion model				Stochastic mean reversion model			
	-Dec 07	Std Err	-Nov04	Std Err	-Dec 07	Std Err	-Nov04	Std Err
κ	0.122	0.006	0.327	0.016	0.137	0.018	0.327	0.015
α	2.960	0.151	0.002	0.124	-0.032	0.014	-0.016	0.003
β	0.407	0.021	0.001	0.017	-	-	-	-
a	0.000	-	0.000	-	0.000	-	0.000	-
c	1.879	5.062	0.000	-	0.484	0.497	0.000	-
d	0.152	0.652	0.034	0.708	-	-	-	-
Sigma1	0.275	0.015	0.223	0.021	0.265	0.013	0.218	0.020
Sigma3	0.927	0.469	0.092	0.212	0.469	0.049	0.194	0.015
Rho31	-0.239	0.138	0.893	2.021	0.014	0.078	0.409	0.079
Front Month	0.026	0.001	0.014	0.003	0.040	0.002	0.014	0.003
1stDec	0.000	-	0.003	0.003	0.003	0.004	0.003	0.003
2ndDec	0.010	0.000	0.006	0.002	0.026	0.003	0.006	0.002
3rdDec	0.000	-	0.005	0.001	0.018	0.003	0.005	0.001
4thDec	0.015	0.002	0.003	0.001	0.000	-	0.003	0.001
5thDec	0.031	0.004	0.005	0.001	0.024	0.002	0.005	0.001
6thDec	0.042	0.004	0.008	0.002	0.061	0.006	0.008	0.002
AIC	-7662.46		-3927.20		-6806.08		-3922.25	

Here, we note that observational errors in three-factor models are very small and that the model replicates the observed futures prices very well. In the two-factor constant mean reversion model, estimates of β using the WTI data up to 2003 and Copper data up to 2007 were $\beta \approx 0.4$, and hence x^3 was observed to have a constant mean reversion level, but in all other periods, both WTI and Copper had β of virtually 0. Therefore, x^3 does not fluctuate with a constant mean reversion level, but is rather more similar to a random walk. The parameters for Copper are significantly different between the data set

up to 2004 and the data set up to 2007. When estimations are made using the data up to 2004, there is only a little more than 2 years data used, and presumably the calculation results in biased parameters that are optimized to these 2 years. The market price of risk is expressed largely in parameter c and d for either WTI or Copper. We also observe that standard errors of Copper's parameters are worse than those of WTI's ones in part due to shortage of data used in estimation. Finally, three-factor models show better fitting results than two-factor models in terms of AIC (Akaike's Information's Criterion).

3.2 Comparison against actual data

This section verifies the degree of correlation between the state variables calculated with the Kalman filter using data through 2007and settlement future prices for NYMEX WTI and LME Copper.

As explained in section 2, the state variables correspond to the term structure of futures. In this case, the state variables are assumed to have the correspondences noted in Table 5 and the analysis seeks to determine the degree of correlation between them.

Table 5: *Correspondence of state variables*

		WTI	Copper
3factor	X1	Front Month Future Price	Front Month Future Price
	X2	(3rd DEC Future Price) - (6th DEC Future Price)*	(2nd DEC Future Price) - (5th DEC Future Price)*
	X3	6th DEC Future Price	5th DEC Future Price
2factor	X1	2nd DEC Future Price	Front Month Future Price
	X3	6th DEC Future Price	5th DEC Future Price

* X2 is compared with the spread between 6th Dec and 3rd Dec for WTI, and the spread between 5th Dec and 2nd Dec for Copper.

Table 6 contains correlations for state variables and logarithmic prices calculated from their 5-business day increments.

Table 6: *Correlations*

		Constant mean reversion model		Stochastic mean reversion model	
		WTI	Copper	WTI	Copper
3factor	X1	0.926	0.923	0.927	0.924
	X2	0.944	0.933	0.941	0.939
	X3	0.980	0.841	0.977	0.821
2factor	X1	0.857	0.885	0.864	0.876
	X3	0.959	0.720	0.986	0.784

Both WTI and Copper have generally high correlations, indicating that the movement of state variables roughly corresponds to actual data. Also, the three-factor models provide higher correlations than the two-factor models.

Next, we examine whether models can reproduce the actual term structures of futures prices. Figure 1 shows the term structures of two-factor and three-factor models against market prices of WTI futures in November 3rd, 2003, November 1st, 2004, November 1st, 2005, November 1st, 2006 and November 1st, 2007, respectively. Also Figure 2 shows results of Copper in December 1st, 2003, December 1st, 2004, December 1st, 2005, December 1st, 2006 and December 3rd, 2007.

Figure 1: *WTI Future term structure*

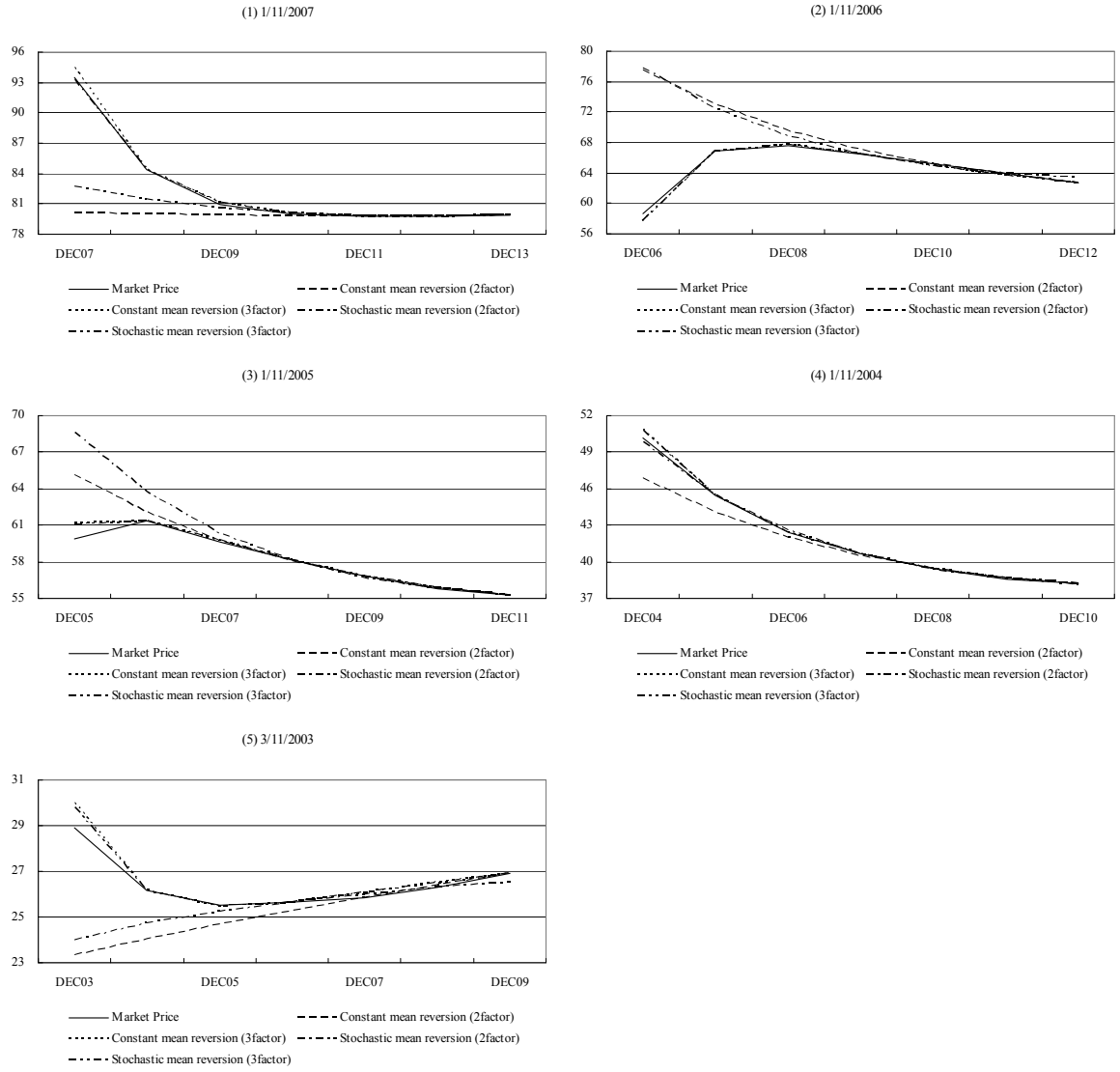
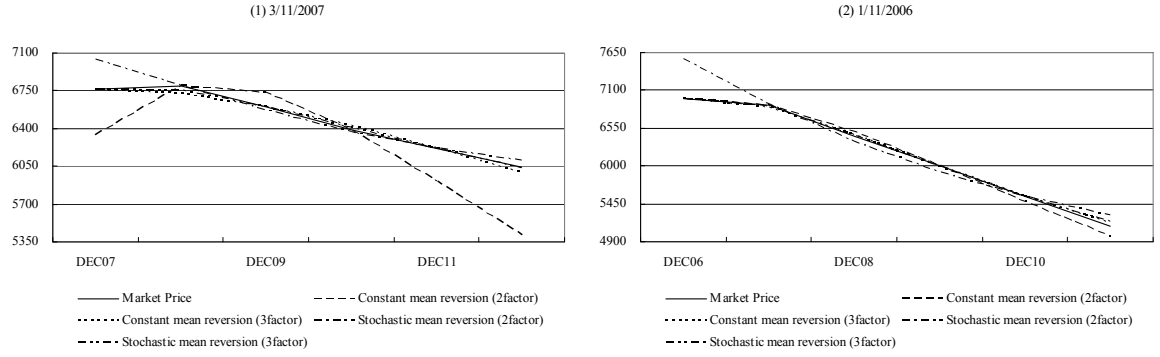
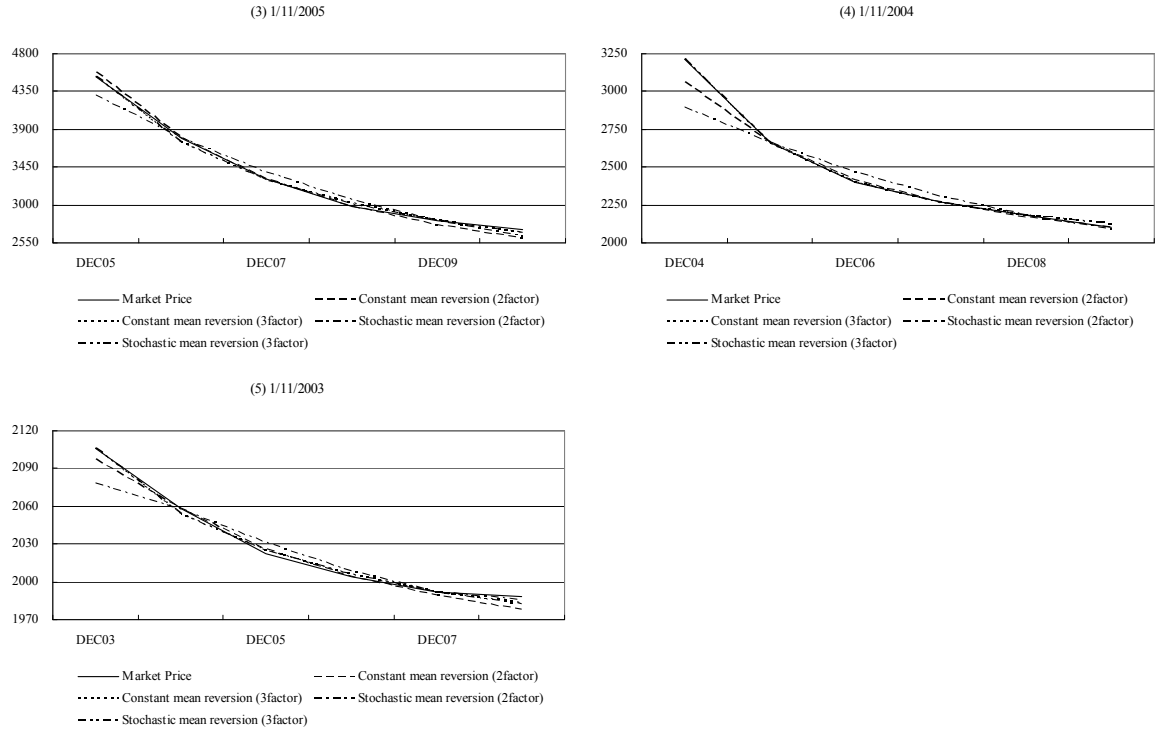


Figure 2: *Copper Future term structure*





We can observe in those cases that three-factor models can replicate the actual term structures well while two-factor models have some difficulty in capturing the actual term structures. In particular, the difference of fitting between the two-factor model and the three-factor model frequently occurs in 2006 and 2007 to the extent observed in the figures (1) and (2) of Table 1 and Table 2.

4. Futures hedging techniques

This section describes a method for building a hedging strategy for 1 unit of a long-term futures contract and observes how the three-factor model described in this paper can be applied to this task.

The equation expressing the futures price uses state variables x^1, x^2 and x^3 so that the shape of the futures price changes according to changes in these state variables (assuming no change in the parameters). Therefore, it is possible in theory to hedge against long-term futures price fluctuations by calculating the deltas of the state variables for the long-term futures price and taking a position $\Phi = (\phi_1, \phi_2, \phi_3)^t$ in the nearer maturity future that cancels out those deltas.

In a three-factor model, there are 3 factors to be hedged and therefore futures with 3 different expirations will be required to build the hedge portfolio. $G_{T_1}(t)$, $G_{T_2}(t)$ and $G_{T_3}(t)$ express nearer maturity futures prices of different expirations, and $G_{T_4}(t)$ the long-term futures price to be hedged. In this case, Φ is the solution to the following simultaneous equation.

$$A\Phi = b,$$

where

$$A = \begin{pmatrix} \dfrac{\partial G_{T_1}(t)}{\partial x_1} & \dfrac{\partial G_{T_2}(t)}{\partial x_1} & \dfrac{\partial G_{T_3}(t)}{\partial x_1} \\ \dfrac{\partial G_{T_1}(t)}{\partial x_2} & \dfrac{\partial G_{T_2}(t)}{\partial x_2} & \dfrac{\partial G_{T_3}(t)}{\partial x_2} \\ \dfrac{\partial G_{T_1}(t)}{\partial x_3} & \dfrac{\partial G_{T_2}(t)}{\partial x_3} & \dfrac{\partial G_{T_3}(t)}{\partial x_3} \end{pmatrix}, \quad b = \begin{pmatrix} \dfrac{\partial G_{T_4}(t)}{\partial x_1} \\ \dfrac{\partial G_{T_4}(t)}{\partial x_2} \\ \dfrac{\partial G_{T_4}(t)}{\partial x_3} \end{pmatrix}.$$

This paper refers to hedging using the hedging portfolio Φ as a "delta hedge". For a two-factor model, it is possible to construct a delta hedge in the similar way by eliminating the second factor x^2 in the corresponding three-factor model. We verify the degree of hedging error against this hedging portfolio when time series data is applied. For the purposes of this paper, the "hedging error rate" is expressed as the final cumulative hedging error divided by the price of the instrument to be hedged at the time the hedge commences.

For comparison, we calculate the hedging error ratio for hedges such as performed by Metallgesellschaft in which an equivalent number of nearer maturity futures is held against the future to be hedged. This paper refers to this hedging method as the "parallel hedge." Metallgesellshcaft hedged its long-term futures with extremely short-term futures of 1-3 contract months. However, given the increased liquidity of current commodities futures markets into the medium-term range, we verify the effectiveness of parallel hedges using futures of up to 6 years for WTI and up to 5 years for Copper.

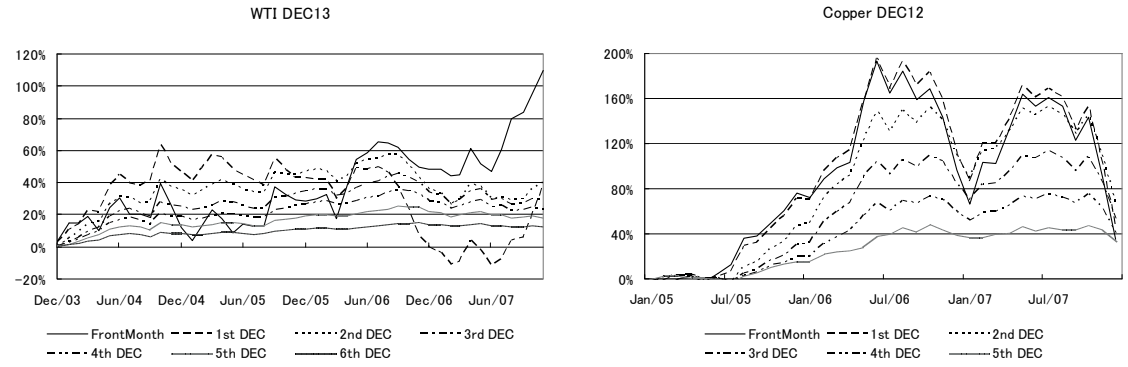
Unless specifically stated to the contrary, the discussion below refers to hedges against the 10th DEC from the front month for the WTI and the 8th DEC for Copper, of which prices are estimated by our models. For the hedging period, it is assumed that the position will be closed with an offsetting trade of the 6th DEC future for the WTI. In other words, a 4-year hedge is entered into that reduces the time to maturity of the instrument to be hedged from 10 years to 6 years. For Copper, it is assumed that the position is closed with an offsetting trade of the 5th DEC future, resulting in a 3-year hedge that reduces the time to maturity of the instrument to be hedged from 8 years to 5 years. For the parallel hedge, futures for listed DECs are used as hedge assets. For the delta hedges of three-factor models, the 1st - 4th - 6th DECs, and 1st - 3 rd - 5th DECs are used for WTI and Copper, respectively; For the delta hedges of two-factor models, 4th - 6th DECs and 3rd - 5th DECs are used for WTI and Copper, respectively. Positions in each futures contract months are adjusted on the 1st business day of the month after reviewing hedging ratios each month. For both the parallel hedge and delta hedge, upon the elapse of 1-year, positions are rolled to the same contract month in the next year. (For example, if a DEC 6 position is used to initiate a hedge on DEC 12, after the elapse of 1-year, the DEC 6 position used in the hedge will be rolled over to DEC 7.) Liquidity declines the more distant the future, but DEC futures have comparatively high liquidity, and given the infrequency with which hedge ratios are changed and the small degree of change in the number of units required for hedging, this is considered a realistic hedge. In selecting futures contract months, this analysis uses combinations that provide the relatively small hedging error rates obtained in section 6. The similar procedure is taken in selecting futures contract months for two-factor models.

4.1 Hedging error rate of the parallel hedge

The paper first verifies the degree of hedging error rate achieved using the parallel hedge. The price of the futures contract month to be hedged is calculated based on the constant mean reversion model and data up to 2007 using parameters and state variables estimated with the Kalman filter.

Figure 3 shows cumulative hedging error rates (the cumulative hedging error divided by the price of the instrument to be hedged at the time the hedge commences) using WTI and Copper time series data. In this case, for the Front Month the cumulative hedging error rate is expressed for a parallel hedge rolled over to the next-expiring contract month each month; for others, the cumulative hedging error rate is expressed for a parallel hedge with a one-year roll using DECs for each year. The futures to be hedged are the WTI DEC 13 and the Copper DEC 12 and the hedge terminates at the most recently available data (2007). The horizontal axis expresses the amount of time elapsed since the commencement of the hedge; the vertical axis, the cumulative hedging error rate. The same notation is used for other graphs in this paper.

Figure 3: *Cumulative Hedging error rates of the parallel hedge*



As can be observed from Figure 3, error is lower the more distant the future used to hedge. Copper has a larger hedging error rate than WTI, indicating that the components in Copper's term structure that change in parallel are smaller than WTI's. However, even using the most distant future with the smallest hedging error rate, the hedging error rates with a parallel hedge were still approximately 12% for WTI and approximately 32% for Copper.

4.2 Hedging error rate of the delta hedge

This section observes the hedging error rate for delta hedges for both the constant mean reversion model and the stochastic mean reversion model.

For each model, parameters estimated from data up to 2007 were used, and for verification purposes, 2 methods were used to estimate state variables in order to estimate long-term futures prices. The first is state variables were estimated using the Kalman filter ("Kalman filter state variables" hereinafter); the second estimation created simultaneous linear equations for the state variables so that the futures price of the model matches the futures price of the futures contract month to be hedged, allowing state variables to be calculated by solving these equations ("simultaneous equation-based state variables" hereinafter). To compare the relative precision of hedging using the model described in this paper, the verifications below

note the results for most distant future, which was the most precise for the parallel hedge. Notations follow the practice used for parallel hedges.

Verifications were performed with different futures to be hedged, and the observed hedging error rates are summarized in Table 7.

Table 7: *Hedging error rates*

		Constant mean reversion model					Stochastic mean reversion model				
		Kalman Fileter		Equations Based			Kalman Fileter		Equations Based		
		3factor	2factor	3factor	2factor	Parallel	3factor	2factor	3factor	2factor	Parallel
WTI	DEC07	1.0%	1.3%	0.7%	1.3%	4.0%	0.7%	0.8%	0.4%	0.4%	3.8%
	DEC08	1.0%	0.6%	0.9%	0.6%	8.4%	0.7%	1.5%	0.5%	0.7%	8.3%
	DEC09	0.8%	0.8%	1.0%	0.8%	3.4%	0.5%	-0.8%	0.8%	0.6%	3.2%
	DEC10	-1.5%	-2.3%	-2.5%	-2.5%	-3.5%	-1.7%	-0.5%	-2.7%	-2.5%	-4.0%
	DEC11	-3.1%	-3.8%	-3.2%	-3.8%	2.7%	-3.1%	-2.8%	-3.1%	-2.6%	2.4%
	DEC12	-3.2%	-4.2%	-3.7%	-4.2%	8.0%	-3.0%	0.4%	-3.6%	-0.9%	7.8%
	DEC13	-0.8%	-1.1%	0.0%	-1.1%	12.1%	-0.3%	0.5%	0.5%	3.2%	12.3%
Copper	DEC10	-0.4%	-1.2%	-0.5%	-2.0%	12.4%	-0.1%	0.0%	0.0%	-1.8%	12.4%
	DEC11	0.0%	-5.2%	0.2%	-4.8%	30.3%	1.2%	2.5%	1.4%	1.4%	30.3%
	DEC12	-3.1%	-20.8%	-6.7%	-20.8%	32.1%	0.9%	4.8%	-3.2%	3.7%	32.0%

Figure 4: *Cumulative hedging error rates of the delta hedge (constant mean reversion model)*

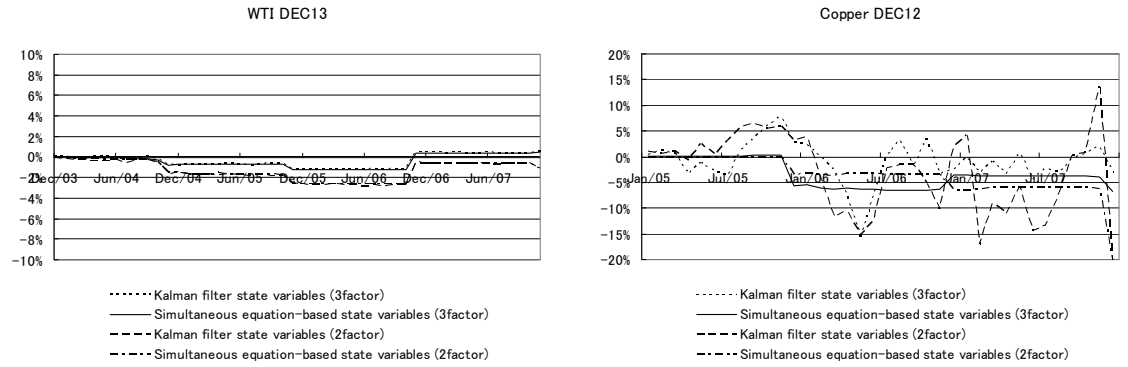
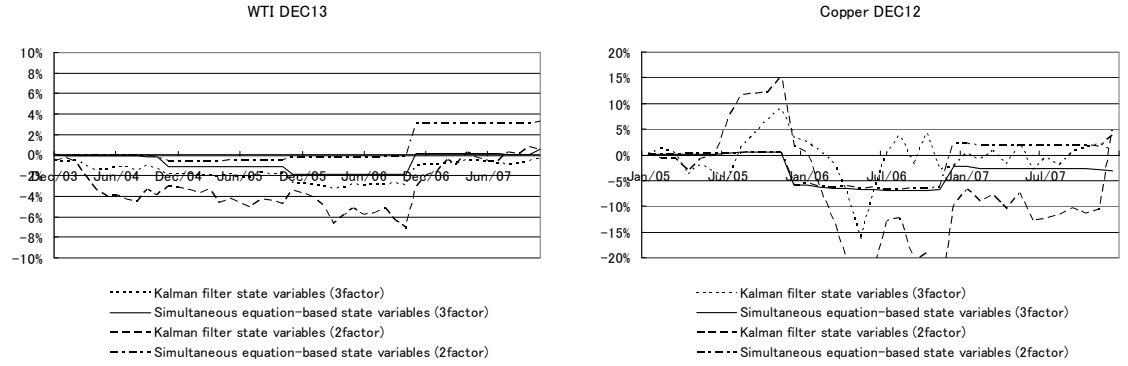


Figure 5: *Cumulative hedging error rates of the delta hedge (stochastic mean reversion model)*



The parallel hedge is able to provide effective hedging when the overall term structure changes in parallel, but generates large hedging error when there are changes in the shape of the term structure. By contrast, the delta hedge works much better than the parallel hedge (see Table 7). Two- and three-factor models provide relatively similar results though three-factor models works better for Copper. However, two-factor models have some difficulty in replicating actual term structures as shown in Figures 1and 2.

Comparing Kalman filter state variables and simultaneous equation-based state variables when performing a delta hedge, for Copper, estimation of Kalman filter state variables produces large hedging error during the term of the hedge, as can be seen in Figure 4 and Figure 5. This is presumably due to differences in whether the model prices are obtained in a manner consistent with the asset prices used in the hedge and the price of the assets to be hedged. When using simultaneous equation-based state variables, model prices (excluding rollover timing) match the prices of the assets used in the hedge and of the assets to be hedged. On the other hand, when using Kalman filter state variables, the actual prices of the assets used in the hedge differs from the model prices, resulting in hedging error when hedging is performed. For WTI, the observational error was small for the futures contract month used in the hedge and virtually equivalent to the simultaneous equation-based state variables, indicating that there is little difference due to the method by which state variables are determined.

Because of the result in the last paragraph, the discussion below uses only state variables that are calculated by solving simultaneous equations for both WTI and copper models.

5. Stability of the delta hedge

The verifications so far have estimated parameters based on data that included the entire hedge period. However, in actual practice, the parameter estimation period and the hedge period differ. Discussions so far have also assumed that the hedges target long-term futures, but general practice is for long-term contracts to be forwards rather than futures, which requires that interest-rate factors also be taken into account.

In this section, we confirm the following settings so as to conduct verifications in a state as close as possible to actual practice.

1. No overlap between the parameters' estimation period and the hedge period
2. Hedging against forwards

In this paper, cases in which the entire hedge period is included in the parameter estimation period are referred to as "In Sample," while hedges in which there are separate parameter estimation periods are referred to as "Out of Sample."

5.1 Out of Sample hedges

To verify the effectiveness of the Out of Sample hedge, this section uses the parameters estimated in section 3 with the data through 2002 or 2003 for WTI and through 2004 for Copper. The future to be hedged is DEC 12 (hedge period from 2002 to 2006) or DEC 13 (hedge period from 2003 to 2007) for WTI and DEC 12 (hedge period from 2004 to 2007) for Copper.

Table 8 summarizes results of verifications using time series data for hedging error rates when hedging long-term futures prices as estimated with the model using these parameters. For purposes of comparison, we have also noted the results for the In Sample estimations in the previous section.

Table 8: *Hedging error rates of the delta hedge (Out of Sample)*

		Constant mean reversion model		Stochastic mean reversion model	
		In sample	Out of sample	In sample	Out of sample
WTI	3factor (DEC12)	-3.7%	-32.1%	-3.6%	-3.4%
	2factor (DEC12)	-4.2%	-33.5%	-0.9%	1.5%
	3factor (DEC13)	0.0%	-7.6%	0.5%	0.2%
	2factor (DEC13)	-1.1%	-16.5%	3.2%	5.3%
Copper	3factor (DEC12)	-6.7%	-6.9%	-3.2%	-1.9%
	2factor (DEC12)	-20.8%	10.0%	3.7%	11.0%

For comparison between the two-factor models and the three-factor models, it is observed that the model producing the smaller absolute error in In Sample also creates the smaller absolute error in Out of Sample for all cases of Table 8.

For the three-factor stochastic mean reversion model, we confirmed that the differences in hedging error rates due to differences in the parameter estimation period were not that large for WTI. For Copper, there were differences in In Sample and Out of Sample hedging error rates, in part due to the differences in the parameters obtained for In Sample and Out of Sample. However, the error is not extreme and even though there are differences in the parameters obtained using the maximum likelihood method, hedging under the model is considered to be relatively stable. On the other hand, for the two-factor stochastic mean reversion model, there are more differences between In sample and Out of sample hedging error rates: This tendency seems stronger for Copper than for WTI.

For the constant mean reversion models, in the Out of Sample WTI, long-term price levels changed due to the sharp increases in oil prices beginning in 2003, virtually eliminating mean reversion. Nonetheless, the long-term futures prices calculated by the models revert to the mean levels observed in the data through 2002 or 2003, increasing the hedging error rates. In light of this, it is likely that the constant mean reversion model is more prone to hedging error when there are changes in mean reversion levels, etc., indicating that it is better to use the stochastic mean reversion model when prices are based on the hedge.

Due to the results above, the discussion in the next subsection uses only the three-factor stochastic mean reversion model.

5.2 Hedging long-term forward contracts

The discussions to this point have assumed that futures would be hedged, but common practice is to trade forwards for the long-term portion that is not traded on exchanges. If interest rates are deterministic or move independently from underlying assets, prices are the same for futures and forwards, but if interest rates are not deterministic, hedges must take account of their movements.

For purposes of simplicity, this discussion assumes that interest rates and underlying assets are independent, describes hedging techniques when the instrument to be hedged is a forward and the assets used in the hedge are futures. The utility of this hedging technique is then verified using time series data.

We consider hedging long-term forwards with short-term futures in the following two steps.

1 Long-term forwards are hedged using long-term futures with the same expiration.

2 Long-term futures are hedged using the delta hedge with the nearer maturity futures as described in section 4.

Because Step 2 is explained in section 4, we explain the hedging technique of Step 1.

The notations used are defined as:

$F_T(t)$: Price at point in time t of forward with expiration T.

$G_T(t)$: Price at point in time t of future with expiration T.

$P_T(t)$: Price at point in time t of zero-coupon bond with expiration T.

In addition, $0 = t_0 < t_1 < \cdots < t_m = T$. The amount of change in the forward profit/loss at point in time T during the period from point in time t_i through t_{i+1} is:

PV at point in time t_{i+1} is expressed as $\big(F_T(t_{i+1}) - F_T(0)\big)P_T(t_{i+1})$. Therefore, the amount of change in PV for the forward during the period from t_i through t_{i+1} is:

(10) $$\big(F_T(t_{i+1}) - F_T(0)\big)P_T(t_{i+1}) - \big(F_T(t_i) - F_T(0)\big)P_T(t_i).$$

In this case, (10) can be reformed as follows:

(11)
$$\begin{aligned}&\big(F_T(t_{i+1}) - F_T(0)\big)P_T(t_{i+1}) - \big(F_T(t_i) - F_T(0)\big)P_T(t_i)\\ &= \big(F_T(t_{i+1}) - F_T(t_i)\big)P_T(t_i) + \big(F_T(t_i) - F_T(0)\big)\big(P_T(t_{i+1}) - P_T(t)\big).\\ &\quad + \big(F_T(t_{i+1}) - F_T(t_i)\big)\big(P_T(t_{i+1}) - P_T(t_i)\big)\end{aligned}$$

Thus, ΔC_m that denotes the accumulation evaluated at time T of the last term on the right side is given by:

$$\Delta C_m \approx \sum_{j=0}^{m-1} e^{\sum_{i=j+1}^{m-1} r_i (t_{i+1} - t_i)} \big(F_T(t_{j+1}) - F_T(t_j)\big)\big(P_T(t_{j+1}) - P_T(t_j)\big),$$

where the instantaneous interest rate in each period $[t_i, t_{i+1}]$ is approximated as a constant r_i. Hereafter, we ignore ΔC_m because it expresses a negligible amount corresponding to the quadratic variation.

Assuming that interest rate is independent of underlying asset prices, the forward price and futures price are equivalent ($F_T(t) = G_T(t)$) and equation (11) can be expressed as shown below:

(12)
$$\begin{aligned}&\big(F_T(t_{i+1}) - F_T(t_i)\big)P_T(t_i) + \big(F_T(t_i) - F_T(0)\big)\big(P_T(t_{i+1}) - P_T(t_i)\big)\\ &= \big(G_T(t_{i+1}) - G_T(t_i)\big)P_T(t_i) + \big(G_T(t_i) - G_T(0)\big)\big(P_T(t_{i+1}) - P_T(t_i)\big).\end{aligned}$$

The first term on the right side expresses the change in the future, according to which a delta hedge is made using futures for the 3 nearer maturity contract months under the method described in section 4. As a result, 1 unit of forwards can be given a proximate hedge using a portfolio comprising futures and zero-coupon bonds, as shown below.

1 A delta hedge using futures for the 3 nearer maturity contract months to hedge $P_T(t)$ units of future $G_T(t)$.

2 Purchase of $G_T(t) - G_T(0)$ units of zero-coupon bond $P_T(t)$.

It is noted that this hedging strategy is essentially the same as Schwartz's hedging strategy (see Schwartz (1997), p.963-964).

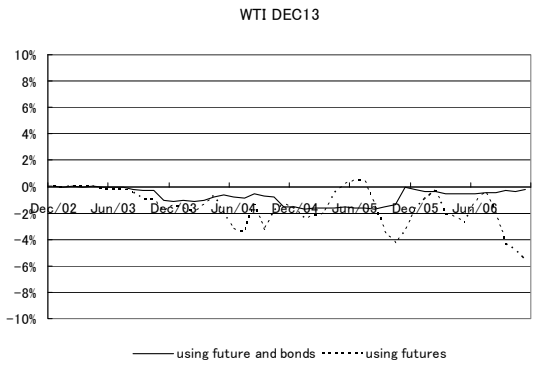
We analyzed the hedging error rate when a 10-year WTI forward contract is hedged according to the method above using WTI futures and zero-coupon bonds for 4 years. Here, forward prices are assumed to be equal to theoretical future prices calculated by our model. Note that the funds for purchase of zero-coupon bonds and cash flow generated by marking futures to market are invested/raised in short-term interest rates. For purposes of the verification, we used In Sample parameters and to simplify, the calculations of the zero-coupon bonds used the 8-year swap rate as spot yield; the calculations of short-term interest rates for investments and funding used the 1M LIBOR, and assume that the interest rate is independent of asset prices, thus forward price is equal to future price.

Table 9 contains hedging error rates due to differences in the forwards to be hedged.

Table 9: *Hedging error rates of the delta hedge*

	DEC07	DEC08	DEC09	DEC10	DEC11	DEC12	DEC13
using futures and bonds	0.2%	0.3%	0.4%	-2.3%	-2.6%	-3.1%	-0.2%
using futures	0.1%	-1.3%	-2.4%	-9.2%	-5.8%	-10.0%	-5.6%

Figure 6: *Cumulative hedging error rates of the delta hedge*



It will be noted that in this verification, which used time series data for interest rates and futures prices, even assuming interest rates and underlying assets to be independent, the use of zero-coupon bonds and futures to hedge forwards and was able to hedge virtually all of the interest rate factors generated by the difference between forwards and futures. However, if interest rates are not hedged, there are cases in which large hedging errors are generated during the hedge period, as can be seen from Figure 6, so the

idea that there does not need to be a hedge on the interest-rate portion is not supported.

6. Measuring the distribution of hedging error rates

The analysis of hedging error rates based on time series data are limited to the one in a few paths. Thus, this section provides a simulation analysis to measure the distribution of hedging error rates resulting from fluctuations in underlying assets. Three-factor stochastic mean reversion models are used in simulation, where In Sample parameters are used, ,and futures are hedged by futures with shorter maturities. Below are the specific procedures for the simulation.

1 Historical daily futures prices were created based on parameters and state variables estimated using the Kalman filter.

2 The error rate between the futures prices based on the model and created in Step 1 vs. actual futures prices quoted on exchanges (for WTI, the front month and the 1st-6th DEC; for Copper, the front month and the 1st-5th DEC) was calculated ((Actual data - Model price)/Model price) and then the mean and covariance of the error rate were obtained. Here, it was assumed that error follows multidimensional normal distribution.

3 Three-dimensional normal random numbers were created and the state variables were caused to fluctuate according to the model so as to create a term structure for futures.

4 Multidimensional normal random numbers according to the distribution described in Step 2 were created for the futures term structure developed in Step 3, multiplied as error and added to the original term structure. (Step 3 model prices + Step 3 model prices * Random numbers following the Step 2 error distribution).

5 Simultaneous equation-based state variables were calculated on the assumption that the term structure created in Step 4 was the term structure actually observed in the market.

6 The term structure created in Step 5 was used to estimate long-term prices, hedges were taken against those prices, and the final hedging error rate measured.

7 Steps 3-6 were repeated for a constant number of times to find the sample mean and sample standard deviation of the hedging error rates obtained.

6.1 Distribution of hedging error rates

We performed 5,000 trials for each combination of futures contract used in the hedge according to the procedures outlined above and calculated the average and standard deviation of the hedging error rates. There was little difference in the hedging error rates due to differences in initial values, so as hedged assets we used DEC 17 for WTI for a period of 4 years beginning 2007 and DEC 15 for Copper for a period of 3 years beginning 2007.

Table 10 contains the means and standard deviations for the obtained hedging error rates. The “contract months” column in the tables refers to which DEC from the front month is used for the hedge. For

example, 1-4-6 refers to the hedge using the 1st, 4th and 6th DECs. Likewise, "6Y Parallel (5Y Parallel)" expresses the hedging error rate when a parallel hedge is entered into using the 6th (5th) DEC. The price of the hedged assets for the parallel hedge is in the price found using 1-4-6 for WTI and 1-3-5 for Copper.

Table 10: *Averages and standard deviations of hedging errors*

WTI					
Contract Months	Average	Standard Deviation	Contract Months	Average	Standard Deviation
1-2-3	-22.6%	20.9%	2-3-5	-0.3%	4.5%
1-2-4	-10.0%	10.3%	2-3-6	0.3%	2.2%
1-2-5	-4.3%	5.0%	2-4-5	-0.1%	4.9%
1-2-6	-1.7%	2.7%	2-4-6	0.4%	2.2%
1-3-4	-4.9%	8.0%	2-5-6	0.4%	1.9%
1-3-5	-1.5%	3.9%	3-4-5	-0.3%	6.0%
1-3-6	-0.3%	2.0%	3-4-6	0.3%	2.5%
1-4-5	-0.5%	4.3%	3-5-6	0.3%	2.0%
1-4-6	0.1%	2.0%	4-5-6	0.3%	2.5%
1-5-6	0.3%	1.8%	Parallel Hedge		
2-3-4	-3.0%	9.3%	6Y Parallel	-0.5%	6.0%
Copper					
Contract Months	Average	Standard Deviation	Contract Months	Average	Standard Deviation
1-2-3	-14.7%	11.5%	2-3-4	0.3%	8.5%
1-2-4	-4.0%	5.9%	2-3-5	0.6%	4.1%
1-2-5	-1.1%	3.7%	2-4-5	-0.2%	3.9%
1-3-4	-0.7%	6.8%	3-4-5	-1.0%	4.7%
1-3-5	0.1%	3.7%	Parallel Hedge		
1-4-5	-0.2%	3.7%	5Y Parallel	0.4%	16.3%

See Appendix 2 for more on the number of future units required due to differences in the selection of futures months at the time the hedge is commenced.

According to the results from the simulation, appropriate selection of the futures contract months for the hedge portfolio when entering into a delta hedge has the potential for a more accurate hedge than the use of a parallel hedge. In particular, in terms of the required amounts (see Appendix 2) and also the relationship between the means and standard deviations of hedging error rates, hedges for the WTI and Copper exhibited efficiency using the 1st - 4th - 6th DECs; and the 1st - 3rd - 5th DECs, respectively. As a more general result, it was found that the futures used to create a hedge portfolio should, to the extent possible, have mutually disparate contract months. This is because when the futures used in the hedge are close to each other the x^1, x^2 and x^3 delta structures are similar and a greater number of futures units is required to offset the delta of the instrument hedged. The greater the number of futures units used in the hedge, the larger the error expressed as differences in the prices of the model and actual futures. Conversely, when the futures are farther away from each other, they have disparate delta structures, making it more likely that hedging will not require as many units.

7. Conclusion

This paper demonstrated that by three-factor Gaussian model with appropriate estimation of parameters,

it was possible to reproduce the term structures of listed commodities futures (NYMEX WTI, LME Copper) during the time period studied and that long-term futures prices could be obtained that were consistent with liquid nearer maturity contracts. It was also found that two-factor Gaussian models have some difficulty in capturing actual term structures of futures.

Furthermore, it went on to propose a hedging technique for long-term futures and forwards contracts, comparing the results from this technique to the results from the simple short-term futures-based hedging strategy used by Metallgesellschaft (parallel hedge) and verified that our proposed strategy was stable in many different circumstances (backwardation, contango, rising prices, declining prices, etc.).

In addition, it found that a stochastic mean reversion model offered more stable hedging than a model with a constant mean reversion level. Also, it observed that the model producing the smaller absolute error in In Sample also creates the smaller absolute error in Out of Sample.

It then used a simulation to measure the hedging error rates obtained due to differences in the contract months of the futures used in the hedge. It was found that the futures used to create a hedge portfolio should, to the extent possible, have mutually disparate contract months.

In sum, the three-factor model with stochastic mean reversion seems useful in practice for pricing long-term futures/forward contracts and for hedging them with appropriate selected liquid instruments.

Future issues include evaluation of option values using the model and structuring of relevant hedging techniques. Commodities generally have average-based options which makes calculation complex. It would be useful to verify hedging techniques and their efficiency.

Acknowledgement

We really thank two anonymous referees for their efforts and precious comments.

References

1. Bessembinder et al.(1995). “Mean Reversion in Equilibrium Asset Prices: Evidence from the future term structure”, *Journal of Finance, 50, 361-375.*
2. Bjerksund, P.(1991) “Contingent claims evaluation when the convenience yield is stochastic: analytical results”, *Working Paper, Norwegian School of Economics and Business Administration.*
3. Black, F.(1976), “The pricing of commodity contracts”, *Journal of Financial Economics, 3,* 167-179.
4. Brenann, M.J. and Schwartz, E.S.(1985). “Evaluating natural resources investments”,*Journal of Business, 58*, 135-157.
5. Brenann, M.J. and Crew, N.I.(1997), “Hedging Long-Maturity Commodity Commitments with Short-Dated Futures Contracts”, *in M. Dempster and S. Pliska(Eds), Mathmatics of*

derivatives securities, pp165-190, Cambridge University Press

6. Bühler, W., Korn, O., and Schöbel, R.(2004), "Hedging Long-term Fowards With Short-Term Futures: A Two-Regime Approach", *Review of Derivatives Research, 7*, 185-212.
7. Casassus,J. and Collin-Dufresne,P.(2005), "Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates" , *Journal of Finance, 60,* 2283-2331.
8. Cortazar, G. and Naranjo, L.(2006), "An N-Factor Gaussian Model of Oil Futures Prices" , *Journal of Futures Markets, 26,* 243-268.
9. Culp, C.L. and Miller, M.H.(1995), "Metallgesellschaft and the Economics of Synthetic Storage", *Journal of Applied Corporate Finance, 7,* 62-76.
10. Gibson, R. and Schwartz, E.S.(1990), "Stochastic convenience yield and the pricing of oil contingent claims", *Journal of Finance, 45,* 959-976.
11. Heath, D., Jarrow, R. and Morton, A.(1992), "Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation", *Econometrica, 60,* 77-105.
12. Hull, J.C. and A. White (1990), "Pricing interest-rate-derivative securities", *Review of Financial Studies, 3,* 573-592.
13. Jamshidian, F. and Fein, M.(1990), "Closed-form solutions for oil futures and European options in the Gibson-Schwartz model" *Working paper, Merrill Lynch Capital Markets*
14. Korn, O.(2005), "Drift Matters: An Analysis of Commodity Derivatives", *Journal of Futures Markets, 25*, 211-241.
15. Mello, A.S. and Parsons, J.E.(1995), "Maturity Structure of a Hedge Matters: Lessons from the Metallgesellschaft Debacle" , *Journal of Applied Corporate Finance, 8,* 106-121.
16. Miltersen, K.R. and Schwartz, E.S.(1998), "Pricing of Options on Commodity Futures with Stochastic Term Structures of Convenience Yields and Interest Rates", *Journal of Financial And Quantitative Analysis 33*, 33-59.
17. Neuberger, A.(1999), "Hedging Long-Term Exposures with Multiple Short-Term Futures Contracts?", *Review of Financial Studies, 12*, 429-459.
18. Schwartz, E.S.(1997), "The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging", *Journal of Finance, Vol52*, 923-973.
19. Schwartz, E.S. and Smith, J.E.(2000), "Short-Term Variations Long-Term Dynamics. in Commodity Prices", *Management Science 46*, 893-911.

Appendix 1

Expectation and covariance matrix of state variables

$$E_t\left[\begin{pmatrix} x_T^1 \\ x_T^2 \\ x_T^3 \end{pmatrix}\right] = \begin{bmatrix} \mu_{11}(x_t^1, x_t^2, x_t^3, T-t) \\ \mu_{22}(x_t^1, x_t^2, x_t^3, T-t) \\ \mu_{33}(x_t^1, x_t^2, x_t^3, T-t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\kappa(T-t)}x_t^1 + \dfrac{\kappa\left(e^{-\gamma(T-t)} - e^{-\kappa(T-t)}\right)}{\kappa-\gamma}x_t^2 + \dfrac{\kappa\left(e^{-\beta(T-t)} - e^{-\kappa(T-t)}\right)}{\kappa-\beta}x_t^3 + \dfrac{\alpha}{\beta}\left(1 - \dfrac{\kappa e^{-\beta(T-t)} - \beta e^{-\kappa(T-t)}}{\kappa-\beta}\right) \\ e^{-\gamma(T-t)}x_t^2 \\ e^{-\beta(T-t)}x_t^3 + \dfrac{\alpha}{\beta}\left(1 - e^{-\beta(T-t)}\right) \end{bmatrix},$$

$$Cov_T\left[\begin{pmatrix} x_T^1 \\ x_T^2 \\ x_T^3 \end{pmatrix}\right] = \begin{bmatrix} \Sigma_{11}(T-t) & \Sigma_{12}(T-t) & \Sigma_{13}(T-t) \\ \Sigma_{12}(T-t) & \Sigma_{22}(T-t) & \Sigma_{23}(T-t) \\ \Sigma_{13}(T-t) & \Sigma_{23}(T-t) & \Sigma_{33}(T-t) \end{bmatrix},$$

$$\Sigma_{11} = {\sigma_1}^2\frac{1-e^{-2\kappa(T-t)}}{2\kappa} + {\sigma_2}^2\left(\frac{\kappa}{\kappa-\gamma}\right)^2\left(\frac{1-e^{-2\gamma(T-t)}}{2\gamma} + \frac{1-e^{-2\kappa(T-t)}}{2\kappa} - 2\frac{1-e^{-(\kappa+\gamma)(T-t)}}{\kappa+\gamma}\right)$$

$$+ {\sigma_3}^2\left(\frac{\kappa}{\kappa-\beta}\right)^2\left(\frac{1-e^{-2\beta(T-t)}}{2\beta} + \frac{1-e^{-2\kappa(T-t)}}{2\kappa} - 2\frac{1-e^{-(\kappa+\beta)(T-t)}}{\kappa+\beta}\right)$$

$$+ 2\rho_{12}\sigma_1\sigma_2\frac{\kappa}{\kappa-\gamma}\left(\frac{1-e^{-(\kappa+\gamma)(T-t)}}{\kappa+\gamma} - \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\right)$$

$$+ 2\rho_{23}\sigma_2\sigma_3\frac{\kappa^2}{(\kappa-\gamma)(\kappa-\beta)}\left(\frac{1-e^{-(\beta+\gamma)(T-t)}}{\beta+\gamma} - \frac{1-e^{-(\kappa+\beta)(T-t)}}{\kappa+\beta} - \frac{1-e^{-(\kappa+\gamma)(T-t)}}{\kappa+\gamma} + \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\right)$$

$$+ 2\rho_{13}\sigma_1\sigma_3\frac{\kappa}{\kappa-\beta}\left(\frac{1-e^{-(\kappa+\beta)(T-t)}}{\kappa+\beta} - \frac{1-e^{-2\kappa(T-t)}}{2\kappa}\right),$$

$$\Sigma_{22} = {\sigma_2}^2\frac{1-e^{-2\gamma(T-t)}}{2\gamma},\quad \Sigma_{33} = {\sigma_3}^2\frac{1-e^{-2\beta(T-t)}}{2\beta},\quad \Sigma_{23} = \rho_{23}\sigma_2\sigma_3\frac{1-e^{-(\beta+\gamma)(T-t)}}{\beta+\gamma},$$

$$\Sigma_{12} = \rho_{12}\sigma_1\sigma_2\frac{1-e^{-(\kappa+\gamma)(T-t)}}{\kappa+\gamma} + {\sigma_2}^2\frac{\kappa}{\kappa-\gamma}\left(\frac{1-e^{-2\gamma(T-t)}}{2\gamma} - \frac{1-e^{-(\kappa+\gamma)(T-t)}}{\kappa+\gamma}\right)$$

$$+ \rho_{23}\sigma_2\sigma_3\frac{\kappa}{\kappa-\beta}\left(\frac{1-e^{-(\beta+\gamma)(T-t)}}{\beta+\gamma} - \frac{1-e^{-(\kappa+\gamma)(T-t)}}{\kappa+\gamma}\right),$$

$$\Sigma_{13} = \rho_{13}\sigma_1\sigma_3\frac{1-e^{-(\kappa+\beta)(T-t)}}{\kappa+\beta} + {\sigma_3}^2\frac{\kappa}{\kappa-\beta}\left(\frac{1-e^{-2\beta(T-t)}}{2\beta} - \frac{1-e^{-(\kappa+\beta)(T-t)}}{\beta+\kappa}\right)$$

$$+ \rho_{23}\sigma_2\sigma_3\frac{\kappa}{\kappa-\gamma}\left(\frac{1-e^{-(\beta+\gamma)(T-t)}}{\beta+\gamma} - \frac{1-e^{-(\kappa+\beta)(T-t)}}{\kappa+\beta}\right).$$

Appendix 2

The number of futures units

	WTI						Copper				
	1stDEC	2ndDEC	3rdDEC	4thDEC	5thDEC	6thDEC	1stDEC	2ndDEC	3rdDEC	4thDEC	5thDEC
1-2-3	1.20	-5.65	5.48	0	0	0	2.29	-8.12	6.84	0	0
1-2-4	0.62	-2.28	0	2.69	0	0	1.06	-2.89	0	2.83	0
1-2-5	0.35	-1.19	0	0	1.87	0	0.55	-1.34	0	0	1.80
1-2-6	0.20	-0.67	0	0	0	1.49	-	-	-	-	-
1-3-4	0.22	0	-3.70	4.50	0	0	0.39	0	-3.76	4.38	0
1-3-5	0.12	0.00	-1.47	0.00	2.37	0	0.20	0	-1.35	0.00	2.15
1-3-6	0.07	0	-0.74	0	0	1.69	-	-	-	-	-
1-4-5	0.06	0	0	-2.96	3.93	0	0.10	0	0	-2.44	3.35
1-4-6	0.03	0	0	-1.12	0	2.11	-	-	-	-	-
1-5-6	0.01	0	0	0	-2.38	3.39	-	-	-	-	-
2-3-4	0	1.29	-5.79	5.53	0	0	0	1.64	-5.90	5.25	0
2-3-5	0	0.64	-2.25	0	2.64	0	0	0.78	-2.13	0	2.36
2-3-6	0	0.34	-1.11	0	0	1.79	-	-	-	-	-
2-4-5	0	0.23	0	-3.52	4.32	0	0	0.29	0	-2.97	3.69
2-4-6	0	0.12	0	-1.31	0	2.21	-	-	-	-	-
2-5-6	0	0.05	0	0	-2.57	3.53	-	-	-	-	-
3-4-5	0	0	1.23	-5.45	5.24	0	0	0	1.27	-4.74	4.48
3-4-6	0	0	0.58	-1.99	0	2.43	-	-	-	-	-
3-5-6	0	0	0.20	0	-3.02	3.84	-	-	-	-	-
4-5-6	0	0	0	1.04	-4.59	4.57	-	-	-	-	-