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**On the Usefulness of the Constrained Planning Problem
in a Model of Money**

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August 2006

Department of Economics
Working Papers Series

Ames, Iowa 50011

On the Usefulness of the Constrained Planning Problem in a Model of Money

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Revised: March 6, 2007

Abstract.   In this paper, we study a decentralized monetary economy with a specified set of markets, rules of trade, an equilibrium concept, and a restricted set of policies and derive a set of equilibrium (monetary) allocations generated by these policies. Next we set up a simpler constrained planning problem in which we restrict the planner to choose from a set that contains the set of equilibrium allocations in the decentralized economy. *If* there is a government policy that allows the decentralized economy to achieve the constrained planner's allocation, then it is the optimal policy choice. To illustrate the power of such analyses, we solve such planning problems in three monetary environments with limited communication. The upshot is that solving constrained planning problems is potentially an extremely "efficient" (easy and quick) way of deriving optimal policies for the corresponding decentralized economies.

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1. Introduction

By their very nature, social planners and the problems they face are fictions. A planner takes as given the primitives of the economy (preferences, technology etc. which together constitute the "physical environment") and tries to maximize agents' welfare subject to the physical environment of the economy. Solutions to these "planning problems" serve as useful benchmarks for comparison with allocations in the decentralized economy; they represent the "best" allocation(s) (often called the "unconstrained first best") that the decentralized economy can hope to achieve via the market or some other mechanism. In this sense, planning problems are fictions created by researchers to facilitate understanding of the importance of the various constraints and frictions in the decentralized economy.

An *alternative* problem of interest would be to take as given a decentralized economy with a specified set of markets, rules of trade, an equilibrium concept, and a policy regime (a restricted set of policies). Such a specification of markets and rules of the game generates a set of equilibrium allocations that are attainable by a policymaker in the decentralized economy from within that policy regime. One can then set up a *much simpler* pseudo-planning problem in which the planner is restricted to choose from a set that contains (and may potentially be larger than) the aforementioned set of equilibrium allocations in the decentralized economy. Call the solution to this constrained planning problem the constrained planning solution. Among policies in the policy regime, *if* there is one that allows the decentralized economy to achieve the same allocation as under the constrained planning solution, it is the optimal policy. The main point of the paper is to demonstrate how the construct of the constrained planning problem may often be a quick, convenient, and insightful way to derive the optimal policy choices for the decentralized economy especially when their direct computation is cumbersome and complicated.

We make our main point in the context of constrained planning problems in monetary environments. Such economies are somewhat special. In a micro-founded monetary economy, there is always a deep friction that motivates the use of money in that environment. If that friction is removed, the economy becomes a non-monetary economy. Since money is a vehicle of market exchange and planners are not interested in exchange, they have no use for money. Hence, solutions to planning problems for monetary economies cannot involve the use of money. In other words, the planning problem corresponding to a decentralized

monetary economy is always *non*-monetary. However, as we demonstrate below, this does not mean that (constrained) planning problems for monetary economies are meaningless fictions.

We present our arguments within the context of a random-relocation overlapping-generations economy with limited communication as the primitive friction.[1] For ease of exposition, consider a baseline two-period lived pure-exchange overlapping generations model in the tradition of Townsend (1987) and Schreft and Smith (1998) where limited communication and stochastic relocation create an endogenous transactions role for fiat money.[2] At the end of each period a deterministic fraction of agents is relocated (the "movers") to a location different from the one they were born in and the only asset they can use to "communicate" with their past is fiat money. This allows money to be held even when dominated in rate of return. The other asset is a commonly accessible linear storage technology with a fixed real return. The "stochastic relocations" act like shocks to agents' portfolio preferences and, in particular, trigger liquidations of some assets at potential losses. They motivate a role for banks that take deposits, hold cash reserves, and make other less liquid investments. The assumption of limited communication disallows banks in location A from contacting and "communicating" with clients *once* they have relocated to location B. As such, if movers arriving on an island (different from their birth island) are to consume, they have to carry cash with them, cash that they have received from the bank on their birth island. The implication is that banks cannot pay movers out of stored goods because they are prohibited from contacting their clientele *after* they have relocated.

In Section 2 below, as a "warm up" exercise, we characterize the set of stationary feasible allocations available to the decentralized economy under a constant money growth policy regime. As is well known (for example, see Bhattacharya, Haslag, and Martin, 2005), the policy that maximizes the lifetime utility of a representative generation (the "golden rule") is a zero money growth policy. We go on to study a constrained planning problem,

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[1] While we choose this special monetary environment to make our points, the methodology suggested in the paper is clearly applicable in a wide range of environments.

[2] The random relocation with limited communication model was popularized by Champ, Smith, and Williamson (1997) and has been used to investigate monetary policy issues in Paal and Smith (2000), Smith (2002), Antinolfi, Huybens, and Keister (2001), Haslag and Martin (forthcoming), Antinolfi and Keister (2006), among others.

one that captures the spirit of the limited communication friction in the sense that the planner (like banks in the decentralized economy) is prevented from paying movers out of stored goods. When the planner is restricted to choose from within the set of equilibrium allocations in the decentralized economy, we find that the constrained planning allocation may be attained by holding the money stock fixed, confirming the optimality of the golden rule monetary policy described above.

To further illustrate the uses of the constrained planning problem, in Section 3 we study a variation of the limited communication economy in which the storage technology yields a stochastic return and the government introduces riskless bonds allegedly to provide better insurance. This is the economy studied in Schreft and Smith (2004) wherein they ask if there is any welfare rationale for a government to supply riskless bonds. By looking for the optimal policy regime in the decentralized economy directly, the authors find: "A positive stock of government debt is optimal only if interest payments on the debt are financed via money creation, agents are not too risk averse, there is a primary government budget deficit, and the economy is operating on the bad side of the Laffer curve. But under these conditions, welfare would be even higher if monetary policy were conducted to put the economy on the good side of the Laffer curve and there were no government bonds. Thus, there is little support for keeping a stock of interest-bearing, risk-free government debt outstanding." Using the construct of the constrained planning problem, we can *easily* show that, generically, there is no insurance role for bonds in such an economy, i.e., the optimal policy regime in the specified decentralized economy does not involve bonds.

As a final use, in Section 4, we employ the construct of the constrained planning problem to compute the optimal monetary policy in the random relocation model with limited communication and stochastic liquidity shocks. Smith (2002) and Antinolfi, Huybens, and Keister (2001), and Antinolfi and Keister (2006) consider settings where relocation shocks are realized after the bank has made its portfolio decisions and the probability of relocation is *itself* random. In such situations, "banking crises" may arise, i.e., if the realized value of the relocation shock is "too high", the bank may run out of all its cash reserves and even be forced to prematurely liquidate storage. Smith (2002) studies the question of optimal monetary policy (restricting attention to stationary constant money growth policies) in such a world. Given the complexity of the problem, he can make limited progress toward

that goal and cannot characterize the optimal policy beyond specifying a range in which it will lie, even under specific assumptions on preferences. By studying the fiction of the constrained planning problem for this economy, we show that one can readily get at a quick definite answer to the question of optimal monetary policy.

2. The random relocation model with limited communication

2.1. Preliminaries. We start by studying an economy consisting of an infinite sequence of two period lived overlapping generations. Time t is discrete and $t = -\infty, ..0, 1, 2, ...\infty$. At each date t, young agents are symmetrically assigned to one of two locations, island A and island B. Each island contains a continuum of young agents with unit mass, and our assumptions will imply that these locations are always symmetric. There is a single good that may be consumed or stored. Each agent is endowed with $y > 0$ units of this good at date t when young and nothing when old.

Agents born at t consume on date $t + 1$; we let c_t denote the consumption of the final good by a representative old agent born at t. All such agents have preferences representable by the utility function $u(c)$ where u is twice-continuously differentiable, strictly increasing, and strictly concave in its arguments. At points below, we will specialize to $u(c) = \left[c^{1-\rho} - 1\right]/(1 - \rho)$, with $\rho > 0$.

The assets available to the agents are storage and fiat currency (money). If $\kappa > 0$ units of the good are placed in storage at any date $t \geq 1$, then $x\kappa$ units are recovered from storage at date $t + 1$, where $x > 1$. Stored goods cannot be transported across locations. [3]The quantity of money in circulation at the end of period $t \geq 1$, per young agent, is denoted M_t. Let $0 < p_t < \infty$ denote the price level at date t. Then the gross real rate of return on money between period t and $t + 1$ is given by $R_{mt} \equiv p_t/p_{t+1}$. We assume that money is a "bad" asset, i.e.,

$$R_{mt} \in (0, x) \ \forall t. \tag{1}$$

holds; eq. (1) will later be mapped into a restriction on the set of constant money growth policies.

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[3]In this part of the paper, we assume that storage cannot be scrapped. Later, in Section 4, we allow storage to be scrapped and transported across locations; if scrapped, the gross rate of return on storage is $r \leq 1$.

In addition to the store-of-value function of money, spatial separation and limited communication generate a transactions role of money as in Townsend (1987). As such, money can be valued even if it is dominated in return by storage. The details are outlined below and follow standard conventions set up in Schreft and Smith (1997).

2.2.   Random relocation and limited communication.   The time line of events is as follows. At birth, agents of an island receive their endowment y plus any transfers from the government (see below). Before the period is up, a θ fraction of the young agents is randomly-selected at each location and is informed that they will have to relocate to the other location. Relocation status is public information. Here we assume that θ is constant over time.

The assumption of limited communication implies that once an agent has physically relocated to an island different from the one she was born in, all communication between her and the agents/entities remaining at her birth island are shut off. In particular, a relocated agent cannot collect the return on any goods she had stored, or that have been stored on her behalf. Currency or the goods recovered from scrapped storage are the only assets that can be transported between locations and limited communication prevents the cross-location exchange of privately issued liabilities such as checks or bonds.

Under the circumstances, there are two strategies an agent can use to transfer income over time. First, she can save on her own, storing some goods and/or acquiring some fiat currency.[4] The problem is that if she is relocated then she must scrap her stored goods; if not relocated, then she is stuck holding fiat currency, a "bad" asset (more below on this). Alternatively, she can deposit her entire endowment in a bank on her birth island. Banks can be thought of as coalitions of ex ante identical young agents born at the same island. Such a bank pools the goods deposited by all the young agents and uses them to acquire a portfolio of stored goods and fiat currency. Banks, however, cannot communicate across locations via establishment of branches. If an agent gets relocated, then she gets a return on her deposit in the same period that takes the form of a fiat currency payment

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[4]In this event, the optimal choice of storage, s, is given by

$$\frac{u'\left(R\ s\right)}{u'\left(R_m\ y + (r - R_m)\ s\right)} \frac{R}{R_m - r} \underset{\geq}{\leq} \frac{\theta}{1 - \theta}, \quad \begin{array}{l} \text{`` < ''} \text{ if } s = 0 \\ \text{`` > ''} \text{ if } s = y \end{array}$$

where y is the agent's endowment.

(whose real value will depend on the following period's price level) funded by the bank's own current holdings of fiat currency. If she does not get relocated, then she gets a return on her deposit next period that is funded by (a) the goods the bank has stored for her and (b) the fiat currency (if any) left after disbursing to relocated agents. Since banks can pool individual risks, it can be checked that the latter strategy always dominates the former and, in equilibrium, all agents deposit their endowments at banks.

To reiterate, limited communication disallows banks on island A from contacting and "communicating" with clients *once* they have relocated to island B. As such, if movers arriving on an island (different from their birth island) are to consume, they have to carry cash with them, cash that they have received from the bank on their birth island.

2.3. The policy regime. The policy regime specifies a restricted set of policies available to the government.[5] For our current purposes, the government conducts monetary policy by changing the nominal stock of fiat currency at a fixed gross rate $\mu > 0$ per period, so that $M_t = \mu\, M_{t-1}$ for all t. If the net money growth rate is positive then the government uses the additional currency it issues to purchase goods, which it gives to current young agents (at the start of a period) in the form of lump-sum transfers. If the net money growth rate is negative, then the government collects lump-sum taxes from the current young agents, which it uses to retire some of the currency. The tax $(-)$ or transfer $(+)$ is denoted τ_t. The budget constraint of the government is

$$\tau_t = \frac{M_t - M_{t-1}}{p_t} = m_t - m_{t-1} R_{m,t-1}, \quad t \geq 1 \tag{2}$$

where $m_t \equiv M_t/p_t$ denote real money balances at date t.

2.4. The bank's problem. As discussed earlier, the asset holdings of young agents are assumed to be costlessly intermediated by banks. The banks hold portfolios of fiat currency and physical assets, which consist of stored goods. Every young agent deposits her after-tax/transfer income in the bank. The bank divides their deposits between stored goods s_t and real balances of fiat currency m_t, so that

$$y + \tau_t = m_t + s_t. \tag{3}$$

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[5] For example, we disallow the availability of a discount window as a policy tool.

The bank announces a return of d_t^m to each mover (one who gets relocated) and d_t^n to each non-mover (one who stays on in the location she was born). These returns satisfy some constraints that are the direct consequence of the assumption of limited communication.

First, relocated agents, of whom there are θ, have to be given money to carry with them; so the bank has to use its own holdings of cash reserves to pay them. Since θ is known in advance, under assumption (1), it can be checked that banks will not want to hold cash reserves to pay non-movers. Define $\gamma_t \equiv \frac{m_t}{y+\tau_t}$ as the ratio of cash reserves to deposits. Then,

$$\theta d_t^m \leq \gamma_t R_{mt} \tag{4}$$

must hold, since money earns a return of $R_{mt} = \frac{p_t}{p_{t+1}}$ between t and $t+1$ (which the bank takes as given). The promised return to the non-movers must also satisfy

$$(1-\theta)\, d_t^n \leq (1-\gamma_t)\, x. \tag{5}$$

In effect then, the bank pays non-movers out of its cache of stored goods, while it pays cash to its own movers that is acquired by selling goods to the movers arriving from the other island. Limited communication, therefore, prevents the bank from implementing the following strategy: store all deposits and pay both movers and non-movers the following period from the return on that storage.

As the bank is managed by ex ante identical young agents born at date t, it chooses return schedules and portfolio allocations so as to maximize the expected utility of a representative depositor, subject to the equality versions of the constraints we have described. It follows from (4) and (5) that

$$c_t^m = \frac{\gamma_t}{\theta} R_{mt}\, (y+\tau_t), \tag{6a}$$

$$c_t^n = \frac{1-\gamma_t}{1-\theta} x\, (y+\tau_t), \tag{6b}$$

where c_t^m and c_t^m denote the old-age consumption of each mover and each non-mover born on date t. Then the bank's problem can be formally written as

$$\max_{\gamma_t \in [0,1]} \left\{ \theta u\left(\frac{\gamma_t}{\theta} R_{mt}\, (y+\tau_t)\right) + (1-\theta)\, u\left(\frac{1-\gamma_t}{1-\theta} x\, (y+\tau_t)\right) \right\}. \tag{7}$$

The first order conditions for this problem are given by

$$R_{mt} \cdot u'(c_t^m) = x \cdot u'(c_t^n) \text{ for } \gamma_t > 0. \tag{8}$$

The bank equates the marginal rate of substitution between the two types to the marginal ratios of opportunity costs for their respective assets. From (1), it follows that since $x > R_{mt}$, $c_t^n > c_t^m$ must hold. The non-movers should get higher consumption because they can get paid out of goods stored at a return higher than that of money.

It is then easily checked that the first order conditions to the problem in (7) for $u(c) = [c^{1-\rho} - 1]/(1-\rho)$ is given by

$$R_{mt} \left(\frac{\gamma_t}{\theta} R_{mt}\right)^{-\rho} = x \left(\frac{1 - \gamma_t}{1 - \theta} x\right)^{-\rho}, \tag{9}$$

the solution to which is given by

$$\gamma_t \equiv \gamma(R_{mt}) = \frac{\theta}{\theta + (1-\theta)(x/R_{mt})^{\frac{1-\rho}{\rho}}}. \tag{10}$$

In keeping with the literature, we consider only stationary equilibria and restrict attention to the set of constant money growth rate policies. Then, as the stationary rate of return on money equals μ^{-1}, (1) restricts the policy set to $\mu \in (x^{-1}, \infty)$.

Note that the choice problem is trivial for all money growth rates $\mu < x^{-1}$, corresponding to $R_m > x$. In such cases, money is not only superior in its rate of return but it also perfectly insures against relocation risk. Each agent will simply sell off her endowment for money and use it next period whether she stays on her home island or moves. Indeed, banks may not even exist under such money growth rates. The same holds true for $\mu = x^{-1}$.[6] In section 2.6 below, we show that $\mu \leq x^{-1}$ would never be part of an optimal monetary policy anyway.

Henceforth, in this section, we shall focus solely on steady states and eliminate the time subscripts.

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[6] If banks do exist for $\mu = x^{-1}$, given the equality of return on the two assets, their choice of cash reserves is indeterminate. They may choose just enough cash reservs to give to only movers or, at another extreme, reserve all of their deposits as cash. In the latter case, everyone will be paid in cash. Alternatively, as an intermediate case, banks may choose to pay non-movers by using both cash reserves and storage.

2.5. Welfare. Finally, steady state welfare (indirect utility as a function of R_m) for CRRA utility can be defined as

$$W\left(R_m\right) = \frac{\{y + \tau\left(R_m\right)\}^{1-\rho}}{1-\rho}\left\{\theta\left[\frac{\gamma\left(R_m\right)}{\theta}R_m\right]^{1-\rho} + \left(1-\theta\right)\left[\frac{1-\gamma\left(R_m\right)}{1-\theta}x\right]^{1-\rho}\right\} - \frac{1}{1-\rho},$$

which, using (9), can be rewritten as

$$W\left(R_m\right) = \frac{\{y + \tau\left(R_m\right)\}^{1-\rho}}{1-\rho}\left\{R_m\left(\frac{\gamma\left(R_m\right)}{\theta}R_m\right)^{-\rho}\right\} - \frac{1}{1-\rho} \tag{11}$$

where $\tau\left(R_m\right)$ and $\gamma\left(R_m\right)$ are obtained from (2) and (10), respectively. Since the gross money growth rate is set to μ, in a steady state, $R_m = 1/\mu$ and so (2) yields $\tau = \left(1 - \frac{1}{\mu}\right)m$. Since $\gamma\left(y + \tau\right) = m$, we have

$$y + \tau = \frac{y}{1 - \left(1 - \frac{1}{\mu}\right)\gamma}. \tag{12}$$

The level of equilibrium real balances (m) in the economy is given by $m = \gamma y / \left[1 - \left(1 - \frac{1}{\mu}\right)\gamma\right]$.Since $I = x\mu$, we can rewrite (10) in steady states as

$$\gamma = \frac{\theta}{\theta + \left(1-\theta\right)\left(x\mu\right)^{\frac{1-\rho}{\rho}}}. \tag{13}$$

The problem of choosing the steady state welfare maximizing money growth rate (henceforth the "golden rule" money growth rate) reduces to

$$\max_{\mu \in \left(x^{-1}, \infty\right)} W\left(\mu\right) \equiv \max_{\mu}\left\{\frac{\left[y + \tau\left(\mu\right)\right]^{1-\rho}}{1-\rho}\left(\frac{1}{\mu}\right)^{1-\rho}\frac{\left(\gamma\left(\mu\right)\right)^{-\rho}}{\theta^{-\rho}} - \frac{1}{1-\rho}\right\}, \tag{14}$$

where $y + \tau\left(\mu\right)$ and $\gamma\left(\mu\right)$ are given by (12) and (13) and (1) restricts $\mu \in \left(x^{-1}, \infty\right)$. We close this section with a fairly well-known result about the golden rule monetary policy in this environment, the proof of which may be found in Bhattacharya, Haslag, and Russell (2005).

Proposition 1. *The golden rule policy is to hold the money stock fixed, i.e., set $\mu = 1$.*

2.6. Constrained planning problem. Continue to focus attention on steady states and consider a psuedo-planning problem, that of a planner who faces the limited communication constraint, at least "in spirit". For emphasis, we reiterate that in the decentralized economy, the movers purchase consumption with money. In their new island, the banks managed by the previous generation have already disbursed goods stored from the previous period to the old non-movers. The banks managed by the current generation use the current deposits to buy the money off the old movers from the other island. In sum, the movers consume a portion of the current deposits and therefore such portion can not be stored.

In the light of the above discussion, the constrained planning problem is formulated as follows. In each period, the planner receives an endowment of y goods in each island. At each island, he is constrained to use a portion of the current endowment to pay the newly arrived people from the other island (those who were born in the previous period and got relocated; there are θ of them at each island). The rest he can store for distributing next period among non-movers born in the current period. The same sequence repeats the next period.

It is important to note here that even though the planner in any period is trading off consumption of the old movers of the *previous* generation against the consumption of future non-movers of the *current* generation (who will consume only in the following period), assuming that the planner cares equally for each generation, this is equivalent to a consumption allocation problem for any single generation.

Formally, let $\tilde{c}^m$ and $\tilde{c}^n$ denote the old-age consumption allocated by the constrained planner to movers and non-movers respectively. Hence, the planner's problem can be written as:

$$\max_{\tilde{c}^m, \tilde{c}^n, s} \quad \theta \, u(\tilde{c}^m) + (1 - \theta) \, u(\tilde{c}^n)$$

subject to

$$\theta \, \tilde{c}^m = y - s \tag{15a}$$

$$(1 - \theta) \, \tilde{c}^n = x \cdot s \tag{15b}$$

Notice that eliminating s from the two constraints yields

$$\theta \, \tilde{c}^m + (1 - \theta) \frac{\tilde{c}^n}{x} = y. \tag{16}$$

In the decentralized economy, the limited communication constraint made it impossible for a bank to pay movers out of stored goods because the bank was prohibited from contacting their clientele after they had relocated to a different island. Limited communication essentially acted as a branch banking restriction. The constrained planning problem captures the essence of the limited communication constraint facing banks. In particular, it prevents the planner from storing all current goods and allocating the returns (equally among movers and non-movers) next period.[7]

From eq. (16), it follows that $\tilde{c}^m \in [0, y/\theta]$ and $\tilde{c}^n \in [0, xy/(1-\theta)]$. The set of feasible allocations available to the constrained planner is depicted by the triangle AOC in Figure 1. Of course, the planner chooses from allocations on the boundary AC.
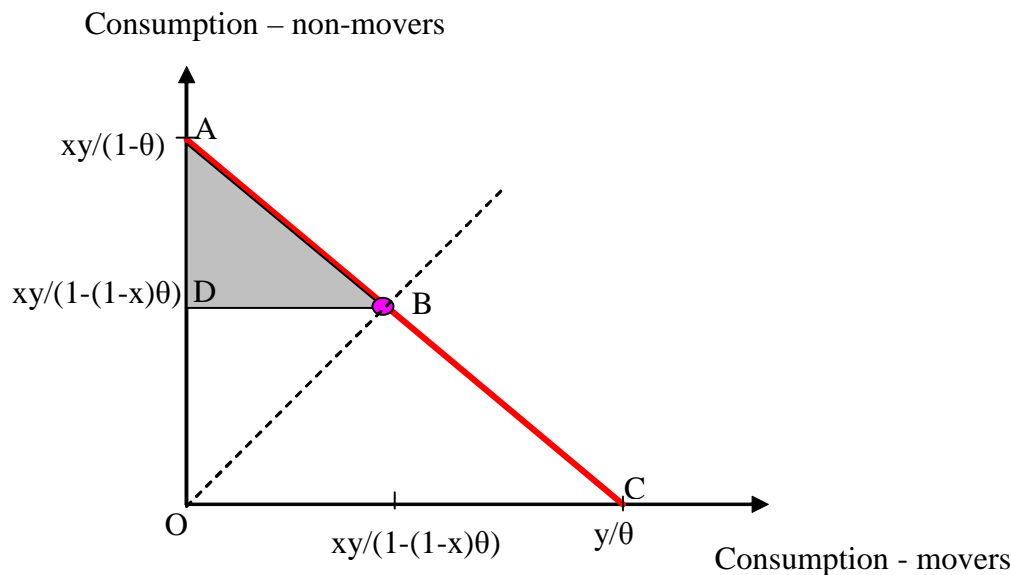

Figure 1: Feasible allocations

How does the planner's set of feasible allocations compare with the set of equilibrium consumption allocations in the decentralized economy? In the decentralized economy recall

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[7]It is important to point out that the constrained planning problem we construct is not the same as a Ramsey problem; the latter would be a problem where the planner maximizes welfare subject to the first order conditions in the decentralized economy, given by (8), and the overall resource constraints.

that $\mu \in \left(x^{-1}, \infty\right)$. In the simple case of *log utility*, it follows from (13) that $\gamma\left(\mu\right) = \theta$. Then, (6a) and (6b) with (12) obtain

$$c^m = \left(\frac{1}{\mu}\right) \frac{y}{1 - \left(1 - \frac{1}{\mu}\right)\theta}, \tag{17a}$$

$$c^n = (x) \frac{y}{1 - \left(1 - \frac{1}{\mu}\right)\theta}. \tag{17b}$$

It then follows that

$$\lim_{\mu \to x^{-1}} c^m = \lim_{\mu \to x^{-1}} c^n = \frac{xy}{1 - (1 - x)\theta}$$

$$\lim_{\mu \to \infty} c^m = 0 \text{ and } \lim_{\mu \to \infty} c^n = \frac{xy}{1 - \theta}$$

Thus allocations under the decentralized equilibrium are restricted to $c^m \in \left(0, \frac{xy}{1-(1-x)\theta}\right)$ and $c^n \in \left(\frac{xy}{1-(1-x)\theta}, \frac{xy}{1-\theta}\right)$. The set of feasible allocations available to the policymaker in the decentralized economy lie in the triangle ADB. The allocations under the decentralized equilibrium fall on the boundary segment AB in Figure 1.[8]

It is straightforward to show that $\mu \leq x^{-1}$ cannot be part of an optimal policy. As $\mu \to x^{-1}$ from the right, $c^n = c^m \to \frac{xy}{1-(1-x)\theta} = \frac{y}{\frac{1-\theta}{x} + \theta} > y$. On the other hand, if $\mu < x^{-1}$ holds, storage is ruled out and only money is used. In that case, goods market equilibrium implies $c^n = c^m = y$. For $\mu = x^{-1}$, if individuals directly made made their portfolio decisions, storage is again ruled out and $c^n = c^m = y$; if instead banks made portfolio decisions, their choice of cash reserves (or storage) is indeterminate. Then, it can be shown that $c^n = c^m \in \left[y, \frac{y}{\frac{1-\theta}{x} + \theta}\right]$. Thus, $\mu \leq x^{-1}$ is clearly sub-optimal.

Notice that the planner's feasible set of consumption allocations contains the equilibrium allocations under the decentralized equilibrium, for any given $\mu \in \left(x^{-1}, \infty\right)$. More specifically, the set of decentralized equilibrium allocations is strictly smaller than the planner's feasibility set because unlike the planner, the policy maker in the decentralized economy is restricted to choosing among equilibrium allocations with positive amounts of storage and money necessitating $\mu \in \left(x^{-1}, \infty\right)$. Since the planner maximizes welfare over a

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[8] Recall that both in the decentralized equilibrium and in the pseudo-planning problem, the total endowment of the young is either stored for the young non-movers or consumed by the old movers. Therefore, in Figure 1, the slope of the planner's feasibility frontier and the slope of the frontier of the decentralized allocations (that are attainable by the policy maker), by construction, are identical.

larger feasibility set, it is possible that her choice of allocations may not be decentralizable by any $\mu \in (x^{-1}, \infty)$.[9] On the other hand, if such μ exists then it must the optimal one, i.e., it must be the golden rule.

The first order conditions to the constrained planner's problem require

$$\frac{u'(\tilde{c}^m)}{u'(\tilde{c}^n)} = x. \tag{18}$$

From the standpoint of the planner, a unit of goods can either be set aside for the newly-relocated agents or it can be stored to yield x next period to be allocated to the non-movers; the value of either option in terms of the marginal utilities of their respective beneficiaries should be equalized at the margin. Comparing the efficiency condition [the stationary version of eq. (8)] under decentralized equilibrium with (18) leads to the optimal monetary policy choice of $R_m = 1$ or $\mu = 1$ (zero net money growth), the same as in Proposition 1. Since $\mu = 1$ lies in (x^{-1}, ∞), the constrained planning solution is decentralizable and is the optimal choice of μ for the policy maker in the decentralized economy. Thus even though the constrained planner's feasible set was larger than that in the decentralized economy (recall, the latter faced an additional constraint that $\mu \in (x^{-1}, \infty)$), the planner chose an allocation that lay inside the feasible set of the decentralized economy.

The optimality of $\mu = 1$ rule can also be seen by comparing the planner's constraints (15a) and (15b) with the bank's constraints (4) and (5) which can be rewritten as

$$\theta c^m \leq m R_m,$$
$$(1 - \theta) c^n = x \cdot (y + \tau - m)$$

where m is now the bank's choice variable. Notice that for $\mu = 1$, $R_m = 1$, and from (12) $\tau = 0$. Then the bank's maximization problem is isomorphic to that of the planner's.

3. RISKLESS BONDS AND INSURANCE IN A STOCHASTIC LIMITED COMMUNICATION ECONOMY

Constrained planning problems have other uses. As we demonstrate below, one can use these to quickly check if certain government interventions ever make sense from the point

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[9]In that case, one can potentially further restrict the constrained planner to choose from allocations that lie in the set of consumption allocations attainable only by $\mu \in (x^{-1}, \infty)$.

of view of a planner. As an example, consider a setting in which the government is contemplating introducing a publicly provided asset into a market economy. To evaluate the desirability of such an action, one can work out the equilibrium conditions in the decentralized economy, pre and post introduction of the asset. Using those, one can compute if equilibrium welfare is raised or lowered by the presence of the asset. In most cases, this becomes a daunting task, especially for general specifications of preferences and technology. As we show below, in some cases, it may be possible to work out a much simpler constrained planning problem and more easily verify if the introduction of the asset could ever be deemed part of an optimal policy by a planner.

In the specific example we study below, the government is considering whether to supply riskless bonds for the benefit of the agents. Schreft and Smith (2004) study a similar question in an economy that is a slight variation on the one explored in the previous section. Specifically, they consider a setting in which x is stochastic with mean $E(x) > 1$. Moreover, while the idiosyncratic relocation shock is realized within the period, the shock to x is realized at the end of the period, only after movers have relocated carrying with them the cash balances reserved for them. Schreft and Smith (2004) are interested in the potential welfare-enhancing role (via "superior access to insurance") played by the introduction of riskless government bonds in this economy.

We investigate this potential by studying the constrained planning problem. Our approach is to ask, suppose we allow the (constrained) planner of Section 2.6 to choose allocations that are made feasible by bonds in the decentralized economy, would he? If there is a true insurance role to be played by these riskless assets, the planner would want to choose those allocations, not otherwise.

The details of the decentralized economy are to be found in Schreft and Smith (2004); some relevant points are sketched below.[10]

3.1. The decentralized equilibrium. Let B_t $\left(b_t \equiv \frac{B_t}{p_t}\right)$ denote the nominal (real) bond holdings by a young agent at date t, I_{t-1} the nominal interest rate on bonds between $t - 1$ and t, and $R_{t-1} \equiv I_{t-1} R_{mt-1}$ be the gross real return on bonds between $t - 1$ and t. Then the government's budget constraint (2) is modified to include potential revenue

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[10] There are other superficial differences between the environment studied in Schreft and Smith (2004) and the one we present below. For example, they allow for a exogenous level of government expenditures that must be financed by currency and bond seigniorage.

collection from the sale of bonds:

$$\tau_t = \frac{M_t - M_{t-1}}{p_t} + \frac{B_t - I_{t-1}B_{t-1}}{p_t} = m_t - m_{t-1}R_{m,t-1} + b_t - R_{t-1}b_{t-1} \tag{19}$$

As bonds purchased on date t are redeemed on date $t+1$, their gross receipts can be given to non-movers only. Moreover, since both money and bonds yield *cash* payments, banks may potentially choose to hold money for the non-movers as well. With these modifications, the bank's constraints can now be written as

$$\theta d_t^m \leq \gamma_{mt}^m R_{mt} \tag{20a}$$

$$(1 - \theta) d_t^m \leq \gamma_{mt}^n R_{mt} + \gamma_{bt}^n R_t + (1 - \gamma_{bt}^n - \gamma_{mt}^n - \gamma_{mt}^m) x_t \tag{20b}$$

where γ_m^j, $j = m, n$ is the cash reserves to deposit ratio held by the bank to pay movers and non-movers respectively and γ_b^n is the bonds to deposit ratio held by the bank to pay non-movers. Notice that the movers continue to get relocated only with money and that remains the only asset they can use to finance consumption once relocated. Non-movers, however, can not only consume stored goods but also potentially finance consumption from gross returns on money and bonds. The bank's problem, analogous to (7) can now be written as

$$\max_{\{\gamma_{mt}^m, \gamma_{mt}^n, \gamma_{bt}^n\} \in [0,1]} \left\{ \begin{array}{l} \theta \, u \left(\frac{\gamma_{mt}^m}{\theta} R_{mt} \left(y + \tau_t \right) \right) + \\ (1 - \theta) \, E \left\{ u \left(\frac{\gamma_{mt}^n R_{mt} + \gamma_{bt}^n R_t + \left(1 - \gamma_{bt}^n - \gamma_{mt}^n - \gamma_{mt}^m \right) x_t}{1 - \theta} \left(y + \tau_t \right) \right) \right\} \end{array} \right\}.$$

The first order conditions to the bank's problem, assuming an interior holding of storage, are

$$u'(c_t^m) R_{mt} = E \left\{ u'(c_t^n) \, x_t \right\} \tag{21a}$$

$$E \left\{ u'(c_t^n) \, x_t \right\} \geq E \left\{ u'(c_t^n) \right\} \, R_{mt}, \text{ with equality if } \gamma_{mt}^n > 0 \tag{21b}$$

$$E \left\{ u'(c_t^n) \, x_t \right\} \geq E \left\{ u'(c_t^n) \right\} \, R_t, \text{ with equality if } \gamma_{bt}^n > 0 \tag{21c}$$

Equations (21b) and (21c) together state that (a) both money and bonds may be held to finance the consumption of the non-movers only if the rate of return on them are equal; otherwise, only the one with the higher return will be held and (b) if money and/or bonds are to be held for the non-movers, its marginal unit must yield the same value in expected terms as does a marginal unit of storage.

Using (21a)-(21c), it is potentially possible to enumerate conditions under which $\gamma_{mt}^n > 0$ and/or $\gamma_{bt}^n > 0$. Subsequent to that, one can compute optimal demands for money, bonds, and storage, clear markets to compute the equilibrium returns to money and bonds (assuming the government budget constraint held), and then compare indirect utility with and without bonds. While all this can be done, it is clear that such an exercise can be tedious and cumbersome. Below, we show how an appropriately constrained planning problem can much more simply generate the principal insight concerning desirability of riskless public debt in this environment.

3.2. A constrained planning problem. Before proceeding further, it is important to note that for a constant money growth rate and for a known distribution of storage return, the bank's *ex ante* portfolio choices of storage, bonds, and money holdings will be time-invariant. Why? Notice that the *ex post* realization of x does not affect the price level because stored goods are directly consumed by non-movers and therefore never enter the market where goods are exchanged for money. If banks hold the same amount of real balances every period, i.e., if money and prices grew at the same rate, the rate of return on money will be fixed at the inverse of money growth rate. With the nominal interest rate and the return on money fixed, real returns on bonds will be fixed as well; with a known distribution of storage returns, the bank's ex-ante portfolio choices of storage, bonds, and money holdings will also remain fixed every period.

It bears emphasis that the only interesting cases to examine in the decentralized equilibrium are the ones where the rate of return on money and bonds fall below the expected return to storage; i.e., $R_m = \mu^{-1} < E(x) > R$ obtains. Otherwise, for its risk-averse non-moving clientele, the bank will always prefer cash reserves over storage; the latter will then never be used in equilibrium. With $R_m < E\{x\}$, as can be seen from (21a) and (21b), non-movers expected consumption will at least equal the consumption of movers. Moreover, with strictly concave preferences, unless $\mu \to \infty$, movers will always receive a strictly positive consumption.

Premutliplying both sides of (20a) and (20b) by $y + \tau$ and using the steady state version of the government's budget constraint (19), $\tau = m(1 - R_m) + b(1 - R)$, the bank's

constraints in a stationary equilibrium are given by

$$\theta c^m = \underbrace{m^m R_m}_{y-h-s} \tag{22a}$$

$$(1-\theta)\, c^n = \underbrace{m^n R_m + bR}_{h} + \left(y - \underbrace{\left(\underbrace{m^n R_m + bR}_{h} \right) - m^m R_m}_{s} \right) x \tag{22b}$$

where m^m and m^n are money balances reserved for movers and non-movers respectively.

To summarize the above two constraints: (a) as argued in Section 2.6, the consumption of the movers is financed solely from the portion of the current deposit base that is not stored, and (b) only non-movers consume stored goods; additionally, if bonds and/or cash is held for them, they get additional consumption which is also financed from the portion of the current deposit base that is *not* stored. Denote s to be the amount that is stored by the bank, and h as the amount that is given to the *non-movers* from the return to money (reserved for non-movers, i.e., $R_m m^n$) and bonds, Rb. The non-movers are paid out of h and the return on stored goods, xs. Goods market equilibrium requires that the remaining endowment $y - h - s$ be given to movers from gross return on money reserved for them. Therefore, by the above definition, $m^m R_m = y - h - s$.

To replicate the features of the decentralized economy, consider the problem of a planner who faces a reformulated limited communication constraint as described below. In each period, the planner receives an endowment of y goods in island A. (The problem for island B is symmetric.) He uses a portion of this to pay the newly arrived people on the island (those who were born in the previous period on island B and got relocated; there are θ of them). To replicate the feature that banks in the decentralized economy buy riskless bonds and possibly hold cash reserves for the non-movers, we constrain the planner set aside h goods to represent the holdings of bonds and money for the non-movers (those who were born in the previous period on island A and have not moved).[11] The rest he stores for prospective non-movers (those who are born in island A in the current period and will not

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[11] To foreshadow, it is easy to see here that from the planner's perspective money and bonds are perfect substitutes – there is nothing that bonds can do for the non-movers that money can not.

move). Then the planner's constraints are identical to (22a) and (22b):

$$\theta \, \tilde{c}^m = y - s - h \tag{23a}$$

$$(1 - \theta) \, \tilde{c}^n = x \cdot s + h \tag{23b}$$

where $\tilde{c}^m$ and $\tilde{c}^n$ denote the old-age consumption allocated by the planner to movers and non-movers respectively. Clearly, the planner's choice set allows $\tilde{c}^m \in \left[0, \frac{y}{\theta}\right]$ and $\tilde{c}^n \in \left[0, \frac{xy}{1-\theta}\right]$. While $\tilde{c}^m = 0$ and $\tilde{c}^n = \frac{xy}{1-\theta}$ corresponds to $s = y$ and $h = 0$, $\tilde{c}^m = \frac{y}{\theta}$ and $\tilde{c}^n = 0$ corresponds to $s = 0$ and $h = 0$. Notice that the *ex post* consumption of non-movers is uncertain while that of the movers is known at the time when h and s are chosen.

While the planner faces the same set of constraints as in the decentralized equilibrium (compare (23a) and (23b) with (22a) and (22b)), the planner's feasible set of consumption allocations for each type is at least as large as what was feasible in the decentralized equilibrium. Thus, once again, the equilibrium allocations of the decentralized economy are contained within the planner's set of feasible allocations under the scheme described above.

The planner's problem is to set the rules described above that maximizes the utility of a representative generation. Formally,

$$\max_{\hat{c}^m, \hat{c}^n, s, h} \quad \theta \, u(\tilde{c}^m) + (1 - \theta) \, E\left(u(\tilde{c}^n)\right)$$

subject to (23a) and (23b). The first order conditions are given by

$$s \quad : \quad u'\left(\tilde{c}^m\right) = E\left\{u'\left(\tilde{c}^n\right) \, x\right\} \tag{24a}$$

$$h \quad : \quad u'\left(\tilde{c}^m\right) \geq E\left\{u'\left(\tilde{c}^n\right)\right\}, \text{ “} = \text{” if } h > 0. \tag{24b}$$

After substituting (23a) and (23b), it can be verified that the planner's welfare function is strictly concave in the choice variables h and s. To that end, note that as the concavity property is preserved under the expectations operator, it suffices to show that the preferences are concave in s and h for any x. The Hessian matrix of the objective function (with respect to s and h) is then given by

$$\begin{vmatrix} \frac{1}{\theta}u''\left(\tilde{c}^m\right) + \frac{1}{1-\theta}u''\left(\tilde{c}^n\right) \, x^2 & \frac{1}{\theta}u''\left(\tilde{c}^m\right) + \frac{1}{1-\theta}u''\left(\tilde{c}^n\right) \, x \\ \frac{1}{\theta}u''\left(\tilde{c}^m\right) + \frac{1}{1-\theta}u''\left(\tilde{c}^n\right) \, x & \frac{1}{\theta}u''\left(\tilde{c}^m\right) + \frac{1}{1-\theta}u''\left(\tilde{c}^n\right) \end{vmatrix}.$$

Clearly, all the elements are negative, and its determinant reduces to

$$\frac{1}{\theta\left(1-\theta\right)}u''\left(\tilde{c}^{m}\right)u''\left(\tilde{c}^{n}\right)\left(x-1\right)^{2}>0.$$

Thus, the matrix is negative definite. It follows that (24a) and (24b) along with (23a) and (3) obtain a unique solution for s and h.

Equation (24a) is the stochastic analog of (18): a unit of goods can either be set aside for the newly-relocated agents or it can be stored to yield x next period to be allocated to the non-movers; at the optimum its expected value should be equal across the two options. Then, comparing (24a) with (21a) yields $R_m = 1$, i.e., the constrained efficient outcome may be decentralized by fixing the money stock, just as before.

The second equation (24b) states that if there are any goods set aside for the non-movers from the current endowment, then the expected marginal utility of the movers and the non-movers must be the same.

Is there a second-best role for bonds in this economy?. It is obvious from the equivalent formulation of the planning problem that in this economy there is nothing that bonds can do for the non-movers that money can not do. Suppose bonds are held in equilibrium and $h > 0$, i.e., (24b) holds with equality. Then, (24a) and (24b) are consistent with (21a) - (21c) if and only if $R_t = R_m = 1$. In other words, the optimal policy in the decentralized equilibrium is to set gross rates of return on money as well as bonds equal to unity, and let the banks decide the amount of holdings.

In general, if storage is "sufficiently" desirable, banks will optimally choose not to hold bonds and money for the non-movers. A sufficient set of conditions on the distribution of x can be obtained from (24b) holding with inequality and where $\tilde{c}^m$, $\tilde{c}^n$, and s are obtained by setting $h = 0$ and solving (23a), (23b), and (24a) together.[12]

4. The random relocation model with limited communication and stochastic liquidity shocks

We close with a final illustration of the usefulness of the constrained planning problem. In this section, we will use the construct to compute the optimal monetary policy in

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[12] Conversely, sufficient restrictions on x can be derived to obtain scenarios where bonds and money is held for non-movers. This requires solving (23a), (23b), (24a), and (24b) holding with equality for $\tilde{c}^m$, $\tilde{c}^n$, s, and h.

the random relocation model with limited communication and stochastic liquidity shocks as studied by Smith (2002). In that paper, the optimal monetary policy is not fully characterized. In Propositions 4 and 5 of that paper, Smith argues against the Friedman rule as optimal policy and proves a range (away from zero but not too high) for the nominal interest rate to be optimal. As we demonstrate below, studying the fiction of the constrained planning problem for this economy readily produces a definite answer to the question of optimal monetary policy.

The economy is very similar to the one studied earlier and identical to the one studied in Smith (2002) and Antinolfi and Keister (2006).[13] The big change relative to the environments studied earlier is that here the fraction of population that is relocated (θ) is itself stochastic, i.i.d., and it is realized after banks have made their portfolio choices. As in Section 2, relocated agents do not remain in contact with their banks, and therefore withdraw their deposits prior to relocation. As before banks can give them currency. In addition, banks can scrap storage to pay to the movers that can be transported across locations. However, this comes at a loss: a unit of storage if scrapped obtains a return $r < 1$.

With uncertain aggregate relocations, banks face uncertain liquidity demand for cash from their depositors. When, the cash demand is sufficiently low, i.e., "too few" people are relocated, banks may choose not to disburse all cash to the movers and instead keep some of it for the non-movers. However, when the cash demand is sufficiently high, i.e., "too many" people are relocated, banks run out of its reserves in that currency, and in the sense of Smith (2002), a liquidity crunch occurs. In case banks also choose to scrap some storage for non-movers, the economy suffers a real loss and, as termed by Smith (2002), a "banking crisis" occurs. Insofar as the choice of the monetary policies determines the opportunity cost of liquidity, this affects the banks' reserve holdings, thereby influencing the probability of liquidity and banking crisis and the welfare of agents.

The banks make portfolio allocations between cash balances and storage as given by the per depositor resource constraint:

$$y + \tau_t \geq m_t + s_t. \tag{25}$$

Banks, at date t, upon receiving deposits decide what the cash reserve to deposit ratio,

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[13] We invite the reader to see Smith (2002) for details.

$\gamma_t \equiv \frac{m_t}{y+\tau_t}$, should be. After γ_t is chosen, θ is revealed. In this environment, banks announce a schedule of returns for the two types of agents that is contingent on θ. Let $d_t^m(\theta)$ and $d_t^n(\theta)$ denote the θ-contingent gross real returns promised to movers and non-movers, respectively. Further, let $\alpha_t(\theta)$ denote the θ-contingent fraction of cash reserves that is used to pay movers, then $(1 - \alpha_t(\theta))$ is the fraction that is disbursed to non-movers. Finally, let $\delta_t(\theta)$ denote the fraction of storage that may be scrapped to pay the movers.[14] The returns to the movers and non-movers must satisfy

$$\text{Movers}: \theta \, d_t^m(\theta) \leq \underbrace{\alpha_t(\theta) \, \gamma_t \, R_{mt}}_{\text{Money}} + \underbrace{\delta_t(\theta)(1-\gamma_t) \, r}_{\text{Scrapped storage}} \tag{26a}$$

$$\text{Non-movers}: (1-\theta) \, d_t^n(\theta) \leq \underbrace{(1-\alpha_t(\theta)) \, \gamma_t \, R_{mt}}_{\text{Money}} + \underbrace{(1-\delta(\theta))(1-\gamma_t) \, x}_{\text{Unscrapped storage}}. \tag{26b}$$

Consider the constraint (26a) first. Once θ is realized, the banks have to pay $d_t^m(\theta)$ to each mover. It uses up a fraction $\alpha_t(\theta)$ of its cash reserves. The bank may in addition scrap $\delta_t(\theta)$ fraction of storage for the movers. Thus, the first (second) term denotes movers' returns funded through cash reserves (scrapped storage). Non-movers' constraint (26b) follows similarly.

The banks' problem can be reformulated as

$$\max_{\gamma_t \in [0,1]} \quad E_\theta \left\{ \max_{\{\alpha_t(\theta) \in [0,1], \delta_t(\theta) \in [0,1]\}} (\theta \, u(c_t^m) + (1-\theta) \, u(c_t^n)) \right\} \tag{27}$$

subject to $c_t^i = d_t^i(\theta)(y+\tau_t)$ for $i = m$ (movers) and n (non-movers), (26a), and (26b). The operator $E_\theta\{.\}$ denotes expectation with respect to the distribution of θ.

It is convenient to conceptualize the bank's problem as a two stage problem and work backwards: 1) in the second stage, given γ_t, the banks choose $\alpha_t(.)$ and $\delta_t(.)$, and 2) given the θ-contingent functions α_t and δ_t, they choose γ_t so as to maximize (27). Given γ_t the second stage first-order-conditions are

$$u'(c_t^n) \, R_{mt} \quad \leq \quad u'(c_t^m) \, R_{mt} \quad \text{``} = \text{''} \quad \text{if } \alpha_t(\theta) < 1 \tag{28a}$$

$$u'(c_t^n) \, x \quad \leq \quad u'(c_t^m) \, r, \quad \text{``} = \text{''} \quad \text{if } \delta_t(\theta) > 0 \tag{28b}$$

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[14] The functions d^m, d^n, α, and δ are assumed to be stationary, which indeed is the case in a stationary equilibrium (verified below).

Given banks' first-stage choices, equations (28a) and (28b) along with (26a), (26b), $c_t^i = d_t^i(\theta)(y + \tau_t)$, and government budget constraint, $\tau_t = m_t - m_{t-1}R_{m,t-1}$ uniquely determine functions $\alpha_t(\theta)$ and $\delta_t(\theta)$.[15]

As these second-stage rules are optimally chosen, the first order condition with respect to γ_t can be obtained by invoking the Envelope Theorem:

$$E_\theta \left\{ \begin{array}{c} \alpha_t(\theta) \ u'(c_t^m) \ R_{mt} \ + \ (1 - \alpha_t(\theta)) \ u'(c_t^n) \ R_{mt} \\ - \ r \ \delta_t(\theta) \ u'(c_t^m) - (1 - \delta_t(\theta)) \ u'(c_t^n) \ x \end{array} \right\} = 0. \tag{29a}$$

Using (28a) and (28b) in (29a), one obtains the essential marginal condition:[16]

$$E_\theta \left\{ u'(c_t^m) \ R_{mt} \right\} = E_\theta \left\{ u'(c_t^n) \ x \right\} \tag{30}$$

It is worth noting that at the point the bank is confronted with its portfolio allocation problem, it does not know whether any cash will be given to non-movers and whether any portion of storage will need to be scrapped for the benefit of the movers. All that it knows for sure is that movers will have to be given cash and non-movers will get to consume some stored goods. In addition, the bank knows the states of the world in which cash will be given to non-movers as well as states in which storage will be scrapped to give to movers. In all such states, the banks is aware that the marginal value of the shared asset (be it money or storage) will have to be equalized across the two types.

Specifically, the bank knows that if cash is given to both non-movers and movers, it will be equalizing consumption across them, i.e., their *ex post* marginal utilities will have to be equalized. Therefore, the bank is aware that a unit of deposits reserved as cash will yield an additional utility of $u'(c_t^m) \frac{R_{mt}}{\theta}$ for a mass θ of movers if all of it went only to movers or $u'(c_t^m) R_{mt}$ for the whole unit mass of agents if it is given to movers as well as to non-movers. For example, suppose a fraction α of this marginal unit of cash goes to finance movers' consumption. The total value of this fraction is $u'(c_t^m) \frac{\alpha}{\theta} R_{mt}$ for θ movers and $u'(c_t^n) \frac{1-\alpha}{1-\theta} R_{mt}$ for $1 - \theta$ non-movers. Since $u'(c_t^n) = u'(c_t^m)$ in all such cases, this is equivalent to $u'(c_t^m) R_{mt}$ for the unit mass of agents.

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[15] For $u(c) \equiv \ln c$, (28a) and (28b) lead to equations (6) and (7) in Smith (2002).

[16] Equation (30) is equivalent to equation (10) in Smith (2002) who works with a logarithmic utility specification. See Propositions 2 and 3 of his paper.

Similarly, if *ex post* storage is distributed *only* to non-movers then *ex ante,* the marginal value of a unit of deposit stored will yield an additional utility of $u'(c_t^m) \frac{R_{mt}}{1-\theta}$ for $1-\theta$ non-movers. But if a fraction δ is scrapped to be given to movers, the utility yield of a marginal unit of storage is $u'(c_t^m) r \frac{\delta}{\theta}$ for θ movers and $u'(c_t^n) x \frac{1-\delta}{1-\theta}$ for $1-\theta$ non-movers. Since the bank ensures at the time of scrapping that $u'(c_t^m) r = u'(c_t^n) x$ holds, the marginal value of a unit of storage amounts to $u'(c_t^n) x$ for a unit mass of agents.

Finally, at the point at which it takes its portfolio allocation decision, it does not know the exact values of c_t^m and c_t^n but takes rational expectations with respect to the distribution of θ. Equation (30) states that the bank equates the expected utility value of a marginal unit of deposit reserved as cash to its expected utility value in storage.

Equations (30) along with (28a), (28b), (26a), (26b), $c_t^i = d_t^i(\theta)(y + \tau_t)$, and government budget constraint, $\tau_t = m_t - m_{t-1} R_{m,t-1}$ uniquely determines γ_t and θ-contingent functions $\alpha_t(\theta)$ and $\delta_t(\theta)$. Notice that the banks ex-ante face an identical problem each period. In a stationary equilibrium, given a time-invariant monetary policy, their portfolio allocation rule γ and functions $\alpha(.)$ and $\delta(.)$ are also time invariant. What is an optimal monetary policy in this environment? Within the set of constant money growth rules, Smith (2002) answers that the optimal money growth rule yields a gross return on money $R_m \in (r, x)$ (See his propositions 4 and 5.).

As we demonstrate below, by studying a constrained planning problem, it is easy to show that the optimal gross money growth rate in this environment is unity which yields a *precise* policy prescription of $R_m = 1$.

4.1. An equivalent planning problem. Before proceeding further, let us review the set of steady state allocations in the decentralized equilibrium discussed above. Premutliplying both sides of (26a) and (26b) by $y + \tau$ and using the steady state government budget constraint $\tau = m(1 - R_m)$, the steady state set of allocations are constrained by

$$\theta \ c^m = \alpha(\theta) \underbrace{m \ R_m}_{y-s} + \delta(\theta) \underbrace{(y - m \ R_m)}_{s} r, \tag{31a}$$

$$(1-\theta) \ c^n = (1-\alpha(\theta)) \underbrace{m \ R_m}_{y-s} + (1-\delta(\theta)) \underbrace{(y - m \ R_m)}_{s} x. \tag{31b}$$

where $\alpha(\theta)$ and $\delta(\theta)$ are, as before, the bank's θ-contingent liquidation rules. Notice that mR_m is the amount of goods that money buys out of the current generation's endowment;

as the endowment is fixed at y, goods market equilibrium requires that the remaining $s \equiv y - m \, R_m$ is stored. Since real balances are chosen by banks and are a function of R_m, it follows that what does not get stored, $m \, R_m$, and conversely what gets stored, $y - m \, R_m$, in any period can be uniquely determined by picking a stationary money growth rate.

To replicate the features of the steady state decentralized equilibrium, consider a psuedo-planning problem, where the limited communication constraint of the decentralized economy is implemented in the following manner. The planner is required to *ex ante* choose: (i) a fixed amount s of endowment to be stored and the rest $y - s$ left unstored every period; (ii) a θ-contingent rule $\tilde{\alpha}(\theta)$ that distributes unstored goods between old movers (numbering θ) and non-movers (numbering $1 - \theta$) born in the previous period; and (iii) a θ-contingent storage scrapping rule $\tilde{\delta}(\theta)$ for the agents who will be relocated in the current period. These rules are symmetric across the two islands. Notice that rule (i) is to replicate the feature that money in the decentralized equilibrium is used bo buy goods deposited by the young. Rules (ii) and (iii) ensure that the planner faces the same fundamental uncertainty concerning the relocation probability as banks do in the decentralized environment. Finally, the *ex ante* constraint on the planner is necessary to replicate both first- and second-stage allocations of the decentralized equilibrium.[17]

Observe from preceding discussion that to differentiate this planner's rules from that of banks we have used a ˜ over the liquidation rules; otherwise, the planner's constraints are identical to (31a) and (31b):

$$\theta \, \tilde{c}^m = \tilde{\alpha}(\theta)(y - s) + \tilde{\delta}(\theta) \, s \, r \tag{32a}$$

$$(1 - \theta) \, \tilde{c}^n = (1 - \tilde{\alpha}(\theta))(y - s) + \left(1 - \tilde{\delta}(\theta)\right) s \, x \tag{32b}$$

where $\tilde{c}^m$ and $\tilde{c}^n$ denote the old-age consumption allocated by the planner to movers and non-movers respectively. Clearly, the planner's choice set allows $\tilde{c}^m \in \left[0, \frac{y}{\theta}\right]$ and $\tilde{c}^n \in \left[0, \frac{xy}{1-\theta}\right]$. While $\tilde{c}^m = 0$ and $\tilde{c}^n = \frac{xy}{1-\theta}$ corresponds to $s = y$ and $\tilde{\delta}(\theta) = 0$ for all θ, $\tilde{c}^m = \frac{y}{\theta}$ and $\tilde{c}^n = 0$ corresponds to $s = 0$ and $\tilde{\alpha}(\theta) = 1$ for all θ. In a decentralized equilibrium, with strictly concave preferences, unless $\mu \to \infty$ and $r = 0$ movers will receive a strictly positive consumption. Further, with $\mu > x^{-1}$, non-movers will always receive at

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[17]Otherwise, if at any date t the planner were allowed to allocate date t endowment between storage (for consumption of agents born at t) and reserves (for consumption of agents born at $t - 1$), the allocations will turn out to be contingent on θ_{t-1}.

least as much consumption as movers. Thus, with the planner free to choose both first-
and second-stage rules described above, his set of feasible allocations contains the set of
decentralized equilibrium allocations.

The constrained planning problem then boils down to

$$
\max_{s \in [0,y]} E_\theta \left\{ \max_{\tilde{\alpha}(\theta) \in [0,1], \tilde{\delta}(\theta) \in [0,1]} (\theta \; u \, (\tilde{c}^m) + (1 - \theta) \; u \, (\tilde{c}^n)) \right\} \tag{33}
$$

subject to (32a) and (32b).

Similar to the banks' second stage rules, the planner's liquidation rules follow

$$
u' \, (\tilde{c}^n) \;\; \leq \;\; u' \, (\tilde{c}^m) \quad \text{`` } = \text{''} \quad \text{if } \tilde{\alpha} \, (\theta) < 1 \tag{34a}
$$

$$
u' \, (\tilde{c}^n) \; x \;\; \leq \;\; u' \, (\tilde{c}^m) \; r, \quad \text{`` } = \text{''} \text{ if } \tilde{\delta} \, (\theta) > 0 \tag{34b}
$$

It is easy to see that $\tilde{\delta} \, (\theta) > 0$ only if $\alpha \, (\theta) = 1$; i.e., only one of the second-stage choices
is interior for any θ. Further, by using (26a) and (26b) to substitute for $\tilde{c}^m$ and $\tilde{c}^n$, it can
be verified that the planner's objective function (33) in the second stage is strictly concave
in both α and δ.[18] Hence, given the planner's first-stage choice of s, equations (34a) and
(34b) along with (26a), (26b) uniquely determine functions $\tilde{\alpha}_t \, (\theta)$ and $\tilde{\delta}_t \, (\theta)$.

Once again, as these second-stage $\theta-$contingent rules are optimally chosen, the first
order condition with respect to s can be obtained by invoking the Envelope Theorem:

$$
E_\theta \left\{ \begin{array}{c} r \, \tilde{\delta} \, (\theta) \; u' \, (\tilde{c}^m) + \left(1 - \tilde{\delta} \, (\theta) \right) \; u' \, (\tilde{c}^n) \; x \\ -\tilde{\alpha} \, (\theta) \; u' \, (\tilde{c}^m) - (1 - \tilde{\alpha} \, (\theta)) \; u' \, (\tilde{c}^n) \end{array} \right\} = 0 \tag{35}
$$

subject to (34a) and (34b). Differentiating (35) further with respect to s verifies that the
planner's objective function is concave in s. Using (34a) and (34b) in (35) obtains

$$
E_\theta \left\{ u' \, (\tilde{c}^m) \right\} \;\; = E_\theta \left\{ u' \, (\tilde{c}^n) \; x \right\} \tag{36}
$$

Equation (36) along with (32a), (32b), (34a) and (34b) thus solve for $s, \tilde{c}^m, \tilde{c}^n, \tilde{\alpha} \, (\theta)$,
and $\tilde{\delta} \, (\theta)$. Further, given that the welfare function is concave in the choice variables, the
solution is also unique.

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[18] Since only one of the choices is interior at a time, it suffices to look at the diagonal elements of the
Hessian matrix, both of which are negative.

Comparing (36) with (30) establishes that $R_m = 1$ is the optimal gross rate of return on money that in a decentralized equilibrium can be obtained by a constant money supply. Moreover, with $R_m = 1$, equations (28a) and (28b) that generate the second-stage rules in the decentralized equilibrium are, in steady state, identical to the planner's second-stage optimality conditions given by (34a) and (34b). Recall that we have already established the equivalence of planner's resource constraint (32a) and (32b) and that of the resource constraint in the decentralized equilibrium (31a) and (31b). Hence, the decentralized allocations under $R_m = 1$ is identical to that chosen by our pseudo-planner.

What is the intuition for this result? In the limited communication environment, money exists to facilitate consumption by the movers. While money does help overcome the limited communication problem, it does so at a cost, the lost return from storage. Recall that under the assumption of fixed endowments and zero net population growth, the gross social return on monetary transactions is unity (the 'biological' interest rate). Hence the social opportunity cost of monetary transactions relative to non-monetary transactions is x. At the point at which the planner makes the portfolio allocation decision, all he knows is that movers have to be paid out of the current endowment and that non-movers must be the only recipients of unscrapped stored goods. Equation (36) captures the idea that the decision of how to allocate a marginal unit of the endowment (to keep aside for the movers or to store it) depends on the marginal valuation of that unit by its ultimate recipients, movers and non-movers.

5. Conclusion

In this paper, we study a monetized decentralized economy with a specified set of markets, rules of trade, and a restricted set of policies and derive the equilibrium set of feasible (monetary) allocations. We then set up a simpler pseudo-planning problem in which we restrict the planner to choose from a set that contains the equilibrium allocations of the decentralized economy. If there is a government policy that allows the decentralized economy to achieve the constrained planner's allocation, then it is the optimal policy choice. To illustrate the power of such analyses, we solve such planning problems in a model with limited communication and demonstrate their use in deriving optimal policies in the corresponding decentralized economies. Overall, whether the suggested approach is useful or not will depend on the specific model environment; for instance, in some cases, it may

not be possible to specify the set of allocations achieved as an equilibrium for some policy choice without computing the equilibrium of the economy for each policy choice.

There are two particular insights for limited communication money models that are worth emphasizing here. First, money only partially resolves the limited communication problem by facilitating the consumption for some agents (movers) that would otherwise be impossible. Yet, a deeper wedge in the form of the relative opportunity costs of consumption between the two types of agents remains: movers consume from the current endowment while non-movers consume stored goods. The best that money can do is not to distort this relative opportunity cost further: a fixed money supply is therefore optimal. Second, when money is already in circulation, providing another nominal asset is redundant. To be precise, there is nothing that nominal bonds can do in this environment that money can not.

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