Opportunity Sets and the Measurement of Information*

Walter Bossert

Department of Economics, University of Nottingham

March 1998

Correspondence Address:

Department of Economics
University of Nottingham
University Park
Nottingham NG7 2RD
UK
Phone: (+44 115) 951 5480
FAX: (+44 115) 951 4159
E-mail: lezwb@unix.ccc.nottingham.ac.uk

* An earlier version of the paper was presented at the University of Caen. I thank Yongsheng Xu and seminar participants for discussions and comments.

Abstract

This paper provides an axiomatic approach to the problem of measuring the information contained in opportunity sets. In many choice situations, the items that can be selected from an opportunity set (the objects of choice) do not coincide with the consequences they induce (the objects a decision-maker ultimately cares about). An informational issue arises because uncertainty regarding the consequences of the choices may be present. An ordinal index of information with a plausible interpretation in this framework is characterized by means of two sets of axioms.

Journal of Economic Literature Classification Nos.: C43, D81.

Keywords: Opportunity Sets, Information, Measurement.

1 Introduction

The problem of ranking opportunity sets in terms of their desirability with respect to certain individual and social criteria has been the subject of investigation of a growing body of literature. An *opportunity set* (or a *menu*) is defined as a set of options from which an individual can make choices. In addition to the *indirect-utility criterion,* which postulates that an agent ranks opportunity sets on the basis of their respective best elements according to her or his preferences on the universal set of options, other criteria have recently been introduced as possible determinants of the desirability of a menu.

For instance, the *freedom of choice* offered by opportunity sets has been used as a criterion to rank these menus. See, for example, Gravel (1994), Gravel, Laslier, and Trannoy (1998), Klemisch-Ahlert (1993), Pattanaik and Xu (1990), Puppe (1995, 1996, 1998), Sen (1988, 1991) for discussions and axiomatic analyses of freedom of choice. The assessment of *distributions* of opportunities has been analyzed in Kranich (1996). Pattanaik and Xu (1997) discuss the importance of *diversity* as a ranking criterion. Rankings of opportunity sets in terms of the overall well-being of an agent rather than in terms of one specific criterion are analyzed in Bossert (1997a), Bossert, Pattanaik, and Xu (1994), and Dutta and Sen (1996). Kreps (1979) provides an approach in terms of *preference for flexibility.*

The purpose of this paper is to propose and explore a new avenue of research within this context. In particular, the ranking of opportunity sets is analyzed from an angle which is quite different from those mentioned above—namely, the *informational contents* of opportunity sets are the subject of investigation. To illustrate why informational issues may be of importance in this framework, consider the following example, taken from Sen (1993). There are two alternative choice situations faced by an individual decision-maker. Suppose first the decision-maker is offered a choice between accepting the invitation of a distant acquaintance for a cup of tea at the acquaintance's place or staying at home. That is, the decision-maker can choose an option from the opportunity set consisting of the two alternatives 'having tea at the acquaintance's place' and 'staying at home.' Alternatively, suppose the acquaintance offers having some cocaine at her or his place as a further option, in which case the decision-maker faces the larger opportunity set which contains, in addition to the two above-mentioned options, the element 'having cocaine at the acquaintance's place.' Sen (1993) argues that it is quite plausible that many decision-makers would accepted the invitation to tea in the first choice situation but select 'staying at home' when faced with the second, and he uses this example to argue against the traditional rational-choice model based on what he refers to as *internal* *consistency* requirements imposed on choice behaviour.

An alternative interpretation of this example seems to be to view the objects that appear as menu items in an opportunity set as being different from the objects the decision-maker ultimately cares about—the *consequences* of his or her choices. For example, it would seem natural to think of the consequences 'having tea at a place where cocaine is being consumed' and 'having tea at a cocaine-free place' as being distinct alternatives, and the choice behaviour outlined above can be explained quite straightforwardly once it is recognized that if 'having tea' and 'staying at home' are the only menu items available for choice, the decision-maker lacks crucial information regarding the possible consequences of her or his actions. In this case, the agent does not know with certainty whether the consequence of choosing tea will be 'having tea at a cocaine-free place' or 'having tea at a place where cocaine is being consumed.' On the other hand, if 'having cocaine' is offered as an additional menu item, the agent can be sure that choosing 'tea' will lead to the latter of those two consequences. Thus, a menu can convey crucial information about the consequences of possible choices. See also Sen's (1993) discussion of the *epistemic value* of a menu, and Bossert (1995b) for alternative definitions and characterizations of rationality in situations where possible consequences do not coincide with menu items.

It is clear that the distinction between choices and consequences is of importance in a variety of decision situations and is, therefore, not restricted to the above example. This observation raises an interesting measurement issue, namely, how the informational contents of opportunity sets should be measured. Basically, the reason why the informational contents may vary considerably across different opportunity sets is that, in many situations, *uncertainty* concerning the possible consequences of choosing particular menu items is present. This measurement problem is addressed in the present paper. In particular, the objective is to establish an index that can be applied to measure the informational contents of opportunity sets. Given an ordinal interpretation of such a measure, this amounts to the construction of an ordering on the set of possible menus with the above-mentioned interpretation.

The following is a brief outline of the model employed in this paper in order to illustrate the methodology being used; a more detailed description follows after introducing the requisite formal definitions. For any given menu, the set of possible consequences is specified for each menu item. A plausible interpretation of these sets of possible consequences is that they contain all those outcomes that cannot be excluded as possible consequences of the choice of a particular menu item. Based on this mapping from choices into consequences, an ordering defined on the set of possible menus is to be established. The approach employed to do so is axiomatic, and two characterizations of a specific ranking of opportunity sets are provided. This ordering compares two menus on the basis of the total number of possible consequences from all possible choices minus the number of elements of the menu.

The above-described index can be defined on the basis of the sets of possible consequences alone. No information regarding, for example, how an agent ranks different outcomes is required, which is a plausible property, given that what is to be measured is the information contained in menus rather than, for example, the quality of the best possible choice or the overall desirability of these menus. Furthermore, there is no need to assume the existence of a probability distribution over the set of possible outcomes that can arise from the choice of a given menu item. In that regard, the model used in this paper is analogous to a class of nonprobabilistic models of choice under uncertainty. See, for example, Bandyopadhyay (1988), Bossert (1989a,b, 1997b), Barberà, Barrett, and Pattanaik (1984), Barberà and Pattanaik (1984), Fishburn (1984), Heiner and Packard (1984), Holzman (1984a,b), Kannai and Peleg (1984), Nitzan and Pattanaik (1984), Pattanaik and Peleg (1984), and some of the contributions to the literature on choice under ignorance—for instance, Arrow and Hurwicz (1972) and Maskin (1979). An approach to the ranking of sets of alternatives based on signed orders is discussed in Fishburn (1992), and Bossert (1995a) analyzes the ranking of sets of outcomes with a fixed number of elements.

However, it should be pointed out that the problem addressed in this paper differs from the standard approach to the ranking sets of possible outcomes in terms of uncertainty. Instead of simply ranking sets of possible outcomes as is the case in nonprobabilisitc decision models, the approach in this paper establishes a ranking of *non-ordered n-tuples* of sets of consequences (where n is not fixed), due to the fact that each menu item in a given opportunity set induces a set of possible outcomes. As a consequence, the axioms employed in this paper and the resulting ranking rules are quite different from those that can be found in the standard decision-theoretic literature on nonprobabilistic choice under uncertainty.

In Section 2, the basic concepts and definitions used in this paper are introduced. The consequences of some basic axioms on the comparison of menus with a fixed number of elements are analyzed in Section 3. Section 4 provides two characterizations of the index proposed in this paper, and Section 5 establishes the independence of the axioms used in the axiomatizations. Concluding remarks are collected in Section 6.

2 Definitions

The set of real numbers is denoted by $\Re$. Let $\mathcal{Z}_+$ (resp. $\mathcal{Z}_{++}$) be the set of nonnegative (resp. positive) integers, and let $\mathcal{Z}^n_+$ (resp. $\mathcal{Z}^n_{++}$) denote the n-fold Cartesian product of $\mathcal{Z}_+$ (resp. $\mathcal{Z}_{++}$). For $n \in \mathcal{Z}_{++}$, $\mathbf{1}_n$ is the vector consisting of n ones. The notation for vector inequalities is as follows. For all $n \in \mathcal{Z}_{++}$ and all $x, y \in \mathcal{Z}^n_+$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i \in \{1, \ldots, n\}$, and $x > y$ if and only if $x \geq y$ and $x \neq y$. For $n \in \mathcal{Z}_{++}$, $x \in \mathcal{Z}^n_{++}$, and $j \in \{1, \ldots, n\}$, define $\hat{x}^j$ by letting

$$\hat{x}^j_i := \begin{cases} x_i + 1 & \text{if } i = j; \\ x_i & \text{if } i \in \{1, \ldots, n\} \setminus \{j\}. \end{cases}$$

Consider an infinite set of items that could appear on a menu, denoted by Σ. $\Pi(\Sigma)$ is the set of all nonempty and finite subsets of Σ. That is, a menu S available to an agent is an element of $\Pi(\Sigma)$. The cardinality of S is denoted by $|S|$ and, for notational convenience, the elements of S are labelled $s_1, \ldots, s_{|S|}$, that is, $S = \{s_1, \ldots, s_{|S|}\}$.

The set of possible consequences that are of relevance for a decision-maker is Ω. Again, Ω is assumed to have infinitely many elements. $\Pi(\Omega)$ is the set of all nonempty and finite subsets of Ω. For $n \in \mathcal{Z}_{++}$, $\Gamma_n(\Omega)$ is the set of all non-ordered n-tuples of nonempty and finite subsets of Ω.

To establish a link between choices of menu items and consequences, a mapping $F\colon \Pi(\Sigma) \to \bigcup_{n \in \mathcal{Z}_{++}} \Gamma_n(\Omega)$ is used, where, for all $S \in \Pi(\Sigma)$, $F(S) \in \Gamma_{|S|}(\Omega)$. Therefore, $F(S)$ can be written as $F(S) = < F_1(S), \ldots, F_{|S|}(S) >$ with $F_i(S) \in \Pi(\Omega)$ for all $i \in \{1, \ldots, |S|\}$. For all $i \in \{1, \ldots, |S|\}$, $F_i(S)$ is the set of possible consequences of choosing menu item s_i from the menu S. Note that $F(S)$ is a *non-ordered* n-tuple because the numbering of elements in S is arbitrary.

It is assumed that F is *surjective,* that is, $F(\Pi(\Sigma)) = \bigcup_{n \in \mathcal{Z}_{++}} \Gamma_n(\Omega)$. The surjectivity of F ensures that any non-ordered n-tuple of sets of possible consequences can be generated by some menu. This assumption is required in order to rule out degenerate cases. See Section 5 for a more detailed discussion.

The measurement issue addressed in this paper is that of finding an index of the information contained in a menu S. Given an ordinal interpretation of such an index, this amounts to establishing an ordering R on $\Pi(\Sigma)$ where, for all $S, T \in \Pi(\Sigma)$, $S\,R\,T$ if and only if S contains at least as much information about consequences as T. The indifference relation and the strict preference relation corresponding to R are denoted by I and P, respectively.

The ordering R^* that is of particular interest in this paper is defined as follows. For all $S, T \in \Pi(\Sigma)$,

$$S\,R^*\,T \;:\Leftrightarrow\; \sum_{i=1}^{|S|} |F_i(S)| - |S| \le \sum_{i=1}^{|T|} |F_i(T)| - |T|.$$

R^* ranks two opportunity sets on the basis of the total number of possible consequences generated by the respective menu, corrected by the number of elements appearing in the menu. This ordering has considerable intuitive appeal. The total number of possible consequences appears to be an adequate measure of the uncertainty involved in the choice to be made. Because a menu item the choice of which unambiguously leads to one specific consequence does not introduce any uncertainty, this total number of consequences should be modified by subtracting the number of menu items. It turns out that, in addition to conforming to the intuition behind this measurement issue, R^* can be given strong axiomatic support. In particular, R^* turns out to be the only information-measurement ordering satisfying two sets of plausible axioms.

3 A Fixed Number of Menu Items

As a first step in the characterizations of R^*, the consequences of some axioms imposing restrictions on the comparison of menus of the same cardinality are examined.

The first axiom is a *certainty indifference* condition. It requires that if each menu item in two opportunity sets of the same cardinality leads to a single possible outcome, then the two menus should be declared indifferent. This is a plausible assumption: if there is no uncertainty regarding the consequences of all possible choices in two opportunity sets, the two menus are indistinguishable in terms of their informational contents.

Certainty Indifference: For all $S, T \in \Pi(\Sigma)$ such that $|S| = |T|$, if $|F_i(S)| = 1$ for all $i \in \{1, \ldots, |S|\}$ and $|F_i(T)| = 1$ for all $i \in \{1, \ldots, |T|\}$, then $S\,I\,T$.

The next axiom is an *independence* condition. It requires that if two menus S and T with the same cardinality are replaced by S' and T', respectively, where the only difference between the former pair and the latter is that one outcome is added as a possible consequence of a choice s_j and a choice t_j for some $j \in \{1, \ldots, |S|\}$, then the relative ranking of S and T is the same as the relative ranking of S' and T'. This type of condition is standard in the literature on the ranking of opportunity sets in terms of freedom of choice and the ranking of sets of uncertain outcomes (see, for example, Barberà, Barrett, and Pattanaik (1984), Barberà and Pattanaik (1984), Bossert (1989a,b), Bossert, Pattanaik, and Xu (1994), Fishburn (1984), Heiner and Packard (1984), Kannai and Peleg (1984), and Pattanaik and Xu (1990)). The following axiom is an adaptation of this principle to the problem investigated in the present paper.

Independence: For all $S, T, S', T' \in \Pi(\Sigma)$ such that $|S| = |S'| = |T| = |T'|$, for all $j \in \{1, \ldots, |S|\}$, for all $\omega \in \Omega \setminus (F_j(S) \cup F_j(T))$, if

$$F_j(S') = F_j(S) \cup \{\omega\} \quad \text{and} \quad F_j(T') = F_j(T) \cup \{\omega\} \quad \text{and}$$
$$F_i(S') = F_i(S) \quad \text{for all } i \in \{1, \ldots, |S|\} \setminus \{j\} \quad \text{and}$$
$$F_i(T') = F_i(T) \quad \text{for all } i \in \{1, \ldots, |T|\} \setminus \{j\},$$

then

$$S\,R\,T \;\Leftrightarrow\; S'\,R\,T'.$$

In the presence of the above axioms, the task of comparing any two opportunity sets S and T on the basis of their informational contents can be simplified considerably. Certainty indifference and independence imply the existence of an ordering $\succeq$ on $\bigcup_{n \in \mathcal{Z}_{++}} \mathcal{Z}^n_{++}$ (with indifference relation $\sim$ and strict preference relation $\succ$) such that the relative ranking of S and T according to R is determined by the ranking of the vectors consisting of the cardinalities of the $F_i(S)$ and $F_i(T)$ according to $\succeq$. In addition, the ordering $\succeq$ must possess certain properties and, in turn, the existence of an ordering of vectors of positive integers with these properties guarantees that the corresponding ordering R satisfies certainty indifference and independence. The above-mentioned properties of the ordering $\succeq$ are the following.

Anonymity: For all $n \in \mathcal{Z}_{++}$, for all $x, y \in \mathcal{Z}^n_{++}$, for all bijections $\rho\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, if $y_i = x_{\rho(i)}$ for all $i \in \{1, \ldots, n\}$, then $x \sim y$.

Additivity: For all $n \in \mathcal{Z}_{++}$, for all $x, y \in \mathcal{Z}^n_{++}$, for all $j \in \{1, \ldots, n\}$,

$$x \succeq y \;\Leftrightarrow\; \hat{x}^j \succeq \hat{y}^j.$$

By repeated application of this property, additivity is easily seen to be equivalent to the following axiom.

Additivity′: For all $n \in \mathcal{Z}_{++}$, for all $x, y \in \mathcal{Z}^n_{++}$, for all $z \in \mathcal{Z}^n_{+}$,

$$x \succeq y \;\Leftrightarrow\; (x + z) \succeq (y + z).$$

The above-described result establishing the consequences of certainty indifference and independence is stated in the following theorem.

Theorem 1 *Suppose F is surjective. R satisfies certainty indifference and independence if and only if there exists an anonymous and additive ordering $\succeq$ on $\bigcup_{n\in\mathcal{Z}_{++}}\mathcal{Z}^n_{++}$ such that, for all $S,T\in\Pi(\Sigma)$,*

$$S\,R\,T \;\Leftrightarrow\; (|F_1(S)|,\ldots,|F_{|S|}(S)|) \succeq (|F_1(T)|,\ldots,|F_{|T|}(T)|). \tag{1}$$

The following lemma will be of use in the proof of this theorem.

Lemma 1 *Suppose F is surjective, and R satisfies certainty indifference and independence. Let $S,T\in\Pi(\Sigma)$ be such that $|S|=|T|$. If, for all $i\in\{1,\ldots,|S|\}$, there exists $j(i)\in\{1,\ldots,|T|\}$ such that $|F_i(S)|=|F_{j(i)}(T)|$, then $S\,I\,T$.*

Proof: Let $S,T\in\Pi(\Sigma)$ be such that $|S|=|T|$, and suppose that, for all $i\in\{1,\ldots,|S|\}$, there exists $j(i)\in\{1,\ldots,|T|\}$ such that $|F_i(S)|=|F_{j(i)}(T)|$. Without loss of generality, we can assume that $j(i)=i$ for all $i\in\{1,\ldots,|S|\}$ because the labelling of the elements in S and T is irrelevant. We proceed by induction over $\sum_{i=1}^{|S|}|F_i(S)|$. Note that $\sum_{i=1}^{|S|}|F_i(S)|=\sum_{i=1}^{|T|}|F_i(T)|$ by assumption. The methodology employed in this proof is analogous to the one used in Pattanaik and Xu's (1990) characterization of the simple cardinality-based ordering for opportunity sets.

The smallest possible value of the above sum is $|S|$ because each $F_i(S)$ contains at least one element. In that case, $F_i(S)$ and $F_i(T)$ must be singletons for all $i\in\{1,\ldots,|S|\}=\{1,\ldots,|T|\}$. By certainty indifference, $S\,I\,T$.

Now suppose the claim is true for $\sum_{i=1}^{|S|}|F_i(S)|=m\geq|S|$. Let $\sum_{i=1}^{|S|}|F_i(S)|=m+1$ and $|F_i(S)|=|F_i(T)|$ for all $i\in\{1,\ldots,|S|\}$. Choose $j\in\{1,\ldots,|S|\}$ such that $|F_j(S)|=|F_j(T)|>1$. Such a j must exist because $m+1>|S|$. Pick any $\omega\in F_j(S)$ and let S' be such that $|S'|=|S|$, $F_j(S')=F_j(S)\setminus\{\omega\}$ and $F_i(S')=F_i(S)$ for all $i\in\{1,\ldots,|S|\}\setminus\{j\}$ (note that the surjectivity of F guarantees that such a menu S' exists). There are two possible cases.

(i) $\omega\in F_j(T)$;
(ii) $\omega\notin F_j(T)$.

In case (i), let T' be such that $|T'|=|T|$, $F_j(T')=F_j(T)\setminus\{\omega\}$, and $F_i(T')=F_i(T)$ for all $i\in\{1,\ldots,|T|\}$. We have

$$|F_i(S')|=|F_i(T')|=|F_i(S)|=|F_i(T)| \quad \text{for all } i\in\{1,\ldots,|S'|\}\setminus\{j\}$$

and

$$|F_j(S')|=|F_j(T')|=|F_j(S)|-1=|F_j(T)|-1.$$

Hence, $\sum_{i=1}^{|S'|} |F_i(S')| = \sum_{i=1}^{|T'|} |F_i(T')| = m$. By the induction hypothesis, $S' \, I \, T'$, and independence implies $S \, I \, T$.

Now consider case (ii). Because $\omega \notin F_j(T)$ and $|F_j(S)| = |F_j(T)|$, it follows that $F_j(T) \setminus F_j(S) \neq \emptyset$. Let $\omega' \in F_j(T) \setminus F_j(S)$. Let T^1 be such that $|T^1| = |T|$, $F_j(T^1) = F_j(T) \setminus \{\omega'\}$, and $F_i(T^1) = F_i(T)$ for all $j \in \{1, \ldots, |T^1|\} \setminus \{j\}$. It follows that $|F_j(S')| = |F_j(T^1)| = |F_j(S)| - 1 = |F_j(T)| - 1$ and $\sum_{i=1}^{|S'|} |F_i(S')| = \sum_{i=1}^{|T^1|} |F_i(T^1)| = m$. By the induction hypothesis,

$$S' \, I \, T^1. \tag{2}$$

Letting T^2 be such that $|T^2| = |T|$, $F_j(T^2) = F_j(T^1) \cup \{\omega\} = (F_j(T) \setminus \{\omega'\}) \cup \{\omega\}$, and $F_i(T^2) = F_i(T^1) = F_i(T)$ for all $i \in \{1, \ldots, |T^2|\}$, (2) and independence imply

$$S \, I \, T^2. \tag{3}$$

Consider any $\omega'' \in F_j(T^1)$, and define T^3 such that $|T^3| = |T|$, $F_j(T^3) = F_j(T^2) \setminus \{\omega''\}$, and $F_i(T^3) = F_i(T)$ for all $i \in \{1, \ldots, |T^3|\}$. Furthermore, let T^4 be such that $|T^4| = |T|$, $F_j(T^4) = F_j(T) \setminus \{\omega''\}$, and $F_i(T^4) = F_i(T)$ for all $i \in \{1, \ldots, |T^4|\} \setminus \{j\}$. By definition, $F_j(T^3) = (F_j(T^1) \cup \{\omega\}) \setminus \{\omega''\}$. Furthermore, $|F_j(T^3)| = |F_j(T^4)| = |F_j(T^1)| = |F_j(T)| - 1$, $|F_i(T^3)| = |F_i(T^4)| = |F_i(T)|$ for all $i \in \{1, \ldots, |T|\}$, and $\sum_{i=1}^{|T^3|} |F_i(T^3)| = \sum_{i=1}^{|T^4|} |F_i(T^4)| = m$. By the induction hypothesis, $T^3 \, I \, T^4$, and independence implies $T^2 \, I \, T$. Together with (3), the transitivity of R implies $S \, I \, T$. ■

Proof of Theorem 1: It is straightforward to verify that any ordering R that can be expressed in terms of an anonymous and additive ordering $\succeq$ as in the theorem statement satisfies certainty indifference and independence.

Conversely, suppose R satisfies these two axioms. Using Lemma 1, an ordering $\succeq$ on $\bigcup_{n \in \mathcal{Z}_{++}} \mathcal{Z}_{++}^n$ with the desired properties is constructed. Let $\{\omega_i \mid i \in \mathcal{Z}_{++}\}$ be a countably infinite set of pairwise distinct elements of Ω. For all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{Z}_{++}^n$, and $y \in \mathcal{Z}_{++}^m$, let $S_x, T_y \in \Pi(\Sigma)$ be such that $|S_x| = n$, $|T_y| = m$, $F_i(S_x) = \{\omega_1, \ldots, \omega_{x_i}\}$ for all $i \in \{1, \ldots, n\}$, and $F_i(T_y) = \{\omega_1, \ldots, \omega_{y_i}\}$ for all $i \in \{1, \ldots, m\}$ (the existence of menus S_x and T_y with these properties is guaranteed by the surjectivity of F). Now define $\succeq$ by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{Z}_{++}^n$, and $y \in \mathcal{Z}_{++}^m$,

$$x \succeq y \;:\Leftrightarrow\; S_x \, R \, T_y.$$

$\succeq$ is an ordering because R is. For all $S, T \in \Pi(\Sigma)$, let $x^S := (|F_1(S)|, \ldots, |F_{|S|}(S)|) \in \mathcal{Z}_{++}^{|S|}$ and $y^T := (|F_1(T)|, \ldots, |F_{|T|}(T)|) \in \mathcal{Z}_{++}^{|T|}$. By Lemma 1 and the definition of $\succeq$,

$$S \, R \, T \;\Leftrightarrow\; S_{x^S} \, R \, T_{y^T} \;\Leftrightarrow\; x^S \succeq y^T \;\Leftrightarrow\; (|F_1(S)|, \ldots, |F_{|S|}(S)|) \succeq (|F_1(T)|, \ldots, |F_{|T|}(T)|).$$

That $\succeq$ is anonymous follows from the observation that the labelling of the elements in a menu is arbitrary.

It remains to be shown that $\succeq$ is additive. Let $n \in \mathcal{Z}_{++}$, $x, y \in \mathcal{Z}^n_{++}$, and $j \in \{1, \ldots, n\}$. Choose $S, T \in \Pi(\Sigma)$ such that $|S| = |T| = n$, $|F_i(S)| = x_i$ for all $i \in \{1, \ldots, n\}$, and $|F_i(T)| = y_i$ for all $i \in \{1, \ldots, n\}$. Pick any $\omega \in \Omega \setminus (F_j(S) \cup F_j(T))$, and let $S', T' \in \Pi(\Sigma)$ be such that $|S'| = |T'| = n$, $F_j(S') = F_j(S) \cup \{\omega\}$, $F_j(T') = F_j(T) \cup \{\omega\}$, $F_i(S') = F_i(S)$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$, and $F_i(T') = F_i(T)$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$. By independence and the definition of $\succeq$,

$$x \succeq y \Leftrightarrow S R T \Leftrightarrow S' R T' \Leftrightarrow \hat{x}^j \succeq \hat{y}^j. \blacksquare$$

Theorem 1 implies that, in the presence of certainty indifference and independence, we can simplify matters by considering the ordering $\succeq$ instead of R without loss of generality. In particular, the ordering $\succeq^*$ on $\bigcup_{n \in \mathcal{Z}_{++}} \mathcal{Z}^n_{++}$ corresponding to R^* is defined by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{Z}^n_{++}$, and $y \in \mathcal{Z}^m_{++}$,

$$x \succeq^* y \;:\Leftrightarrow\; \sum_{i=1}^{n} x_i - n \leq \sum_{i=1}^{m} y_i - m.$$

For each of the axioms imposed on R in this paper (in addition to certainty indifference and independence), an equivalent formulation in terms of $\succeq$ can be defined, and characterizations of an ordering R can be provided by characterizing the corresponding ordering $\succeq$.

To conclude this section, a plausible *expansion monotonicity* property is introduced, and the class of orderings satisfying this axiom in addition to certainty indifference and independence is characterized. Expansion monotonicity requires that if a menu T differs from a menu S only in that some $F_j(T)$ contains all elements in $F_j(S)$ and one further element ω (all other sets of possible consequences being the same for S and T), then T must be worse than S. This is a very weak requirement and is motivated by the argument that the additional possible consequence that is present in T leads to a higher degree of uncertainty and, thus, to a lower degree of information about consequences.

Expansion Monotonicity: For all $S, T \in \Pi(\Sigma)$ such that $|S| = |T|$, if there exist $j \in \{1, \ldots, |S|\}$ and $\omega \in \Omega \setminus F_j(S)$ such that $F_j(T) = F_j(S) \cup \{\omega\}$ and $F_i(T) = F_i(S)$ for all $i \in \{1, \ldots, |S|\} \setminus \{j\}$, then

$$S P T.$$

As is the case for certainty indifference and independence, expansion monotonicity only imposes a restriction on the ranking of menus with a fixed number of elements.

Adding expansion monotonicity to certainty indifference and independence, we obtain a characterization of the class of orderings R that rank any two menus of fixed cardinality on the basis of the total number of possible consequences. Before stating and proving this result, the property of $\succeq$ corresponding to expansion monotonicity is defined.

Monotonicity: For all $n \in \mathcal{Z}_{++}$, for all $x, y \in \mathcal{Z}^n_{++}$, if $x < y$, then $x \succ y$.

The proof of the following lemma is straightforward and omitted.

Lemma 2 *Suppose F is surjective, and R satisfies certainty indifference and independence. Let $\succeq$ be the corresponding ordering introduced in Theorem 1. R satisfies expansion monotonicity if and only if $\succeq$ satisfies monotonicity.*

We obtain

Theorem 2 *Suppose F is surjective. R satisfies certainty indifference, independence, and expansion monotonicity if and only if there exists an anonymous and additive ordering $\succeq$ on $\bigcup_{n \in \mathcal{Z}_{++}} \mathcal{Z}^n_{++}$ such that (1) is satisfied for all $S, T \in \Pi(\Sigma)$ and, for all $S, T \in \Pi(\Sigma)$ such that $|S| = |T|$,*

$$S \, R \, T \;\Leftrightarrow\; \sum_{i=1}^{|S|} |F_i(S)| \leq \sum_{i=1}^{|T|} |F_i(T)|. \tag{4}$$

Proof: That any ordering R satisfying (4) satisfies certainty indifference, independence, and expansion monotonicity can be verified easily (note that none of these properties imposes any restriction on the comparison of menus of different cardinalities).

Now suppose R satisfies these axioms. By Theorem 1 and Lemma 2, it follows that $\succeq$ is well-defined and satisfies anonymity, additivity, and monotonicity. It is sufficient to show that, for all $n \in \mathcal{Z}_{++}$, $x, y \in \mathcal{Z}^n_{++}$,

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \;\Rightarrow\; x \sim y \tag{5}$$

and

$$\sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i \;\Rightarrow\; x \succ y. \tag{6}$$

If $n = 1$, (5) follows from reflexivity and (6) follows from monotonicity. Now suppose $n \geq 2$.

To prove (5), let $x, y \in \mathcal{Z}^n_{++}$ be such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Define $z_n := n \sum_{i=1}^n x_i$, $z_1 := x_n - y_1 + z_n$ and, if $n \geq 3$, $z_k := x_{k-1} - y_k + z_{k-1}$ for all $k \in \{2, \ldots, n-1\}$. It follows that $z \in \mathcal{Z}^n_{++}$, $x_n + z_n = y_1 + z_1$, and $x_k + z_k = y_{k+1} + z_{k+1}$ for all $k \in \{1, \ldots, n-1\}$.

Therefore, because $\succeq$ is anonymous, $(x + z) \sim (y + z)$ and, by additivity$'$ (which is equivalent to additivity), we obtain $x \sim y$.

To show that (6) is true, suppose $x, y \in \mathcal{Z}^n_{++}$ are such that $\sum_{i=1}^n x_i < \sum_{i=1}^n y_i$. Let $x' := (\sum_{i=1}^n x_i - (n-1), \mathbf{1}_{n-1})$ and $y' := (\sum_{i=1}^n y_i - (n-1), \mathbf{1}_{n-1})$. By (5), $x' \sim x$ and $y' \sim y$. Monotonicity implies $x' \succ y'$ and, hence, $x \succ y$. ■

An interesting feature of Theorem 2 is that the independence property implies that the restriction of R to menus of a fixed cardinality can be represented by the sum of the cardinalities of the sets of possible consequences generated by a menu. Note that this is a quite strong conclusion—not only an additively separable representation is implied but full additivity. Furthermore, unlike in the usual characterizations of additive functions, the domain of a representation of R is a discrete set, which explains why the proof technique employed in the above theorem differs from the usual separability arguments that are commonly used when characterizing representations with Euclidean spaces as domains.

4 Two Characterizations of R^*

Theorem 2 can be extended to obtain a characterization of R^* by strengthening certainty indifference and independence appropriately. All that is needed is to require the conclusions of these axioms to be satisfied not only for comparisons involving menus with a fixed number of items but for menus of arbitrary cardinality. Accordingly, *extended* certainty indifference is defined as follows.

Extended certainty indifference: For all $S, T \in \Pi(\Sigma)$, if $|F_i(S)| = 1$ for all $i \in \{1, \ldots, |S|\}$ and $|F_i(T)| = 1$ for all $i \in \{1, \ldots, |T|\}$, then $S\,I\,T$.

Extended certainty indifference is as plausible an axiom as certainty indifference. If there is no uncertainty associated with any of the choices that can be made in two menus, the informational contents of these menus should be the same. Note that, in conjunction with expansion monotonicity, this also implies that the informational contents are maximal for menus that only generate singletons of possible consequences which, again, is a very natural property for an index of information.

Analogously, *extended* independence requires the relative ranking of any two menus to be invariant with respect to additions of a possible consequence to the set of possible consequences of choosing one specific menu item in each.

Extended independence: For all $S, T, S', T' \in \Pi(\Sigma)$ such that $|S| = |S'|$ and $|T| = |T'|$, for all $j \in \{1, \ldots, \min\{|S|, |T|\}\}$, for all $\omega \in \Omega \setminus (F_j(S) \cup F_j(T))$, if

$$F_j(S') = F_j(S) \cup \{\omega\} \quad \text{and} \quad F_j(T') = F_j(T) \cup \{\omega\} \quad \text{and}$$
$$F_i(S') = F_i(S) \quad \text{for all } i \in \{1, \ldots, |S|\} \setminus \{j\} \quad \text{and}$$
$$F_i(T') = F_i(T) \quad \text{for all } i \in \{1, \ldots, |T|\} \setminus \{j\},$$

then

$$S \, R \, T \;\Leftrightarrow\; S' \, R \, T'.$$

The corresponding properties of the ordering $\succeq$ are the following.

Unity Indifference: For all $n, m \in \mathcal{Z}_{++}$, $\mathbf{1}_n \sim \mathbf{1}_m$.

Extended additivity: For all $n, m \in \mathcal{Z}_{++}$, for all $x \in \mathcal{Z}^n_{++}$, for all $y \in \mathcal{Z}^m_{++}$, for all $j \in \{1, \ldots, \min\{n, m\}\}$,

$$x \succeq y \;\Leftrightarrow\; \hat{x}^j \succeq \hat{y}^j.$$

The following lemma is immediate and is stated without a proof.

Lemma 3 *Suppose F is surjective, and R satisfies certainty indifference and independence. Let $\succeq$ be the corresponding ordering introduced in Theorem 1.*

(a) R satisfies extended certainty indifference if and only if $\succeq$ satisfies unity indifference.

(b) R satisfies extended independence if and only if $\succeq$ satisfies extended additivity.

The following theorem provides the first characterization of R^*.

Theorem 3 *Suppose F is surjective. R satisfies expansion monotonicity, extended certainty indifference, and extended independence if and only if $R = R^*$.*

Proof: It is straighforward to verify that R^* satisfies the required axioms. Conversely, suppose R is an ordering satisfying the axioms in the theorem statement. Again, $\succeq$ is well-defined and must satisfy anonymity, monotonicity, unity indifference, and extended additivity, and it is sufficient to show that $\succeq = \succeq^*$. Let $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{Z}^n_{++}$ and $y \in \mathcal{Z}^m_{++}$. If $n = 1$, $x = (\sum_{i=1}^n x_i) = (\sum_{i=1}^n x_i - (n-1))$ and, by reflexivity,

$$\left(\sum_{i=1}^{n} x_i - (n-1) \right) \sim x. \tag{7}$$

Now suppose $n \geq 2$. By unity indifference, $\mathbf{1}_1 \sim \mathbf{1}_n$. Repeated application of extended additivity yields $(\sum_{i=1}^n x_i - (n-1)) \sim (\sum_{i=1}^n x_i - (n-1), \mathbf{1}_{n-1})$. By the fixed-cardinality result of Theorem 2, $(\sum_{i=1}^n x_i - (n-1), \mathbf{1}_{n-1}) \sim x$. Hence, by transitivity, $(\sum_{i=1}^n x_i - (n-1)) \sim x$. Therefore, (7) must be true for any value of n.

Analogously, it follows that $(\sum_{i=1}^m y_i - (m-1)) \sim y$. Therefore, using monotonicity,

$$\begin{aligned} x \succeq y \;&\Leftrightarrow\; \left(\sum_{i=1}^n x_i - (n-1)\right) \succeq \left(\sum_{i=1}^m y_i - (m-1)\right) \\ &\Leftrightarrow\; \sum_{i=1}^n x_i - (n-1) \le \sum_{i=1}^m y_i - (m-1) \\ &\Leftrightarrow\; \sum_{i=1}^n x_i - n \le \sum_{i=1}^m y_i - m \end{aligned}$$

and, therefore, $\succeq = \succeq^*$. ■

There is an interesting alternative to strengthening independence in order to arrive at a characterization of R^*. Consider the following situation involving two menus S and T. Suppose S and T are such that T contains one more element than S, the sets of possible consequences generated by the first $|S|$ items are identical in S and in T, and the additional element in T generates an outcome with certainty. In that case, the additional element in T does not introduce any further uncertainty as compared to S and, from an informational viewpoint, it seems natural to consider S and T to be indifferent. This is captured in the following *certainty extension* condition.

Certainty extension: For all $S, T \in \Pi(\Sigma)$ such that $|T| = |S| + 1$, if $F_i(T) = F_i(S)$ for all $i \in \{1, \ldots, |S|\}$ and $|F_{|T|}(T)| = 1$, then $S\, I\, T$.

The corresponding property of $\succeq$ is the following *extension* axiom.

Extension: For all $n \in \mathcal{Z}_{++}$, for all $x \in \mathcal{Z}_{++}^n$, $x \sim (x, 1)$.

Certainty extension is a strengthening of extended certainty indifference. This is shown in the following lemma.

Lemma 4 *Suppose F is surjective. If R satisfies certainty extension, then R satisfies extended certainty indifference.*

Proof: Suppose $S, T \in \Pi(\Sigma)$ are such that $|F_i(S)| = 1$ for all $i \in \{1, \ldots, |S|\}$ and $|F_i(T)| = 1$ for all $i \in \{1, \ldots, |T|\}$. Let $S^0, S', T' \in \Pi(\Sigma)$ be such that $|S^0| = 2$, $|S'| = |T'| = 1$, $F(S^0) = < F_1(S), F_1(T) >$, $F(S') = < F_1(S) >$, and $F(T') = < F_1(T) >$. By certainty extension, $S^0\, I\, S'$ and $S^0\, I\, T'$. Therefore, $S'\, I\, T'$. Repeated application of certainty extension yields $S\, I\, S'$ and $T\, I\, T'$. Therefore, by transitivity, $S\, I\, T$. ■

It is straightforward to verify that extension is the property of $\succeq$ corresponding to certainty extension. This observation is stated in the following lemma, the proof of which is omitted.

Lemma 5 *Suppose F is surjective, and R satisfies certainty indifference and independence. Let $\succeq$ be the corresponding ordering introduced in Theorem 1. R satisfies certainty extension if and only if $\succeq$ satisfies extension.*

The following result provides an alternative characterization of R^*.

Theorem 4 *Suppose F is surjective. R satisfies independence, expansion monotonicity, and certainty extension if and only if $R = R^*$.*

Proof: That R^* satisfies the axioms is straightforward. Again, we can consider $\succeq$ instead of R in order to simplify exposition. Suppose $\succeq$ satisfies anonymity, additivity, monotonicity, and extension. Let $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{Z}^n_{++}$, and $y \in \mathcal{Z}^m_{++}$. The case $n = m$ is taken care of in Theorem 2. Suppose $n \neq m$. Without loss of generality, let $n > m$. By repeated application of extension, $y \sim (y, \mathbf{1}_{n-m})$. Therefore,

$$x \succeq y \;\Leftrightarrow\; x \succeq (y, \mathbf{1}_{n-m}). \tag{8}$$

Using Theorem 2,

$$x \succeq (y, \mathbf{1}_{n-m}) \;\Leftrightarrow\; \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{m} y_i + (n-m) \;\Leftrightarrow\; \sum_{i=1}^{n} x_i - n \leq \sum_{i=1}^{m} y_i - m$$

and, using (8), it follows that $\succeq = \succeq^*$. ■

5 Independence of the Axioms

Before establishing the independence of the axioms imposed on R in the above characterization results, it is shown that the surjectivity of F is essential. Suppose F is such that, for all $S \in \Pi(\Sigma)$ and all $i \in \{1, \ldots, |S|\}$, $|F_i(S)| = 2$. In that case, the universal indifference relation (the relation that declares all elements of $\Pi(\Sigma)$ pairwise indifferent) satisfies all of the axioms introduced in this paper. Note that the only axiom that requires a strict preference to be established between any two menus is expansion monotonicity, which is vacuously satisfied if F only generates non-ordered n-tuples of sets of possible outcomes with cardinality two.

To establish the independence of the axioms used in Theorem 2, consider the following examples.

Choose an arbitrary $\bar{\omega} \in \Omega$, and define the function $f \colon \Pi(\Omega) \to \Re$ by

$$f(A) := \begin{cases} |A| & \text{if } \bar{\omega} \notin A; \\ |A| + 1 & \text{if } \bar{\omega} \in A \end{cases}$$

for all $A \in \Pi(\Omega)$. Now define the ordering R^1 on $\Pi(\Sigma)$ by letting, for all $S, T \in \Pi(\Sigma)$,

$$S\, R^1\, T \;:\Leftrightarrow\; \sum_{i=1}^{|S|} f(F_i(S)) \leq \sum_{i=1}^{|T|} f(F_i(T)).$$

R^1 satisfies independence and expansion monotonicity but violates certainty indifference.

Now let, for all $S, T \in \Pi(\Sigma)$,

$$S\, R^2\, T \;:\Leftrightarrow\; \sum_{i=1}^{|S|} |F_i(S)|^2 - |S| \leq \sum_{i=1}^{|T|} |F_i(T)|^2 - |T|.$$

This ordering satisfies certainty indifference and expansion monotonicity but violates independence.

The universal indifference relation on $\Pi(\Sigma)$ satisfies certainty indifference and independence but violates expansion monotonicity.

Now consider Theorem 3.

Define, for all $S, T \in \Pi(\Sigma)$,

$$S\, R^3\, T \;:\Leftrightarrow\; \sum_{i=1}^{|S|} |F_i(S)| \leq \sum_{i=1}^{|T|} |F_i(T)|.$$

This ordering satisfies expansion monotonicity and extended independence but violates extended certainty indifference (note that certainty indifference is satisfied by this example).

Define, for all $S, T \in \Pi(\Sigma)$,

$$S\, R^4\, T \;:\Leftrightarrow\; \frac{1}{|S|} \sum_{i=1}^{|S|} |F_i(S)| \leq \frac{1}{|T|} \sum_{i=1}^{|T|} |F_i(T)|.$$

R^4 satisfies extended certainty indifference and expansion monotonicity but violates extended independence. Independence is satisfied.

Again, the universal indifference relation can be used to establish that expansion monotonicity is not implied by the other axioms used in Theorem 3.

Finally, the independence of the axioms used in Theorem 4 is established.

The relation R^2 satisfies expansion monotonicity and certainty extension but violates independence.

The universal indifference relation satisfies independence and certainty extension but violates expansion monotonicity.

The relation R^4 satisfies independence and expansion monotonicity but violates certainty extension (note that extended certainty indifference is satisfied).

6 Concluding Remarks

There is an interesting formal connection between the relation R^* and *critical-level utilitarian* social-evaluation principles that can be used to assess policies with population consequences. Critical-level utilitarianism declares a state of the world $\bar{A}$ at least as good as a state $\hat{A}$ from a social viewpoint if and only if

$$\sum_{i\in\bar{N}}[\bar{u}_i - \alpha] \geq \sum_{i\in\hat{N}}[\hat{u}_i - \alpha] \tag{9}$$

where $\bar{N}$ (resp. $\hat{N}$) is the set of individuals alive in $\bar{A}$ (resp. $\hat{A}$), the $\bar{u}_i$ and $\hat{u}_i$ are the individual lifetime utilities of those alive in the respective state, and $\alpha \in \Re$ is a *critical level* of lifetime utility. The critical level is interpreted as that level of utility which, if experienced by an additional individual, leads to a state that is indifferent to the inital state, provided none of the existing individuals are affected by this population augmentation. See, for example, Blackorby, Bossert, and Donaldson (1995) and Blackorby and Donaldson (1984) for discussions of critical-level utilitarian social-evaluation principles. (9) can be rewritten as

$$\sum_{i\in\bar{N}}\bar{u}_i - |\bar{N}|\alpha \geq \sum_{i\in\hat{N}}\hat{u}_i - |\hat{N}|\alpha$$

and, therefore, the inverse relation of R^* has the same formal structure as critical-level utilitarianism with a critical level of $\alpha = 1$. Despite this formal similarity, there are important differences between these criteria that should be pointed out. Whereas the critical level in (9) is an ethical parameter the choice of which is by no means an obvious task, the corresponding value of one in the definition of R^* comes about very naturally by considering the interpretation of R^* as a ranking of non-ordered n-tuples of sets of possible outcomes in a choice situation. Furthermore, the set of possible utility values is usually assumed to be the set $\Re$, whereas the set of objects to be ranked by R^* is a discrete set, which makes the proof techniques employed in axiomatizations of these rankings very different. In addition, the standard axiomatizations in population ethics usually provide characterizations of a more general class of orderings, namely, critical-level *generalized* utilitarian rules, which allow for an increasing and continuous transformation to be applied to all utility numbers. In contrast, the independence condition used in the characterizations of R^* generate additivity rather than merely additive separability. As a technical note, the extension axiom used in the proof of Theorem 4 is akin to Blackorby and Donaldson's (1984) *critical-level population principle.*

The above-described relationship between the ordering characterized in this paper and the critical-level utilitarian population principles suggests that there is further scope for connections to be made across those different areas of research.

References

Arrow, K.J. and Hurwicz, L. (1972), "An optimality criterion for decision-making under ignorance," in: Carter, C.F. and Ford, J.L. (eds.), *Uncertainty and Expectations in Economics: Essays in Honour of G.S.L. Shackle,* Basil Blackwell, Oxford, pp. 1–11.

Bandyopadhyay, T. (1988), "Extensions of an order on a set to the power set: some further observations," *Mathematical Social Sciences* **15**, 81–85.

Barberà, S. (1977), "The manipulation of social choice mechanisms that do not leave "too much" to chance," *Econometrica* **45**, 1573–1588.

Barberà, S., Barrett, C.R., and Pattanaik, P.K. (1984), "On some axioms for ranking sets of alternatives," *Journal of Economic Theory* **33**, 301–308.

Barberà, S. and Pattanaik, P.K. (1984), "Extending an order on a set to the power set: some remarks on Kannai and Peleg's approach," *Journal of Economic Theory* **32**, 185–191.

Blackorby, C., Bossert, W., and Donaldson, D. (1995), "Intertemporal population ethics: critical-level utilitarian principles," *Econometrica* **63**, 1303–1320.

Blackorby, C. and Donaldson, D. (1984), "Social criteria for evaluating population change," *Journal of Public Economics* **25**, 13–33.

Bossert, W. (1989a), "On the extension of preferences over a set to the power set: an axiomatic characterization of a quasi-ordering," *Journal of Economic Theory* **49**, 84–92.

Bossert, W. (1989b), "On the extension of preferences over a set to the power set: a generalization," *mimeo,* Universität Karlsruhe.

Bossert, W. (1995a), "Preference extension rules for ranking sets of alternatives with a fixed cardinality," *Theory and Decision* **39**, 301–317.

Bossert, W. (1995b), "Choices, consequences, and rationality," *Working Paper* **9501**, University of Waterloo, Department of Economics.

Bossert, W. (1997a), "Opportunity sets and individual well-being," *Social Choice and*

Welfare **14**, 97–112.

Bossert, W. (1997b), "Uncertainty aversion in nonprobabilistic decision models," *Mathematical Social Sciences* **34**, 191–203.

Bossert, W., Pattanaik, P.K., and Xu, Y. (1994), "Ranking opportunity sets: an axiomatic approach," *Journal of Economic Theory* **63**, 326–345.

Dutta, B. and Sen, A. (1996), "Ranking opportunity sets and Arrow impossibility theorems: correspondence results," *Journal of Economic Theory* **71**, 90–101.

Fishburn, P.C. (1984), "Comment on the Kannai-Peleg impossibility theorem for extending orders," *Journal of Economic Theory* **32**, 176–179.

Fishburn, P.C. (1992), "Signed orders and power set extensions," *Journal of Economic Theory* **56**, 1–19.

Gravel, N. (1994), "Can a ranking of opportunity sets attach an intrinsic importance to freedom of choice?," *American Economic Review* **84**, 454–458.

Gravel, N., Laslier, J.-F., and Trannoy, A. (1998), "Individual freedom of choice in a social setting," in: Laslier, J.-F., Fleurbaey, M., Gravel, N., and Trannoy, A. (eds.), *Freedom in Economics: New Perspectives in Normative Analysis,* Routledge, London, pp. 76–92.

Heiner, R. and Packard, D. (1984), "A uniqueness result for extending orders; with applications to collective choice as inconsistency resolution," *Journal of Economic Theory* **32**, 180–184.

Holzman, R. (1984a), "A note on the redundancy of an axiom in the Pattanaik-Peleg characterization of the lexicographic maximin extension," *Social Choice and Welfare* **1**, 123–125.

Holzman, R. (1984b), "An extension of Fishburn's theorem on extending orders," *Journal of Economic Theory* **32**, 192–196.

Kannai, Y. and Peleg, B. (1984), "A note on the extension of an order on a set to the power set," *Journal of Economic Theory* **32**, 172–175.

Klemisch-Ahlert, M. (1993), "Freedom of choice: a comparison of different rankings of opportunity sets," *Social Choice and Welfare* **10**, 189–207.

Kranich, L. (1996), "Equitable opportunities: an axiomatic approach," *Journal of Economic Theory* **71**, 131–147.

KREPS, D. (1979), "A representation theorem for 'preference for flexibility,'" *Econometrica* **47**, 565–577.

MASKIN, E. (1979), "Decision-making under ignorance with implications for social choice," *Theory and Decision* **11**, 319–337.

NITZAN, S. AND PATTANAIK, P.K. (1984), "Median-based extensions of an ordering over a set to the power set: an axiomatic characterization," *Journal of Economic Theory* **34**, 252–261.

PATTANAIK, P.K. AND PELEG, B. (1984), "An axiomatic characterization of the lexicographic maximin extension of an ordering over a set to the power set," *Social Choice and Welfare* **1**, 113–122.

PATTANAIK, P.K. AND XU, Y. (1990), "On ranking opportunity sets in terms of freedom of choice," *Recherches Economiques de Louvain* **56**, 383–390.

PATTANAIK, P.K. AND XU, Y. (1997), "On diversity and freedom of choice," *Discussion Paper* **97/18**, University of Nottingham, Department of Economics.

PUPPE, C. (1995), "Freedom of choice and rational decisions," *Social Choice and Welfare* **12**, 137–153.

PUPPE, C. (1996), "An axiomatic approach to 'preference for freedom of choice,'" *Journal of Economic Theory* **68**, 174–199.

PUPPE, C. (1998), "Individual freedom and social choice," in: LASLIER, J.-F., FLEURBAEY, M., GRAVEL, N., AND TRANNOY, A. (eds.), *Freedom in Economics: New Perspectives in Normative Analysis,* Routledge, London, pp. 49–68.

SEN, A. (1988), "Freedom of choice: concept and content," *European Economic Review* **32**, 269–294.

SEN, A. (1991), "Welfare, preference and freedom," *Journal of Econometrics* **50**, 15–29.

SEN, A. (1993), "Internal consistency of choice," *Econometrica* **61**, 495–521.