# Quantile-Based Nonparametric Inference for First-Price Auctions

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**Abstract**

We propose a quantile-based nonparametric approach to inference on the probability density function (PDF) of the private values in first-price sealed-bid auctions with independent private values. Our method of inference is based on a fully nonparametric kernel-based estimator of the quantiles and PDF of observable bids. Our estimator attains the optimal rate of Guerre et al. (2000), and is also asymptotically normal with the appropriate choice of the bandwidth.
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## 1 Introduction

Following the seminal article of Guerre et al. (2000), GPV hereafter, there has been an enormous interest in nonparametric approaches to auctions.[1] By removing the

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[1] See a recent survey by Athey and Haile (2007).

need to impose tight functional form assumptions, the nonparametric approach provides a more flexible framework for estimation and inference. Moreover, the sample sizes available for auction data can be sufficiently large to make the nonparametric approach empirically feasible.[2] This paper contributes to this literature by providing a fully nonparametric framework for making inferences on the density of bidders' valuations $f(v)$. The need to estimate the density of valuations arises in a number of economic applications, as for example the problem of estimating a revenue-maximizing reserve price.[3]

As a starting point, we briefly discuss the estimator proposed in GPV. For the purpose of introduction, we adopt a simplified framework. Consider a random, i.i.d. sample $b_{il}$ of bids in first-price auctions each of which has $n$ *risk-neutral* bidders; $l$ indexes auctions and $i = 1, \ldots, n$ indexes bids in a given auction. GPV assume independent private values (IPV). In equilibrium, the bids are related to the valuations via the equilibrium bidding strategy $B$: $b_{il} = B(v_{il})$. GPV show that the inverse bidding strategy is identified directly from the observed distribution of bids:

$$v = \xi(b) \equiv b + \frac{1}{n-1}\frac{G(b)}{g(b)}, \tag{1}$$

where $G(b)$ is the cumulative distribution function (CDF) of bids in an auction with $n$ bidders, and $g(b)$ is the corresponding density. GPV propose to use nonparametric estimators $\hat{G}$ and $\hat{g}$. When $b = b_{il}$, the left-hand side of (1) will then give what GPV call the pseudo-values $\hat{v}_{il} = \hat{\xi}(b_{il})$. The CDF $F(v)$ is estimated as the empirical CDF, and the PDF $f(v)$ is estimated by the method of kernels, both using $\hat{v}_{il}$ as observations. GPV show that, with the appropriate choice of the bandwidth, their estimator converges to the true value at the optimal rate (in the minimax sense; Khasminskii (1979)). However, the asymptotic distribution of this estimator is as yet unknown, possibly because both steps of the GPV method are nonparametric with estimated values $\hat{v}_{il}$ entering the second stage.

[2] For example, List et al. (2004) study bidder collusion in timber auctions using thousands of auctions conducted in the Province of British Columbia, Canada. Samples of similar size are also available for highway procurement auctions in the United States (e.g., Krasnokutskaya (2009)).

[3] Several previous articles have studied that problem, see Paarsch (1997), Haile and Tamer (2003), and Li et al. (2003). In the supplement to this paper, we discuss how the approach developed here can be used for construction of confidence sets for the optimal reserve price. The supplement is available as Marmer and Shneyerov (2010) from the UBC working papers series and the authors' web-sites.

The estimator $\hat{f}(v)$ proposed in this paper avoids the use of pseudo-values. It builds instead on the insight of Haile et al. (2003).[4] They show that the quantiles of the distribution of valuations can be expressed in terms of the quantiles, PDF, and CDF of bids. We show below that this relation can be used for estimation of $f(v)$. Consider the $\tau$-th quantile of valuations $Q(\tau)$ and the $\tau$-th quantile of bids $q(\tau)$. The latter can be easily estimated from the sample by a variety of methods available in the literature. As for the quantile of valuations, since the inverse bidding strategy $\xi(b)$ is monotone, equation (1) implies that $Q(\tau)$ is related to $q(\tau)$ as follows:

$$Q(\tau) = q(\tau) + \frac{\tau}{(n-1)\,g(q(\tau))}, \tag{2}$$

providing a way to estimate $Q(\tau)$ by a plug-in method. The CDF $F(v)$ can then be recovered by inverting the quantile function, $F(v) = Q^{-1}(v)$.

Our estimator $\hat{f}(v)$ is based on a simple idea that by differentiating the quantile function we can recover the density: $Q'(\tau) = 1/f(Q(\tau))$, and therefore $f(v) = 1/Q'(F(v))$. Taking the derivative in (2) and using the fact that $q'(\tau) = 1/g(q(\tau))$, we obtain, after some algebra, our basic formula:

$$f(v) = \left( \frac{n}{n-1} \frac{1}{g(q(F(v)))} - \frac{1}{n-1} \frac{F(v)\,g'(q(F(v)))}{g^3(q(F(v)))} \right)^{-1}. \tag{3}$$

Note that all the quantities on the right-hand side, i.e. $g(b)$, $g'(b)$, $q(\tau)$, $F(v) = Q^{-1}(v)$ can be estimated nonparametrically, for example, using kernel-based methods. Once this is done, we can plug them in (3) to obtain our nonparametric estimator.

The expression in (3) can be also derived using the following relationship between the CDF of values and the CDF of bids:

$$F(v) = G(B(v)).$$

Applying the change of variable argument to the above identity, one obtains

$$\begin{aligned} f(v) &= g(B(v))\,B'(v) \\ &= g(B(v))\,/\xi'(B(v)) \end{aligned}$$

[4] The focus of Haile et al. (2003) is a test of common values. Their model is therefore different from the IPV model, and requires an estimator that is different from the one in GPV. See also Li et al. (2002).

$$= \left( \frac{n}{n-1} \frac{1}{g(B(v))} - \frac{1}{n-1} \frac{F(v) g'(B(v))}{g^3(B(v))} \right)^{-1}.$$

Note however, that from the estimation perspective, the quantile-based formula appears to be more convenient, since the bidding strategy function $B$ involves integration of $F$ (see equation (4) below). Furthermore, replacing $B(v)$ with appropriate quantiles has no effect on the asymptotic distribution of the estimator.

Our framework results in the estimator of $f(v)$ that is both consistent and asymptotically normal, with an asymptotic variance that can be easily estimated. Moreover, we show that, with an appropriate choice of the bandwidth sequence, the proposed estimator attains the minimax rate of GPV.

In a Monte Carlo experiment, we compare finite sample biases and mean squared errors of our quantile-based estimator with that of the GPV's estimator. Our conclusion is that neither estimator strictly dominates the other. The GPV estimator is more efficient when the PDF of valuations has a positive derivative at the point of estimation and the number of bidders tends to be large. On the other hand, the quantile-based estimator is more efficient when the PDF of valuations has a negative derivative and the number of bidders is small. The Monte Carlo results suggest that the proposed estimator will be more useful when there are sufficiently many independent auctions with a small number of bidders.[5]

The rest of the paper is organized as follows. Section 2 introduces the basic setup. Similarly to GPV, we allow the number of bidders to vary from auctions to auction, and also allow auction-specific covariates. Section 3 presents our main results. Section 4 discusses the bootstrap-based approach to inference on the PDF of valuations. In Section 5, we extend our framework to the case of auctions with binding reserve price. We report Monte Carlo results in Section 6. Section 7 concludes. The proofs of the main results are given in the Appendix. The supplement to this paper contains the proof of the bootstrap result in Section 4, some additional Monte Carlo results, as well as an illustration of how the approach developed here can be applied for conducting inference on the optimal reserve price.

[5] We thank a referee for pointing this out.

## 2 Definitions

The econometrician observes a random sample $\{(b_{il}, x_l, n_l) : l = 1, \ldots, L; i = 1, \ldots n_l\}$, where $b_{il}$ is the equilibrium bid of *risk-neutral* bidder $i$ submitted in auction $l$ with $n_l$ bidders, and $x_l$ is the vector of auction-specific covariates for auction $l$. The corresponding unobservable valuations of the object are given by $\{v_{il} : l = 1, \ldots, L; i = 1, \ldots n_l\}$. We make the following assumption similar to Assumptions A1 and A2 of GPV (see also footnote 14 in their paper).

**Assumption 1** **(a)** $\{(n_l, x_l) : l = 1, \ldots, L\}$ *are i.i.d.*

**(b)** *The marginal PDF of $x_l$, $\varphi$, is strictly positive and continuous on its compact support $\mathcal{X} \subset R^d$, and admits up to $R \geq 2$ continuous derivatives on its interior.*

**(c)** *The distribution of $n_l$ conditional on $x_l$ is denoted by $\pi(n|x)$ and has support $\mathcal{N} = \{\underline{n}, \ldots, \bar{n}\}$ for all $x \in \mathcal{X}$, $\underline{n} \geq 2$.*

**(d)** *$\{v_{il} : l = 1, \ldots, L; i = 1, \ldots, n_l\}$ are i.i.d. and independent of the number of bidders conditional on $x_l$ with the PDF $f(v|x)$ and CDF $F(v|x)$.*

**(e)** *$f(\cdot|x)$ is strictly positive and bounded away from zero and admits up to $R-1$ continuous derivatives on its support, a compact interval $[\underline{v}(x), \overline{v}(x)] \subset \mathbb{R}_+$ for all $x \in \mathcal{X}$; $f(v|\cdot)$ admits up to $R$ continuous partial derivatives on $Interior(\mathcal{X})$ for all $v \in [\underline{v}(x), \overline{v}(x)]$.*

**(f)** *For all $n \in \mathcal{N}$, $\pi(n|\cdot)$ is strictly positive and admits up to $R$ continuous derivatives on the interior of $\mathcal{X}$.*

Under Assumption 1(c), the equilibrium bids are determined by

$$b_{il} = v_{il} - \frac{1}{(F(v_{il}|x_l))^{n-1}} \int_{\underline{v}}^{v_{il}} (F(u|x_l))^{n-1}\, du, \tag{4}$$

(see, for example, GPV). Let $g(b|n,x)$ and $G(b|n,x)$ be the PDF and CDF of $b_{il}$, conditional on both $x_l = x$ and the number of bidders $n_l = n$. Since $b_{il}$ is a function of $v_{il}$, $x_l$, and $F(\cdot|x_l)$, the bids $\{b_{il}\}$ are also i.i.d. conditional on $(n_l, x_l)$. Furthermore, by Proposition 1(i) and (iv) of GPV, for all $n = \underline{n}, \ldots, \overline{n}$ and $x \in \mathcal{X}$, $g(\cdot|n,x)$ has the compact support $\left[\underline{b}(n,x), \overline{b}(n,x)\right]$ for some $\underline{b}(n,x) < \overline{b}(n,x)$, and $g(\cdot|n,\cdot)$ admits up to $R$ continuous bounded partial derivatives.

The $\tau$-th quantile of $F(v|x)$ is defined as

$$Q(\tau|x) = F^{-1}(\tau|x) \equiv \inf_v \{v : F(v|x) \geq \tau\}.$$

The $\tau$-th quantile of $G$,

$$q(\tau|n,x) = G^{-1}(\tau|n,x),$$

is defined similarly. The quantiles of the distributions $F(v|x)$ and $G(b|n,x)$ are related through the following conditional version of equation (2):

$$Q(\tau|x) = q(\tau|n,x) + \frac{\tau}{(n-1)\, g(q(\tau|n,x)|n,x)}. \tag{5}$$

Note that the expression on the left-hand side does not depend on $n$, since by Assumption 1(d) and as it is usually assumed in the literature, the distribution of valuations is the same regardless of the number of bidders.

The true distribution of the valuations is unknown to the econometrician. Our objective is to construct a valid asymptotic inference procedure for the unknown $f$ using the data on observable bids. Differentiating (5) with respect to $\tau$, we obtain the following equation relating the PDF of valuations with functionals of the distribution of the bids:

$$\begin{aligned}\frac{\partial Q(\tau|x)}{\partial \tau} &= \frac{1}{f(Q(\tau|x)|x)} \\ &= \frac{n}{n-1}\frac{1}{g(q(\tau|n,x)|n,x)} - \frac{\tau g^{(1)}(q(\tau|n,x)|n,x)}{(n-1)\, g^3(q(\tau|n,x)|n,x)},\end{aligned} \tag{6}$$

where $g^{(k)}(b|n,x) = \partial^k g(b|n,x)/\partial b^k$. Substituting $\tau = F(v|x)$ in equation (6) and using the identity $Q(F(v|x)|x) = v$, we obtain the following equation that represents the PDF of valuations in terms of the quantiles, PDF and derivative of PDF of bids:

$$\begin{aligned}\frac{1}{f(v|x)} = {} & \frac{n}{n-1}\frac{1}{g(q(F(v|x)|n,x)|n,x)} \\ & - \frac{1}{n-1}\frac{F(v|x)\, g^{(1)}(q(F(v|x)|n,x)|n,x)}{g^3(q(F(v|x)|n,x)|n,x)}.\end{aligned} \tag{7}$$

Note that the overidentifying restriction of the model is that $f(v|x)$ is the same for all $n$.

In this paper, we suggest a nonparametric estimator for the PDF of valuations based on equations (5) and (7). Such an estimator requires nonparametric estimation of the conditional CDF and quantile functions, PDF and its derivative.

Let $K$ be a kernel function. We assume that the kernel is compactly supported and of order $R$.

**Assumption 2** *$K$ is compactly supported on $[-1,1]$, has at least $R$ derivatives on its support, the derivatives are Lipschitz, and $\int K(u)\,du = 1$, $\int u^k K(u)\,du = 0$ for $k = 1, \ldots, R-1$.*

To save on notation, denote

$$K_h(z) = \frac{1}{h} K\left(\frac{z}{h}\right),$$

and for $x = (x_1, \ldots, x_d)'$, define

$$K_{*h}(x) = \frac{1}{h^d} K_d\left(\frac{x}{h}\right) = \frac{1}{h^d} \prod_{k=1}^{d} K\left(\frac{x_k}{h}\right).$$

Consider the following estimators:

$$\hat{\varphi}(x) = \frac{1}{L} \sum_{l=1}^{L} K_{*h}(x_l - x), \tag{8}$$

$$\hat{\pi}(n|x) = \frac{1}{\hat{\varphi}(x) L} \sum_{l=1}^{L} 1(n_l = n) K_{*h}(x_l - x),$$

$$\hat{G}(b|n,x) = \frac{1}{\hat{\pi}(n|x)\hat{\varphi}(x) nL} \sum_{l=1}^{L} \sum_{i=1}^{n_l} 1(n_l = n) 1(b_{il} \le b) K_{*h}(x_l - x),$$

$$\hat{q}(\tau|n,x) = \hat{G}^{-1}(\tau|n,x) \equiv \inf_b \left\{ b : \hat{G}(b|n,x) \ge \tau \right\},$$

$$\hat{g}(b|n,x) = \frac{1}{\hat{\pi}(n|x)\hat{\varphi}(x) nL} \sum_{l=1}^{L} \sum_{i=1}^{n_l} 1(n_l = n) K_h(b_{il} - b) K_{*h}(x_l - x), \tag{9}$$

where $1(S)$ is an indicator function of a set $S \subset \mathbb{R}$.[6,7]

The derivatives of the density $g(b|n,x)$ are estimated simply by the derivatives of $\hat{g}(b|n,x)$:

$$\hat{g}^{(k)}(b|n,x) = \frac{(-1)^k}{\hat{\pi}(n|x)\,\hat{\varphi}(x)\,nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n)\, K_h^{(k)}(b_{il} - b)\, K_{*h}(x_l - x), \qquad (10)$$

where

$$K_h^{(k)}(u) = \frac{1}{h^{1+k}} K^{(k)}\left(\frac{u}{h}\right),$$

and $K^{(k)}(u)$ denotes the $k$-th derivative of $K(u)$.

Our approach also requires nonparametric estimation of $Q$, the conditional quantile function of valuations. An estimator for $Q$ can be constructed using the relationship between $Q$, $q$ and $g$ given in (5). A similar estimator was proposed by Haile et al. (2003) in a different context. In our case, the estimator of $Q$ will be used to construct $\hat{F}$, an estimator of the conditional CDF of valuations. The CDF $F$ is related to the quantile function $Q$ through

$$F(v|x) = Q^{-1}(v|x) = \sup_{\tau\in[0,1]} \{\tau : Q(\tau|x) \leq v\}, \qquad (11)$$

and therefore $\hat{F}$ can be obtained by inverting the estimator of the conditional quantile function. However, since an estimator of $Q$ based on (5) involves kernel estimation of the PDF $g$, it will be inconsistent for the values of $\tau$ that are close to zero and one because of the asymptotic bias in $\hat{g}$ at the boundaries. In particular, such an estimator of $Q$ can exhibit large oscillations for $\tau$ near one by taking on very small values, which due to supremum in (11), might proliferate and bring an upward bias into the estimator of $F$. A solution to this problem that we pursue in this paper is to use a monotone version of the estimator of $Q$. First, we define a preliminary

[6] We estimate the CDF of bids by a conditional version of the empirical CDF. In a recent paper, Li and Racine (2008) discuss a smooth estimator of the CDF (and a corresponding quantile estimator) obtained by integrating the kernel PDF estimator. We, however, adopt the non-smooth empirical CDF approach in order for our estimator to be comparable with that of GPV; both estimator can be modified by using the smooth conditional CDF estimator.

[7] The quantile estimator $\hat{q}$ is constructed by inverting the estimator of the conditional CDF of bids. This approach is similar to that of Matzkin (2003).

estimator, $\hat{Q}^p$:

$$\hat{Q}^p(\tau|n,x) = \hat{q}(\tau|n,x) + \frac{\tau}{(n-1)\,\hat{g}(\hat{q}(\tau|n,x)\,|n,x)}. \tag{12}$$

Next, we choose some $\tau_0 \in (0,1)$ sufficiently far from 0 and 1, for example, $\tau_0 = 1/2$. We define a monotone version of the estimator of $Q$ as follows:

$$\hat{Q}(\tau|n,x) = \begin{cases} \sup_{t\in[\tau_0,\tau]} \hat{Q}^p(t|n,x), & \tau_0 \leq \tau < 1, \\ \inf_{t\in[\tau,\tau_0]} \hat{Q}^p(t|n,x), & 0 \leq \tau < \tau_0. \end{cases} \tag{13}$$

The estimator of the conditional CDF of the valuations based on $\hat{Q}(\tau|n,x)$ is then given by

$$\hat{F}(v|n,x) = \sup_{\tau\in[0,1]} \left\{\tau : \hat{Q}(\tau|n,x) \leq v\right\}. \tag{14}$$

Since $\hat{Q}(\cdot|n,x)$ is monotone, $\hat{F}$ is not affected by $\hat{Q}^p(\tau|n,x)$ taking on small values near $\tau = 1$. Furthermore, in our framework, inconsistency of $\hat{Q}(\tau|n,x)$ near the boundaries does not pose a problem, since we are interested in estimating $F$ only on a compact inner subset of its support.

Using (7), for a given $n$ we propose to estimate $f(v|x)$ by the plug-in method, i.e. by replacing $g(b|n,x)$, $q(\tau|n,x)$, and $F(v|x)$ in (7) with $\hat{g}(b|n,x)$, $\hat{q}(\tau|n,x)$, and $\hat{F}(v|n,x)$. That is our estimator $\hat{f}(v|n,x)$ is given by the reciprocal of

$$\begin{aligned} &\frac{n}{n-1}\frac{1}{\hat{g}\left(\hat{q}\left(\hat{F}(v|n,x)\,|n,x\right)|n,x\right)} - \\ &\qquad - \frac{1}{n-1}\frac{\hat{F}(v|n,x)\,\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v|n,x)\,|n,x\right)|n,x\right)}{\hat{g}^3\left(\hat{q}\left(\hat{F}(v|n,x)\,|n,x\right)|n,x\right)}. \end{aligned} \tag{15}$$

While the PDF of valuations does not depend on the number of bidders $n$, the estimator defined by (15) does, and therefore we have a number of estimators for $f(v|x)$: $\hat{f}(v|n,x)$, $n = \underline{n},\ldots,\overline{n}$. The estimators $\hat{f}(v|\underline{n},x),\ldots,\hat{f}(v|\bar{n},x)$ can be averaged to obtain:

$$\hat{f}(v|x) = \sum_{n=\underline{n}}^{\bar{n}} \hat{w}(n,x)\,\hat{f}(v|n,x), \tag{16}$$

where the weights $\hat{w}(n,x)$ satisfy

$$\hat{w}(n,x) \to_p w(n,x) > 0,$$
$$\sum_{n=\underline{n}}^{\bar{n}} w(n,x) = 1.$$

In the next section, we discuss how to construct optimal weights that minimize the asymptotic variance of $\hat{f}(v|x)$.

We also suggest estimating the conditional CDF of $v$ using the average of $\hat{F}(v|n,x)$, $n = \underline{n}, \ldots, \overline{n}$:

$$\hat{F}(v|x) = \sum_{n=\underline{n}}^{\bar{n}} \hat{w}(n,x) \hat{F}(v|n,x). \tag{17}$$

# 3 Asymptotic properties

In this section, we discuss uniform consistency and asymptotic normality of the estimator of $f$ proposed in the previous section. The consistency of the estimator of $f$ follows from the uniform consistency of its components.

It is well known that kernel estimators can be inconsistent near the boundaries of the support, and therefore we estimate the PDF of valuations at the points that lie away from the boundaries of $[\underline{v}(x), \bar{v}(x)]$. The econometrician can choose quantile values $\tau_1$ and $\tau_2$ such that

$$0 < \tau_1 < \tau_2 < 1,$$

in order to cut off the boundaries of the support where estimation is problematic. While $\underline{v}(x)$ and $\bar{v}(x)$ are unknown, consider instead the following interval of $v$'s for selected $\tau_1$ and $\tau_2$:

$$\hat{\Lambda}(x) = \left[ \max_{n=\underline{n},\ldots,\bar{n}} \hat{Q}(\tau_1|n,x), \min_{n=\underline{n},\ldots,\bar{n}} \hat{Q}(\tau_2|n,x) \right]. \tag{18}$$

**Remark.** Since according to Lemma 1(g) below, $\hat{Q}(\tau|n,x)$ consistently estimates $Q(\tau|x)$ for $\tau \in [\tau_1 - \varepsilon, \tau_2 + \varepsilon]$ and all $n = \underline{n}, \ldots, \bar{n}$ , the boundaries of $\hat{\Lambda}(x)$ satisfy $\max_{n=\underline{n},\ldots,\bar{n}} \hat{Q}(\tau_1|n,x) \to_p Q(\tau_1|x)$ and $\min_{n=\underline{n},\ldots,\bar{n}} \hat{Q}(\tau_2|n,x) \to_p Q(\tau_2|x)$. Thus, the boundaries of $\hat{\Lambda}(x)$ consistently estimate the boundaries of $\Lambda(x) = [Q(\tau_1|x), Q(\tau_2|x)]$, the interval between the $\tau_1$ and $\tau_2$ quantiles of the distribution of bidders' valuations.

We also show in Theorems 1 and 2 below that our estimator of $f$ is uniformly consistent and asymptotically normal when $f$ is estimated at the points from $\hat{\Lambda}(x)$.

In practice, $\tau_1$ and $\tau_2$ can be selected as follows. Since by Assumption 2 the length of the support of $K$ is two, and following the discussion on page 531 of GPV, when there are no covariates one can choose $\tau_1$ and $\tau_2$ such that

$$[\hat{q}(\tau_1|n), \hat{q}(\tau_2|n)] \subset \left(\hat{b}_{\min}(n) + h, \hat{b}_{\max}(n) - h\right)$$

for all $n \in \mathcal{N}$, where $\hat{b}_{\min}(n)$ and $\hat{b}_{\max}(n)$ denote the minimum and maximum bids respectively in the sample of auctions with $n$ bidders. When there are covariates available and $f$ is estimated conditional on $x_l = x$, one can replace $\hat{b}_{\min}(n)$ and $\hat{b}_{\max}(n)$ with the corresponding minimum and maximum bids in the neighborhood of $x$ as defined on page 541 of GPV.

Next, we present a lemma that provides uniform convergence rates for the components of the estimator $\hat{f}$. In the case of the estimators of $g$ and its derivatives, uniform consistency is established on the following interval. Since the bidding function is monotone, by Proposition 2.1 of GPV, there is an inner compact interval of the support of the bids distribution, say $[b_1(n,x), b_2(n,x)]$,[8] such that

$$\begin{aligned}[q(\tau_1|n,x), q(\tau_2|n,x)] \subset (b_1(n,x), b_2(n,x)), \text{ and}\\ [b_1(n,x), b_2(n,x)] \subset \left(\underline{b}(n,x), \bar{b}(n,x)\right).\end{aligned} \tag{19}$$

**Lemma 1** *Under Assumptions 1 and 2, for all $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$,*

**(a)** $\hat{\pi}(n|x) - \pi(n|x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

**(b)** $\hat{\varphi}(x) - \varphi(x) = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

**(c)** $\sup_{b \in \left[\underline{b}(n,x), \bar{b}(n,x)\right]} |\hat{G}(b|n,x) - G(b|n,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$.

**(d)** $\sup_{\tau \in [\varepsilon, 1-\varepsilon]} |\hat{q}(\tau|n,x) - q(\tau|n,x)| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right)$, *for any* $0 < \varepsilon < 1/2$.

[8] The knowledge of $b_1(n,x)$ and $b_2(n,x)$ is not required for construction of our estimator.

**(e)** $\sup_{\tau\in[\varepsilon,1-\varepsilon]}(\lim_{t\downarrow\tau}\hat{q}(t|n,x)-\hat{q}(\tau|n,x)) = O_p\left(\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}\right)$, *for any* $0<\varepsilon<1/2$.

**(f)** $\sup_{b\in[b_1(n,x),b_2(n,x)]}|\hat{g}^{(k)}(b|n,x)-g^{(k)}(b|n,x)| = O_p\left(\left(\frac{Lh^{d+1+2k}}{\log L}\right)^{-1/2}+h^R\right)$, $k=0,\ldots,R$, *where* $[b_1(n,x),b_2(n,x)]$ *is defined in (19).*

**(g)** $\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}|\hat{Q}(\tau|n,x)-Q(\tau|x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right)$, *for some* $\varepsilon>0$ *such that* $\tau_1-\varepsilon>0$ *and* $\tau_2+\varepsilon<1$.

**(h)** $\sup_{v\in\hat{\Lambda}(x)}|\hat{F}(v|n,x)-F(v|x)| = O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right)$, *where* $\hat{\Lambda}(x)$ *is defined in (18).*

**Remarks.** 1. Parts (a), (b), and (f) of the lemma follow from Lemmas B.1 and B.2 of Newey (1994) which show that kernel estimators of $k$-order derivatives of smooth functions of $d$ variables are uniformly consistent with the rate $(Lh^{d+2k}/\log L)^{-1/2}+h^R$, where $R$ is the degree of smoothness. The conditional CDF estimator $\hat{G}(\cdot|n,x)$ in part (c) of Lemma 1 is a step function which involves kernel smoothing only with respect to $x$. It therefore does not fit in Newey's framework and his Lemma B.1 does not apply in that case. However, precisely because there is no kernel smoothing with respect to $b$, one should expect to see the uniform convergence rate of $(Lh^d/\log L)^{-1/2}+h^R$ for $\hat{G}(b|n,x)$. In the proof of part (c) in the Appendix, we verify this claim using the covering numbers results (Pollard, 1984, Chapter II). A similar result appears in GPV. In their Lemma B2, they derive the uniform convergence rate for $\hat{G}(\cdot|n,x)$ on an expanding subset of $\left[\underline{b}(n,x),\bar{b}(n,x)\right]$ that does not include the neighborhoods of the boundaries. In our case, uniform convergence of $\hat{G}(\cdot|n,x)$ on the entire support $\left[\underline{b}(n,x),\bar{b}(n,x)\right]$ is useful for establishing the uniform convergence rate of $\hat{q}(\cdot|n,x)$.

2. In part (d) of the lemma, we show that the quantile estimator $\hat{q}(\cdot|n,x)$ inherits the uniform convergence rate of its corresponding empirical CDF $\hat{G}(\cdot|n,x)$. The result is established using the following argument (to save on notation, we will suppress $n$ and $x$ here). Since $G(b)$ is a continuous CDF and by the properties of quantiles (van der Vaart, 1998, Lemma 21.1), write $G(\hat{q}(\tau))-G(q(\tau))=G(\hat{q}(\tau))-\hat{G}(\hat{q}(\tau))+\hat{G}(\hat{q}(\tau))-\tau$. Since $g(q(\tau))$ is bounded away from zero, an application of the mean-value theorem implies then that the uniform distance between $\hat{q}(\cdot)$ and $q(\cdot)$ can be bounded by the uniform distance between $\hat{G}(\cdot)$ and $G(\cdot)$ and the size of the largest jump in $\hat{G}(\cdot)$ (the later is of order $(Lh^d)^{-1}$).

3. Arguments similar to those in the previous remark are also used in the proof of part (h) (recall that $\hat{F}(\cdot|n,x)$ is defined as the inverse function of $\hat{Q}(\cdot|n,x)$). The jumps in $\hat{Q}(\cdot|n,x)$ depend on those of $\hat{q}(\cdot|n,x)$ and are shown to be of order $(Lh^d/\log L)^{-1/2}$ using the results in Deheuvels (1984) (see the proof of part (e) of the lemma).

As it follows from Lemma 1, the estimator of the derivative of $g(\cdot|n,x)$ has the slowest rate of convergence among all the components of $\hat{f}$. Consequently, it determines the uniform convergence rate of $\hat{f}$.

**Theorem 1** *Let $\hat{\Lambda}(x)$ be as defined in (18). Under Assumptions 1 and 2, and for all $x \in Interior(\mathcal{X})$, $\sup_{v\in\hat{\Lambda}(x)} \left|\hat{f}(v|x) - f(v|x)\right| = O_p\left(\left(\frac{Lh^{d+3}}{\log L}\right)^{-1/2} + h^R\right)$.*

**Remarks.** 1. The theorem also holds when $\hat{\Lambda}(x)$ is replaced by an inner closed subset of $[\underline{v}(x), \overline{v}(x)]$, as in Theorem 3 of GPV. Estimation of $\Lambda(x)$ has no effect on the result of our theorem because the event

$$E_L(n,x) = \left\{v \in \hat{\Lambda}(x) : \hat{q}\left(\hat{F}(v|n,x)|n,x\right) \in [b_1(n,x), b_2(n,x)]\right\} \tag{20}$$

satisfies $P(E_L(n,x)) \to 1$ as $L \to \infty$ for all $n \in \mathcal{N}$ and $x \in Interior(\mathcal{X})$ by the results in Lemma 1.

2. One of the implications of theorem is that our estimator achieves the optimal rate of GPV. Consider the following choice of the bandwidth parameter: $h = c(L/\log L)^{-\alpha}$. By choosing $\alpha$ so that $\left(Lh^{d+3}/\log L\right)^{-1/2}$ and $h^R$ are of the same order, one obtains $\alpha = 1/(d+3+2R)$ and the rate $(L/\log L)^{-R/(d+3+2R)}$, which is the same as the optimal rate established in Theorem 3 of GPV.

Next, we discuss asymptotic normality of the proposed estimator. We make following assumption.

**Assumption 3** *$Lh^{d+1} \to \infty$, and $\left(Lh^{d+1+2k}\right)^{1/2} h^R \to 0$.*

The rate of convergence and asymptotic variance of the estimator of $f$ are determined by $\hat{g}^{(1)}(b|n,x)$, the component with the slowest rate of convergence. Hence, Assumption 3 will be imposed with $k = 1$ which limits the possible choices of the bandwidth for kernel estimation. For example, if one follows the rule $h = cL^{-\alpha}$, then $\alpha$ has to be in the interval $(1/(d+3+2R), 1/(d+1))$. As usual for asymptotic normality, there is some under smoothing relatively to the optimal rate.

**Lemma 2** *Let* $[b_1(n,x), b_2(n,x)]$ *be as in (19). Then, under Assumptions 1-3, for all* $b \in [b_1(n,x), b_2(n,x)]$, $x \in Interior(\mathcal{X})$, *and* $n \in \mathcal{N}$,

**(a)** $\left(Lh^{d+1+2k}\right)^{1/2}\left(\hat{g}^{(k)}(b|n,x) - g^{(k)}(b|n,x)\right) \rightarrow_d N(0, V_{g,k}(b,n,x))$, *where*

$$V_{g,k}(b,n,x) = K_k g(b|n,x) / (n\pi(n|x)\varphi(x)),$$

*and* $K_k = \left(\int K^2(u)\,du\right)^d \int \left(K^{(k)}(u)\right)^2 du.$

**(b)** $\hat{g}^{(k)}(b|n_1,x)$ *and* $\hat{g}^{(k)}(b|n_2,x)$ *are asymptotically independent for all* $n_1 \neq n_2$, $n_1, n_2 \in \mathcal{N}$.

Now, we present the main result of the paper. Using the result in (70) in the Appendix, we have the following decomposition:

$$\begin{aligned}\hat{f}(v|n,x) - f(v|x) &= \frac{F(v|x) f^2(v|x)}{(n-1) g^3(q(F(v|x)|n,x)|n,x)} \\ &\times \left(\hat{g}^{(1)}(q(F(v|x)|n,x)|n,x) - g^{(1)}(q(F(v|x)|n,x)|n,x)\right) + o_p\left(\left(Lh^{d+3}\right)^{-1/2}\right).\end{aligned} \tag{21}$$

Lemma 2, the definition of $\hat{f}(v|n,x)$, and the decomposition in (21) lead to the following theorem.

**Theorem 2** *Let* $\hat{\Lambda}(x)$ *be as defined in (18). Under Assumptions 1, 2, and 3 with* $k=1$, *for* $v \in \hat{\Lambda}(x)$, $x \in Interior(\mathcal{X})$, *and* $n \in \mathcal{N}$,

$$\left(Lh^{d+3}\right)^{1/2}\left(\hat{f}(v|n,x) - f(v|x)\right) \rightarrow_d N(0, V_f(v,n,x)),$$

*where*

$$V_f(v,n,x) = \frac{K_1 F^2(v|x) f^4(v|x)}{n(n-1)^2 \pi(n|x)\varphi(x) g^5(q(F(v|x)|n,x)|n,x)},$$

*and* $K_1$ *is as defined in Lemma 2. Furthermore,* $\hat{f}(v|\underline{n},x), \ldots, \hat{f}(v|\bar{n},x)$ *are asymptotically independent.*

**Remarks.** 1. The theorem also holds for fixed $v$'s in an inner closed subset of $[\underline{v}(x), \overline{v}(x)]$. Estimation of $\hat{\Lambda}(x)$ has no effect on the asymptotic distribution of $\hat{f}(v|n,x)$ by the same reason as in Remark 1 after Theorem 1.

2. Our approach can be used for estimation of the conditional PDF of values at quantile $\tau$, $f(Q(\tau|x))$. In this case, the estimator, say $\hat{f}(Q(\tau|x)|n,x)$, is given by

$$\hat{f}(Q(\tau|x)|n,x) = \left(\frac{n}{n-1}\frac{1}{\hat{g}(\hat{q}(\tau|n,x)|n,x)} - \frac{1}{n-1}\frac{\tau\hat{g}^{(1)}(\hat{q}(\tau|n,x)|n,x)}{\hat{g}^3(\hat{q}(\tau|n,x)|n,x)}\right)^{-1},$$

and $(Lh^{d+3})^{1/2}\left(\hat{f}(Q(\tau|x)|n,x) - f(Q(\tau|x)|x)\right) \to_d N(0, V_f(Q(\tau|x), n, x))$.

By Lemma 1, the asymptotic variance $V_f(v,n,x)$ can be consistently estimated by the plug-in estimator which replaces the unknown $F, f, \varphi, \pi, g$, and $q$ in the expression for $V_f(v,n,x)$ with their consistent estimators.

Using asymptotic independence of $\hat{f}(v|\underline{n},x), \ldots, \hat{f}(v|\bar{n},x)$, the optimal weights for the averaged PDF estimator of $f(v|x)$ in (16) can be obtained by solving a GLS-type problem. As usual, the optimal weights are inversely related to the variances $V_f(v,n,x)$:

$$\begin{aligned}
\hat{w}^*(n,x) &= \left(1/\hat{V}_f(v,n,x)\right) / \left(\sum_{j=\underline{n}}^{\bar{n}} 1/\hat{V}_f(v,j,x)\right) \\
&= \frac{n(n-1)^2\hat{\pi}(n|x)\hat{g}^5\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right)}{\sum_{j=\underline{n}}^{\bar{n}} j(j-1)^2\hat{\pi}(j|x)\hat{g}^5\left(\hat{q}\left(\hat{F}(v|n,x)|j,x\right)|j,x\right)},
\end{aligned}$$

and the asymptotic variance of the optimal weighted estimator is therefore given by

$$V_f(v,x) = \frac{K_1F^2(v|x)f^4(v|x)}{\sum_{n=\underline{n}}^{\bar{n}} n(n-1)^2\pi(n|x)g^5(q(F(v|x)|n,x)|n,x)}. \qquad (22)$$

In small samples, the accuracy of the normal approximation can be improved by taking into account the variance of the second-order term multiplied by $h^2$. To make the notation simple, consider the case of a single value $n$. We can expand the decomposition in (21) to obtain that $(Lh^{d+3})^{1/2}\left(\hat{f}(v|x,n) - f(v|x)\right)$ is given by

$$\frac{Ff^2}{(n-1)g^3}(Lh^{d+3})^{1/2}\left(\hat{g}^{(1)} - g^{(1)}\right) + h\left(\frac{3f}{g} - \frac{2nf^2}{(n-1)g^2}\right)(Lh^d)^{1/2}(\hat{g} - g) + o_p(h),$$

where, $F$ is the conditional CDF evaluated at $v$, and $g$, $g^{(1)}$, $\hat{g}$, $\hat{g}^{(1)}$ are the conditional density (given $x$ and $n$), its derivative, and their estimators evaluated at

$q\left(F\left(v|x\right)|n,x\right)$. According to this decomposition, one can improve the accuracy of the asymptotic approximation in small samples by using the following variance estimator instead of $\hat{V}_f$:[9]

$$\tilde{V}_f = \hat{V}_f + h^2\left(\frac{3\hat{f}}{\hat{g}} - \frac{2n\hat{f}^2}{(n-1)\,\hat{g}^2}\right)^2 \hat{V}_{g,0}.$$

Note that the second summand in the expression for $\tilde{V}_f$ is $O_p\left(h^2\right)$ and negligible in large samples.

# 4 Bootstrap

The results in the previous section suggest that a confidence interval for $f = f\left(v|x\right)$, for some chosen $x \in Interior(\mathcal{X})$ and $v \in \hat{\Lambda}(x)$, can be constructed using the usual normal approximation. In this section, we discuss an alternative approach based on the bootstrap percentile method.[10] The bootstrap percentile method approximates the distribution of $\hat{f} - f$ by that of $\hat{f}^\dagger - \hat{f}$, where $\hat{f} = \hat{f}(v|x)$ and $\hat{f}^\dagger$ is the bootstrap analogue of $\hat{f}$ computed using bootstrap data resampled from the original data. Note that the distribution of $\hat{f}^\dagger - \hat{f}$ can be approximated by simulations.

To generate bootstrap samples, first we draw randomly with replacement $L$ auctions from the original sample of auctions $\{(n_l, x_l) : l = 1, \ldots, L\}$. In the second step, we draw bids randomly with replacement from the bids data corresponding to each selected auction. Thus, if auction $\bar{l}$ is selected in the first step, in the second step we draw $n_{\bar{l}}$ bids from $\{b_{i\bar{l}} : i = 1, \ldots, n_{\bar{l}}\}$.

Let $M$ be the number of bootstrap samples. For each bootstrap sample $m = 1, \ldots, M$, we compute $\hat{f}^\dagger_m$, the bootstrap analogue of $\hat{f}$. Note that $\hat{f}^\dagger_m$ is computed the same way as $\hat{f}$ but using the data in bootstrap sample $m$ instead of the original data. Let $\phi^\dagger_\tau$ be the $\tau$ empirical quantile of $\{\hat{f}^\dagger_m : m = 1, \ldots, M\}$. The bootstrap percentile confidence interval is constructed as

$$CI^{BP}_{1-\alpha} = \left[\phi^\dagger_{\alpha/2}, \phi^\dagger_{1-\alpha/2}\right]. \tag{23}$$

[9] There is no covariance term because $\int K\left(u\right) K^{(1)}\left(u\right) du = 0$.

[10] See, for example, Shao and Tu (1995) for a general discussion of the bootstrap methods.

Let $H_{f,L}$ denote the CDF of $\sqrt{Lh^{d+3}}\left(\hat{f}-f\right)$ and $H^{\dagger}_{f,L}$ be the conditional CDF of $\sqrt{Lh^{d+3}}\left(\hat{f}^{\dagger}_{m}-\hat{f}\right)$ given the original data:

$$\begin{aligned} H_{f,L}\left(u\right) &= P\left(\sqrt{Lh^{d+3}}\left(\hat{f}-f\right)\leq u\right), \\ H^{\dagger}_{f,L}\left(u\right) &= P^{\dagger}\left(\sqrt{Lh^{d+3}}\left(\hat{f}^{\dagger}_{m}-\hat{f}\right)\leq u\right), \end{aligned}$$

where $P^{\dagger}(\cdot)$ denotes the conditional probability given the original sample of auctions $\{(b_{1l},\ldots,b_{n_l l},n_l,x_l):l=1,\ldots,L\}$. The asymptotic validity of $CI^{BP}_{1-\alpha}$ is implied by the result of the following theorem.[11]

**Theorem 3** *Suppose that Assumptions 1, 2, and 3 with $k=1$ hold. Then, as $L\to\infty$, $\sup_{u\in\mathbb{R}}|H_{f,L}(u)-H^{\dagger}_{f,L}(u)|\to_p 0$.*

# 5 Binding reserve prices

We have so far assumed that there is no reserve price. Alternatively, we could have assumed that there is a reserve price, but it is non-binding. However, in real world auctions, sellers often use binding reserve prices to increase their expected revenues, so it is useful to extend our results in this direction.

Let $r$ be the reserve price. As in GPV, we assume that only the bidders with $v_{il}\geq r$ submit bids. In this section, we use $n_l$ to denote the number of actual observed bidders in auction $l$. Let $\bar{n}$ denote the unobserved number of potential bidders. We make the following assumption identical to Assumption A5 in GPV.

**Assumption 4** **(a)** *The number of potential bidders $\bar{n}\geq 2$ is constant.*

**(b)** *The reserved price $r$ is a possibly unknown deterministic $R$ continuously differentiable function $Res(\cdot)$ of the auction characteristics $x$.*

**(c)** *The reserve price is binding in the sense that, for some $\varepsilon>0$, $\underline{v}(x)+\varepsilon\leq Res(x)\leq\bar{v}(x)-\varepsilon$ for all $x\in\mathcal{X}$.*

[11] In the supplement, we compare the accuracy of the bootstrap percentile method with that of the asymptotic normal approximation in Monte Carlo simulations, and find that the bootstrap is more accurate.

Our estimation method easily extends to this environment. Let

$$F^*(v|x) \equiv \frac{F(v|x) - F(r|x)}{1 - F(r|x)}$$

be the distribution of valuations conditional on participation, and let $f^*(v|x)$ be its density. Note that the parent density $f(v|x)$ is related to $f^*(v|x)$ as

$$f(v|x) = (1 - F(r|x)) f^*(v|x). \tag{24}$$

Our estimator for $f(v|x)$ is based on (24): we separately estimate $F(r|x)$ and $f^*(v|x)$. We estimate $F(r|x)$ as a nonparametric regression exactly as in GPV:[12]

$$\hat{F}(r|x) = 1 - \frac{1}{\hat{n}Lh^d\hat{\varphi}(x)} \sum_{l=1}^{L} n_l K_{*h}(x - x_l),$$

where again as in GPV,

$$\hat{n} = \max_{l=1,\ldots,L} n_l$$

is the estimator of the number of potential bidders $\bar{n}$. Note that by standard results,

$$\hat{n} = \bar{n} + O(L^{-1}) \tag{25}$$

and

$$\hat{F}(r|x) = F(r|x) + O_p\left(\left(Lh^d\right)^{-1/2}\right). \tag{26}$$

We now describe how our approach can be extended to estimation of $f^*(v|x)$. Let $G(b|x)$ be the CDF of bids conditional on $x$ and on having a valuation above the reserve price, $v_{il} \geq r$. Let $g(b|x)$ be the corresponding PDF. By the law of total probability,

$$\begin{aligned} G(b|x) &= \sum_{n=\underline{n}}^{\bar{n}} \pi(n|x) G(b|n,x), & (27) \\ g(b|x) &= \sum_{n=\underline{n}}^{\bar{n}} \pi(n|x) g(b|n,x). & (28) \end{aligned}$$

[12]See the third equation on page 550 of GPV.

The estimators $\hat{G}(b|x)$ and $\hat{g}(b|x)$ then can be constructed by the plug-in method using our previously derived estimators $\hat{n}$, $\hat{\pi}(n|x)$, $\hat{G}(b|n,x)$, and $\hat{g}(b|n,x)$.[13]

With $\hat{G}(b|x)$ and $\hat{g}(b|x)$ in hand, we estimate the density $f^*(v|x)$ by following exactly the same steps as in the case without reserve price. Since the inverse bidding strategy under a binding reserve price is given by

$$\xi(b|x) = b + \frac{1}{\bar{n}-1}\frac{(1-F(r|x))\,G(b|x)+F(r|x)}{(1-F(r|x))\,g(b|x)},$$

the valuation quantile for the participants becomes

$$Q^*(\tau|x) = q(\tau|x) + \frac{1}{\bar{n}-1}\frac{(1-F(r|x))\,\tau+F(r|x)}{(1-F(r|x))\,g(q(\tau|x)\,|x)}, \tag{29}$$

where $Q^*(\tau|x)$ is the quantile function of $F^*(v|x)$. Let $\hat{Q}^{*p}(\tau|x)$ be the plug-in estimator of $Q^*(\tau|x)$ based on (29), $\hat{Q}^*(\tau|x)$ be its monotone version as in (13), and $\hat{F}^*(v|x)$ be the corresponding estimator of the CDF $F^*(v|x)$ as in (14). The estimator $\hat{f}^*(v|x)$ is derived parallel to (15), as the reciprocal of

$$\left(1+\frac{1-\hat{F}(r|x)}{\hat{n}-1}\right)\frac{1}{\hat{g}\left(\hat{q}\left(\hat{F}^*(v|x)\,|x\right)|x\right)}$$
$$-\frac{1}{\hat{n}-1}\frac{\left(1-\hat{F}(r|x)\right)\hat{F}^*(v|x)\,\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}^*(v|x)\,|x\right)|x\right)}{\hat{g}^3\left(\hat{q}\left(\hat{F}^*(v|x)\,|x\right)|x\right)}.$$

Similarly to $\hat{\Lambda}(x)$ in Section 3, define $\hat{\Lambda}^*(x) = \left[\hat{Q}^*(\tau_1|x), \hat{Q}^*(\tau_2|x)\right]$, where $0 < \tau_1 < \tau_2 < 1$ are chosen by the econometrician. Note that by construction, $v > Res(x)$ with probability approaching one for all $v \in \hat{\Lambda}^*(x)$. As before, the asymptotics of $\hat{f}^*(v|x)$ are driven by $\hat{g}^{(1)}$, the term with the slowest convergence rate. All the steps in our previous results routinely transfer to this setting.[14] In particular, we have an exact analogue to Lemma 1, and parallel to (21), the delta-method expansion for the

[13] Assumption 4(a) implies that $G(b|x)$ does not depend on $n$. Note that in the present setting, $n_l$ are draws from the Binomial dostribution, $n_l|x \sim Binomial(\bar{n}, 1-F(r|x))$, and $\pi(n|x)$ are the corresponding Binomial probabilities.

[14] Since we pick the inner quantiles $0 < \tau_1 < \tau_2 < 1$, we only use the bid observations sufficiently far from the boundary $\underline{b}(n,x) = r$. We therefore do not need to transform the bids as in GPV to avoid the singularity of $g(b|x)$ when $b \downarrow r$.

estimator $\hat{f}^*(v|x)$ for $v \in \hat{\Lambda}^*(x)$ takes the form

$$\hat{f}^*(v|x) - f^*(v|x) = \frac{(1-F(r|x))\,F^*(v|x)\,f^{*2}(v|x)}{(\bar{n}-1)\,g^3(q(F^*(v|x)|x)|x)} \times \left(\hat{g}^{(1)}(q(F^*(v|x)|x)|x) - g^{(1)}(q(F^*(v|x)|x)|x)\right) + o_p\left(\left(Lh^{d+3}\right)^{-1/2}\right).$$

The estimator $\hat{f}^*(v|x)$ therefore satisfies

$$\left(Lh^{d+1}\right)^{1/2}\left(\hat{f}^*(v|x) - f^*(v|x)\right) \to_d N(0, V_{f^*}(v,x)), \tag{30}$$

for $v \in \hat{\Lambda}^*(x)$. The asymptotic variance is given by

$$V_{f^*}(v,x) = \left(\frac{F^*(v|x)\,f^{*2}(v|x)}{(\bar{n}-1)\,g^3(q(F^*(v|x)|x)|x)}\right)^2 V_{g,1}(q(F^*(v|x)|x), x),$$

where from (28),

$$V_{g,1}(b,x) = \sum_{n=1}^{\bar{n}} \pi(n|x)^2\, V_{g,1}(b,n,x).$$

The asymptotic variance $V_{f^*}$ can be consistently estimated by the plug-in method.

From (24), the estimator of $f(v|x)$ for $v \in \hat{\Lambda}^*(x)$ is given by

$$\hat{f}(v|x) \equiv \left(1 - \hat{F}(r|x)\right)\hat{f}^*(v|x),$$

Combining (30) and (26), we have the following asymptotic normality result.

**Theorem 4** *Under Assumptions 1, 2, 3 with $k = 1$, and 4, for $v \in \hat{\Lambda}^*(x)$ and $x \in Interior(\mathcal{X})$,*

$$\left(Lh^{d+1}\right)^{1/2}\left(\hat{f}(v|x) - f(v|x)\right) \to_d N(0, V_f(v,x)),$$

*where $V_f(v,x) = (1-F(r|x))^2\, V_{f^*}(v,x)$.*

# 6 Monte Carlo experiments

In this section, we compare the finite sample performance of our estimator with that of the GPV's estimator in terms of bias and mean squared error (MSE). We consider the case with no covariates ($d = 0$). The true CDF of valuations used in our simulations is given by

$$F(v) = \begin{cases} 0, & v < 0, \\ v^{\alpha}, & 0 \leq v \leq 1, \\ 1, & v > 1, \end{cases} \tag{31}$$

where $\alpha > 0$. Such a choice of $F$ is convenient because the corresponding bidding strategy is easy to compute:

$$B(v) = \left(1 - \frac{1}{\alpha(n-1)+1}\right) v. \tag{32}$$

In our simulations, we consider the values $\alpha = 1/2, 1$, and 2. When $\alpha = 1$, the distribution of valuations is uniform over the interval $[0,1]$, $\alpha = 1/2$ corresponds to the case of a downward-sloping PDF of valuations, and $\alpha = 2$ corresponds to the upward-sloping PDF.

We report the results for $v = 0.4, 0.5, 0.6$, and the number of bidders $n = 3$ and 5. The number of auctions $L$ is chosen so that the total number of observations in a simulated sample, $nL$, is the same for all values of $n$. In this case, the differences in simulations results observed across $n$ cannot be attributed to varying sample size. We set $nL = 4200$. Each Monte Carlo experiment has $10^3$ replications.

Similarly to GPV, we use the tri-weight kernel function for the kernel estimators, and the normal rule-of-thumb bandwidth in estimation of $g$:

$$h_1 = 1.06\hat{\sigma}_b (nL)^{-1/5},$$

where $\hat{\sigma}_b$ is the estimated standard deviation of bids. The MSE optimal bandwidth for derivative estimation is of order $L^{-1/7}$ (Pagan and Ullah, 1999, Page 56). Therefore, for estimation of $g^{(1)}$ we use the following bandwidth:

$$h_2 = 1.06\hat{\sigma}_b (nL)^{-1/7}.$$

In each Monte Carlo replication, we generate randomly $nL$ valuations $\{v_i : i = 1, \ldots, nL\}$ from the CDF in (31), and then compute the corresponding bids according to (32). The computation of the quantile-based estimator $\hat{f}(v)$ involves several steps. First, we estimate the quantile function of bids $q(\tau)$. Let $b_{(1)}, \ldots, b_{(nL)}$ denote the ordered sample of bids. We set $\hat{q}\left(\frac{i}{nL}\right) = b_{(i)}$. Second, we estimate the PDF of bids $g(b)$ using (9). To construct our estimator, $g$ needs to be estimated at all points $\left\{\hat{q}\left(\frac{i}{nL}\right) : i = 1, \ldots, nL\right\}$. Given the estimates $\hat{g}$, we compute $\left\{\hat{Q}^p\left(\frac{i}{nL}\right) : i = 1, \ldots, nL\right\}$ using (12), its monotone version according to (13), and $\hat{F}(v)$ according to (14). Let $\lceil x \rceil$ denote the nearest integer greater than or equal to $x$; we compute $\hat{q}\left(\hat{F}(v)\right)$ as $\hat{q}\left(\frac{\lceil nL\hat{F}(v)\rceil}{nL}\right)$. Next, we compute $\hat{g}\left(\hat{q}\left(\hat{F}(v)\right)\right)$ and $\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v)\right)\right)$ using (9) and (10) respectively, and $\hat{f}(v)$ as the reciprocal of (15).

To compute the GPV's estimator of $f(v)$, in the first step we compute the pseudo-valuations $\hat{v}_{il}$ according to equation (1), with $G$ and $g$ replaced by their estimators. In the second step, we estimate $f(v)$ by the kernel method from the sample $\{\hat{v}_{il}\}$ obtained in the first-step. To avoid the boundary bias effect, GPV suggest trimming of the observations that are too close to the estimated boundary of the support. Note that no explicit trimming is necessary for our estimator, since implicit trimming occurs from our use of quantiles instead of pseudo-valuations.[15]

In their simulations, GPV use the bandwidths of order $(nL)^{-1/5}$ in the first and second steps of estimation. We found, however, that using a bandwidth of order $(nL)^{-1/7}$ in the second step significantly improves the performance of their estimator in terms of bias and MSE. To compute the GPV's estimator, we therefore use $h_1$ as the first step bandwidth (for estimation of $G$ and $g$), and $h_2$ at the second step. Similarly to the quantile-based estimator, the GPV's estimator is implemented with the tri-weight kernel.

The results are reported in Table 1. In most cases, the GPV's estimator has a smaller bias. This can be due to the fact that the GPV's estimator is obtained by kernel smoothing of the data, while the quantile-based estimator is a nonlinear function of the estimated CDF, PDF and its derivative. In terms of MSE, however, there is no clear winner, and the relative efficiency of the estimators depends on the underlying distribution of the valuations and the number of bidders in the auction. The GPV's estimator is more efficient when the number of bidders is relatively large and PDF has a positive slope. On the other hand, our estimator is more attractive when the number of bidders is small and the PDF has a negative slope.[16]

[15] In our simulations, we found that trimming has no effect on the estimator of GPV: essentially the same estimates were obtained with and without trimming.

[16] Additional results, including the simulations for $n = 2, 4, 6$, and 7, are reported in the supplement.

# 7 Concluding remarks

In this paper, we have assumed that the bidders are risk-neutral. It would be important to extend our method to the case of risk-averse bidders. Guerre et al. (2009) consider nonparametric identification of a first-price auction with risk-averse bidders each of whom has an unknown utility function $U(\cdot)$, and find that exclusion restrictions are necessary to achieve the identification of model primitives. They show that under risk aversion, the bids and valuations are linked as

$$v = \xi_\lambda (b|n) \equiv b + \lambda^{-1} \left( \frac{1}{n-1} \frac{G(b|n)}{g(b|n)} \right),$$

where $\lambda^{-1}(\cdot)$ is the inverse of $U(\cdot)/U'(\cdot)$.[17] Consequently, the quantiles of bids and valuations are now linked as $Q(\tau|n) = \xi_\lambda (q(\tau|n)|n)$. Assuming that the variation in $n$ is exogenous, the valuation quantiles $Q(\tau|n)$ do not depend on $n$. Guerre et al. (2009) show that $\lambda(\cdot)$ (and hence $U(\cdot)$) is identifiable through this restriction, and in the concluding section of their paper, discuss some strategies for the nonparametric estimation of $\lambda$. At this point, it is not known whether these approaches can lead to a consistent estimator $\hat{\lambda}$. However, when such an estimator becomes available, it might be possible to extend the approach of our paper to accommodate risk aversion. Such an extension is left for future work.

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[17] See their equation (4) on page 1198.

# Appendix of proofs

**Proof of Lemma 1.** For part (c), define

$$G^0(b,n,x) = n\pi(n|x)\,G(b|n,x)\,\varphi(x),$$

and its estimator

$$\hat{G}^0(b,n,x) = \frac{1}{L}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n)\,1(b_{il} \le b)\,K_{*h}(x_l - x). \tag{33}$$

Next,

$$\begin{aligned}
E\hat{G}^0(b,n,x) &= E\left(1(n_l = n)\,K_{*h}(x_l - x)\sum_{i=1}^{n_l} 1(b_{il} \le b)\right)\\
&= nE\left(1(n_l = n)\,1(b_{il} \le b)\,K_{*h}(x_l - x)\right)\\
&= nE\left(\pi(n|x_l)\,G(b|n,x_l)\,K_{*h}(x_l - x)\right)\\
&= n\int \pi(n|u)\,G(b|n,u)\,K_{*h}(u - x)\,\varphi(u)\,du\\
&= \int G^0(b,n,x+hu)\,K_d(u)\,du.
\end{aligned}$$

By Assumption 1(e) and Proposition 1(iii) of GPV, $G(b|n,\cdot)$ admits up to $R$ continuous bounded derivatives. Then, as in the proof of Lemma B.2 of Newey (1994), there exists a constant $c > 0$ such that

$$\begin{aligned}
&\left|G^0(b,n,x) - E\hat{G}^0(b,n,x)\right|\\
\le\ & ch^R\left(\int |K_d(u)|\,\|u\|^R\,du\right)\left\|vec\left(D_x^R G^0(b,n,x)\right)\right\|,
\end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm and $D_x^R G^0$ denotes the $R$-th partial derivative of $G^0$ with respect to $x$. It follows then that

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}\left|G^0(b,n,x) - E\hat{G}^0(b,n,x)\right| = O\left(h^R\right). \tag{34}$$

Next, we show that

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}|\hat{G}^0(b,n,x)-E\hat{G}^0(b,n,x)|=O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2}\right). \tag{35}$$

We follow the approach of Pollard (1984). Fix $n\in\mathcal{N}$ and $x\in Interior(\mathcal{X})$, and consider a class of functions $\mathcal{Z}$ indexed by $h$ and $b$, with a representative function

$$z_l(b,n,x)=\sum_{i=1}^{n_l}1(n_l=n)\,1(b_{il}\le b)\,h^dK_{*h}(x_l-x).$$

By the result in Pollard (1984) (Problem 28), the class $\mathcal{Z}$ has polynomial discrimination. Theorem 37 in Pollard (1984) (see also Example 38) implies that for any sequences $\delta_L$, $\alpha_L$ such that $L\delta_L^2\alpha_L^2/\log L\to\infty$ and $Ez_l^2(b,n,x)\le\delta_L^2$,

$$\alpha_L^{-1}\delta_L^{-2}\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}|\frac{1}{L}\sum_{l=1}^{L}z_l(b,n,x)-Ez_l(b,n,x)|\to 0 \tag{36}$$

almost surely. We claim that this implies the result in (35). The proof is by contradiction. Suppose not. Then there exist a sequence $\gamma_L\to\infty$ and a subsequence of $L$ such that along this subsequence,

$$\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}|\hat{G}^0(b,n,x)-E\hat{G}^0(b,n,x)|\ge\gamma_L\left(\frac{Lh^d}{\log L}\right)^{-1/2}. \tag{37}$$

on a set of events $\Omega'\subset\Omega$ with a positive probability measure. Now if we let $\delta_L^2=h^d$ and $\alpha_L=(\frac{Lh^d}{\log L})^{-1/2}\gamma_L^{1/2}$, then the definition of $z$ implies that, along the subsequence on a set of events $\Omega'$,

$$\begin{aligned}
&\alpha_L^{-1}\delta_L^{-2}\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}|\frac{1}{L}\sum_{l=1}^{L}z_l(b,n,x)-Ez_l(b,n,x)|\\
=&\left(\frac{Lh^d}{\log L}\right)^{1/2}\gamma_L^{-1/2}h^{-d}\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}|\frac{1}{L}\sum_{l=1}^{L}z_l(b,n,x)-Ez_l(b,n,x)|\\
=&\left(\frac{Lh^d}{\log L}\right)^{1/2}\gamma_L^{-1/2}\sup_{b\in\left[\underline{b}(n,x),\bar{b}(n,x)\right]}|\hat{G}^0(b,n,x)-E\hat{G}^0(b,n,x)|
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{Lh^d}{\log L}\right)^{1/2} \gamma_L^{-1/2} \gamma_L \left(\frac{Lh^d}{\log L}\right)^{-1/2} \\
&= \gamma_L^{1/2} \to \infty,
\end{aligned}
$$

where the inequality follows by (37), a contradiction to (36). This establishes (35), so that (34), (35) and the triangle inequality together imply that

$$
\sup_{b \in \left[\underline{b}(n,x), \bar{b}(n,x)\right]} \left|\hat{G}^0(b,n,x) - G^0(b,n,x)\right| = O_p\left(\left(\frac{Lh^d}{\log L}\right)^{-1/2} + h^R\right). \tag{38}
$$

To complete the proof, recall that from the definitions of $G^0(b,n,x)$ and $\hat{G}^0(b,n,x)$,

$$
G(b|n,x) = \frac{G^0(b,n,x)}{\pi(n|x)\varphi(x)} \text{ and } \hat{G}(b|n,x) = \frac{\hat{G}^0(b,n,x)}{\hat{\pi}(n|x)\hat{\varphi}(x)},
$$

so that by the mean-value theorem, $\left|\hat{G}(b|n,x) - G(b|n,x)\right|$ is bounded by

$$
\begin{aligned}
&\left\|\left(\frac{1}{\tilde{\pi}(n,x)\tilde{\varphi}(x)}, \frac{\tilde{G}^0(b,n,x)}{\tilde{\pi}^2(n,x)\tilde{\varphi}(x)}, \frac{\tilde{G}^0(b,n,x)}{\tilde{\pi}(n,x)\tilde{\varphi}^2(x)}\right)\right\| \times \\
&\qquad \times \left\|\left(\hat{G}^0(b,n,x) - G^0(b,n,x), \hat{\pi}(n|x) - \pi(n|x), \hat{\varphi}(x) - \varphi(x)\right)\right\|,
\end{aligned} \tag{39}
$$

where $\left\|\left(\tilde{G}^0 - G^0, \tilde{\pi} - \pi, \tilde{\varphi} - \varphi\right)\right\| \leq \left\|\left(\hat{G}^0 - G^0, \hat{\pi} - \pi, \hat{\varphi} - \varphi\right)\right\|$. Further, by Assumption 1(b) and (c) and the results in parts (a) and (b) of the lemma, with the probability approaching one $\tilde{\pi}$ and $\tilde{\varphi}$ are bounded away from zero. The desired result follows from (38), (39) and parts (a) and (b) of the lemma.

For part (d) of the lemma, since $\hat{G}(\cdot|n,x)$ is monotone by construction,

$$
\begin{aligned}
P\left(\hat{q}(\varepsilon|n,x) \leq \underline{b}(n,x)\right) &= P\left(\inf_b \left\{b : \hat{G}(b|n,x) \geq \varepsilon\right\} \leq \underline{b}(n,x)\right) \\
&= P\left(\hat{G}\left(\underline{b}(n,x)|n,x\right) \geq \varepsilon\right) \\
&= o(1),
\end{aligned}
$$

where the last equality is by the result in part (c). Similarly,

$$
P\left(\hat{q}(1-\varepsilon|n,x) \geq \bar{b}(n,x)\right) = P\left(\hat{G}\left(\bar{b}(n,x)|n,x\right) \leq 1-\varepsilon\right)
$$

$$= \quad o(1).$$

Hence, for all $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$, $\underline{b}(n,x) < \hat{q}(\varepsilon|n,x) < \hat{q}(1-\varepsilon|n,x) < \overline{b}(n,x)$ with probability approaching one. Since the distribution $G(b|n,x)$ is continuous in $b$, $G(q(\tau|n,x)|n,x) = \tau$, and for $\tau \in [\varepsilon, 1-\varepsilon]$, we can write the identity

$$G(\hat{q}(\tau|n,x)|n,x) - G(q(\tau|n,x)|n,x) = G(\hat{q}(\tau|n,x)|n,x) - \tau. \tag{40}$$

Next, we have that with probability one,

$$\begin{aligned} 0 \quad &\leq \quad \hat{G}(\hat{q}(\tau|n,x)|n,x) - \tau \leq \frac{\left(\sup_{u\in\mathbb{R}} K(u)\right)^d}{N(n,x)}, \text{ where} && (41) \\ N(n,x) \quad &= \quad \sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n)\, 1(K_{*h}(x_l - x) > 0). && (42) \end{aligned}$$

The first inequality in (41) is by Lemma 21.1(ii) of van der Vaart (1998). The second inequality in (41) holds (with probability one) because $\hat{G}(\cdot|n,x)$ is a weighted empirical CDF of a continuous random variable ($\hat{G}(\cdot|n,x)$ is a step function, $b_{il}$ is continuously distributed, and therefore with probability one, the size of each step of $\hat{G}(\cdot|n,x)$ is inversely related to the number of observations with non-zero weights used in its construction). Let $B_h(x) = \left\{u \in \mathbb{R}^d : K_{*h}(u-x) > 0\right\}$. We have

$$\begin{aligned} EN(n,x) \quad &= \quad P(n_l = n, K_{*h}(x_l - x) > 0)\, nL \\ &= \quad nL \int_{B_h(x)} \pi(n|u)\, \varphi(u)\, du \\ &\leq \quad nL \left(\sup_{x\in\mathcal{X}} \varphi(x)\right)\left(\int_{B_h(x)} du\right). && (43) \end{aligned}$$

By a similar argument, we have

$$EN(n,x) \geq nL\left(\inf_{x\in\mathcal{X}} \pi(n|x)\right)\left(\inf_{x\in\mathcal{X}} \varphi(x)\right)\left(\int_{B_h(x)} du\right). \tag{44}$$

Further,

$$\begin{aligned} Var(N(n,x)) \quad &\leq \quad P(n_l = n, K_{*h}(x_l - x) > 0)\, nL \\ &= \quad O\left(Lh^d\right). && (45) \end{aligned}$$

It follows now by Assumptions 1(b),(f) and from (43)-(45) that there is a constant $c_{n,x} > 0$ such that

$$\begin{aligned} N(n,x) &= EN(n,x) + O_p\left(\left(Lh^d\right)^{1/2}\right) \\ &= Lh^d\left(c_{n,x} + O_p\left(\left(Lh^d\right)^{-1/2}\right)\right). \end{aligned} \tag{46}$$

By the results in parts (a) and (b) and (46),

$$\hat{G}\left(\hat{q}\left(\tau|n,x\right)|n,x\right) = \tau + O_p\left(\left(Lh^d\right)^{-1}\right) \tag{47}$$

uniformly over $\tau$. Combining (40) and (47), and applying the mean-value theorem to the left-hand side of (40), we obtain

$$\begin{aligned} &\hat{q}\left(\tau|n,x\right) - q\left(\tau|n,x\right) = \\ &\qquad = \frac{G\left(\hat{q}\left(\tau|n,x\right)|n,x\right) - \hat{G}\left(\hat{q}\left(\tau|n,x\right)|n,x\right)}{g\left(\widetilde{q}\left(\tau|n,x\right)|n,x\right)} + O_p\left(\left(Lh^d\right)^{-1}\right), \end{aligned} \tag{48}$$

where $\widetilde{q}$ lies between $\hat{q}$ and $q$ for all $(\tau,n,x)$. By Proposition 1(ii) of GPV, $g\left(b|n,x\right) > c_g > 0$ for all $b \in \left[\underline{b}\left(n,x\right), \bar{b}\left(n,x\right)\right]$, and the result in part (d) follows from (48) and part (c) of the lemma.

Next, we prove part (e) of the lemma. Let $N(n,x)$ be as defined in (42). Consider the ordered sub-sample of bids $b_{(1)} \leq \ldots \leq b_{(N(n,x))}$ with $n_l = n$ and $K_{*h}\left(x_l - x\right) > 0$. Then,

$$0 \leq \lim_{t \downarrow \tau} \hat{q}\left(t|n,x\right) - \hat{q}\left(\tau|n,x\right) \leq \max_{j=2,\ldots,N(n,x)} \left(b_{(j)} - b_{(j-1)}\right).$$

By the results of Deheuvels (1984),

$$\max_{j=2,\ldots,N} \left(b_{(j)} - b_{(j-1)}\right) = O_p\left(\left(\frac{N(n,x)}{\log N(n,x)}\right)^{-1}\right). \tag{49}$$

The result of part (e) follows from (49) and (46).

To prove part (f), note that by Assumption 1(e) and Proposition 1(iv) of GPV, $g\left(\cdot|n,\cdot\right)$ admits up to $R$ continuous bounded partial derivatives. Let

$$g_0^{(k)}\left(b,n,x\right) = \pi\left(n|x\right) g^{(k)}\left(b|n,x\right) \varphi\left(x\right), \tag{50}$$

and define

$$\hat{g}_0^{(k)}(b,n,x) = \frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l} 1(n_l = n) K_h^{(k)}(b_{il} - b) K_{*h}(x_l - x). \tag{51}$$

We can write the estimator $\hat{g}(b|n,x)$ as $\hat{g}(b|n,x) = \hat{g}_0(b,n,x)/(\hat{\pi}(n|x)\hat{\varphi}(x))$, so that $\hat{g}^{(k)}(b|n,x) = \hat{g}_0^{(k)}(b,n,x)/(\hat{\pi}(n|x)\hat{\varphi}(x))$. By Lemma B.3 of Newey (1994), the estimator $\hat{g}_0^{(k)}(b,n,x)$ is uniformly consistent in $b$ over $[b_1(n,x), b_2(n,x)]$. By the results in parts (a) and (b), the estimators $\hat{\pi}(n|x)$ and $\hat{\varphi}(x)$ converge at the rate faster than that of $\hat{g}_0^{(k)}(b,n,x)$. The desired result follows by the same argument as in the proof of part (c), equation (39).

For part (g), let $c_g$ be as in the proof of part (d) of the lemma. First, we consider the preliminary estimator, $\hat{Q}^p(\tau|n,x)$. We have that $\left|\hat{Q}^p(\tau|n,x) - Q(\tau|x)\right|$ is bounded by

$$\begin{aligned}
& |\hat{q}(\tau|n,x) - q(\tau|n,x)| + \frac{|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(q(\tau|n,x)|n,x)|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g} \\
\leq\ & |\hat{q}(\tau|n,x) - q(\tau|n,x)| + \frac{|g(\hat{q}(\tau|n,x)|n,x) - g(q(\tau|n,x)|n,x)|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g} \\
& + \frac{|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(\hat{q}(\tau|n,x)|n,x)|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g} \\
\leq\ & \left(1 + \frac{\sup_{b\in[b_1(n,x),b_2(n,x)]}\left|g^{(1)}(b|n,x)\right|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g}\right)|\hat{q}(\tau|n,x) - q(\tau|n,x)| \\
& + \frac{|\hat{g}(\hat{q}(\tau|n,x)|n,x) - g(\hat{q}(\tau|n,x)|n,x)|}{\hat{g}(\hat{q}(\tau|n,x)|n,x)\,c_g}.
\end{aligned} \tag{52}$$

By continuity of the distributions, we can pick $\varepsilon > 0$ small enough so that

$$q(\tau_1 - \varepsilon|n,x) > b_1(n,x) \text{ and } q(\tau_2 + \varepsilon|n,x) < b_2(n,x).$$

Define

$$E_L(n,x) = \{\hat{q}(\tau_1 - \varepsilon|n,x) \geq b_1(n,x), \hat{q}(\tau_2 + \varepsilon|n,x) \leq b_2(n,x)\}.$$

By the result in part (d), $P(E_L^c(n,x)) = o(1)$. Hence, it follows from part (f) of the lemma that the estimator $\hat{g}(\hat{q}(\tau|n,x)|n,x)$ is bounded away from zero with probability approaching one. Consequently, by Assumption 1(e) and part (d) of the lemma that the first summand on the right-hand side of (52) is $O_p(\beta_L)$ uniformly over $[\tau_1-\varepsilon,\tau_2+\varepsilon]$, where $\beta_L=\left(Lh^{d+1+2k}/\log L\right)^{-1/2}+h^R$. Next,

$$\begin{aligned}
& P\left(\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\beta_L^{-1}\left|\hat{g}\left(\hat{q}\left(\tau|n,x\right)|n,x\right)-g\left(\hat{q}\left(\tau|n,x\right)|n,x\right)\right|>M\right)\\
\leq\; & P\left(\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\beta_L^{-1}\left|\hat{g}\left(\hat{q}\left(\tau|n,x\right)|n,x\right)-g\left(\hat{q}\left(\tau|n,x\right)|n,x\right)\right|>M,E_L\left(n,x\right)\right)\\
& +P\left(E_L^c\left(n,x\right)\right)\\
\leq\; & P\left(\sup_{b\in[b_1(n,x),b_2(n,x)]}\beta_L^{-1}\left|\hat{g}\left(b|n,x\right)-g\left(b|n,x\right)\right|>M\right)+o\left(1\right). \qquad (53)
\end{aligned}$$

It follows from part (f) of the lemma and (53) that

$$\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}|\hat{Q}^p\left(\tau|n,x\right)-Q\left(\tau|x\right)|=O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right). \qquad (54)$$

Further, by construction, $\hat{Q}\left(\tau|n,x\right)-\hat{Q}^p\left(\tau|n,x\right)\geq 0$ for $\tau\geq\tau_0$. We can choose $\tau_0\in[\tau_1,\tau_2]$. Since $\hat{Q}^p\left(\cdot|n,x\right)$ is left-continuous, there exists $\tau'\in[\tau_0,\tau]$ such that $\hat{Q}^p\left(\tau'|n,x\right)=\sup_{t\in[\tau_0,\tau]}\hat{Q}^p\left(t|n,x\right)$. Since $Q\left(\cdot|x\right)$ is nondecreasing,

$$\begin{aligned}
& \hat{Q}\left(\tau|n,x\right)-\hat{Q}^p\left(\tau|n,x\right)\\
=\; & \hat{Q}^p\left(\tau'|n,x\right)-\hat{Q}^p\left(\tau|n,x\right)\\
\leq\; & \hat{Q}^p\left(\tau'|n,x\right)-Q\left(\tau'|x\right)+Q\left(\tau|x\right)-\hat{Q}^p\left(\tau|n,x\right)\\
\leq\; & \sup_{t\in[\tau_0,\tau]}\left(\hat{Q}^p\left(t|n,x\right)-Q\left(t|x\right)\right)+Q\left(\tau|x\right)-\hat{Q}^p\left(\tau|n,x\right)\\
\leq\; & 2\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\left|\hat{Q}^p\left(\tau|n,x\right)-Q\left(\tau|x\right)\right|\\
=\; & O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right),
\end{aligned}$$

where the last result follows from (54). Using a similar argument for $\tau<\tau_0$, we conclude that

$$\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\left|\hat{Q}\left(\tau|n,x\right)-\hat{Q}^p\left(\tau|x\right)\right|=O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right). \qquad (55)$$

The result of part (g) follows from (54) and (55).

Lastly, we prove part (h). Let $\varepsilon$ be as in part(g). By Lemma 21.1(ii) of van der Vaart (1998), $\hat{F}\left(\hat{Q}\left(\tau|n,x\right)|n,x\right) \geq \tau$, where the inequality becomes strict only at the points of discontinuity, and therefore

$$\hat{F}\left(\hat{Q}\left(\tau_1|n,x\right)|n,x\right) \geq \tau_1 > \tau_1 - \varepsilon$$

for all $n$. Further, since $\hat{Q}\left(\cdot|n,x\right)$ is non-decreasing,

$$\begin{aligned} & P\left(\hat{F}\left(\hat{Q}\left(\tau_2|n,x\right)|n,x\right) < \tau_2 + \varepsilon\right) \\ = \ & P\left(\sup_{t\in[0,1]}\left\{t : \hat{Q}\left(t|n,x\right) \leq \hat{Q}\left(\tau_2|n,x\right)\right\} < \tau_2 + \varepsilon\right) \\ \geq \ & P\left(\hat{Q}\left(\tau_2|n,x\right) < \hat{Q}\left(\tau_2 + \varepsilon|n,x\right)\right) \\ \to \ & 1, \end{aligned}$$

where the last result is by part (g) of the lemma and because $Q(\tau_2|x) < Q\left(\tau_2 + \varepsilon|x\right)$. Thus, for all $v \in \hat{\Lambda}\left(x\right)$,

$$\hat{F}\left(v|n,x\right) \in \left[\tau_1 - \varepsilon, \tau_2 + \varepsilon\right] \tag{56}$$

with probability approaching one. Therefore, using the same argument as in part (g), equation (53), it is sufficient to consider only $v \in \hat{\Lambda}\left(x\right)$ such that $\hat{F}\left(v|n,x\right) \in \left[\tau_1 - \varepsilon, \tau_2 + \varepsilon\right]$. Since by Assumption 1(f), $Q\left(\cdot|x\right)$ is continuously differentiable on $\left[\tau_1 - \varepsilon, \tau_2 + \varepsilon\right]$, for such $v$'s by the mean-value theorem we have that,

$$\begin{aligned} Q\left(\hat{F}\left(v|n,x\right)|x\right) - v &= Q\left(\hat{F}\left(v|n,x\right)|x\right) - Q\left(F\left(v|x\right)\right) \\ &= \frac{1}{f\left(Q\left(\tilde{\tau}\left(v,n,x\right)|n,x\right)|x\right)}\left(\hat{F}\left(v|n,x\right) - F\left(v|x\right)\right), \end{aligned} \tag{57}$$

where $\tilde{\tau}\left(v,n,x\right)$ is between $\hat{F}\left(v|n,x\right)$ and $F\left(v|x\right)$.

By Lemma 21.1(iv) of van der Vaart (1998), $\hat{Q}\left(\hat{F}\left(v|n,x\right)|n,x\right) \leq v$, and equality can fail only at the points of discontinuity of $\hat{Q}$. Hence,

$$\sup_{v\in\hat{\Lambda}(x)}\left(v - \hat{Q}\left(\hat{F}\left(v|n,x\right)|n,x\right)\right) \leq \sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\left(\lim_{t\downarrow\tau}\hat{Q}\left(t|n,x\right) - \hat{Q}\left(\tau|n,x\right)\right)$$

$$+O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right); \tag{58}$$

however,

$$\begin{aligned}
&\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\left(\lim_{t\downarrow\tau}\hat{Q}\left(t|n,x\right)-\hat{Q}\left(\tau|n,x\right)\right)\\
\leq &\left(1+\frac{\sup_{b\in[b_1(n,x),b_2(n,x)]}\left|\hat{g}^{(1)}\left(b|n,x\right)\right|}{\hat{g}^2\left(\hat{q}\left(\tau|n,x\right)|n,x\right)}\right)\sup_{\tau\in[0,1]}\left(\lim_{t\downarrow\tau}\hat{q}\left(t|n,x\right)-\hat{q}\left(\tau|n,x\right)\right)\\
= &\ O_p\left(\left(\frac{Lh^d}{\log(Lh^d)}\right)^{-1}\right),
\end{aligned} \tag{59}$$

where the inequality follows from the definition of $\hat{Q}$ and by continuity of $K$, and the equality (59) follows from part (e) of the lemma. Note that, as shown in the proof of part (g), $\hat{g}\left(\hat{q}\left(\tau|n,x\right)|n,x\right)$ is bounded away from zero with probability approaching one. Combining (57)-(59), and by Assumption 1(e) we obtain that there exists a constant $c>0$ such that $\sup_{v\in\hat{\Lambda}(x)}\left|\hat{F}\left(v|n,x\right)-F\left(v|x\right)\right|$ is bounded by

$$\begin{aligned}
&c\sup_{v\in\hat{\Lambda}(x)}\left|Q\left(\hat{F}\left(v|n,x\right)|x\right)-\hat{Q}\left(\hat{F}\left(v|n,x\right)|n,x\right)\right|+O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right)\\
\leq &\ c\sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\left|Q\left(\tau|x\right)-\hat{Q}\left(\tau|n,x\right)\right|+O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right)\\
= &\ O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right),
\end{aligned}$$

where the equality follows from part (g) of the lemma. ■

**Proof of Theorem 1.** Let $E_L(n,x)$ be as defined in (20). By Lemma 1(d),(f) and (h), $P(E_L(n,x))\to 1$ as $L\to\infty$ for all $n\in\mathcal{N}$, $x\in Interior(\mathcal{X})$, and therefore using the same argument as in the proof of Lemma 1(g) equation (53), it is sufficient to consider only $v$'s from $E_L(n,x)$. Next,

$$\begin{aligned}
&\left|\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)|n,x\right)-g^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)\right|\\
\leq &\sup_{b\in[b_1(n,x),b_2(n,x)]}\left|\hat{g}^{(1)}\left(b|n,x\right)-g^{(1)}\left(b|n,x\right)\right|
\end{aligned}$$

$$+g^{(2)}\left(\widetilde{q}\left(v,n,x\right)\right)\left|\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)-q\left(F\left(v|x\right)|n,x\right)\right|. \tag{60}$$

where $\widetilde{q}$ is the mean value between $\hat{q}$ and $q$. Further, $g^{(2)}$ is bounded by Assumption 1(e) and Proposition 1(iv) of GPV, and

$$\begin{aligned}&\left|\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)-q\left(F\left(v|x\right)|n,x\right)\right|\\ \leq\ & \sup_{\tau\in[\tau_1-\varepsilon,\tau_2+\varepsilon]}\left|\hat{q}\left(\tau|n,x\right)-q\left(\tau|n,x\right)\right|+\frac{1}{c_g}\sup_{v\in\hat{\Lambda}(x)}\left|\hat{F}\left(v|n,x\right)-F\left(v|x\right)\right|, \end{aligned}\tag{61}$$

where $c_g$ as in the proof of Lemma 1(d). By (60), (61) and Lemma 1(d),(f),(h),

$$\begin{aligned}&\sup_{v\in\hat{\Lambda}(x)}\left|\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)|n,x\right)-g^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)\right|\\ =\ & O_p\left(\left(\frac{Lh^{d+3}}{\log L}\right)^{-1/2}+h^R\right).\end{aligned}\tag{62}$$

By a similar argument,

$$\begin{aligned}&\hat{f}\left(v|n,x\right)-f\left(v|n,x\right)\\ =\ & \frac{F\left(v|x\right)\widetilde{f}^2\left(v|n,x\right)}{\left(n-1\right)g^3\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)}\\ &\times\left|\hat{g}^{(1)}\left(\hat{q}\left(\hat{F}\left(v|n,x\right)|n,x\right)|n,x\right)-g^{(1)}\left(q\left(F\left(v|x\right)|n,x\right)|n,x\right)\right|\\ &+O_p\left(\left(\frac{Lh^{d+1}}{\log L}\right)^{-1/2}+h^R\right),\end{aligned}\tag{63}$$

uniformly in $v\in\hat{\Lambda}\left(x\right)$, where $\widetilde{f}\left(v|x\right)$ as in (15) but with some mean value $\widetilde{g}^{(1)}$ between $g^{(1)}$ and its estimator $\hat{g}^{(1)}$. The desired result follows from (16), (62), and (63). ■

**Proof of Lemma 2.** Consider $g_0^{(k)}(b,n,x)$ and $\hat{g}_0^{(k)}(b,n,x)$ defined in (50) and (51) respectively. It follows from parts (a) and (b) of Lemma 1,

$$\begin{aligned}&\left(Lh^{d+1+2k}\right)^{1/2}\left(\hat{g}^{(k)}\left(b|n,x\right)-g^{(k)}\left(b|n,x\right)\right)\\ &\qquad=\frac{1}{\pi\left(n|x\right)\varphi\left(x\right)}\left(Lh^{d+1+2k}\right)^{1/2}\left(\hat{g}_0^{(k)}\left(b,n,x\right)-g_0^{(k)}\left(b,n,x\right)\right)+o_p(1).\end{aligned}\tag{64}$$

By the same argument as in the proof of part (f) of Lemma 1 and Lemma B2 of Newey

(1994), $E\hat{g}_0^{(k)}(b,n,x) - g_0^{(k)}(b,n,x) = O\left(h^R\right)$ uniformly in $b \in [b_1(n,x), b_2(n,x)]$ for all $x \in Interior(\mathcal{X})$ and $n \in \mathcal{N}$. Then, by Assumption 3, it remains to establish asymptotic normality of

$$\left(nLh^{d+1+2k}\right)^{1/2}\left(\hat{g}_0^{(k)}(b,n,x) - E\hat{g}_0^{(k)}(b,n,x)\right).$$

Define

$$\begin{aligned} w_{il,n} &= h^{(d+1+2k)/2} 1(n_l = n) K_h^{(k)}(b_{il} - b) K_{*h}(x_l - x), \\ \overline{w}_{L,n} &= (nL)^{-1} \sum_{l=1}^{L} \sum_{i=l}^{n_l} w_{il,n}, \end{aligned}$$

so that

$$\left(nLh^{d+1+2k}\right)^{1/2}\left(\hat{g}_0^{(k)}(b,n,x) - E\hat{g}_0^{(k)}(b,n,x)\right) = (nL)^{1/2}\left(\overline{w}_{L,n} - E\overline{w}_{L,n}\right). \qquad (65)$$

By the Liapunov CLT (see, for example, Corollary 11.2.1 on page 427 of Lehmann and Romano (2005)),

$$(nL)^{1/2}\left(\overline{w}_{L,n} - E\overline{w}_{L,n}\right) / \left(nLVar\left(\overline{w}_{L,n}\right)\right)^{1/2} \to_d N(0,1), \qquad (66)$$

provided that $Ew_{il,n}^2 < \infty$, and for some $\delta > 0$,

$$\lim_{L\to\infty} \frac{1}{L^{\delta/2}} E\left|w_{il,n}\right|^{2+\delta} = 0. \qquad (67)$$

The condition in (67) follows from the Liapunov's condition (equation (11.12) on page 427 of Lehmann and Romano (2005)) and because $w_{il,n}$ are i.i.d. Next, $Ew_{il,n}$ is given by

$$\begin{aligned} & h^{(d+1+2k)/2} E\left(\pi(n|x_l) \int K_h^{(k)}(u-b)\, g(u|n,x_l)\, du K_{*h}(x_l - x)\right) \\ = {} & h^{(d+1+2k)/2} \int \pi(n|y) K_{*h}(y-x) \varphi(y) \int K_h^{(k)}(u-b)\, g(u|n,y)\, dudy \\ = {} & h^{(d+1)/2} \int \pi(n|hy+x) K_d(y) \varphi(hy+x) \\ & \times \int K^{(k)}(u)\, g(hu+b|n,hy+x)\, dudy \end{aligned}$$

$$\rightarrow \quad 0.$$

Further, $Ew_{il,n}^2$ is given by

$$\begin{aligned}
& h^{d+1+2k} \int \pi(n|y) K_{*h}^2 (y-x) \varphi(y) \int \left(K_h^{(k)}(u-b)\right)^2 g(u|n,y)\, dudy \\
= & \int \pi(n|hy+x) K_d^2(y) \varphi(hy+x) \\
& \times \int \left(K^{(k)}(u)\right)^2 g(hu+b|n,hy+x)\, dudy.
\end{aligned}$$

Hence, $nLVar(\overline{w}_{L,n})$ converges to

$$\pi(n|x) g(b|n,x) \varphi(x) \left(\int K^2(u)\, du\right)^d \int \left(K^{(k)}(u)\right)^2 du. \tag{68}$$

Lastly, $E|w_{il,n}|^{2+\delta}$ is given by

$$\begin{aligned}
& h^{(d+1+2k)(1+\delta/2)} \\
& \times \int \pi(n|y) |K_{*h}(y-x)|^{2+\delta} \varphi(y) \int \left|K_h^{(k)}(u-b)\right|^{2+\delta} g(u|n,y)\, dudy \\
= & h^{-(d+1)\delta/2} \int \pi(n|hy+x) |K_d(y)|^{2+\delta} \varphi(hy+x) \\
& \times \int \left|K^{(k)}(u)\right|^{2+\delta} g(hu+b|n,hy+x)\, dudy \\
\leq & h^{-(d+1)\delta/2} c_g \sup_{u\in[-1,1]} |K(u)|^{d(2+\delta)} \sup_{x\in\mathcal{X}} \varphi(x) \sup_{u\in[-1,1]} \left|K^{(k)}(u)\right|^{2+\delta},
\end{aligned} \tag{69}$$

where $c_g$ as in the proof of Lemma 1(d). The condition (67) is satisfied by Assumptions 1(b) and 3, and (69). It follows now from (64)-(69),

$$\begin{aligned}
& \left(nLh^{d+3}\right)^{1/2} \left(\hat{g}^{(k)}(b|n,x) - g^{(k)}(b|n,x)\right) \\
& \qquad \rightarrow_d N\left(0, \frac{g(b|n,x)}{\pi(n|x)\varphi(x)} \left(\int K^2(u)\, du\right)^d \int \left(K^{(k)}(u)\right)^2 du\right).
\end{aligned}$$

To prove part (b), note that the asymptotic covariance of $\overline{w}_{L,n_1}$ and $\overline{w}_{L,n_2}$ involves the product of two indicator functions, $1(n_l = n_1) 1(n_l = n_2)$, which is zero for $n_1 \neq n_2$. The joint asymptotic normality and asymptotic independence of $\hat{g}^{(k)}(b|n_1,x)$ and

$\hat{g}^{(k)}(b|n_2, x)$ follows then by the Cramér-Wold device. ■

**Proof of Theorem 2.** Let $E_L(n,x)$ be as defined in (20). For all $z \in R$,

$$P\left(\left(Lh^{d+3}\right)^{1/2}\left(\hat{f}(v|n,x) - f(v|x)\right) \leq z\right) =$$
$$= P\left(\left(Lh^{d+3}\right)^{1/2}\left(\hat{f}(v|n,x) - f(v|x)\right) \leq z, E_L(n,x)\right) + R_n,$$

where $0 \leq R_n \leq P(E_L^c(n,x)) = o(1)$, by Lemma 1(d) and (56) in the proof of Lemma 1(h). Therefore, it suffices to consider only $v$'s from $E_L(n,x)$. For such $v$'s,

$$\begin{aligned} & \hat{g}^{(1)}\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right)|n,x\right) - g^{(1)}(q(F(v|x)|n,x)|n,x) \\ = \; & \hat{g}^{(1)}(q(F(v|x)|n,x)|n,x) - g^{(1)}(q(F(v|x)|n,x)|n,x) \\ & + \hat{g}^{(2)}(\widetilde{q}(v,n,x)|n,x)\left(\hat{q}\left(\hat{F}(v|n,x)|n,x\right) - q(F(v|x)|n,x)\right), \end{aligned} \tag{70}$$

where $\widetilde{q}$ is the mean value. It follows from Lemma 1(d) and (f) that the second summand on the right-hand side of the above equation is $o_p\left(\left(Lh^{d+3}\right)^{-1/2}\right)$. One arrives at (21), and the desired result follows immediately from (21), Theorem 1, and Lemma 2. ■

**Proof of Theorem 3.** We provide only an outline of the proof here. The detailed proof is found in the supplement Marmer and Shneyerov (2010). First, one can show that a bootstrap version of Lemma 1 holds, and from those results it can be shown that

$$\begin{aligned} \hat{f}^{\dagger}(v|x) - \hat{f}(v|x) = & \frac{F(v|x) f^2(v|n,x)}{(n-1) g^3(q(F(v|x)|n,x)|n,x)} \\ & \times \left(\hat{g}^{\dagger(1)}(q(F(v|x)|n,x)) - \hat{g}^{(1)}(q(F(v|x)|n,x))\right) + e_L^{\dagger}, \end{aligned} \tag{71}$$

where $\hat{g}^{\dagger(1)}(b|n,x)$ is the bootstrap analogue of $\hat{g}^{(1)}(b|n,x)$, and $e_L^{\dagger}$ is the reminder term satisfying $P^{\dagger}((Lh^{d+3})^{1/2}|e_L^{\dagger}| > \varepsilon) \to_p 0$ for all $\varepsilon > 0$. Let $\Phi$ denote the standard normal CDF. By Theorem 1 in Mammen (1992) and Lemma 2(a),

$$P^{\dagger}\left(\left(Lh^{d+3}\right)^{1/2}\left(\hat{g}^{\dagger(1)}(b|n,x) - \hat{g}^{(1)}(b|n,x)\right) \leq u\right) \to_p \Phi\left(\frac{u}{V_{g,1}^{1/2}(b,n,x)}\right), \tag{72}$$

where $V_{g,1}(b,n,x)$ is defined in Lemma 2(a). The desired result then follows from (71) and (72) by Pólya's Theorem (Shao and Tu, 1995, page 447). ■

# References


Athey, S., Haile, P. A., 2007. Nonparametric approaches to auctions. In: Heckman, J. J., Leamer, E. E. (Eds.), Handbook of Econometrics. Vol. 6, Part 1. Elsevier, Amsterdam, Ch. 60, pp. 3847–3965.

Deheuvels, P., 1984. Strong limit theorems for maximal spacings from a general univariate distribution. Annals of Probability 12 (4), 1181–1193.

Guerre, E., Perrigne, I., Vuong, Q., 2000. Optimal nonparametric estimation of first-price auctions. Econometrica 68 (3), 525–74.

Guerre, E., Perrigne, I., Vuong, Q., 2009. Nonparametric identification of risk aversion in first-price auctions under exclusion restrictions. Econometrica 77 (4), 1193–1227.

Haile, P. A., Hong, H., Shum, M., 2003. Nonparametric tests for common values in first-price sealed bid auctions, NBER Working Paper 10105.

Haile, P. A., Tamer, E., 2003. Inference with an incomplete model of english auctions. Journal of Political Economy 111 (1), 1–51.

Khasminskii, R. Z., 1979. A lower bound on the risks of nonparametric estimates of densities in the uniform metric. Theory of Probability and its Applications 23 (4), 794–798.

Krasnokutskaya, E., 2009. Identification and estimation in procurement auctions under unobserved auction heterogeneity. Review of Economic Studies forthcoming.

Lehmann, E. L., Romano, J. P., 2005. Testing Statistical Hypotheses, 3rd Edition. Springer, New York.

Li, Q., Racine, J., 2008. Nonparametric estimation of conditional cdf and quantile functions with mixed categorical and continuous data. Journal of Business and Economic Statistics 26 (4), 423–434.

Li, T., Perrigne, I., Vuong, Q., 2002. Structural estimation of the affiliated private value auction model. RAND Journal of Economics 33 (2), 171–193.

Li, T., Perrigne, I., Vuong, Q., 2003. Semiparametric estimation of the optimal reserve price in first-price auctions. Journal of Business & Economic Statistics 21 (1), 53–65.

List, J., Daniel, M., Michael, P., 2004. Inferring treatment status when treatment assignment is unknown: with an application to collusive bidding behavior in canadian softwood timber auctions, Working Paper, University of Chicago.

Mammen, E., 1992. Bootstrap, wild bootstrap, and asymptotic normality. Probability Theory and Related Fields 93 (4), 439–455.

Marmer, V., Shneyerov, A., 2010. Supplement to "Quantile-based nonparametric inference for first-price auctions", Working Paper, University of British Columbia.

Matzkin, R. L., 2003. Nonparametric estimation of nonadditive random functions. Econometrica 71 (5), 1339–1375.

Newey, W. K., 1994. Kernel estimation of partial means and a general variance estimator. Econometric Theory 10 (2), 233–253.

Paarsch, H. J., 1997. Deriving an estimate of the optimal reserve price: An application to British Columbian timber sales. Journal of Econometrics 78 (2), 333–357.

Pagan, A., Ullah, A., 1999. Nonparametric Econometrics. Cambridge University Press, New York.

Pollard, D., 1984. Convergence of Stochastic Processes. Springer-Verlag, New York.

Shao, J., Tu, D., 1995. The Jackknife and Bootstrap. Springer-Verlag, New York.

van der Vaart, A. W., 1998. Asymptotic Statistics. Cambridge University Press, Cambridge.

Table 1: The simulated bias and MSE of the quantile-based (QB) and GPV's estimators for different points of density estimations ($v$), numbers of bidders ($n$), and different values of the distribution parameter $\alpha$, for sample size $nL = 4200$

| | Bias | | MSE | |
|---|---|---|---|---|
| $v$ | QB | GPV | QB | GPV |
| | | $\alpha = 1/2, n = 3$ | | |
| 0.4 | -0.0302 | **-0.0110** | **0.0299** | 0.0572 |
| 0.5 | -0.0323 | **0.0030** | **0.0352** | 0.0770 |
| 0.6 | -0.0596 | **-0.0094** | **0.0393** | 0.0781 |
| | | $\alpha = 1/2, n = 5$ | | |
| 0.4 | -0.0142 | **-0.0053** | **0.0156** | 0.0195 |
| 0.5 | -0.0077 | **0.0035** | **0.0208** | 0.0261 |
| 0.6 | -0.0278 | **-0.0039** | **0.0211** | 0.0273 |
| | | $\alpha = 1, n = 3$ | | |
| 0.4 | -0.0063 | **0.0045** | **0.0194** | 0.0245 |
| 0.5 | **-0.0056** | 0.0147 | **0.0284** | 0.0371 |
| 0.6 | -0.0342 | **-0.0059** | **0.0402** | 0.0519 |
| | | $\alpha = 1, n = 5$ | | |
| 0.4 | -0.0017 | **0.0013** | 0.0087 | **0.0078** |
| 0.5 | **0.0026** | 0.0088 | 0.0124 | **0.0113** |
| 0.6 | -0.0138 | **-0.0035** | 0.0171 | **0.0156** |
| | | $\alpha = 2, n = 3$ | | |
| 0.4 | -0.0037 | **0.0028** | 0.0113 | **0.0106** |
| 0.5 | -0.0166 | **-0.0084** | 0.0194 | **0.0188** |
| 0.6 | -0.0137 | **0.0029** | 0.0310 | **0.0299** |
| | | $\alpha = 2, n = 5$ | | |
| 0.4 | **-0.0008** | 0.0014 | 0.0054 | **0.0040** |
| 0.5 | -0.0075 | **-0.0054** | 0.0080 | **0.0062** |
| 0.6 | -0.0041 | **0.0011** | 0.0127 | **0.0097** |