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Testing for the Null Hypothesis of Cointegration with Structural Breaks*

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Abstract

In this paper we propose residual-based tests for the null hypothesis of cointegration with structural breaks against the alternative of no cointegration. The Lagrange Multiplier test is proposed and its limiting distribution is obtained for the case in which the timing of a structural break is known. Then the test statistic is extended in two ways to deal with a structural break of unknown timing. The first test statistic, a plug-in version of the test statistic for known timing, replaces the true break point by the estimated one. We also propose a second test statistic where the break point is chosen to be most favorable for the null hypothesis. We show the limiting properties of both statistics under the null as well as the alternative. Critical values are calculated for the tests by simulation methods. Finite-sample simulations show that the empirical size of the test is close to the nominal one unless the regression error is very persistent and that the test rejects the null when no cointegrating relationship with a structural break is present.

Key words: Cointegration, Integrated time series, No cointegration, Structural breaks.

JEL Classification: C22, C32.

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1 Introduction

Cointegration has been the subject of intensive research after Granger (1983) and Engle & Granger (1987) introduced the concept. A number of tests for cointegration have been proposed since then. The three most commonly used tests concerning cointegration are the residual-based test for the null hypothesis of no cointegration by Engle & Granger (1987) and Phillips & Ouliaris (1990), the residual-based test for the null hypothesis of cointegration by Shin (1994) and the cointegrating rank test by Johansen (1988, 1991). Each has a different purpose yet complements the others.

These tests for cointegration have been generalized to accommodate structural breaks of unknown timing, reflecting the recent upsurge of research on structural breaks.[1] The residual-based test for the null of no cointegration against the alternative of cointegration with structural breaks of unknown timing is proposed by Gregory & Hansen (1996). Quintos (1997) and Seo (1998) consider tests for the null of cointegration against the alternative of cointegration with structural breaks of unknown timing by extending the approach of Johansen (1988, 1991). Inoue (1999) and Lütkepohl et al. (2004) have developed the cointegrating rank test allowing structural breaks of unknown timing in the trend and the level respectively. However, no residual-based test for the null hypothesis of cointegration with structural breaks against the alternative of no cointegration has yet been established.

The above-mentioned tests for cointegration with structural breaks specify "no cointegration" or "cointegration without any structural break" as the null. Hence rejection of these null hypotheses is often understood as the existence of cointegration with structural breaks. However,from the view of classical hypothesis testing, if we are primarily concerned about cointegration with structural breaks, it seems a more natural choice for the null hypothesis. Thus in this paper we propose a test for the null of cointegration with structural breaks against the alternative of no cointegration.

The proposed test is a residual-based test derived from single equation models. It is an extension of the test for the null of cointegration by Shin (1994), just as the test by Gregory & Hansen (1996) is an extension of the test for the null of no cointegration by Engle & Granger (1987) and Phillips & Ouliaris (1990). The Lagrange Multiplier (LM) test statistic is presented and the limiting distribution is derived for the case in which a structural break occurs at known timing. We show that the limiting distribution of the test statistic is free of nuisance parameter dependencies except for the number of I(1) regressors and the location of the structural break. Then we develop two test statistics for the case in which the break point is unknown. The first test statistic is a plug-in version of the test statistic for known timing that replaces the true break point by the estimated one. The second test statistic we propose is derived from the idea that the break point is chosen to give the most favorable result for the null. We show the limiting properties of both statistics under the null as well as the alternative hypotheses. Critical values are calculated for the tests by simulation methods. Finite-sample simulations show that the empirical size of the test is close to the nominal one unless the regression error is very persistent and that the test rejects the null when no cointegrating relationship with a structural break is present.

The rest of the paper is organized as follows. Section 2 describes three types of single-equation cointegration regression models with a structural break. In Section 3 we present test statistics for the null hypothesis of cointegration with a structural break of known timing against the alternative of no cointegration. The tests are generalized to the case where a structural break occurs at unknown timing in Section 4. Section 5 provides some simulation results and Section 6 concludes. All proofs are in the Appendix.

2 Models

In this section we consider single-equation cointegrating regression models with structural breaks. The observed data is $y_t = (y_{1t}, y_{2t}')'$ where y_{1t} is a scalar and y_{2t} is an $(m \times 1)$-vector, i.e. $y_{2t} = (y_{21,t}, y_{22,t}, \dots, y_{2m,t})'$. It is useful to define the dummy variable

$$\varphi_{t\tau} = \begin{cases} 0 & \text{if} \quad t \leq [n\tau], \\ 1 & \text{if} \quad t > [n\tau], \end{cases}$$

where $[s]$ denotes the largest integer not exceeding s. That is, $\varphi_{t\tau} = 1\{t > [n\tau]\}$ where $1\{\cdot\}$ denotes the indicator function. τ and $[n\tau]$ represent the break fraction and the break date, respectively. Following Gregory & Hansen (1996), three forms of structural breaks are considered:

Model 1: Level shift

$$y_{1t} = \mu_1 + \mu_2 \varphi_{t\tau} + \beta' y_{2t} + e_t, \quad t = 1, \dots, n. \tag{1}$$

Model 2: Level shift with trend

$$y_{1t} = \mu_1 + \mu_2 \varphi_{t\tau} + \alpha t + \beta' y_{2t} + e_t, \quad t = 1, \dots, n. \tag{2}$$

Model 3: Regime shift

$$y_{1t} = \mu_1 + \mu_2 \varphi_{t\tau} + \beta_1' y_{2t} + \beta_2' y_{2t} \varphi_{t\tau} + e_t, \quad t = 1, \dots, n. \tag{3}$$

where in each case

$$\begin{aligned} e_t &= \gamma_t + v_{1t}, \\ \gamma_t &= \gamma_{t-1} + u_t, \quad \gamma_0 = 0. \end{aligned}$$

Here u_t is i.i.d. $(0, \sigma_u^2)$. This formulation of the error process e_t has been frequently used in tests for parameter constancy, stationarity, and cointegration (see, eg. Nabeya & Tanaka, 1988, Kwiatkowski et al., 1992 and Shin, 1994). Our null hypothesis of cointegration with structural breaks corresponds to e_t being stationary, i.e. $\sigma_u^2 = 0$. Note that $e_t = v_{1t}$ under the null hypothesis and assume u_t is independent of v_{1t}.

Our test for the null of cointegration with structural breaks is residual-based. If the timing of the break is known, regression model (1), (2) or (3) is estimated by ordinary least squares (OLS) depending on the hypothesis of interest, and then we test for stationarity of the regression error. In the next section, the test statistic is presented and its limiting properties are analyzed.

3 Testing the null of cointegration with structural breaks of known timing

In this section we propose a test statistic for the null of cointegration with structural breaks when the breaks occur at known timing. We begin by proposing a test statistic and developing its limiting properties in a simple setting.

3.1 When regressors are strictly exogenous

For the moment, we assume that the regressors are strictly exogenous. Before moving on to the test, we will analyze limiting properties of the OLS estimators for coefficients in models 1, 2 and 3 because they play important roles in developing limiting distributions of the test statistic. To do so, we shall place some assumptions on innovation sequences and introduce some notation. Define $\Delta y_{2t} = v_{2t}$. Let $v_t = (v_{1t}, v_{2t}')'$ and assume that v_t satisfies the following assumption.

ASSUMPTION 3.1 (a) *$\{v_t\}$ is mean-zero and strong mixing with mixing coefficients of size $-p\beta/(p-\beta)$ and $\mathbf{E}|v_t|^p < \infty$ for some $p > \beta > 5/2$.*

(b) *y_0 is a random vector with $\mathbf{E}|y_0| < \infty$*

Under Assumption 3.1, the multivariate invariance principle holds with long–run variance Ω:

$$n^{-1/2} \sum_{t=1}^{[nr]} v_t \Rightarrow B(r), \quad 0 \le r \le 1 \tag{4}$$

where $B(r) = (B_1(r), B_2(r)')'$ is an $(m+1)$-dimensional Brownian motion with covariance matrix Ω and $B_1(r)$ and $B_2(r)$ denote Brownian motions of 1 and m dimensions, respectively (see Herrndorf, 1984 and Phillips & Durlauf, 1986 for the proof). We assume that Ω can be written as

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{21}' \\ \omega_{21} & \Omega_{22} \end{bmatrix} = \lim_{n\to\infty} n^{-1}\mathbf{E}(\xi_n \xi_n'), \tag{5}$$

$$= \Sigma + \Lambda + \Lambda', \tag{6}$$

where

$$\xi_s = \sum_{t=1}^{s} v_t \tag{7}$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21}' \\ \sigma_{21} & \Sigma_{22} \end{bmatrix} = \lim_{n\to\infty} n^{-1}\sum_{t=1}^{n}\mathbf{E}(v_t v_t'), \tag{8}$$

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \Lambda_{22} \end{bmatrix} = \lim_{n\to\infty} n^{-1}\sum_{t=2}^{n}\sum_{j=1}^{t-1}\mathbf{E}(v_j v_t'). \tag{9}$$

We assume that covariance matrices ω_{11} and Ω_{22} of $B_1(r)$ and $B_2(r)$ are positive definite. This implies that the elements of y_{2t} are not cointegrated and also rules out multicointegration (see Granger & Lee (1989) for further explanations of the concept of multicointegration).

Let the least squares estimator of b be $\hat{b}_\tau$, where b is a vector that consists of the coefficient vectors in each model. For example, $b = (\mu_1, \mu_2, \beta')'$ for model 1. Note that the OLS estimator $\hat{b}_\tau$ depends on τ because it is a function of $\varphi_{t\tau}$. Define "$\Rightarrow$" as denoting weak convergence of the associated probability measures with respect to the uniform metric over either $\tau \in [0,1]$ or $\mathcal{T}$ where $\mathcal{T} = [\underline{\tau}, \bar{\tau}]$, $0 < \underline{\tau} < \bar{\tau} < 1$. Since the results shown in the following lemma do not represent a mere pointwise convergence, we will refer them as holding "uniformly over τ" (see Gregory & Hansen, 1996 for further explanations). The limiting properties of the OLS estimator of $\hat{b}_\tau$ are given in the following lemma.

LEMMA 3.1 *Let Assumption 3.1 hold. Assume $\omega_{21} = 0$; that is, y_{2t} is strictly exogenous with respect to v_{1t}. Then under the null hypothesis as $n \to \infty$,*

$$D_n(\hat{b}_\tau - b) \Rightarrow \left(\int_0^1 X_\tau(r)X_\tau(r)'dr\right)^{-1}\int_0^1 X_\tau(r)dB_1(r)$$

uniformly over $\tau \in [0,1]$ where b, D_n and X_τ depend on the model. If the least squares estimator is from OLS estimation of model 1, then

$$b = (\mu_1, \mu_2, \beta')'$$

$$D_n = \text{diag}\left(n^{1/2}, n^{1/2}, nI_m\right),$$

$$X_\tau(r) = (1, \varphi_\tau(r), B_2(r)')'$$

where I_m is an m-dimensional identity matrix and $\varphi_\tau(r) = 1\{r > \tau\}$. If the least squares estimator is from OLS estimation of model 2, then

$$\begin{aligned} b &= (\mu_1, \mu_2, \alpha, \beta')' \\ D_n &= \operatorname{diag}\left(n^{1/2}, n^{1/2}, n^{3/2}, nI_m\right), \\ X_\tau(r) &= (1, \varphi_\tau(r), r, B_2(r)')'. \end{aligned}$$

If the least squares estimator is from OLS estimation of model 3, then

$$\begin{aligned} b &= (\mu_1, \mu_2, \beta_1', \beta_2')' \\ D_n &= \operatorname{diag}\left(n^{1/2}, n^{1/2}, nI_m, nI_m\right), \\ X_\tau(r) &= (1, \varphi_\tau(r), B_2(r)', B_2(r)'\varphi_\tau(r))'. \end{aligned}$$

Next we describe how to compute the test statistic. For a given change point τ, estimate one of the models 1–3 by OLS according to our hypothesis of interest. Denote the residual by $\hat{e}_{t\tau}$. Note that the residual depends on the choice of change point τ. Following Shin (1994), the Lagrange Multiplier (LM) test can be written as

$$V_{n\tau} = n^{-2} \sum_{t=1}^{n} \hat{S}_{t\tau}^2 / \hat{\omega}_{11\tau} \tag{10}$$

where $\hat{S}_{t\tau} = \sum_{s=1}^{t} \hat{e}_{s\tau}$ and $\hat{\omega}_{11\tau}$ is any consistent estimator of ω_{11}. $\hat{\omega}_{11\tau}$ depends on τ because it in turn depends on the residual $\hat{e}_{t\tau}$. One of many valid candidates for $\hat{\omega}_{11\tau}$ is the standard semiparametric estimator (see e.g., Newey & West, 1987, Andrews, 1991 and Shin, 1994). It is defined by

$$\hat{\omega}_{11\tau} = n^{-1} \sum_{t=1}^{n} \hat{e}_{t\tau}^2 + 2n^{-1} \sum_{s=1}^{\ell} k(s/\ell) \sum_{t=s+1}^{n} \hat{e}_{t\tau}\hat{e}_{t-s,\tau} \tag{11}$$

where $k(\cdot)$ is a kernel function and ℓ is a bandwidth parameter.

The test statistic given by (10) leads to the following limiting distribution as the sample size n goes to infinity.

THEOREM 3.1 *Suppose the conditions in Lemma 3.1 are satisfied. Then under the null hypothesis as $n \to \infty$*

$$V_{n\tau} \Rightarrow \int_0^1 Q_\tau^2(r)dr$$

uniformly over $\tau \in [0, 1]$ where

$$Q_\tau(r) = W_1(r) - \left(\int_0^r W_\tau(s)ds\right)' \left(\int_0^1 W_\tau(s)W_\tau(s)'ds\right)^{-1} \left(\int_0^1 W_\tau(s)dW_1(s)\right)$$

and $W_\tau(r)$ depends on the model. If the residuals are from OLS estimation of model 1, then

$$W_\tau(r) = (1, \varphi_\tau(r), W_2(r)')'$$

where $W_2(r)$ is an m-dimensional standard Brownian motion independent of the scalar valued standard Brownian motion $W_1(r)$. If the residuals are from OLS estimation of model 2, then

$$W_\tau(r) = (1, \varphi_\tau(r), r, W_2(r)')'.$$

If the residuals are from OLS estimation of model 3, then

$$W_\tau(r) = (1, \varphi_\tau(r), W_2(r)', W_2(r)'\varphi_\tau(r))'.$$

Theorem 3.1 shows that the limiting distributions of the test statistics depend only on the timing of break τ and the number of I(1) regressors m.

3.2 When regressors are not strictly exogenous

Next we generalize the results in the last section to the case where regressors are not strictly exogenous. It is well known that the exogeneity assumption made in the last section is overly restrictive. Thus our generalization is of practical importance.

We employ the asymptotically efficient estimation technique developed by Saikkonen (1991) to extend the results of the last section. In the following, we show that we can construct a test statistic whose limiting distribution is free of nuisance paramters as a result of this efficient estimation technique. First we show how this technique works under the null hypothesis. Note that the regression error e_t in (1)–(3) is equal to v_{1t} under the null. Consider the following modified regression models:

Model 1': Level shift (C)

$$y_{1t} = \mu_1 + \mu_2\varphi_{t\tau} + \beta' y_{2t} + \sum_{i=-K}^{K} \pi_i' \Delta y_{2,t-i} + \varepsilon_t^*, \quad t = 1, \ldots, n. \tag{12}$$

Model 2': Level shift with trend (C/T)

$$y_{1t} = \mu_1 + \mu_2\varphi_{t\tau} + \alpha t + \beta' y_{2t} + \sum_{i=-K}^{K} \pi_i' \Delta y_{2,t-i} + \varepsilon_t^*, \quad t = 1, \ldots, n. \tag{13}$$

Model 3': Regime shift (C/S)

$$y_{1t} = \mu_1 + \mu_2\varphi_{t\tau} + \beta_1' y_{2t} + \beta_2' y_{2t}\varphi_{t\tau} + \sum_{i=-K}^{K} \pi_i' \Delta y_{2,t-i} + \varepsilon_t^*, \quad t = 1, \ldots, n \tag{14}$$

where π_i is an $(m \times 1)$ parameter vector for $-K \leq i \leq K$ and $\Delta y_{2t} = y_{2t} - y_{2,t-1}$. These are regression models where the leads and lags of Δy_{2t} are added to models 1–3. Note that the regression error here is not v_{1t} but ε_t^*. The relationship between them is characterized below. Researchers who are familiar with the technique of Saikkonen (1991) may wonder whether we might need the leads and lags of $\Delta y_{2t} \varphi_{t\tau}$ in addition to those of Δy_{2t}. In fact we do not, and the reason will be explained after we introduce some assumptions and describe some basic results. To derive the limiting distribution of the OLS estimator in models 1'–3', we need to make the following assumption on the error process v_t in (1)–(3):

ASSUMPTION 3.2 (a) $\{v_t\}$ *is strictly stationary with spectral density matrix* $f_{vv}(\lambda)$ *bounded away from zero so that*

$$f_{vv}(\lambda) \geq \alpha I_n, \quad \lambda \in [0, \pi],$$

where $\alpha > 0$.

(b) *The covariance function of* v_t *is absolutely summable*

$$\sum_{j=-\infty}^{\infty} ||\Gamma(j)|| < \infty,$$

where $\Gamma(j) = \mathbf{E}(v_t v_{t+j}')$ *and* $|| \cdot ||$ *is the standard Euclidean norm.*

(c) *Denote the fourth cumulants of* ε_t *by* $\kappa_{ijkl}(m_1, m_2, m_{,3})$ *(for a definition, see Brillinger, 1981, chap. 2). We assume*

$$\sum\sum\sum_{m_1,m_2,m_3=-\infty}^{\infty} |\kappa_{ijkl}(m_1, m_2, m_3)| < \infty.$$

It is well known that we can deduce under Assumption 3.1 that

$$v_{1t} = \sum_{j=-\infty}^{\infty} \pi_j' v_{2,t-j} + \varepsilon_t$$

where

$$\sum_{j=-\infty}^{\infty} ||\pi_j|| < \infty$$

and ε_t is a stationary process with the property that

$$\mathbf{E}(v_{2t}\varepsilon_{t+k}) = 0, \quad k = 0, \pm 1, \pm 2, \ldots \quad . \tag{15}$$

See Brillinger (1981) for more details. Furthermore,

$$2\pi f_{\varepsilon\varepsilon}(0) = \omega_{11} - \omega_{21}' \Omega_{22}^{-1} \omega_{21}$$

where $f_{\varepsilon\varepsilon}(\lambda)$ is the spectral density of ε at frequency λ. We shall now explain why we do not have to include the leads and lags of $\Delta y_{2t}\varphi_{t\tau}$. The key requirement for our asymptotically efficient estimation technique is that ε_t be exogenous with respect to the regressors. Note that (15) means that ε_t is strictly exogenous with respect to v_{2t}. This in turn implies that ε_t is strictly exogenous with respect to not only Δy_{2t} but also to $\Delta y_{2t}\varphi_{t\tau}$. Thus including the leads and lags of Δy_{2t} suffices to ensure that ε_t is strictly exogenous with respect to regressors that include both y_{2t} and $y_{2t}\varphi_{t\tau}$.

Observe that ε_t^* in (12)–(14) can be represented as

$$\varepsilon_t^* = \varepsilon_t + \sum_{|j|>K}^{\infty} \pi_j' v_{2,t-j}.$$

If $\pi_j = 0$ for $|j| > K$, then ε_t^* is strictly exogenous with respect to v_{2t}. This makes it relatively easy to derive the limiting distributions of the OLS estimators of the coefficients and the accosiated test statistics. However, this is not the case in general. Thus we also need to make an assumption on the truncation parameter K.

ASSUMPTION 3.3 *K tends to infinity with n at a suitable rate:*

$$(a) \qquad K^3/n \to 0,$$
$$(b) \qquad n^{1/2} \sum_{j>|K|}^{\infty} ||\pi_j|| \to 0.$$

First we will show the limiting properties of the OLS estimator of the coefficients as we did in the last section. For a given change point τ, we estimate one of models 1'–3' by OLS according to our hypothesis of interest. Let $\tilde{b}_\tau$ be the OLS estimator of b based on the modified regression model. For example, $b = (\mu_1, \mu_2, \beta')'$ for model 1'. Also let $\tilde{\pi}_i$ be the least squares estimator[2] of π_i.

LEMMA 3.2 *Let Assumptions 3.2 and 3.3 hold. Also suppose that the process $(\varepsilon_t, v_{2t}')'$ satisfies Assumption 3.1 imposed on v_t. Then under the null hypothesis as $n \to \infty$,*

$$D_n(\tilde{b}_\tau - b) \Rightarrow \left(\int_0^1 X_\tau(r) X_\tau(r)' dr \right)^{-1} \int_0^1 X_\tau(r) dB_{1\cdot 2}(r)$$

uniformly over $\tau \in [0,1]$ where $B_{1\cdot 2}(r) \equiv B_1(r) - \omega_{21}' \Omega_{22}^{-1} B_2(r)$. b, D_n and X_τ depend on the model and are as defined in Lemma 3.1. In addition, we have

$$\sum_{j=-K}^{K} ||\tilde{\pi}_j - \pi_j||^2 = O_p\left(\frac{K}{n}\right).$$

Next we propose a test statistic and show its limiting properties under both the null and the alternative hypotheses. Denote the residual based on OLS estimation of the modified regression models by $\tilde{e}_{t\tau}$. Note that this residual depends on the choice of break fraction τ. The test statistic is given by

$$\tilde{V}_{n\tau} = n^{-2} \sum_{t=1}^{n} \tilde{S}_{t\tau}^2 / \tilde{\omega}_{1\cdot 2\tau} \tag{16}$$

where $\tilde{S}_{t\tau} = \sum_{s=1}^{t} \tilde{e}_{s\tau}$ and $\tilde{\omega}_{1\cdot 2\tau}$ is any consistent estimator of $\omega_{1\cdot 2} = \omega_{11} - \omega_{21}' \Omega_{22} \omega_{21}$. The subscript τ of $\tilde{\omega}_{1\cdot 2\tau}$ is meant to imply that the residual is from OLS estimation of the model in which the change point τ is known. To derive the limiting properties of the test statistic under the alternative, we need to specify what kind of consistent estimator we are using. We employ the following estimator:

$$\tilde{\omega}_{1\cdot 2\tau} = n^{-1} \sum_{t=1}^{n} \tilde{e}_{t\tau}^2 + 2n^{-1} \sum_{s=1}^{\ell} k(s/\ell) \sum_{t=s+1}^{n} \tilde{e}_{t\tau} \tilde{e}_{t-s,\tau} \tag{17}$$

where $\tilde{e}_{t\tau}$ is the residual obtained from the modified regression, $k(\cdot)$ is a kernel function and ℓ is a bandwidth parameter. We assume that ℓ goes to infinity as the sample size n goes to infinity and $\ell = o(n^{1/2})$.

THEOREM 3.2 *Suppose the conditions in Lemma 3.2 are satisfied. Then (i) under the null hypothesis $\tilde{V}_{n\tau}$ has the same limiting distribution as $V_{n\tau}$ uniformly over $\tau \in [0, 1]$. (ii) Under the alternative, $\tilde{V}_{n\tau} = O_p(n/\ell)$.*

The first part of Theorem 3.2 implies that the test statistics based on models 1'–3' have the same limiting distributions as those in Theorem 3.1 even if the I(1) regressors are not strictly exogenous. Critical values of the tests are calculated for $m = 1$–5 in Tables 1–5, respectively. They are based on the representation of the limiting distributions in Theorem 3.1 where m is the number of I(1) regressors. Each table is calculated for values of $\tau = 0.1$–0.9. The critical values are obtained from $50,000$ replications at sample size $n = 2,000$. The second part of Theorem 3.2 shows that the test is consistent. Indeed, the test statistic diverges to infinity at a rate (n/l) under the alternative. This result is analogous to that in Kwiatkowski et al. (1992) and Shin (1994), where the rate of divergence of the test statistics depends on the bandwidth parameter ℓ. The consistency of the test proposed in Theorem 3.1 is straightforward, a special case of the second part of Theorem 3.2.

[Table 1–5 About Here]

4 Testing the null of cointegration with structural breaks of unknown timing

In this section, we generalize the results given in the last section to the case where a structural break occurs at unknown timing. When the location of the break point (or, equivalently, break fraction τ) is unknown, we can employ two strategies. One is to first estimate the break fraction and then construct the test statistic by replacing the known fraction with the estimated one. The other is to construct the test statistic for all possible break points and then take the infimum of those statistics. In the framework of testing for stationarity with a structural break, the first method is suggested by Kurozumi (2002) while the second is proposed by Busetti & Harvey (2001).

To present the first test statistic and derive its asymptotic distribution, we need to begin by estimating the break fraction. Two types of estimators for the break fraction are present in the literature. One is the pseudo-Gaussian maximum likelihood estimator (MLE) proposed by Bai et al. (1998) and the other is the least squares estimator developed by Kurozumi & Arai (2005). Whereas Bai et al. (1998) show detailed limiting properties (including the limiting distribution) of the pseudo-Gaussian MLE under restrictive assumptions, Kurozumi & Arai (2005) only show particular limting properties of the least squares estimator under much less restrictive assumptions. In this paper we employ the second estimator because some of the assumptions made in Bai et al. (1998) are not very realistic. Moreover, as we shall see in the proof of Theorem 4.1, the properties given in Kurozumi & Arai (2005) are sufficient for our purpose. The estimator is defined by

$$\hat{\tau} = \arg \inf_{\tau \in \mathcal{T}} SSR_n(\tau) \tag{18}$$

where $SSR_n(\tau) = \sum_{t=1}^{n} \hat{e}_{t\tau}^2$ and $\hat{e}_{t\tau}$ is defined as before. The limiting properties of this estimator are derived under the following assumption:[3]

ASSUMPTION 4.1 *β_2 in (12), (13) and (14) shrinks to zero as the sample size n goes to infinity at the rate of $n^{1/2}$, i.e. $\beta_2 = \beta_{2n} = n^{-1/2}\beta_{2o}$ where β_{2o} is a vector of constants.*

Assumption 4.1 embodies the idea that the post-break coefficients of the integrated regressors shrink to the pre-break coefficients at a suitable rate. This is equivalent to considering that the magnitude of a shift is small and converges to zero as the sample size goes to infinity. Bai et al. (1998) convincingly explain three reasons for assuming this shift to be small. First, we can show some analytical properties of the estimated break point. This becomes important when we develop the limiting properties of the test statistics. Second, if we can consistently estimate the break point for a small shift, we should be able to estimate it consistently for a large shift. Third, if a shift in the coefficients for the I(1) regressors does not converge to zero much faster than that for the intercept, the limiting behavior of the estimated break point will be dominated by the I(1) coefficients (see Bai et al., 1998 for more detailed explanations).

Now we are ready to propose a test statistic and show its limiting distribution. Let $\hat{\tau}$ be the estimated break fraction defined in (18) and estimate one of models 1–3 by OLS using modified regression model (12), (13) or (14). Denote the residual by $\tilde{e}_{t\hat{\tau}}$. Note that this residual depends on the estimated change point $\hat{\tau}$. Then the test statistic is given by

$$\tilde{V}_{n\hat{\tau}} = n^{-2} \sum_{t=1}^{n} \tilde{S}_{t\hat{\tau}}^2 / \tilde{\omega}_{1\cdot 2\hat{\tau}} \tag{19}$$

where $\tilde{S}_{t\hat{\tau}} = \sum_{s=1}^{t} \tilde{e}_{s\hat{\tau}}$ and $\tilde{\omega}_{1\cdot 2\hat{\tau}}$ is any consistent estimator of $\omega_{1\cdot 2} = \omega_{11} - \omega_{21}'\Omega_{22}\omega_{21}$. As an example, the standard semiparametric estimator based on $\tilde{e}_{t\hat{\tau}}$ would satisfy the requirement of consistency. The subscript $\hat{\tau}$ in $\tilde{\omega}_{1\cdot 2\hat{\tau}}$ implies that the residual is from OLS estimation of the model where the break fraction τ is estimated. The next theorem derives the limiting properties of the test statistic $\tilde{V}_{n\hat{\tau}}$.

THEOREM 4.1 *Suppose the conditions in Lemma 3.2 are satisfied. In addition, suppose Assumption 4.1 holds. Then, (i) under the null hypothesis as* $n \to \infty$

$$\tilde{V}_{n\hat{\tau}} - \tilde{V}_{n\tau} \xrightarrow{p} 0$$

uniformly over τ*. (ii) Under the alternative,* $\tilde{V}_{n\hat{\tau}} = O_p(n/l)$*.*

The first part of Theorem 4.1 implies that the test statistic has the same limiting distribution given in Theorem 3.2, even if we use the estimated break fraction to construct it. Thus even if we do not know the timing of the structural break, by constructing the test statistic using the estimated break fraction, we can conduct a test for cointegration with structural breaks based on the same critical values as if the true break fraction was known. The second part of Theorem 4.1 shows that the test is consistent against the alternative.

Next we introduce the second type of test statistic. When the location of a break point is unknown in the context of testing for stationarity, Busetti and Harvey (2001) propose an approach to construct a test statistic such that the break point is chosen to be most favorable for the null hypothesis.[4] We apply this approach to the test of cointegration with structural breaks. As in Busetti & Harvey (2001), we assume the following instead of Assumption 4.1:

ASSUMPTION 4.2 *We assume* $\mu_2 = \mu_{2n} = o(n^{-1/2})$ *for models 1 and 2, and* $\mu_2 = \mu_{2n} = o_p(n^{-1/2})$ *and* $\beta_2 = \beta_{2n} = o(n^{-1})$ *for model 3.*

Assumption 4.2 implies that we need to impose stronger conditions on how μ_{2n} and β_{2n} shrink to zero as the sample size n goes to infinity. The test statistic is proposed by

$$\underline{V_n} \stackrel{\text{def}}{=} \inf_{\underline{\tau}\le\tau\le\bar{\tau}} \tilde{V}_{n\tau} = \inf_{\underline{\tau}\le\tau\le\bar{\tau}} \left(n^{-2}\sum_{t=1}^{n} \tilde{S}_{t\tau}^2/\tilde{\omega}_{1\cdot 2\tau} \right).$$

THEOREM 4.2 *Let the conditions in Lemma 3.2 be satisfied. In addition, suppose Assumption 4.2 holds. Then, (i) under the null hypothesis,*

$$\inf_{\underline{\tau}\le\tau\le\bar{\tau}} \tilde{V}_{n\tau} \Rightarrow \inf_{\underline{\tau}\le\tau\le\bar{\tau}} \int_0^1 Q_\tau^2(r)dr,$$

where $Q_\tau(r)$ is given in Theorem 3.1. (ii) Under the alternative, the test statistic is $O_p(n/\ell)$.

5 Simulation Evidence

In this section we investigate finite sample properties of the test statistics. The data generating processes in our experiments are

$$\begin{aligned} y_{1t} &= 1 + \mu_2\varphi_{t\tau} + 2y_{2t} + e_t \qquad \text{(model 1)},\\ y_{1t} &= 1 + \mu_2\varphi_{t\tau} + 0.2t + 2y_{2t} + e_t \qquad \text{(model 2)},\\ y_{1t} &= 1 + \mu_2\varphi_{t\tau} + 2y_{2t} + \beta_2 y_{2t}\varphi_{t\tau} + e_t \qquad \text{(model 3)} \end{aligned}$$

where the dimension of y_{2t} (m) is equal to one,

$$\begin{aligned} e_t &= \gamma_t + v_{1t},\\ y_{2t} &= y_{2,t-1} + v_{2t},\\ \gamma_t &= \gamma_{t-1} + u_t,\\ v_t &= Av_{t-1} + \epsilon_t, \end{aligned}$$

with $A = \text{diag}\{a_{11}, a_{22}\}$, $\epsilon_t \sim i.i.d.N(0, I_2)$ and $u_t \sim i.i.d.N(0, \sigma_u^2)$ independent of $\{\epsilon_s\}$ for all s. We set $a_{11} = a_{22} = 0$, ± 0.4 or ± 0.8, $\sigma_u^2 = 0$, 0.01, 0.1, 1 or 10, and sample size $n = 100$ or 200. The break fraction τ is primarily set at 0.5 and other values are discussed briefly without providing results.[5] The values of μ_2 and β_2 are discussed and specified later. We use the semiparametric consistent estimator defined by (17) with the Bartlett kernel $k(s/\ell) = 1 - s/(\ell+1)$. We use three kinds of bandwidth parameter ℓ: $\ell_4 = [4(n/100)^{1/4}]$ and $\ell_{12} = [12(n/100)^{1/4}]$ as used by Schwert (1989) and Kwiatkowski et al. (1992), while the third choice ℓ_a is a truncated version of the data-dependent method proposed by Andrews (1991) and used by Arai (2004) and Kurozumi (2002):[6]

$$\ell_a = \min\left(1.1447\left\{\frac{4\hat{\rho}^2 n}{(1+\hat{\rho})^2(1-\hat{\rho})^2}\right\}^{1/3}, 1.1447\left\{\frac{4\times 0.9^2 n}{(1+0.9)^2(1-0.9)^2}\right\}^{1/3}\right),$$

where $\hat{\rho}$ is the coefficient estimated by the first order autoregression of ε_t^*. The number of leads and lags used to estimate the parameters is determined by a F test for the significance of leads and lags with a maximum lag length ℓ_4.[7] The level of significance is 0.05 and the number of replications is 1,000 in all experiments.

For the known break point case the test statistics are invariant to the true values of the coefficients so μ_2 and β_2 can be set to zero without loss of generality. Results are tabulated in Tables 6 and 7. On the whole, the empirical size of the test is close to the nominal one unless $|a_{ii}|$ is large, in which case the test tends to be oversized, especially when $n = 100$. The bandwidth of ℓ_4 is enough for the test to have an empirical size close to 0.05 when a_{ii} is small, but a bandwidth such as ℓ_{12} or ℓ_a is required for the empirical size of the test to be close to the nominal one for large a_{ii}. The test tends to be more powerful for larger n and smaller $|a_{ii}|$. We also note that power does not necessarily increase as σ_u^2 increases, especially when σ_u^2 is greater than 1 and ℓ_a is used. The reason is that a unit root process γ_t dominates the other stationary components when σ_u^2 is large (even in small samples), and then the longer bandwidth in ℓ_a tends to be chosen. Hence the power of the test decreases as predicted by the second part of Theorem 3.2.

[Tables 6 and 7 around here]

For the unknown break point case the finite-sample properties of the test statistics are affected by the magnitude of the break and so we consider two cases: for models 1 and 2 we set $\mu_2 = 1.1$ for a small shift and $\mu_2 = 2.2$ for a large shift for all sample sizes. For model 3 $\{\mu_2, \beta_2\}$ are set equal to $\{0.4, 0.1\}$ and $\{0.8, 0.2\}$ for a small and large shift when $n = 100$, while they are $\{0.2, 0.09\}$ and $\{0.4, 0.18\}$ when $n = 200$. These parameters are chosen so that the magnitude of the change is about equal to a half or one standard deviation of $y_{1,t+1}$, for a respective small or large shift. Note that the component of variation in $y_{1,t+1}$ given y_{1t} is $2v_{2,t+1} + e_{t+1}$, with variance $\text{Var}(2v_{2,t+1} + e_{t+1}) = 4 + 1 = 5$ when $\{v_t\}$ is an i.i.d. sequence, such that a half standard deviation is $0.5 \times \sqrt{5} \simeq 1.1$. Then, for example, since the magnitude of the break is μ_1 for models 1 and 2, it is set equal to 1.1 for the small shift case. For model 3, the magnitude of the break is $\mu_2 + \beta_2 y_{2,51}$ when $n = 100$ and since the standard deviation of $\beta_2 y_{2,51}$ is $\sqrt{51}\beta_2$, we set μ_2 and β_2 such that $\mu_2 + \sqrt{51}\beta_2 \simeq 1.1$ for the small magnitude case.

[Tables 8 and 9 around here]

Tables 8 and 9 summarize the results of experiments when we use $\mathcal{T} = \mathcal{T}_1 = [0.05, 0.95]$. We report only the small shift case because the results for the large shift case turns out to be very similar in our unreported simulation results. The size of the test is close to the nominal one when ℓ_a is used (except for the case where $a_{ii} = -0.8$), but the power of the test is low when a_{ii} is large and $n = 100$. On the whole, the bandwidth parameter ℓ_4 is not a suitable choice for large a_{ii}. We also conducted simulations for $\mathcal{T} = \mathcal{T}_2 = [0.15, 0.85]$. Unreported results show that the small sample properties of the proposed tests are not significantly affected by the choice of $\mathcal{T}$.

[Tables 10 and 11 about here]

The results for the inf-type test are summarized in Tables 10 and 11.[8] The empirical sizes are rarely below 0.10. Since the size distortions are very large even for $n = 200$, we do not recommend using these statistics when structural changes are considered in the model.

We now study the effect produced by the location of the break point. We conduct the same experiments as above for different values of τ: 0.05, 0.25, 0.75 and 0.95. The results are roughly symmetric around $\tau = 0.5$. There is no strong tendency for the finite-sample properties of a specific value of τ to be any better than those for other values. For example, the differences in the empirical sizes are never larger than 0.04 unless we use ℓ_{12} in model 3 with known timing. These properties are preserved in the case when the break point is unknown. Hence we conclude that the effect of the location of the break point is small.

6 Concluding Remarks

In this paper we have developed residual-based tests for the null hypothesis of cointegration with structural breaks against the alternative hypothesis of no cointegration. The LM test statistic is derived and its limiting distribution is obtained for the case where the timing of a structural break is known. Then it is generalized to accommodate a structural break of unknown timing in two ways. The limiting properties of both statistics are studied under the null as well as the alternative. Finite-sample simulations show that the empirical size of the test is close to the nominal one unless the regression error is very persistent. Additionally, the test rejects the null when no cointegrating relationship with a structural break is present. It is also revealed in our limited set of simulations that the "Inf-type" statistic proposed for the case where a structural break occurs at unknown timing suffers large size distortions and hence is not very useful in practice.

APPENDIX A

Throughout the proof, $\Rightarrow$ denotes weak convergence of the associated probability measure with respect to the uniform metric over $\tau \in [0,1]$ or $\tau \in \mathcal{T}$ where $\mathcal{T} = [\underline{\tau}, \bar{\tau}]$, $0 < \underline{\tau} < \bar{\tau} < 1$. Remember that $\xi_t = \sum_{s=1}^{t} v_s$ is the partial sum of the innovations v_t and $B(r)$ is an $(m+1)$-dimensional Brownian motion. Partition $\xi_t = (\xi_{1t}, \xi_{2t}')'$ and $B(r) = (B_1(r), B_2(r)')'$ in conformity with v_t. The following lemma, which we state without proof, is fundamental for our proof.

LEMMA 6.1 (Gregory & Hansen, 1996) *Under Assumption 3.1, the following joint weak convergence holds*

$$(a) \quad \frac{1}{n^{3/2}} \sum_{t=[n\tau]}^{n} \xi_t \Rightarrow \int_{\tau}^{1} B(r)dr,$$

$$(b) \quad \frac{1}{n^2} \sum_{t=[n\tau]}^{n} \xi_t \xi_t \Rightarrow \int_{\tau}^{1} B(r)B(r)'dr,$$

$$(c) \quad \frac{1}{n} \sum_{t=[n\tau]}^{n} \xi_t v_{t+1}' \Rightarrow \int_{\tau}^{1} B(r)dB(r)' + (1-\tau)\Lambda.$$

Following Gregory & Hansen (1996), we refer to results such as (a), (b) and (c) in Lemma 6.1 as holding "uniformly over τ".

Proof of Lemma 3.1: We provide a rigorous proof for model 3. A proof for model 1 is a special case of that for model 3 and that for model 2 is a simple extension of that for model 3. For model 3, we have $b = (\mu_1, \mu_2, \beta_1', \beta_2')'$ and $D_n = \left(n^{1/2}, n^{1/2}, nI_m, nI_m\right)$ as defined in Section 3. Then the OLS estimator of b is given by

$$\hat{b}_\tau = \left(\sum_{t=1}^{n} X_{t\tau} X_{t\tau}'\right)^{-1} \left(\sum_{t=1}^{n} X_{t\tau} y_{1t}\right)$$

where $X_{t\tau} = (1, \varphi_{t\tau}, y_{2t}', y_{2t}'\varphi_{t\tau})'$. This implies that under the null hypothesis

$$D_n\left(\hat{b}_\tau - b\right) = D_n \left(\sum_{t=1}^{n} X_{t\tau} X_{t\tau}'\right)^{-1} D_n D_n^{-1} \left(\sum_{t=1}^{n} X_{t\tau} v_{1t}\right)$$

or equivalently

$$D_n\left(\hat{b}_\tau - b\right) =$$

$$\begin{pmatrix} 1 & n^{-1}\sum_{t=1}^{n}\varphi_{t\tau} & n^{-3/2}\sum_{t=1}^{n}y_{2t}' & n^{-3/2}\sum_{t=1}^{n}y_{2t}'\varphi_{t\tau} \\ n^{-1}\sum_{t=1}^{n}\varphi_{t\tau} & n^{-1}\sum_{t=1}^{n}\varphi_{t\tau} & n^{-3/2}\sum_{t=1}^{n}y_{2t}'\varphi_{t\tau} & n^{-3/2}\sum_{t=1}^{n}y_{2t}'\varphi_{t\tau} \\ n^{-3/2}\sum_{t=1}^{n}y_{2t} & n^{-3/2}\sum_{t=1}^{n}y_{2t}\varphi_{t\tau} & n^{-2}\sum_{t=1}^{n}y_{2t}y_{2t}' & n^{-2}\sum_{t=1}^{n}y_{2t}y_{2t}'\varphi_{t\tau} \\ n^{-3/2}\sum_{t=1}^{n}y_{2t}\varphi_{t\tau} & n^{-3/2}\sum_{t=1}^{n}y_{2t}\varphi_{t\tau} & n^{-2}\sum_{t=1}^{n}y_{2t}y_{2t}'\varphi_{t\tau} & n^{-2}\sum_{t=1}^{n}y_{2t}y_{2t}'\varphi_{t\tau} \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} n^{-1/2}\sum_{t=1}^{n}v_{1t} \\ n^{-1/2}\sum_{t=1}^{n}\varphi_{t\tau}v_{1t} \\ n^{-1}\sum_{t=1}^{n}y_{2t}v_{1t} \\ n^{-1}\sum_{t=1}^{n}y_{2t}\varphi_{t\tau}v_{1t} \end{pmatrix}$$

where we used the facts that $e_t = v_{1t}$ under the null and $\varphi_{t\tau}^2 = \varphi_{t\tau}$. It follows from Lemma 6.1 that

$$D_n\left(\hat{b}_\tau - b_\tau\right) \Rightarrow$$
$$\begin{pmatrix} 1 & \int_0^1 \varphi_\tau(r)dr & \int_0^1 B_2(r)'dr & \int_0^1 B_2(r)'\varphi_\tau(r)dr \\ \int_0^1 \varphi_\tau(r)dr & \int_0^1 \varphi_\tau(r)dr & \int_0^1 B_2(r)'\varphi_\tau(r)dr & \int_0^1 B_2(r)'\varphi_\tau(r)dr \\ \int_0^1 B_2(r)dr & \int_0^1 B_2(r)\varphi_\tau(r)dr & \int_0^1 B_2(r)B_2(r)'dr & \int_0^1 B_2(r)B_2(r)'\varphi_\tau(r)dr \\ \int_0^1 B_2(r)\varphi_\tau(r)dr & \int_0^1 B_2(r)\varphi_\tau(r)dr & \int_0^1 B_2(r)B_2(r)'\varphi_\tau(r)dr & \int_0^1 B_2(r)B_2(r)'\varphi_\tau(r)dr \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} B_1(1) \\ \int_0^1 \varphi_\tau(r)dB_1(r) \\ \int_0^1 B_2(r)dB_1(r) \\ \int_0^1 B_2(r)\varphi_\tau(r)dB_1(r) \end{pmatrix}$$
$$= \left(\int_0^1 X_\tau(r)X_\tau(r)'dr\right)^{-1}\left(\int_0^1 X_\tau(r)dB_1(r)\right)$$

where $X_\tau(r) = (1, \varphi_\tau(r), B_2(r)', B_2(r)'\varphi_\tau(r))'$ for model 3, giving the required results. □

Proof of Theorem 3.1: Using the notation given in the proof of Lemma 3.1, $\hat{S}_{t\tau}$ can be expressed as

$$\hat{S}_{t\tau} = \sum_{s=1}^{t}\hat{e}_{s\tau} = \sum_{s=1}^{t}\left(y_{1s} - \hat{b}_\tau' X_{s\tau}\right) = \sum_{s=1}^{t}v_{1s} - \left(\hat{b}_\tau - b\right)'\sum_{s=1}^{t}X_{s\tau}$$

by noting that $e_t = v_{1t}$ under the null. It follows from Lemma 6.1 and Lemma 3.1 that

$$\begin{aligned} n^{-1/2}\hat{S}_{[nr]\tau} &= n^{-1/2}\sum_{t=1}^{[nr]}v_{1t} - \left(\hat{b}_\tau - b\right)' D_n D_n^{-1} n^{-1/2}\sum_{t=1}^{[nr]}X_{t\tau} \\ &\Rightarrow B_1(r) - \left(\int_0^r X_\tau(s)ds\right)'\left(\int_0^1 X_\tau(s)X_\tau(s)'ds\right)^{-1}\left(\int_0^1 X_\tau(s)dB_1(s)\right) \\ &\stackrel{\text{def}}{=} Q_{X\tau}(r). \end{aligned}$$

Define $\Omega_X = \text{diag}\,(1, 1, \Omega_{22}, \Omega_{22})$ and remember that

$$Q_\tau(r) = W_1(r) - \left(\int_0^r W_\tau(s)ds\right)' \left(\int_0^1 W_\tau(s)W_\tau(s)'ds\right)^{-1} \left(\int_0^1 W_\tau(s)dW_1(s)\right).$$

where $W_\tau(r) = (1, \varphi_\tau(r), W_2(r)', W_2(r)'\varphi_\tau(r))'$ and $W_2(r)$ is an m-dimensional standard Brownian motion independent of a scalar-valued Brownian motion $W_1(r)$. Then we get

$$\begin{aligned} & Q_{X\tau}(r) \overset{d}{=} \omega_{11}^{1/2} W_1(r) \\ & -\omega_{11}^{1/2} \left(\int_0^r W_\tau(s)ds\right)' \Omega_X^{1/2}\Omega_X^{-1/2} \left(\int_0^1 W_\tau(s)W_\tau(s)'ds\right)^{-1} \Omega_X^{-1/2}\Omega_X^{1/2} \left(\int_0^1 W_\tau(s)dW_1(s)\right) \\ = \;& \omega_{11}^{1/2} Q_\tau(r) \end{aligned}$$

where "$\overset{d}{=}$" signifies equality in distribution. Since $\hat{\omega}_{11\tau}$ is a consistent estimator of ω_{11}, we have

$$V_{n\tau} = n^{-2} \sum_{t=1}^{n} \hat{S}_{t\tau}^2 / \hat{\omega}_{11\tau} \Rightarrow \int_0^1 Q_\tau^2(r)dr.$$

If we use the semiparametric estimator given in (11), its consistency can be shown as in Shin (1994) under general regularity conditions. □

Proof of Lemma 3.2: Note that we now have $n - 2K$ observations, but we will use n instead of $n - 2K$ without loss of generality. Denote

$$\begin{aligned} \pi &= (\pi'_{-K}, \pi'_{-K+1}, \ldots, \pi'_{K-1}, \pi'_K)', \\ \gamma &= (b', \pi')', \\ Z_{tK} &= (\Delta y'_{2,t-K}, \Delta y'_{2,t-K+1}, \ldots, \Delta y'_{2,t+K-1}, \Delta y'_{2,t+K})', \\ U_{t\tau} &= (X'_{t\tau}, Z'_{tK})' \\ D_n^* &= \text{diag}(n^{1/2} I_m, \ldots, n^{1/2} I_m) \\ \text{and} \quad \tilde{D}_n &= \text{diag}(D_n, D_n^*) \end{aligned}$$

where $X_{t\tau}$ is defined as in Lemma 3.1 and dependence of $U_{t\tau}$ on K is suppressed for simplicity. Also let the OLS estimator of π and γ be $\tilde{\pi}_\tau$ and $\tilde{\gamma}_\tau$ respectively. We have

$$\begin{aligned} \tilde{D}_n(\tilde{\gamma}_\tau - \gamma) &= \tilde{D}_n \left(\sum_{t=1}^n U_{t\tau}U'_{t\tau}\right)^{-1} \tilde{D}_n \tilde{D}_n^{-1} \left(\sum_{t=1}^n U_{t\tau}\varepsilon_t^*\right) \\ &= \tilde{R}^{-1} \tilde{D}_n^{-1} \left(\sum_{t=1}^n U_{t\tau}\varepsilon_t^*\right) \end{aligned} \qquad (20)$$

where

$$\tilde{R} = \begin{pmatrix} D_n^{-1} \sum_{t=1}^n X_{t\tau}X'_{t\tau} D_n{}^{-1} & D_n^{-1} \sum_{t=1}^n X_{t\tau}Z'_{tK} D_n^{*}{}^{-1} \\ D_n^{*}{}^{-1} \sum_{t=1}^n Z_{tK}X'_{t\tau} D_n{}^{-1} & D_n^{*}{}^{-1} \sum_{t=1}^n Z_{tK}Z'_{tK} D_n^{*}{}^{-1} \end{pmatrix}.$$

Define

$$R = \begin{pmatrix} D_n^{-1}\sum_{t=1}^n X_{t\tau}X_{t\tau}'D_n{}^{-1} & 0 \\ 0 & \mathbf{E}\left(Z_{tK}Z_{tK}'\right) \end{pmatrix}.$$

Lemma A4 of Saikkonen (1991) shows that

$$\left|\left|\tilde{R}^{-1} - R^{-1}\right|\right|_1 = O_p\left(K/T^{1/2}\right)$$

where $||\cdot||_1$ is the matrix norm $||A||_1 = \sup\{||Ax|| : ||x|| \leq 1\}$ and $||\cdot||$ is the standard Euclidean norm. Define $\eta_{tK} = \sum_{j>|K|}\pi_j v_{2,t-j}$. Since R is block diagonal, (20) implies

$$\begin{aligned} D_n\left(\tilde{b}_\tau - b\right) &= D_n\left(\sum_{t=1}^n X_{t\tau}X_{t\tau}'\right)^{-1} D_n D_n^{-1}\left(\sum_{t=1}^n X_{t\tau}\varepsilon_t^*\right) + o_p(1) \\ &= D_n\left(\sum_{t=1}^n X_{t\tau}X_{t\tau}'\right)^{-1} D_n D_n^{-1}\left(\sum_{t=1}^n X_{t\tau}\varepsilon_t\right) \\ &\quad + D_n\left(\sum_{t=1}^n X_{t\tau}X_{t\tau}'\right)^{-1} D_n D_n^{-1}\left(\sum_{t=1}^n X_{t\tau}\eta_{tK}\right) + o_p(1). \end{aligned} \quad (21)$$

It was shown in the proof of Lemma 3.1 that

$$D_n\left(\sum_{t=1}^n X_{t\tau}X_{t\tau}'\right)^{-1} D_n = O_p(1). \quad (22)$$

Define $w_t = (\varepsilon_t, v_{2t}')'$. Note that we have assumed that $(\varepsilon_t, v_{2t}')'$ satisfies Assumption 3.1. Then the multivariate invariance principle holds:

$$n^{-1/2}\sum_{t=1}^{[nr]} w_t \Rightarrow \tilde{B}(r)$$

where $\tilde{B}(r) = (B_{1\cdot 2}(r), B_2(r)')'$ is an $(m+1)$-dimensional Brownian motion with covariance matrix $\tilde{\Omega} = \text{diag}(\omega_{1\cdot 2}, \Omega_{22})$, $\omega_{1\cdot 2} = \omega_{11} - \omega_{21}'\Omega_{22}\omega_{21}$, and $B_{1\cdot 2}(r)$ and $B_2(r)$ denote Brownian motions of 1 and m dimensions respectively. By Lemma 6.1, we have

$$D_n^{-1}\sum_{t=1}^n X_{t\tau}\varepsilon_t \Rightarrow \int_0^1 X_\tau(r)dB_{1\cdot 2}(r) \quad (23)$$

uniformly over τ. (A12) in Saikkonen (1991) also shows that

$$\left|\left|D_n^{-1}\sum_{t=1}^n X_{t\tau}\eta_{tK}\right|\right| = o_p(1). \quad (24)$$

Combining (21), (22), (23) and (24) gives the result required for the first part of Lemma 3.2. The second part of Lemma 3.2 is shown by Saikkonen (1991, pp.21). □

Proof of Theorem 3.2: Using the notation given in the proof of Lemma 3.2, $\tilde{S}_{t\tau}$ can be expressed as

$$\begin{aligned}\tilde{S}_{[nr]\tau} &= \sum_{t=1}^{[nr]} \tilde{e}_{t\tau} \\ &= \sum_{t=1}^{[nr]} \left(y_{1t} - \tilde{b}_\tau' X_{t\tau} - \tilde{\pi}_\tau' Z_{tK} \right) \\ &= \sum_{t=1}^{[nr]} \varepsilon_t + \sum_{t=1}^{[nr]} \eta_{tK} - \left(\tilde{b}_\tau - b_\tau \right)' \sum_{t=1}^{[nr]} X_{t\tau} - (\tilde{\pi}_\tau - \pi)' \sum_{t=1}^{[nr]} Z_{tK}.\end{aligned}$$

It follows from Lemma 6.1, and Lemma 3.2 that

$$\begin{aligned}n^{-1/2}\tilde{S}_{[nr]\tau} &= n^{-1/2}\sum_{t=1}^{[nr]} \varepsilon_t + n^{-1/2}\sum_{t=1}^{[nr]} \eta_{tK} \\ &\quad - \left(\hat{b}_\tau - b \right)' D_n D_n^{-1} n^{-1/2} \sum_{t=1}^{[nr]} X_{t\tau} - (\tilde{\pi}_\tau - \pi)' n^{-1/2} \sum_{t=1}^{[nr]} Z_{tK} \\ &\Rightarrow B_{1\cdot 2}(r) - \left(\int_0^r X_\tau(s)ds \right)' \left(\int_0^1 X_\tau(s) X_\tau(s)' ds \right)^{-1} \left(\int_0^1 X_\tau(s) dB_{1\cdot 2}(s) \right) \\ &\stackrel{\text{def}}{=} \tilde{Q}_{X\tau}(r),\end{aligned}$$

if we have

$$\begin{aligned}\sup_{0\le r\le 1} \left| n^{-1/2} \sum_{t=1}^{[nr]} \eta_{tK} \right| &= o_p(1), \\ \sup_{0\le r\le 1} \left\| (\tilde{\pi}_\tau - \pi)' n^{-1/2} \sum_{t=1}^{[nr]} Z_t \right\| &= o_p(1). \qquad (25)\end{aligned}$$

These were shown in the proof of Theorem 2 by Shin (1994, p.113). Define $\Omega_X = \operatorname{diag}(1, 1, \Omega_{22}, \Omega_{22})$ and remember that

$$Q_\tau(r) = W_1(r) - \left(\int_0^r W_\tau(s)ds \right)' \left(\int_0^1 W_\tau(s) W_\tau(s)' ds \right)^{-1} \left(\int_0^1 W_\tau(s) dW_1(s) \right).$$

Then we get

$$\begin{aligned}&\tilde{Q}_{X\tau}(r) \stackrel{d}{=} \omega_{1\cdot 2}^{1/2} W_1(r) \\ &-\omega_{1\cdot 2}^{1/2} \left(\int_0^r W_\tau(s)ds \right)' \Omega_X^{1/2}\Omega_X^{-1/2} \left(\int_0^1 W_\tau(s) W_\tau(s)' ds \right)^{-1} \Omega_X^{-1/2}\Omega_X^{1/2} \left(\int_0^1 W_\tau(s) dW_1(s) \right) \\ =\ & \omega_{1\cdot 2}^{1/2} Q_\tau(r).\end{aligned}$$

Since $\hat{\omega}_{1\cdot 2}$ is a consistent estimator of $\omega_{1\cdot 2}$, we have

$$\tilde{V}_{n\tau} = n^{-2} \sum_{t=1}^{n} \tilde{S}_{t\tau}^2 / \tilde{\omega}_{1\cdot 2} \Rightarrow \int_0^1 Q_\tau^2(r) dr,$$

proving the first part of Theorem 3.1.

To show the second part, we derive the convergence rate of $(\tilde{b}_\tau - b)$ and $(\tilde{\pi}_j - \pi_j)$ under the alternative. As in the proof of Lemma 3.2, we have

$$\tilde{D}_n(\tilde{\gamma}_\tau - \gamma) = \tilde{R}^{-1}\tilde{D}_n^{-1}\sum_{t=1}^{n} U_{t\tau}(\gamma_t + \varepsilon_t^*)$$

under the alternative. Since $\tilde{R}$ is the same as under the null hypothesis, we can investigate $\tilde{b}_\tau$ separately from $\tilde{\pi}_\tau$ as we did in the proof of Lemma 3.2. Noting that

$$D_n^{-1}\sum_{t=1}^{n} X_{t\tau}(\gamma_t + \varepsilon_t^*) = D_n^{-1}\sum_{t=1}^{n} X_{t\tau}(\gamma_t + \varepsilon_t + \eta_{tK}),$$

we can see that $D_n^{-1}\sum_{t=1}^{n} X_{t\tau}\gamma_t$ is of order n and dominates the other terms. Then we have

$$D_n(\tilde{b}_\tau - b) = O_p(n), \tag{26}$$

under the alternative.

To derive the convergence rate of $(\tilde{\pi}_j - \pi_j)$ under the alternative, let us define $\tilde{R}_{22} = D_n^{*-1}\sum_{t=1}^{n} Z_{tK}Z_{tK}' D_n^{*-1}$ and $R_{22} = E(Z_{tK}Z_{tK}')$. Since $\tilde{R}$ is asymptotically block diagonal, $||D_n^*(\tilde{\pi}_\tau - \pi)||$ is asymptotically equivalent to $||\tilde{R}_{22}^{-1}D_n^{*-1}\sum_{t=1}^{n} Z_{tK}(\gamma_t + \varepsilon_t^*)||$. Note that

$$\begin{aligned} &||\tilde{R}_{22}^{-1}D_n^{*-1}\sum_{t=1}^{n} Z_{tK}(\gamma_t + \varepsilon_t^*)|| \\ \leq\ & ||\tilde{R}_{22}^{-1} - R_{22}^{-1}||_1 ||D_n^{*-1}\sum_{t=1}^{n} Z_{tK}(\gamma_t + \varepsilon_t^*)|| + ||R_{22}^{-1}||_1 ||D_n^{*-1}\sum_{t=1}^{n} Z_{tK}(\gamma_t + \varepsilon_t^*)||. \end{aligned}$$

Since

$$n^{-1/2}\sum_{t=1}^{n} v_{2,t-j}(\gamma_t + \varepsilon_t^*) = O_p(n^{1/2}),$$

we have

$$||D_n^{*-1}\sum_{t=1}^{n} Z_{tK}(\gamma_t + \varepsilon_t^*)|| = O_p(K^{1/2}n^{1/2}).$$

Using this result and the following results shown by Saikkonen (1991)

$$||\tilde{R}_{22}^{-1} - R_{22}^{-1}||_1 = O_p(K/n^{1/2}) \quad \text{and} \quad ||R_{22}^{-1}||_1 = O_p(1),$$

we can show that

$$||D_n^*(\tilde{\pi}_\tau - \pi)|| = O_p(K^{1/2}n^{1/2}).$$

Then it follows by noting $||D_n^*(\tilde{\pi}_\tau - \pi)|| = n^{1/2}(\sum_{j=-K}^{K} ||\tilde{\pi}_j - \pi_j||^2)^{1/2}$ that

$$\sum_{j=-K}^{K} ||\tilde{\pi}_j - \pi_j||^2 = O_p(K). \tag{27}$$

Next, we consider the partial sum of the regression residuals $\tilde{e}_{t\tau}$,

$$\begin{aligned}\sum_{t=1}^{[nr]} \tilde{e}_{t\tau} &= \sum_{t=1}^{[nr]} (y_{1t} - \tilde{b}_\tau' X_{t\tau} - \tilde{\pi_\tau}' Z_{tK}) \\ &= \sum_{t=1}^{[nr]} \left\{ \gamma_t + \varepsilon_t + \eta_{tK} - (\tilde{b}_\tau - b)' D_n D_n^{-1} X_{t\tau} - (\tilde{\pi_\tau} - \pi)' Z_{tK} \right\}. \qquad (28)\end{aligned}$$

Observe that we have by (27)

$$\begin{aligned}\sup_{0 \le r \le 1} |(\tilde{\pi}_\tau - \pi)' Z_{[nr]K}| &\le \sup_{0 \le r \le 1} \left| \left\{ (\tilde{\pi}_\tau - \pi)'(\tilde{\pi}_\tau - \pi)(Z_{[nr]K}' Z_{[nr]K}) \right\}^{1/2} \right| \\ &= \sup_{0 \le r \le 1} \left| \left(\sum_{j=-K}^{K} ||\tilde{\pi}_j - \pi_j||^2 \sum_{j=-K}^{K} ||v_{2,[nr]-j}||^2 \right)^{1/2} \right| \\ &\le (2K+1)^{1/2} \sup_{0 \le r \le 1} ||v_{2[nr]}|| \left(\sum_{j=-K}^{K} ||\tilde{\pi}_j - \pi_j||^2 \right)^{1/2} \\ &= O_p(K). \qquad (29)\end{aligned}$$

Then it follows by (26) and (29) that the first and fourth terms in (28) dominate the other terms and are of order $n^{3/2}$, such that $\sum_{t=1}^{[nr]} \tilde{e_{t\tau}} = O_p(n^{3/2})$. Then we have

$$\sum_{t=1}^{n} \left(\sum_{j=1}^{t} \tilde{e_{j\tau}} \right)^2 = O_p(n^4).$$

In the same way as Kwiatkowski et al. (1992) and Phillips (1991), we can also see that $n^{-1} \sum_{t=1}^{n-j} \tilde{e}_{t\tau} \tilde{e}_{t+j,\tau} = O_p(n)$ and then $\tilde{\omega}_{1\cdot 2\tau} = O_p(\ell n)$ under some general regularity conditions on $k(\cdot)$. Then the order of $\tilde{V}_{n\tau}$ becomes $n^{-2} \times O_p(n^4) \times O_p(\ell^{-1} n^{-1}) = O_p(n/\ell)$. □

Proof of Theorem 4.1: We note that b is now defined as $b = (\mu_1, \mu_{2n}, \beta_1', \beta_{2n}')'$. Let $\tilde{b}_\tau$ and $\tilde{\pi}_\tau$ be defined as in the proof of Lemma 3.2 and also let $\tilde{b}_{\hat{\tau}}$ and $\tilde{\pi}_{\hat{\tau}}$ be the OLS estimates of b and π obtained using the estimated change point $\hat{\tau}$. First we show the following lemma.

LEMMA 6.2 *Let Assumption 3.1 and 4.1 hold. Then we have, as* $n \to \infty$

(i) $n^{-1/2} \sum_{t=1}^{[nr]} (\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \xrightarrow{p} 0$,

(ii) $n^{-1} \sum_{t=1}^{[nr]} y_{2t} (\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \xrightarrow{p} 0$,

(iii) $n^{-3/2} \sum_{t=1}^{[nr]} y_{2t} y_{2t}' (\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \xrightarrow{p} 0$,

(iv) $n^{-1/2} \sum_{t=1}^{[nr]} v_{1t} (\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \xrightarrow{p} 0$,

(v) $n^{-1} \sum_{t=1}^{[nr]} y_{2t} v_{1t} (\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \xrightarrow{p} 0$,

(vi) $D_n \left(\tilde{b}_{\hat{\tau}} - \tilde{b}_\tau \right) \xrightarrow{p} 0$.

The convergences (i)-(v) hold uniformly over $r \in [0, 1]$.

Proof of Lemma 6.2: (i) Without loss of generality we shall assume that $n\tau$ and $n\hat{\tau}$ are integers. We have

$$\left| n^{-1/2} \sum_{t=1}^{[nr]} (\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \right| \leq \left| n^{1/2}(\hat{\tau} - \tau) \right| \xrightarrow{p} 0$$

by Proposition 3 of Kurozumi & Arai (2005).

(ii) We get

$$\left| n^{-1} \sum_{t=1}^{[nr]} y_{2t}(\varphi_{t\hat{\tau}} - \varphi_{t\tau}) \right| \leq \sup_{0\leq r\leq 1} \left| \frac{y_{2[nr]}}{n^{1/2}} \right| \left| n^{1/2}(\hat{\tau} - \tau) \right| \xrightarrow{p} 0$$

because $\sup_{0\leq r\leq 1} \left| y_{2[nr]}/n^{1/2} \right| = O_p(1)$.

(iii)-(v) can be proved in the same way as (ii). To prove (vi), observe that

$$D_n \left(\tilde{b}_{\hat{\tau}} - \tilde{b}_{\tau} \right) = D_n \left\{ \left(\sum_{t=1}^{n} X_{t\hat{\tau}} X'_{t\hat{\tau}} \right)^{-1} \sum_{t=1}^{n} X_{t\hat{\tau}} y_{1t} - \left(\sum_{t=1}^{n} X_{t\tau} X'_{t\tau} \right)^{-1} \sum_{t=1}^{n} X_{t\tau} y_{1t} \right\} + o_p(1) \quad (30)$$

by the argument used to prove Lemma 3.2. The required result follows if we show

$$D_n^{-1} \left(\sum_{t=1}^{n} X_{t\hat{\tau}} X'_{t\hat{\tau}} - \sum_{t=1}^{n} X_{t\tau} X'_{t\tau} \right) D_n^{-1} \xrightarrow{p} 0 \quad (31)$$

and

$$D_n^{-1} \sum_{t=1}^{n} \left(X_{t\hat{\tau}} - X_{t\tau} \right) v_{1t} \xrightarrow{p} 0. \quad (32)$$

Noting that

$$X_{t\hat{\tau}} X'_{t\hat{\tau}} - X_{t\tau} X'_{t\tau} = (X_{t\hat{\tau}} - X_{t\tau}) X'_{t\hat{\tau}} + X_{t\tau} (X_{t\hat{\tau}} - X_{t\tau})'$$

and

$$(X_{t\hat{\tau}} - X_{t\tau})' = [0, (\varphi_{t\hat{\tau}} - \varphi_{t\tau}), 0, y'_{2t}(\varphi_{t\hat{\tau}} - \varphi_{t\tau})]',$$

we can show (31) using Lemma 6.2 (i), (ii) and (iii). Similarly, (32) follows by Lemma 6.2 (iv) and (v).

To prove the main result, it is sufficient to show that

$$\sup_{0\leq r\leq 1} \left| n^{-1/2} \tilde{S}_{[nr]\hat{\tau}} - n^{-1/2} S_{[nr]\tau} \right| \xrightarrow{p} 0 \quad (33)$$

and

$$\tilde{\omega}_{1\cdot 2\hat{\tau}} - \omega_{1\cdot 2\tau} \xrightarrow{p} 0. \quad (34)$$

Using the notation given above, $\tilde{S}_{[nr]\tau}$ and $\tilde{S}_{[nr]\hat{\tau}}$ can be written as

$$\tilde{S}_{[nr]\tau} = \sum_{s=1}^{[nr]} \tilde{e}_{s\tau} = \sum_{s=1}^{[nr]} \left(y_{1s} - \tilde{b}'_{\tau} X_{s\tau} - \tilde{\pi}'_{\tau} Z_{sK} \right)$$

and

$$\tilde{S}_{[nr]\hat{\tau}} = \sum_{s=1}^{[nr]} \tilde{e}_{s\hat{\tau}} = \sum_{s=1}^{[nr]} \left(y_{1s} - \tilde{b}_{\hat{\tau}}' X_{s\hat{\tau}} - \tilde{\pi}_{\hat{\tau}}' Z_{sK} \right).$$

Then it follows that

$$\begin{aligned} n^{-1/2}\tilde{S}_{[nr]\hat{\tau}} - n^{-1/2}\tilde{S}_{[nr]\tau} \quad = \quad & -\left(\tilde{b}_{\hat{\tau}} - b\right)' D_n \left\{ n^{-1/2} D_n^{-1} \sum_{s=1}^{[nr]} (X_{s\hat{\tau}} - X_{s\tau}) \right\} \\ & -b' n^{-1/2} \sum_{s=1}^{[nr]} (X_{s\hat{\tau}} - X_{s\tau}) \\ & -\left(\tilde{b}_{\hat{\tau}} - \tilde{b}_{\tau}\right)' D_n \left\{ n^{-1/2} D_n^{-1} \sum_{s=1}^{[nr]} X_{s\tau} \right\} \\ & -\left(\tilde{\pi}_{\hat{\tau}} - \tilde{\pi}_{\tau}\right)' n^{-1/2} \sum_{s=1}^{[nr]} Z_{sK}. \qquad (35) \end{aligned}$$

To show (33), it suffices to show that each term in (35) converges to zero in probability uniformly over $r \in [0,1]$. It follows by (i), (ii) and (vi) of Lemma 6.2 that the first term in (35) vanishes in probability as $n \to \infty$. For the second term in (35), we have by Assumption 4.1, (i) and (ii) of Lemma 6.2 that

$$b' n^{-1/2} \sum_{s=1}^{[nr]} (X_{s\hat{\tau}} - X_{s\tau}) = \mu_{2n} n^{-1/2} \sum_{s=1}^{[nr]} (\varphi_{s\hat{\tau}} - \varphi_{s\tau}) + \beta_{2n}' n^{-1/2} \sum_{s=1}^{[nr]} y_{2t} (\varphi_{s\hat{\tau}} - \varphi_{s\tau}) \xrightarrow{p} 0.$$

The third term of (35) converges to zero in probability by (a) of Lemma 6.1 and (vi) of Lemma 6.2. To show the convergence of the fourth term in (35), observe that the CLT and an argument similar to the one used to show Lemma 6.2 (vi) give

$$||D_n^* \left(\tilde{\pi}_{\hat{\tau}} - \tilde{\pi}_{\tau}\right)|| = o_p\left(K^{1/2}/n^{1/2}\right)$$

and

$$\left|\left| n^{-1/2} \sum_{s=1}^{[nr]} Z_{sK} \right|\right| = O_p\left(K^{1/2}\right).$$

Thus the fourth term also converges to zero in probability. Observing that all convergences shown above are uniform in r shows (33). (34) can be shown by noting that $n^{1/2}(\hat{\tau} - \tau) = o_p(1)$ by Proposition 3 of Kurozumi & Arai (2005) and using standard arguments (see Shin (1994)). This finishes the proof of the first part of Theorem 4.1.

Proof of (ii): Let τ be an arbitrary break fraction and τ_o be the true one. Then the model can be expressed as

$$y_{1t} = b' X_{t\tau} + \pi' Z_{tK} + \gamma_t + \varepsilon_t^* - b'(X_{t\tau} - X_{t\tau_o}),$$

where $b'(X_{t\tau}-X_{t\tau_o})=\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})$. We have by Assumption 4.1 and Lemma 6.1 that

$$\sum_{t=1}^{n} b'(X_{t\tau}-X_{t\tau_o})=O_p(n) \quad (36)$$

and

$$\sum_{t=1}^{n} y_{2t}b'(X_{t\tau}-X_{t\tau_o})=O_p(n^{3/2}). \quad (37)$$

It can be deduced from (36) and (37) that

$$\begin{aligned} D_n^{-1}\sum_{t=1}^{n} X_{t\tau}\{\gamma_t+\varepsilon_t^*-b'(X_{t\tau}-X_{t\tau_o})\} &= D_n^{-1}\sum_{t=1}^{n} X_{t\tau}\gamma_t+o_p(n) \\ &= O_p(n). \end{aligned}$$

As in the proof of Lemma 3.2, we can show that $\tilde{R}$ becomes asymptotically block diagonal and that (27) holds. Then we have by (27), (36) and (37) that

$$\begin{aligned} \sum_{t=1}^{[nr]}\tilde{e}_{t\tau} &= \sum_{t=1}^{[nr]}(y_{1t}-\tilde{b}_\tau' X_{t\tau}-\tilde{\pi}_\tau' Z_{tK}) \\ &= \sum_{t=1}^{[nr]}\Big\{\gamma_t+\varepsilon_t^*-(\tilde{b}_\tau-b)'X_{t\tau}-(\tilde{\pi}_\tau-\pi)'Z_{tK}-b'(X_{t\tau}-X_{t\tau_o})\Big\} \\ &= \sum_{t=1}^{[nr]}\gamma_t-(\tilde{b}_\tau-b)'D_nD_n^{-1}\sum_{t=1}^{[nr]}X_{t\tau}+o_p(n^{3/2}) \\ &= O_p(n^{3/2}) \end{aligned}$$

under the alternative. This implies that $\sum_{t=1}^{n}(\sum_{j=1}^{t}\tilde{e}_{j\tau})^2$ is of order n^4. We can also show that $\tilde{\omega}_{1\cdot 2\tau}=O_p(\ell n)$ as in the known break point case. Therefore $\tilde{V}_{n\tau}=O_p(n/\ell)$. Since this relation holds for any τ, the theorem is proved. □

Proof of Theorem 4.2: In this proof, let τ_o and τ be the true break fraction and the fraction that is used for estimation, respectively. We first show that $(\tilde{b}_\tau-b)$ has the same asymptotic properties as $(\tilde{b}_{\tau_o}-b)$ in Lemma 3.2. Since the model is expressed as

$$y_{1t}=b'X_{t\tau}+\pi'Z_{tK}+\varepsilon_t^*-b'(X_{t\tau}-X_{t\tau_o}),$$

we can proceed the same way we did in Lemma 3.2, with ε_t^* replaced by $\varepsilon_t^*-b'(X_{t\tau}-X_{t\tau_o})$. Since

$$b'(X_{t\tau}-X_{t\tau_o})=\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o}),$$

we can see that

$$D_n^{-1}\sum_{t=1}^{n} X_{t\tau}(X_{t\tau}-X_{t\tau_o})'b = \begin{bmatrix} n^{-1/2}\sum_{t=1}^{n}\{\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})\} \\ n^{-1/2}\sum_{t=1}^{n}\varphi_{t\tau}\{\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})\} \\ n^{-1}\sum_{t=1}^{n}y_{2t}\{\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})\} \\ n^{-1}\sum_{t=1}^{n}y_{2t}\varphi_{t\tau}\{\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})\} \end{bmatrix} \xrightarrow{p} 0,$$

where the convergence is established because

$$\begin{aligned} &\left|n^{-1/2}\sum_{t=1}^{n}\{\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})+\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})\}\right| \\ \leq\ & n^{-1/2}\sum_{t=1}^{n}|\mu_{2n}(\varphi_{t\tau}-\varphi_{t\tau_o})| + n^{-1/2}\sum_{t=1}^{n}|\beta_{2n}'y_{2t}(\varphi_{t\tau}-\varphi_{t\tau_o})| \\ \leq\ & n^{1/2}|\mu_{2n}| + \sup_{0\leq r\leq 1}\left|\frac{y_{2[nr]}}{n^{1/2}}\right| n|\beta_{2n}| \xrightarrow{p} 0 \end{aligned}$$

by Assumption 4.2. This implies that $D_n(\tilde{b}_\tau - b)$ has the same limiting distribution as that in Lemma 3.2. Similarly, we can show that $\sum_{j=-K}^{K}||\tilde{\pi}_j-\pi_j||^2 = O_p(K/n)$ in the same way we did in Lemma 3.2.

Next, note that

$$\begin{aligned} \tilde{e}_{t\tau} &= y_{1t} - \tilde{b}_\tau' X_{t\tau} - \tilde{\pi}_\tau' Z_{tK} \\ &= \varepsilon_t + \eta_{tK} - (\tilde{b}_\tau - b)'X_{t\tau} - (\tilde{\pi}_\tau - \pi)'Z_{tK} - b_{\tau_o}'(X_{t\tau}-X_{t\tau_o}) \end{aligned}$$

and that $(\tilde{b}_\tau - b)$ has the same asymptotic properties as $(\tilde{b}_{\tau_o} - b)$ in Lemma 3.2. Then, just as in the proof of Theorem 3.2, we can show that $\tilde{S}_{[nr]\tau}$ converges weakly to $\tilde{Q}_{X_\tau}(r)$ and $\tilde{\omega}_{1\cdot 2\tau}$ converges to $\omega_{1\cdot 2}$ in probability, implying that $\tilde{V}_{n\tau} \Rightarrow \int_0^1 Q_\tau^2(r)dr$. Then the theorem is established using the continuous mapping theorem.

Consistency of the test statistic can be proved in the same way as in Theorem 3.2. □

Notes

[1]Tests for cointegration with structural breaks of "known" timing are proposed by Saikkonen & Lütkepohl (2000), Hansen (2003) and Lütkepohl et al. (2003).

[2]Although $\tilde{\pi}_i$ depends on the break fraction τ, we suppress the dependence for simplicity.

[3] To prove Theorem 4.1 by using the pseudo-Gaussian MLE, we need to assume the following: **ASSUMPTION 4.1'***(a) The I(1) regressor y_{2t} satisfies:*

$$\mathbf{E}\left(\frac{ty_{2i,t}^2}{y_{2i,1}^2+y_{2i,2}^2+\cdots+y_{2i,t}^2}\right)\le M \qquad \text{for all } t\ge 1 \text{ and } i=1,\ldots,m.$$

(b) ε_t is independent of the regressors for all leads and lags. (c) $\pi_i = 0$ for all $i > |K|$ where K is a finite constant. (d) μ_2 and β_2 in (12), (13) and (14) depend on the sample size n. We denote them by μ_{2n} and β_{2n}. We assume $\mu_{2n} = \delta_n\mu_0$ for the models 1 and 2, and for the model 3, $\mu_{2n} = \mu_o\delta_n$, $\beta_{2n} = n^{-1/2}\beta_o\delta_n$ where δ_n is a scalar such that $\delta_n = o(n^{-\rho})$ for $0 < \rho < 1/4$, μ_0 is a constant scalar and β_0 is a constant m-vector. These assumptions are obviously more restrictive than Assumption 4.1. Assumption 4.1' (b) would not be very realistic among others as we note in the last section.

[4]We also consider the analogous approach to constructing a test statistic so as to give the least favorable result for the null as in Zivot & Andrews (1992). The test statistic is defined as $\sup_{\underline{\tau}\le\tau\le\bar{\tau}}\tilde{V}_{n\tau}$ where $\tilde{V}_{n\tau}$ is as in (19) and we can show results similar to those in Theorem 4.2. However, we do not present them in this and subsequent sections because unreported simulation experiments show that the finite-sample performance of such a "sup-type" test statistic is dominated by the "inf-type" one.

[5]All unreported results are available upon request.

[6]There are typos in the expression of $\ell_A K$ in Kurozumi (2002, pp.81). The numerator of the second argument in parentheses must be multiplied by 4 as the above expression.

[7]We use a maximum lag length less than ℓ_4 when τ is close to the end points in order to obtain enough observations for estimation.

[8]Asymptotic critical values are calculated by employing the method of MacKinnon (1991).

References

Andrews, D. W. K. (1991) Heteroskedasticity and autocorrelation consistent covariance matrix estimation. *Econometrica* **59**, 817–858.

Arai, Y. (2004) Testing for linearity in regressions with I(1) processes. Discussion paper CIRJE-F-303, Faculty of Economics, University of Tokyo.

Bai, J., R. L. Lumsdaine, & J. H. Stock (1998) Testing for and dating common breaks in multivariate time series. *Review of Economic Studies* **65**, 395–432.

Brillinger, D. R. (1981) *Time Series: Data Analysis and Theory*New York: Holt, Rinehart and Winston.

Busetti, F. & A. Harvey (2001) Testing for the presence of a random walk in series with structural breaks. *Journal of Time Series Analysis* **22**, 127–150.

Engle, R. F. & C. W. J. Granger (1987) Co–integration and error correction: Representation, estimation and testing. *Econometrica* **55**, 547–557.

Granger, C. W. J. (1983) Co–integrated and error–correcting models. Discussion Paper 83–13, Department of Economics, University of California, San Diego.

Granger, C. W. J. & T.-H. Lee (1989) Multicointegration. *Advances in Econometrics* **8**, 71–84.

Gregory, A. W. & B. E. Hansen (1996) Residual–based tests for cointegration in models with regime shifts. *Journal of Econometrics* **70**, 99–126.

Hansen, P. R. (2003) Structural changes in the cointegrated vector autoregressive model. *Journal of Econometrics* **114**, 261–295.

Herrndorf, N. (1984) A functional central limit theorem for weakly dependent sequences of random variables. *Annals of Probability* **12**, 141–153.

Inoue, A. (1999) Tests of cointegrating rank with a trend–break. *Journal of Econometrics* **90**, 215–237.

Johansen, S. (1988) Statistical analysis of cointegration vectors. *Journal of Economic Dynamics and Control* **12**, 231–254.

Johansen, S. (1991) Estimation and hypothesis testing of cointegration vectors in Gaussian vector autoregressive models. *Econometrica* **59**, 1551–1580.

Kurozumi, E. (2002) Testing for stationarity with a break. *Journal of Econometrics* **108**, 63–99.

Kurozumi, E. & Y. Arai (2005) Efficient estimation and inference in cointegrating regressions with structural change. Discussion Paper No. 2004-9, Graduate School of Economics, Hitotsubashi University (http://www.econ.hit-u.ac.jp/˜kurozumi/paper/efficientjan05.pdf).

Kwiatkowski, D., P. C. B. Phillips, P. Schmidt, & Y. Shin (1992) Testing the null hypothesis of stationarity against the alternative of a unit root. *Journal of Econometrics* **54**, 159–178.

Lütkepohl, H., P. Saikkonen, & C. Trenkler (2003) Comparison of tests for the cointegrating rank of a VAR process with a structural shift. *Journal of Econometrics* **113**, 201–229.

Lütkepohl, H., P. Saikkonen, & C. Trenkler (2004) Testing for the cointegrating rank of a VAR processes with level shift at unknown time. *Econometrica* **72**, 647–662.

MacKinnon, J. G. (1991) Critical values for cointegration tests. In R. F. Engle & C. W. J. Granger (eds.), *Long-Run Economic Relationships*, New York, pp. 267–276. Oxford University Press.

Nabeya, S. & K. Tanaka (1988) Asymptotic theory of a test for the constancy of regression coefficients against the random walk alternative. *Annals of Statistics* **16**, 218–235.

Newey, W. K. & K. D. West (1987) A simple, positive semidefinite, heteroskedasticity and autocorrelated consistent covariance matrix. *Econometrica* **55**, 586–591.

Phillips, P. C. B. (1991) Spectral regression for cointegrated time series. In W. Barnett (ed.), *Nonparametric and Semiparametric Methods in Economics and Statistics*, pp. 413–435. New York: Cambridge University Press.

Phillips, P. C. B. & S. N. Durlauf (1986) Multiple time series regression with integrated processes. *Review of Economic Studies* **53**, 473–496.

Phillips, P. C. B. & S. Ouliaris (1990) Asymptotic properties of residual based tests for cointegration. *Econometrica* **58**, 165–193.

Quintos, C. E. (1997) Stability tests in error correction models. *Journal of Econometrics* **82**, 289–315.

Saikkonen, P. (1991) Asymptotically efficient estimation of cointegration regressions. *Econometric Theory* **7**, 1–21.

Saikkonen, P. & H. Lütkepohl (2000) Testing for the cointegrating rank of a VAR process with structural shifts. *Journal of Business and Economic Statistics* **18**, 451–464.

Schwert, G. W. (1989) Tests for unit roots: A monte carlo investigation. *Journal of Business and Economic Statistics* **7**, 147–159.

Seo, B. (1998) Tests for structural change in cointegrated systems regressions. *Econometric Theory* **14**, 222–259.

Shin, Y. (1994) A residual–based test of the null of cointegration against alternative of no cointegration. *Econometric Theory* **10**, 91–115.

Zivot, E. & D. W. K. Andrews (1992) Further evidence on the great crash, the oil-price shock, and the unit-root hypothesis. *Journal of Business and Economic Statistics* **10**, 251–270.

Table 1: Percentiles for the null distribution of the test statistic ($m = 1$)

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
a. Model 1							
$\tau = 0.1$	0.01826	0.02550	0.03111	0.06967	0.19117	0.25936	0.44825
$\tau = 0.2$	0.01745	0.02417	0.02896	0.06197	0.15999	0.21613	0.35836
$\tau = 0.3$	0.01751	0.02373	0.02846	0.05860	0.13928	0.18173	0.29459
$\tau = 0.4$	0.01733	0.02385	0.02851	0.05757	0.12828	0.16218	0.24215
$\tau = 0.5$	0.01755	0.02393	0.02855	0.05742	0.12435	0.15452	0.22353
$\tau = 0.6$	0.01765	0.02393	0.02855	0.05770	0.12674	0.16041	0.24412
$\tau = 0.7$	0.01753	0.02401	0.02864	0.05864	0.13812	0.17948	0.29220
$\tau = 0.8$	0.01757	0.02422	0.02912	0.06203	0.15842	0.21490	0.35681
$\tau = 0.9$	0.01820	0.02551	0.03093	0.06959	0.19136	0.25810	0.44634
b. Model 2							
$\tau = 0.1$	0.01395	0.01849	0.02151	0.03999	0.08163	0.10088	0.14716
$\tau = 0.2$	0.01399	0.01811	0.02109	0.03793	0.07311	0.08829	0.12646
$\tau = 0.3$	0.01400	0.01818	0.02111	0.03841	0.07433	0.08934	0.12567
$\tau = 0.4$	0.01396	0.01838	0.02150	0.03953	0.08009	0.09949	0.15045
$\tau = 0.5$	0.01400	0.01839	0.02157	0.03997	0.08453	0.10649	0.16299
$\tau = 0.6$	0.01392	0.01837	0.02155	0.03959	0.08053	0.10019	0.14566
$\tau = 0.7$	0.01397	0.01820	0.02117	0.03858	0.07492	0.08902	0.12445
$\tau = 0.8$	0.01380	0.01808	0.02108	0.03806	0.07340	0.08889	0.12824
$\tau = 0.9$	0.01387	0.01837	0.02155	0.03980	0.08107	0.10063	0.15110
c . Model 3							
$\tau = 0.1$	0.01777	0.02495	0.03041	0.06868	0.18930	0.25684	0.44560
$\tau = 0.2$	0.01641	0.02251	0.02709	0.05840	0.15357	0.20784	0.34951
$\tau = 0.3$	0.01595	0.02148	0.02552	0.05170	0.12574	0.16678	0.27798
$\tau = 0.4$	0.01564	0.02111	0.02497	0.04869	0.10877	0.13943	0.21848
$\tau = 0.5$	0.01564	0.02091	0.02466	0.04791	0.10375	0.12913	0.19226
$\tau = 0.6$	0.01572	0.02113	0.02495	0.04876	0.10783	0.13789	0.21751
$\tau = 0.7$	0.01595	0.02156	0.02564	0.05188	0.12455	0.16442	0.27529
$\tau = 0.8$	0.01662	0.02262	0.02723	0.05815	0.15210	0.20626	0.34447
$\tau = 0.9$	0.01773	0.02489	0.03028	0.06859	0.18954	0.25650	0.44203

Table 2: Percentiles for the null distribution of the test statistic ($m = 2$)

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
a. Model 1							
$\tau = 0.1$	0.01610	0.02170	0.02590	0.05360	0.13500	0.18020	0.31100
$\tau = 0.2$	0.01570	0.02100	0.02490	0.04930	0.11720	0.15620	0.26110
$\tau = 0.3$	0.01560	0.02090	0.02460	0.04810	0.10710	0.13790	0.22770
$\tau = 0.4$	0.01550	0.02090	0.02460	0.04790	0.10300	0.12990	0.20270
$\tau = 0.5$	0.01540	0.02070	0.02460	0.04780	0.10330	0.12950	0.19090
$\tau = 0.6$	0.01550	0.02080	0.02460	0.04790	0.10420	0.13090	0.20100
$\tau = 0.7$	0.01550	0.02080	0.02470	0.04820	0.10840	0.14010	0.22790
$\tau = 0.8$	0.01560	0.02100	0.02480	0.04950	0.11740	0.15570	0.26290
$\tau = 0.9$	0.01580	0.02150	0.02570	0.05390	0.13520	0.18070	0.32580
b. Model 2							
$\tau = 0.1$	0.01270	0.01660	0.01920	0.03460	0.06910	0.08500	0.12550
$\tau = 0.2$	0.01270	0.01650	0.01900	0.03350	0.06390	0.07740	0.11080
$\tau = 0.3$	0.01280	0.01660	0.01920	0.03390	0.06520	0.07830	0.10960
$\tau = 0.4$	0.01270	0.01660	0.01920	0.03440	0.06800	0.08360	0.12280
$\tau = 0.5$	0.01280	0.01670	0.01920	0.03450	0.06970	0.08660	0.13490
$\tau = 0.6$	0.01280	0.01670	0.01930	0.03430	0.06810	0.08400	0.12400
$\tau = 0.7$	0.01280	0.01660	0.01910	0.03390	0.06490	0.07830	0.11070
$\tau = 0.8$	0.01270	0.01650	0.01910	0.03350	0.06390	0.07700	0.10830
$\tau = 0.9$	0.01270	0.01660	0.01920	0.03470	0.06900	0.08570	0.12560
c . Model 3							
$\tau = 0.1$	0.01540	0.02070	0.02470	0.05210	0.13230	0.17780	0.30750
$\tau = 0.2$	0.01400	0.01860	0.02190	0.04380	0.10780	0.14370	0.24730
$\tau = 0.3$	0.01330	0.01760	0.02050	0.03880	0.08780	0.11430	0.19490
$\tau = 0.4$	0.01300	0.01700	0.01980	0.03630	0.07680	0.09760	0.15800
$\tau = 0.5$	0.01300	0.01670	0.01950	0.03560	0.07330	0.09230	0.14310
$\tau = 0.6$	0.01300	0.01690	0.01970	0.03630	0.07700	0.09880	0.15880
$\tau = 0.7$	0.01330	0.01750	0.02040	0.03890	0.08960	0.11730	0.20060
$\tau = 0.8$	0.01390	0.01860	0.02190	0.04400	0.10750	0.14300	0.24950
$\tau = 0.9$	0.01500	0.02060	0.02470	0.05220	0.13180	0.17810	0.31890

Table 3: Percentiles for the null distribution of the test statistic ($m = 3$)

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
a. Model 1							
$\tau = 0.1$	0.01430	0.01900	0.02230	0.04350	0.10150	0.13190	0.22610
$\tau = 0.2$	0.01420	0.01860	0.02170	0.04100	0.09080	0.11780	0.19160
$\tau = 0.3$	0.01430	0.01860	0.02170	0.04050	0.08640	0.10910	0.17190
$\tau = 0.4$	0.01410	0.01860	0.02180	0.04040	0.08490	0.10640	0.16080
$\tau = 0.5$	0.01410	0.01860	0.02170	0.04050	0.08490	0.10620	0.15810
$\tau = 0.6$	0.01410	0.01860	0.02170	0.04060	0.08500	0.10690	0.16290
$\tau = 0.7$	0.01420	0.01860	0.02170	0.04060	0.08620	0.10970	0.17620
$\tau = 0.8$	0.01420	0.01870	0.02190	0.04130	0.09040	0.11730	0.19860
$\tau = 0.9$	0.01430	0.01900	0.02230	0.04340	0.10070	0.13210	0.22470
b. Model 2							
$\tau = 0.1$	0.01190	0.01520	0.01760	0.03050	0.05890	0.07190	0.10510
$\tau = 0.2$	0.01180	0.01520	0.01740	0.03010	0.05570	0.06660	0.09470
$\tau = 0.3$	0.01190	0.01520	0.01750	0.03030	0.05650	0.06810	0.09690
$\tau = 0.4$	0.01190	0.01510	0.01750	0.03040	0.05770	0.07050	0.10580
$\tau = 0.5$	0.01190	0.01510	0.01740	0.03040	0.05860	0.07260	0.11000
$\tau = 0.6$	0.01190	0.01520	0.01740	0.03030	0.05780	0.07050	0.10400
$\tau = 0.7$	0.01190	0.01520	0.01750	0.03030	0.05680	0.06850	0.09620
$\tau = 0.8$	0.01180	0.01510	0.01740	0.02990	0.05550	0.06710	0.09530
$\tau = 0.9$	0.01180	0.01520	0.01750	0.03050	0.05830	0.07190	0.10610
c . Model 3							
$\tau = 0.1$	0.01350	0.01790	0.02110	0.04160	0.09800	0.12780	0.22020
$\tau = 0.2$	0.01210	0.01590	0.01850	0.03500	0.07950	0.10410	0.17610
$\tau = 0.3$	0.01150	0.01490	0.01720	0.03090	0.06560	0.08460	0.13870
$\tau = 0.4$	0.01130	0.01430	0.01640	0.02870	0.05680	0.07140	0.11380
$\tau = 0.5$	0.01120	0.01420	0.01630	0.02810	0.05430	0.06760	0.10390
$\tau = 0.6$	0.01120	0.01440	0.01650	0.02870	0.05720	0.07150	0.11480
$\tau = 0.7$	0.01160	0.01490	0.01720	0.03080	0.06560	0.08460	0.14240
$\tau = 0.8$	0.01220	0.01600	0.01860	0.03520	0.07970	0.10460	0.18030
$\tau = 0.9$	0.01340	0.01780	0.02100	0.04160	0.09750	0.12850	0.22190

Table 4: Percentiles for the null distribution of the test statistic ($m = 4$)

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
a. Model 1							
$\tau = 0.1$	0.01300	0.01690	0.01980	0.03690	0.07960	0.10210	0.16730
$\tau = 0.2$	0.01300	0.01680	0.01950	0.03540	0.07320	0.09270	0.15050
$\tau = 0.3$	0.01290	0.01680	0.01950	0.03530	0.07100	0.08890	0.13800
$\tau = 0.4$	0.01290	0.01690	0.01950	0.03530	0.07080	0.08740	0.13500
$\tau = 0.5$	0.01290	0.01680	0.01950	0.03540	0.07110	0.08830	0.13250
$\tau = 0.6$	0.01290	0.01680	0.01950	0.03530	0.07080	0.08800	0.13440
$\tau = 0.7$	0.01290	0.01670	0.01950	0.03520	0.07150	0.08940	0.13860
$\tau = 0.8$	0.01290	0.01680	0.01960	0.03540	0.07360	0.09300	0.14830
$\tau = 0.9$	0.01290	0.01700	0.01980	0.03680	0.07990	0.10240	0.16810
b. Model 2							
$\tau = 0.1$	0.01090	0.01390	0.01600	0.02720	0.05100	0.06170	0.08950
$\tau = 0.2$	0.01110	0.01390	0.01590	0.02700	0.04890	0.05910	0.08460
$\tau = 0.3$	0.01100	0.01400	0.01600	0.02710	0.04960	0.05970	0.08460
$\tau = 0.4$	0.01100	0.01400	0.01600	0.02720	0.05060	0.06110	0.08930
$\tau = 0.5$	0.01110	0.01410	0.01610	0.02720	0.05110	0.06190	0.09150
$\tau = 0.6$	0.01110	0.01400	0.01600	0.02710	0.05060	0.06150	0.08950
$\tau = 0.7$	0.01100	0.01390	0.01600	0.02710	0.04950	0.05960	0.08530
$\tau = 0.8$	0.01100	0.01400	0.01600	0.02690	0.04890	0.05880	0.08360
$\tau = 0.9$	0.01100	0.01400	0.01600	0.02720	0.05110	0.06240	0.08990
c . Model 3							
$\tau = 0.1$	0.01190	0.01570	0.01840	0.03480	0.07610	0.09810	0.16230
$\tau = 0.2$	0.01090	0.01390	0.01620	0.02920	0.06190	0.07970	0.12980
$\tau = 0.3$	0.01020	0.01290	0.01480	0.02560	0.05120	0.06430	0.10440
$\tau = 0.4$	0.00990	0.01240	0.01420	0.02370	0.04420	0.05430	0.08430
$\tau = 0.5$	0.00990	0.01240	0.01400	0.02310	0.04230	0.05140	0.07580
$\tau = 0.6$	0.01000	0.01250	0.01420	0.02370	0.04420	0.05450	0.08380
$\tau = 0.7$	0.01030	0.01300	0.01490	0.02550	0.05120	0.06470	0.10310
$\tau = 0.8$	0.01090	0.01390	0.01610	0.02930	0.06180	0.07980	0.13110
$\tau = 0.9$	0.01210	0.01580	0.01840	0.03470	0.07630	0.09810	0.16170

Table 5: Percentiles for the null distribution of the test statistic ($m = 5$)

	0.01	0.05	0.1	0.5	0.9	0.95	0.99
a. Model 1							
$\tau = 0.1$	0.01200	0.01540	0.01790	0.03200	0.06500	0.08240	0.13340
$\tau = 0.2$	0.01200	0.01540	0.01770	0.03110	0.06120	0.07630	0.11950
$\tau = 0.3$	0.01200	0.01540	0.01770	0.03110	0.06050	0.07460	0.11310
$\tau = 0.4$	0.01200	0.01540	0.01770	0.03100	0.06050	0.07440	0.11090
$\tau = 0.5$	0.01200	0.01540	0.01770	0.03110	0.06030	0.07430	0.11170
$\tau = 0.6$	0.01200	0.01530	0.01770	0.03100	0.06040	0.07360	0.11010
$\tau = 0.7$	0.01190	0.01530	0.01770	0.03100	0.06040	0.07460	0.11420
$\tau = 0.8$	0.01190	0.01530	0.01770	0.03100	0.06130	0.07600	0.11980
$\tau = 0.9$	0.01210	0.01550	0.01790	0.03200	0.06500	0.08200	0.13300
b. Model 2							
$\tau = 0.1$	0.01030	0.01300	0.01480	0.02450	0.04460	0.05390	0.07790
$\tau = 0.2$	0.01030	0.01300	0.01480	0.02450	0.04350	0.05190	0.07270
$\tau = 0.3$	0.01030	0.01310	0.01490	0.02450	0.04420	0.05300	0.07470
$\tau = 0.4$	0.01040	0.01300	0.01480	0.02460	0.04450	0.05340	0.07660
$\tau = 0.5$	0.01030	0.01310	0.01480	0.02460	0.04460	0.05360	0.07730
$\tau = 0.6$	0.01040	0.01300	0.01480	0.02460	0.04460	0.05340	0.07620
$\tau = 0.7$	0.01030	0.01300	0.01480	0.02460	0.04420	0.05300	0.07420
$\tau = 0.8$	0.01040	0.01300	0.01480	0.02450	0.04370	0.05250	0.07370
$\tau = 0.9$	0.01040	0.01300	0.01480	0.02470	0.04490	0.05380	0.07740
c . Model 3							
$\tau = 0.1$	0.01090	0.01410	0.01640	0.02970	0.06170	0.07800	0.12690
$\tau = 0.2$	0.00980	0.01250	0.01430	0.02490	0.05040	0.06310	0.10190
$\tau = 0.3$	0.00920	0.01150	0.01310	0.02180	0.04160	0.05160	0.08150
$\tau = 0.4$	0.00890	0.01110	0.01250	0.02020	0.03630	0.04420	0.06620
$\tau = 0.5$	0.00890	0.01090	0.01230	0.01970	0.03430	0.04130	0.06040
$\tau = 0.6$	0.00890	0.01100	0.01250	0.02010	0.03600	0.04380	0.06500
$\tau = 0.7$	0.00920	0.01150	0.01310	0.02190	0.04160	0.05170	0.08040
$\tau = 0.8$	0.00980	0.01250	0.01420	0.02500	0.05020	0.06360	0.10190
$\tau = 0.9$	0.01100	0.01430	0.01650	0.02980	0.06150	0.07820	0.12690

Table 6: Rejection frequencies of the tests when the break point is known ($n = 100$)

a_{ii}	σ_u^2	Model 1			Model 2			Model 3		
$n = 100$		ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a
-0.8	0	0.006	0.073	0.097	0.011	0.125	0.159	0.089	0.249	0.272
	0.01	0.321	0.350	0.371	0.211	0.314	0.335	0.365	0.459	0.469
	0.1	0.695	0.516	0.652	0.550	0.447	0.518	0.678	0.542	0.647
	1	0.771	0.515	0.823	0.660	0.438	0.775	0.756	0.548	0.834
	10	0.788	0.525	0.476	0.666	0.423	0.447	0.759	0.547	0.584
-0.4	0	0.029	0.074	0.038	0.042	0.106	0.043	0.050	0.134	0.057
	0.01	0.391	0.339	0.395	0.250	0.303	0.255	0.334	0.363	0.340
	0.1	0.698	0.501	0.811	0.559	0.420	0.682	0.647	0.483	0.739
	1	0.777	0.515	0.603	0.649	0.434	0.514	0.724	0.506	0.601
	10	0.791	0.519	0.431	0.658	0.424	0.426	0.724	0.512	0.502
0	0	0.048	0.072	0.042	0.053	0.103	0.044	0.063	0.130	0.051
	0.01	0.303	0.272	0.321	0.189	0.247	0.201	0.250	0.294	0.265
	0.1	0.656	0.478	0.683	0.501	0.389	0.553	0.566	0.444	0.610
	1	0.762	0.504	0.518	0.639	0.410	0.441	0.679	0.475	0.507
	10	0.784	0.517	0.419	0.654	0.411	0.414	0.701	0.487	0.470
0.4	0	0.085	0.083	0.084	0.079	0.110	0.078	0.091	0.125	0.091
	0.01	0.209	0.180	0.199	0.145	0.170	0.137	0.186	0.215	0.177
	0.1	0.564	0.403	0.496	0.399	0.318	0.346	0.451	0.388	0.414
	1	0.742	0.487	0.451	0.598	0.390	0.395	0.644	0.440	0.447
	10	0.774	0.504	0.412	0.635	0.398	0.403	0.670	0.443	0.452
0.8	0	0.245	0.110	0.108	0.196	0.122	0.114	0.214	0.150	0.129
	0.01	0.256	0.127	0.118	0.209	0.138	0.133	0.228	0.155	0.139
	0.1	0.429	0.248	0.232	0.281	0.178	0.176	0.322	0.212	0.211
	1	0.655	0.415	0.356	0.466	0.293	0.288	0.489	0.331	0.324
	10	0.724	0.466	0.379	0.539	0.336	0.328	0.567	0.363	0.365

Table 7: Rejection frequencies of the tests when the break point is known ($n = 200$)

a_{ii}	σ_u^2	Model 1			Model 2			Model 3		
$n = 200$		ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a
-0.8	0	0.006	0.051	0.064	0.013	0.084	0.109	0.072	0.188	0.211
	0.01	0.685	0.618	0.625	0.521	0.483	0.488	0.639	0.588	0.596
	0.1	0.919	0.717	0.881	0.860	0.582	0.770	0.896	0.676	0.844
	1	0.947	0.730	0.869	0.915	0.569	0.826	0.941	0.666	0.884
	10	0.961	0.732	0.575	0.922	0.570	0.502	0.945	0.665	0.607
-0.4	0	0.028	0.049	0.038	0.035	0.075	0.046	0.030	0.077	0.046
	0.01	0.712	0.580	0.708	0.545	0.447	0.542	0.631	0.513	0.622
	0.1	0.921	0.705	0.964	0.866	0.558	0.935	0.880	0.629	0.940
	1	0.956	0.726	0.689	0.911	0.567	0.581	0.929	0.643	0.635
	10	0.962	0.728	0.553	0.921	0.573	0.493	0.932	0.645	0.546
0	0	0.056	0.055	0.054	0.056	0.081	0.058	0.054	0.072	0.051
	0.01	0.626	0.499	0.664	0.457	0.380	0.491	0.546	0.440	0.575
	0.1	0.905	0.691	0.865	0.828	0.531	0.778	0.852	0.597	0.797
	1	0.950	0.718	0.606	0.904	0.566	0.517	0.922	0.628	0.564
	10	0.957	0.724	0.547	0.915	0.569	0.482	0.926	0.631	0.532
0.4	0	0.112	0.075	0.090	0.093	0.082	0.085	0.091	0.085	0.078
	0.01	0.460	0.357	0.425	0.353	0.274	0.311	0.409	0.319	0.364
	0.1	0.860	0.636	0.700	0.747	0.485	0.556	0.784	0.535	0.610
	1	0.938	0.713	0.559	0.889	0.552	0.478	0.889	0.602	0.518
	10	0.954	0.718	0.537	0.913	0.556	0.465	0.920	0.607	0.512
0.8	0	0.345	0.122	0.119	0.311	0.128	0.123	0.297	0.117	0.116
	0.01	0.424	0.182	0.176	0.377	0.158	0.151	0.370	0.167	0.157
	0.1	0.715	0.438	0.376	0.593	0.320	0.295	0.616	0.370	0.340
	1	0.898	0.645	0.486	0.838	0.460	0.400	0.834	0.519	0.438
	10	0.940	0.690	0.504	0.896	0.504	0.423	0.888	0.548	0.461

Table 8: Rejection frequencies of the tests when the break point is unknown ($n = 100$)

a_{ii}	σ_u^2	Model 1			Model 2			Model 3		
$n = 100$		ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a
-0.8	0	0.016	0.071	0.095	0.026	0.144	0.179	0.144	0.214	0.219
	0.01	0.200	0.220	0.238	0.101	0.195	0.218	0.214	0.225	0.234
	0.1	0.388	0.252	0.356	0.270	0.243	0.268	0.385	0.274	0.355
	1	0.460	0.239	0.607	0.419	0.287	0.671	0.457	0.283	0.614
	10	0.485	0.235	0.254	0.462	0.299	0.314	0.477	0.270	0.308
-0.4	0	0.025	0.060	0.029	0.039	0.119	0.041	0.057	0.112	0.060
	0.01	0.220	0.206	0.226	0.101	0.153	0.105	0.145	0.162	0.154
	0.1	0.380	0.222	0.516	0.305	0.245	0.394	0.353	0.249	0.466
	1	0.456	0.229	0.337	0.422	0.279	0.388	0.435	0.252	0.366
	10	0.461	0.230	0.229	0.449	0.290	0.282	0.446	0.258	0.258
0	0	0.040	0.066	0.031	0.052	0.112	0.039	0.049	0.099	0.035
	0.01	0.155	0.168	0.164	0.068	0.127	0.062	0.107	0.142	0.095
	0.1	0.350	0.217	0.388	0.245	0.226	0.300	0.331	0.242	0.367
	1	0.451	0.219	0.260	0.401	0.259	0.292	0.416	0.243	0.280
	10	0.454	0.217	0.208	0.431	0.285	0.273	0.433	0.237	0.233
0.4	0	0.054	0.074	0.057	0.056	0.130	0.056	0.064	0.092	0.061
	0.01	0.092	0.094	0.092	0.062	0.116	0.063	0.096	0.119	0.098
	0.1	0.255	0.176	0.212	0.160	0.164	0.141	0.241	0.203	0.215
	1	0.425	0.228	0.228	0.357	0.242	0.238	0.389	0.237	0.238
	10	0.441	0.219	0.200	0.404	0.270	0.259	0.401	0.230	0.223
0.8	0	0.116	0.076	0.072	0.112	0.102	0.064	0.137	0.090	0.081
	0.01	0.121	0.074	0.076	0.112	0.104	0.079	0.125	0.093	0.082
	0.1	0.181	0.097	0.098	0.150	0.128	0.105	0.179	0.121	0.117
	1	0.319	0.156	0.166	0.252	0.176	0.150	0.276	0.180	0.173
	10	0.358	0.184	0.178	0.329	0.228	0.202	0.330	0.186	0.186

Table 9: Rejection frequencies of the tests when the break point is unknown ($n = 200$)

a_{ii}	σ_u^2	Model 1			Model 2			Model 3		
$n = 200$		ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a
-0.8	0	0.010	0.051	0.060	0.022	0.076	0.094	0.181	0.225	0.234
	0.01	0.485	0.395	0.407	0.288	0.254	0.271	0.392	0.307	0.318
	0.1	0.693	0.366	0.601	0.653	0.298	0.491	0.666	0.357	0.569
	1	0.759	0.381	0.651	0.778	0.323	0.779	0.778	0.388	0.711
	10	0.777	0.383	0.293	0.784	0.332	0.295	0.789	0.377	0.328
-0.4	0	0.024	0.041	0.036	0.039	0.078	0.045	0.027	0.058	0.041
	0.01	0.504	0.351	0.501	0.327	0.231	0.318	0.353	0.262	0.345
	0.1	0.703	0.354	0.816	0.667	0.296	0.808	0.684	0.347	0.818
	1	0.762	0.381	0.379	0.767	0.319	0.399	0.764	0.362	0.380
	10	0.768	0.381	0.267	0.791	0.330	0.267	0.765	0.361	0.287
0	0	0.042	0.047	0.043	0.052	0.083	0.045	0.048	0.069	0.046
	0.01	0.388	0.265	0.435	0.238	0.175	0.261	0.288	0.225	0.329
	0.1	0.658	0.335	0.567	0.597	0.275	0.584	0.624	0.319	0.574
	1	0.758	0.370	0.311	0.757	0.310	0.310	0.742	0.354	0.310
	10	0.767	0.380	0.259	0.778	0.317	0.266	0.755	0.352	0.268
0.4	0	0.085	0.053	0.071	0.073	0.070	0.068	0.071	0.074	0.067
	0.01	0.236	0.160	0.201	0.154	0.115	0.133	0.235	0.175	0.204
	0.1	0.567	0.293	0.364	0.476	0.218	0.306	0.550	0.277	0.351
	1	0.731	0.348	0.258	0.725	0.292	0.267	0.707	0.326	0.272
	10	0.758	0.362	0.253	0.764	0.311	0.251	0.744	0.341	0.248
0.8	0	0.182	0.077	0.077	0.175	0.068	0.069	0.228	0.092	0.092
	0.01	0.198	0.073	0.070	0.197	0.067	0.067	0.264	0.129	0.129
	0.1	0.393	0.154	0.144	0.326	0.125	0.115	0.430	0.216	0.208
	1	0.640	0.281	0.196	0.639	0.237	0.206	0.633	0.277	0.218
	10	0.722	0.327	0.221	0.738	0.269	0.231	0.694	0.290	0.218

Table 10: Rejection frequencies of the Inf-type tests when the break point is unknown ($n = 100$)

a_{ii}	σ_u^2	Model 1			Model 2			Model 3		
$n = 100$		ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a
	0	0.034	0.327	0.361	0.118	1.000	0.954	0.067	0.530	0.519
	0.01	0.341	0.553	0.551	0.327	1.000	0.953	0.305	0.676	0.623
-0.8	0.1	0.701	0.662	0.704	0.732	0.998	0.893	0.731	0.782	0.753
	1	0.824	0.693	0.912	0.911	0.992	0.956	0.856	0.790	0.927
	10	0.831	0.704	0.684	0.919	0.989	0.897	0.868	0.800	0.760
	0	0.078	0.313	0.086	0.217	1.000	0.267	0.057	0.463	0.076
	0.01	0.400	0.520	0.407	0.456	1.000	0.476	0.325	0.629	0.336
-0.4	0.1	0.712	0.660	0.794	0.784	0.994	0.792	0.715	0.747	0.784
	1	0.824	0.695	0.736	0.905	0.992	0.920	0.841	0.778	0.786
	10	0.825	0.698	0.694	0.920	0.990	0.920	0.842	0.774	0.733
	0	0.105	0.294	0.096	0.218	1.000	0.138	0.082	0.431	0.064
	0.01	0.316	0.455	0.318	0.369	0.998	0.291	0.247	0.556	0.232
0	0.1	0.648	0.622	0.696	0.712	0.992	0.715	0.646	0.711	0.680
	1	0.799	0.682	0.660	0.892	0.991	0.861	0.807	0.753	0.713
	10	0.826	0.694	0.705	0.919	0.989	0.927	0.824	0.757	0.728
	0	0.115	0.261	0.115	0.225	0.993	0.215	0.130	0.404	0.128
	0.01	0.223	0.359	0.223	0.283	0.993	0.272	0.218	0.451	0.214
0.4	0.1	0.527	0.553	0.486	0.560	0.994	0.558	0.521	0.648	0.493
	1	0.759	0.669	0.626	0.853	0.989	0.839	0.749	0.731	0.655
	10	0.812	0.691	0.698	0.906	0.991	0.920	0.805	0.733	0.727
	0	0.247	0.226	0.172	0.456	0.964	0.499	0.284	0.367	0.226
	0.01	0.258	0.233	0.175	0.439	0.968	0.506	0.292	0.362	0.233
0.8	0.1	0.376	0.342	0.274	0.539	0.977	0.584	0.372	0.442	0.314
	1	0.633	0.545	0.501	0.761	0.985	0.800	0.583	0.581	0.501
	10	0.724	0.605	0.601	0.828	0.988	0.877	0.665	0.615	0.587

Table 11: Rejection frequencies of the Inf-type tests when the break point is unknown ($n = 200$)

a_{ii}	σ_u^2	Model 1			Model 2			Model 3		
$n = 100$		ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a	ℓ_4	ℓ_{12}	ℓ_a
	0	0.026	0.158	0.197	0.067	0.590	0.734	0.070	0.231	0.274
	0.01	0.709	0.669	0.680	0.671	0.825	0.850	0.620	0.647	0.659
-0.8	0.1	0.949	0.783	0.896	0.960	0.887	0.921	0.937	0.813	0.895
	1	0.973	0.794	0.958	0.987	0.900	0.996	0.975	0.823	0.965
	10	0.977	0.782	0.722	0.994	0.900	0.914	0.979	0.821	0.800
	0	0.096	0.175	0.117	0.214	0.550	0.275	0.061	0.161	0.076
	0.01	0.765	0.646	0.764	0.756	0.795	0.761	0.677	0.621	0.677
-0.4	0.1	0.955	0.766	0.980	0.969	0.889	0.984	0.945	0.792	0.975
	1	0.970	0.790	0.800	0.991	0.897	0.921	0.972	0.814	0.845
	10	0.976	0.782	0.718	0.994	0.900	0.929	0.978	0.808	0.778
	0	0.140	0.177	0.139	0.267	0.523	0.234	0.105	0.170	0.098
	0.01	0.673	0.559	0.696	0.632	0.730	0.669	0.568	0.535	0.605
0	0.1	0.931	0.745	0.914	0.945	0.857	0.949	0.919	0.761	0.904
	1	0.963	0.779	0.742	0.990	0.894	0.895	0.969	0.800	0.786
	10	0.975	0.776	0.719	0.991	0.896	0.940	0.976	0.804	0.779
	0	0.181	0.166	0.161	0.277	0.426	0.263	0.183	0.184	0.155
	0.01	0.502	0.414	0.451	0.477	0.582	0.448	0.450	0.397	0.409
0.4	0.1	0.874	0.678	0.759	0.894	0.795	0.812	0.865	0.680	0.742
	1	0.959	0.767	0.717	0.987	0.882	0.892	0.961	0.781	0.744
	10	0.973	0.765	0.719	0.991	0.901	0.945	0.972	0.789	0.779
	0	0.431	0.171	0.166	0.597	0.393	0.378	0.484	0.225	0.221
	0.01	0.469	0.212	0.202	0.637	0.399	0.398	0.508	0.285	0.270
0.8	0.1	0.720	0.439	0.416	0.805	0.596	0.599	0.723	0.459	0.445
	1	0.933	0.702	0.645	0.970	0.815	0.872	0.919	0.664	0.653
	10	0.959	0.735	0.687	0.985	0.872	0.947	0.955	0.733	0.727