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Regression-based seasonal unit root tests

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Regression-Based Seasonal Unit Root Tests*

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Abstract

The contribution of this paper is three-fold. Firstly, a characterisation theorem of the sub-hypotheses comprising the seasonal unit root hypothesis is presented which provides a precise formulation of the alternative hypotheses against which regression-based seasonal unit root tests test. Secondly, it proposes regression-based tests for the seasonal unit root hypothesis which allow a general seasonal aspect for the data and are similar both exactly and asymptotically with respect to initial values and seasonal drift parameters. Thirdly, limiting distribution theory is given for these statistics where, in contrast to previous papers in the literature, in doing so it is not assumed that unit roots hold at all of the zero and seasonal frequencies. This is shown to alter the large sample null distribution theory for regression t-statistics for unit roots at the complex frequencies, but interestingly to not affect the limiting null distributions of the regression t-statistics for unit roots at the zero and Nyquist frequencies and regression F-statistics for unit roots at the complex frequencies. Our results therefore have important implications for how tests of the seasonal unit root hypothesis should be conducted in practice. Associated simulation evidence on the size and power properties of the statistics presented in this paper is given which is consonant with the predictions from the large sample theory.

Keywords: Seasonal unit root tests; seasonal drifts; characterisation theorem.

JEL Classification: C22.

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1 Introduction

This paper considers testing for *seasonal* unit roots in a univariate time-series process; that is, whether or not the *data generating process* (DGP) for the time series admits unit roots at the zero, Nyquist and harmonic seasonal frequencies. We allow a general seasonal aspect for the data and permit the drift parameter to vary across the seasons under the null hypothesis that the data is generated by a seasonal random walk DGP. Differential seasonal drift allows the amplitude of the variation across the seasons of the deterministic component in the level of the time-series to vary (linearly) through time in order that its relative magnitude may be maintained. Constant drift implies that the amplitude is constant over time. The tests considered in this paper are similar, both exactly and asymptotically, with respect to the possibility of differential seasonal drift as well as to initial conditions.

Existing tests for seasonal unit roots are predicated on more restricted specifications for the deterministics or are confined to consideration of particular values for the seasonal aspect of the data. Hylleberg *et al.* (1990), HEGY henceforth, develop regression-based tests for seasonal unit roots in a quarterly context as separate t- and F-tests for unit roots at the zero, Nyquist and annual frequencies. They consider regressions which may include an intercept, seasonal intercepts and a trend variable. Depending on the DGP when seasonal unit roots are present and which regression formulation is adopted, the corresponding statistics will be similar, both exactly and asymptotically, with respect to particular nuisance parameters, for example, initial values and the value of the drift parameter in the DGP. Ghysels *et al.* (1994) (GLN) provide critical values for other F-statistics of interest in the quarterly context. Beaulieu and Miron (1993) (BM) discuss the corresponding test statistics appropriate for monthly data. Smith and Taylor (1998) (ST1) and Taylor (1998) have generalized the regression-based approach of HEGY and BM to allow for differential seasonal drift in quarterly and monthly scenarios respectively. However, the HEGY tests are not similar (either *exactly* or *asymptotically*) with respect to drift which displays seasonal variation. As noted by GLN (p.436), the safer strategy in applications is to include potentially irrelevant deterministics in order to avoid erroneous inferences.

Section 2 of the paper sets out the problem for a univariate time series whose DGP is specified as a general autoregression (AR). Section 3 presents a decomposition of the AR polynomial in terms of polynomials which isolate the possible unit roots corresponding to each spectral frequency. Section 4 develops a regression-based approach to testing for seasonal unit roots and explores the impact of the decomposition of section 3 on testing for seasonal unit roots. Section 5 details representations for the asymptotic null distributions of the seasonal unit root statistics. Here, and in contrast to previous papers in the literature, we do not assume in deriving our results that the unit root null hypothesis holds at all of the zero and seasonal frequencies. Relative to the case where unit roots are present at all frequencies, this is shown to alter the large sample null distribution theory for regression t-statistics for unit roots at the complex frequencies, but to not affect the limiting null distributions of the regression t-statistics for unit roots at the zero and Nyquist frequencies and regression F-statistics for unit roots at the complex frequencies. The limiting null distributions of joint F-statistics for the null hypothesis of unit roots at all of the seasonal frequencies, and at both the zero and all of the seasonal frequencies are also both unaffected. A Monte Carlo investigation into the finite sample properties of the statistics is provided in Section 5 and shows that the limiting distribution theory provides a useful prediction for small sample behaviour. Our work therefore provides a useful complement to Taylor (2003) who showed that the limiting null distributions of the seasonal unit root tests of Canova and Hansen (1992) are not pivotal in the presence of unattended unit roots at the zero or seasonal frequencies. Our results enable us to make firm recommendations on which of the available statistics should (and which should not) be used in practice. Section 6 concludes the paper. An appendix contains the proofs of our main results and some general results on circulant matrices.

2 The Problem

Let S denote the seasonal order of the data; for example, if the time-series is quarterly, $S = 4$, or, if monthly, $S = 12$. The model for the univariate time series process $\{x_{St+s}\}$ is given by:

$$\alpha(L)\left[x_{St+s} - \gamma_s^* - \delta_s^*(St+s)\right] = u_{St+s}, s = 1-S, ..., 0, t = 1, 2, \ldots, \tag{2.1}$$

where $\alpha(z)$ is an autoregressive (AR) polynomial of order S, $\alpha(z) \equiv 1 - \sum_{j=1}^{S} \alpha_j^{**} z^j$. We adopt the following convention for the lag operator L here and throughout the paper; *viz.* L operates on the process $\{x_{St+s}\}$ in the standard manner, $L^{Sj+k}x_{St+s} = x_{S(t-j)+s-k}$, whereas, for the purely seasonally varying coefficients, $L^{Sj+k}\gamma_s^*(\equiv \gamma_{s-Sj-k}^*) = \gamma_{s-k}^*$, $L^{Sj+k}\delta_s^*(\equiv \delta_{s-Sj-k}^*) = \delta_{s-k}^*$ if $1-S \leq s-k \leq 0$ and γ_{S+s-k}^* and δ_{S+s-k}^* otherwise, $k = 0, ..., S-1$, $j = 1, 2, ...$ The specification (2.1) allows for the presence of differential seasonal intercept and time-trend terms *via* γ_s^* and δ_s^* respectively, $s = 1-S, ..., 0$. The error process $\{u_{St+s}\}$ in (2.1) is assumed to follow an $AR(p)$ process; viz.,

$$\phi(L)u_{St+s} = \epsilon_{St+s}, \tag{2.2}$$

where $\phi(z) \equiv 1 - \sum_{i=1}^{p} \phi_i z^i$ is a stationary (the roots of $\phi(z) = 0$ all lie outside the unit circle $|z| = 1$) AR polynomial of order p, $0 \leq p < \infty$, and $\{\epsilon_{St+s}\}$ a martingale difference sequence (MDS) with constant conditional variance, σ^2; see Fuller (1996, Theorem 5.3.5,pp.236-37) for precise assumptions on $\{\epsilon_{St+s}\}$. For notational convenience, we define S^* as $(S/2) - 1$ (if S is even) and $[S/2]$ (if S is odd), where $[S/2]$ is the integer part of $S/2$.

Combining the $AR(S)$ process (2.1) for $\{x_{St+s}\}$ with the $AR(p)$ process (2.2) for the error process $\{u_{St+s}\}$, we may re-cast the $AR(S+p)$ process $\{x_{St+s}\}$ alternatively as:

$$\alpha^*(L)x_{St+s} = \gamma_s^{**} + \delta_s^{**}(St+s) + \epsilon_{St+s}, s = 1-S, ..., 0, t = 1, 2, \ldots, \tag{2.3}$$

where the $AR(S+p)$ polynomial $\alpha^*(z) \equiv \alpha(z)\phi(z) = 1 - \sum_{i=1}^{S+p} \alpha_i^* z^i$, $\gamma_s^{**} \equiv (\gamma_s^* - \sum_{j=1}^{S+p} \alpha_j^* \gamma_{s-j}^*) + \sum_{j=1}^{S+p} j\alpha_j^* \delta_{s-j}^*$ and $\delta_s^{**} \equiv (\delta_s^* - \sum_{j=1}^{S+p} \alpha_j^* \delta_{s-j}^*)$.

This paper is concerned with regression-based tests for seasonal unit roots in the autoregressive $AR(S)$ lag polynomial $\alpha(z)$; that is, the null hypothesis of interest is

$$H_0 : \alpha(z) = 1 - z^S; \tag{2.4}$$

the corresponding alternative hypothesis H_1 is stated in section 3.

The hypothesis H_0 of (2.4) induces $\gamma_s^{**} = S(\delta_s^* - \sum_{j=1}^{p} \phi_j \delta_{s-j}^*)$ and $\delta_s^{**} = 0$ in (2.3) as $\gamma_s^* \equiv \gamma_{s-S}^*$ and $\delta_s^* \equiv \delta_{s-S}^*$, $s = 1 - S, ..., 0$. Thus, the DGP for $\{x_{St+s}\}$ is that of a seasonal unit root process with seasonal drifts

$$\Delta_S x_{St+s} = \gamma_s^{**} + u_{St+s}, s = 1 - S, ..., 0, t = 1, 2, \ldots, \tag{2.5}$$

where $\Delta_S \equiv 1 - L^S$ is the seasonal difference operator. Solving (2.5) for the level process $\{x_{St+s}\}$ results in

$$x_{St+s} = x_s + \gamma_s^{**} t + \sum_{v=1}^{t} u_{Sv+s}, s = 1 - S, ..., 0, t = 1, 2, \ldots. \tag{2.6}$$

Under H_0, there is no deterministic trend present in the DGP of the differenced process $\{\Delta_S x_{St+s}\}$ and, therefore, deterministic *linear* trends in the level process $\{x_{St+s}\}$ through the presence of the differential seasonal drifts, γ_s^{**}, $s = 1 - S, ..., 0$, indicated by (2.6). Moreover, the specification (2.1)-(2.2) and (2.3) ensures that the deterministic trending behaviour of the level process $\{x_{St+s}\}$ is linear under both H_0 and the alternative hypothesis when $\alpha(z)$ is a stationary $AR(S)$ polynomial as is evident from a comparison of (2.3) and (2.6). Consequently, we can see from a comparison of (2.3) and (2.6) that the level process $\{x_{St+s}\}$ will display similar deterministic *linear* trending behaviour in both trend-stationary and seasonal difference-stationary environments, that is, whether or not the seasonal unit root hypothesis holds. Moreover, from (2.6), the amplitude of the seasonal variation of the deterministic component $x_s + \gamma_s^{**} t$ of $\{x_{St+s}\}$ is permitted to vary through time in a linear fashion, $s = 1 - S, ..., 0$, $t = 1, 2, \ldots$.

We may also identify other scenarios of interest within (2.1), and hence (2.3), which are special cases thereof: (i) no intercept, no time-trend: $\gamma_s^* = 0$, $\delta_s^* = 0$, $s = 1-S, ..., 0$; (ii) intercept, no time-trend: $\gamma_s^* = \gamma^*$, $s = 1 - S, ..., 0$, $\delta_s^* = 0$, $s = 1 - S, ..., 0$; (iii) seasonal intercepts, no time-trend: $\delta_s^* = 0$, $s = 1 - S, ..., 0$; (iv) intercept, time-trend: $\gamma_s^* = \gamma^*$, $s = 1 - S, ..., 0$, $\delta_s^* = \delta^*$, $s = 1 - S, ..., 0$; (v) seasonal intercepts, time-trend: $\delta_s^* = \delta^*$, $s = 1 - S, ..., 0$.

3 A Characterisation Theorem

In formulating regression-based tests for seasonal unit roots, previous studies employ an alternative representation for the polynomial $\alpha^*(z) \equiv \alpha(z)\phi(z)$; see, for example, HEGY (Proposition, p.221) and the local expansion approach of ST1 ((2.9), p.272) and Smith and Taylor (1999, (2.11)), ST2 henceforth. This representation for $\alpha^*(z)$ employs filters which remove the possibility of seasonal unit roots in $\{x_{St+s}\}$ apart from at the seasonal frequency of interest. However, although the decomposition for $\alpha^*(z)$ given in these papers enables one to identify particular resultant regressors with specific sub-hypotheses of H_0 of (2.4) defined at the seasonal frequencies, the corresponding sub-hypotheses forming the alternative against which the null hypothesis H_0 is being tested remain unclear. Proposition 3.1 below resolves this ambiguity and enables an exact description of the alternative hypotheses examined by the test statistics of earlier research and those discussed here.

We denote the seasonal frequencies as $\omega_k \equiv 2\pi k/S$, $k = 0, ..., [S/2]$. Therefore, letting $i \equiv \sqrt{(-1)}$, we factorise the $AR(S)$ polynomial $\alpha(L)$ at the seasonal frequencies as:

$$\alpha(L) = \prod{}_{k=0}^{[S/2]} \omega_k(L), \tag{3.1}$$

where the lag polynomial

$$\omega_0(L) \equiv (1 - \alpha_0 L) \tag{3.2}$$

associates the parameter α_0 with the zero frequency $\omega_0 \equiv 0$, the lag polynomials $\omega_k(L)$, $k = 1, ..., S^*$, in (3.1) correspond to the conjugate seasonal frequencies $(\omega_k, 2\pi - \omega_k)$, respectively, whose roots occur as the conjugate pair $\alpha_k \pm \beta_k i$, and are defined by

$$\begin{aligned}\omega_k(L) &\equiv [1 - (\alpha_k + \beta_k i)\exp(i\omega_k) L]\,[1 - (\alpha_k - \beta_k i)\exp(-i\omega_k) L] \\ &= 1 - 2(\alpha_k \cos\omega_k - \beta_k \sin\omega_k) L + (\alpha_k^2 + \beta_k^2) L^2,\end{aligned} \tag{3.3}$$

with corresponding parameters α_k and β_k, $k = 1, ..., S^*$, together with

$$\omega_{S/2}(L) \equiv (1 + \alpha_{S/2} L), \tag{3.4}$$

with the Nyquist frequency $\omega_{S/2} \equiv \pi$ parameter $\alpha_{S/2}$ when S is even.

Consequently, the null hypothesis H_0 of (2.4) may be partitioned as

$$H_0 = \left(\cap_k H_{0,k}^{\alpha}\right) \cap \left(\cap_k H_{0,k}^{\beta}\right) \tag{3.5}$$

where $H_{0,k}^{\alpha} : \alpha_k = 1, k = 0, ..., [S/2]$ and $H_{0,k}^{\beta} : \beta_k = 0, k = 1, ..., S^*$. The hypothesis $H_{0,0}^{\alpha} : \alpha_0 = 1$ corresponds to a unit root at the zero-frequency $\omega_0 = 0$, while $H_{0,S/2}^{\alpha} : \alpha_{S/2} = 1$ yields a unit root at the Nyquist frequency $\omega_{S/2} = \pi$. A unit root at the conjugate seasonal frequencies $(\omega_k, 2\pi - \omega_k)$ is obtained under $H_{0,k} \equiv H_{0,k}^{\alpha} \cap H_{0,k}^{\beta}$, that is, $H_{0,k} : \alpha_k = 1, \beta_k = 0$, $k = 1, ..., S^*$.

The alternative hypothesis, denoted H_1, may be succinctly stated as $H_1 = \cup_{k=0}^{[S/2]} H_{1,k}$ where the sub-hypotheses $H_{1,0} : \alpha_0 < 1$, $H_{1,k} : \alpha_k^2 + \beta_k^2 < 1$, $k = 1, ..., S^*$, and

$H_{1,S/2} : \alpha_{S/2} < 1$, if S is even. Notice, therefore, that the maintained hypothesis $H_0 \cup H_1$ excludes the possibility of unit roots at all frequencies other than ω_k, $k = 0, ..., [S/2]$, and permits the possibility of a single unit root at frequencies ω_k, $k = 0, [S/2]$, and a complex conjugate pair of unit roots at each of the harmonic seasonal frequencies ω_k, $k = 1, ..., S^*$.

Under $H_{0,k} = H^{\alpha}_{0,k} \cap H^{\beta}_{0,k}$, the polynomial $\omega_k(z)$ of (3.3) becomes

$$\begin{aligned}\omega^0_k(z) &\equiv [1 - \exp(i\omega_k)z][1 - \exp(-i\omega_k)z] \\ &= 1 - 2\cos\omega_k z + z^2,\end{aligned}$$

$k = 1, ..., S^*$. Similarly, under $H^{\alpha}_{0,0} : \alpha_0 = 1$, $\omega_0(z)$ of (3.2) reduces to $\omega^0_0(z) \equiv 1 - z$, and, if S is even, under $H^{\alpha}_{0,S/2} : \alpha_{S/2} = 1$, $\omega_{S/2}(z)$ of (3.4) is $\omega^0_{S/2}(z) \equiv 1 + z$.

A more convenient parameterisation for the null hypothesis H_0 of (2.4) and its constituent sub-hypotheses $H_{0,k}$, $k = 0, ..., [S/2]$, than that given in (3.2)-(3.4) may be obtained in terms of deviations of *length* and *phase* from the seasonal unit roots $\exp(\pm i\omega_k)$, $k = 0, ..., [S/2]$. Define the length $r_k \equiv (\alpha^2_k + \beta^2_k)^{1/2}$ and the phase shift $\theta_k \equiv \tan^{-1}(\beta_k/\alpha_k)$. Hence, from (3.3),

$$\begin{aligned}\omega_k(z) &= [1 - r_k \exp[i(\theta_k + \omega_k)]z][1 - r_k \exp[-i(\theta_k + \omega_k)]z] \\ &= 1 - 2r_k \cos(\theta_k + \omega_k)z + r^2_k z^2. \qquad (3.6)\end{aligned}$$

Therefore, from (3.6), we may express

$$\omega_k(z) = \omega^0_k(z) + \gamma_k[-(\cos\omega_k - z)z] + \delta_k[\sin\omega_k z],$$

where

$$\gamma_k = r^2_k - 1, \delta_k = (\sin\omega_k)^{-1}\left((r^2_k - 1)\cos\omega_k - 2[r_k \cos(\theta_k + \omega_k) - \cos\omega_k]\right), \qquad (3.7)$$

$k = 1, ..., S^*$. For frequencies ω_0 and $\omega_{S/2}$, $\omega_0(z) = \omega^0_0(z) - \gamma_0 z$, where $\gamma_0 \equiv r_0 - 1$, and $\omega_{S/2}(z) = \omega^0_{S/2}(z) + \gamma_{S/2} z$, where $\gamma_{S/2} \equiv r_{S/2} - 1$. In (3.7), the parameter γ_k represents the deviation of the length of the kth root from unity and δ_k is a combination of the length deviation γ_k and the phase shift θ_k, $k = 1, ..., S^*$. Thus, the seasonal unit root hypothesis $H_{0,k}$ may be re-expressed as $H_{0,k} = H^{\gamma}_{0,k} \cap H^{\delta}_{0,k}$, with

$$H^{\gamma}_{0,k} : \gamma_k = 0, H^{\delta}_{0,k} : \delta_k = 0,$$

where the sub-hypothesis $H^{\gamma}_{0,k} : \gamma_k = 0$ is the unit root length restriction $r^2_k = 1$ and, conditional on $H^{\gamma}_{0,k} : \gamma_k = 0$, $H^{\delta}_{0,k} : \delta_k = 0$ is the phase shift restriction $\theta_k = 0$ in (3.6) as, under $\gamma_k = 0$, $\delta_k = -2[\cos(\theta_k + \omega_k) - \cos\omega_k]/\sin\omega_k$, $k = 1, ..., S^*$. Correspondingly, the alternative hypothesis $H_{1,k}$ may be simply expressed as $H^{\gamma}_{1,k} : \gamma_k < 0$, $k = 0, ..., [S/2]$.

The following proposition defines a decomposition of the $AR(S + p)$ polynomial $\alpha^*(z)$ in terms of polynomials $\Delta^0_k(z)$, defined below, associated with the seasonal frequencies ω_k, $k = 0, ..., [S/2]$. In particular, the polynomial $\Delta^0_k(z)$ involves seasonal filters which, under the maintained hypothesis $H_0 \cup H_1$, remove the possible presence of seasonal unit roots at all seasonal frequencies ω_j, $j \neq k$, apart from ω_k, $k = 0, ..., [S/2]$.

Proposition 1 *(Characterisation Theorem.) The polynomial* $\alpha^*(z) \equiv \alpha(z)\phi(z)$ *may be expressed as*

$$\alpha^*(z) = \phi^*(z)\Delta_S(z) - \pi_0^* z\Delta_0^0(z) - \pi_{S/2}^* z\Delta_{S/2}^0(z)$$
$$-\sum_{k=0}^{[S/2]} \left(\pi_{k,\alpha}^*[-(\cos\omega_k - z)z] + \pi_{k,\beta}^*[\sin\omega_k z]\right)\Delta_k^0(z), \quad (3.8)$$

dropping the term $-\pi_{S/2}^* z\Delta_{S/2}^0(z)$ *if* S *is odd, where*

$$\pi_0^* \equiv -\Lambda_0^*\gamma_0, \pi_{S/2}^* \equiv \Lambda_{S/2}^*\gamma_{S/2},$$
$$\pi_{k,\alpha}^* \equiv \mathcal{R}e(\Lambda_k^*)\gamma_k - \mathcal{I}m(\Lambda_k^*)\delta_k, \pi_{k,\beta}^* \equiv \mathcal{R}e(\Lambda_k^*)\delta_k + \mathcal{I}m(\Lambda_k^*)\gamma_k, \quad (3.9)$$

$k = 1, ..., S^*$, $\phi^*(z) \equiv 1 - \sum_{i=1}^{p} \phi_i^* z^i$ *is a stationary polynomial, the parameters* γ_k *and* δ_k *are defined in (3.7),* $\Delta_k^0(z) \equiv -\prod_{j\neq k,j=0}^{[S/2]} \omega_k^0(z)$ *and*

$$\Lambda_k^* \equiv \frac{\Delta_k[\exp(-i\omega_k)]}{\Delta_k^0[\exp(-i\omega_k)]}\phi[\exp(-i\omega_k)],$$

$\Delta_k(z) \equiv -\prod_{j\neq k,j=0}^{[S/2]} \omega_k(z)$, $k = 0, ..., [S/2]$.

Remark 1: Observe that

$$\Delta_0^0(z) \equiv -\left(1 + z + ... + z^{S-1}\right), \Delta_{S/2}^0(z) \equiv -\left(1 - z + z^2 - ... - z^{S-1}\right),$$

$$\Delta_k^0(z) = -\frac{1}{\sin\omega_k}\sum_{j=0}^{S-1} \sin[(j+1)\omega_k]z^j, \quad (3.10)$$

$k = 1, ..., S^*$; see ST2 (equation (2.16)). □

It is immediate from Proposition 1 that the sub-hypotheses $H_{0,k} = H_{0,k}^\gamma \cap H_{0,k}^\delta$ of a seasonal unit root at frequency ω_k, $k = 0, ..., [S/2]$, may be simply re-stated from (3.9) in the form

$$H_{0,0} : \pi_0^* = 0, H_{0,S/2} : \pi_{S/2}^* = 0,$$
$$H_{0,k} : \pi_{k,\alpha}^* = \pi_{k,\beta}^* = 0, \quad (3.11)$$

$k = 1, ..., S^*$. However, the corresponding sub-hypotheses $H_{1,k}^\gamma : \gamma_k < 0$ of the alternative hypothesis $H_1 = \cup_{k=0}^{[S/2]} H_{1,k}^\gamma$ may not be simply expressed in terms of the coefficients $\pi_{k,\alpha}^*$ and $\pi_{k,\beta}^*$, $k = 1, ..., S^*$, of (3.9).

The polynomials $z\Delta_0^0(z)$, $z\Delta_{S/2}^0(z)$, $-(\cos\omega_k - z)z\Delta_k^0(z)$ and $\sin\omega_k z\Delta_k^0(z)$, $k = 1, ..., S^*$, in (3.8) are precisely those employed in earlier research when constructing regression-based t- and F-tests for seasonal unit roots; see *inter alia* HEGY, GLN, BM and ST1. However, Proposition 1 emphasises that the corresponding regression coefficients $\pi_{k,\alpha}^*$ and $\pi_{k,\beta}^*$ of (3.9) are *both* (linear) functions of the length deviation parameter γ_k and the composite length deviation and phase shift parameter δ_k, $k = 1, ..., S^*$. This observation holds even in the special case when $\Delta_k(z) = \Delta_k^0(z)$, that is, there are unit roots at all other seasonal frequencies $j \neq k$, and, thus, $\mathcal{R}e(\Lambda_k^*) = \mathcal{R}e(\phi[\exp(-i\omega_k)])$ and $\mathcal{I}m(\Lambda_k^*) = \mathcal{I}m(\phi[\exp(-i\omega_k)])$. However, if the order of the AR polynomial $\phi(z)$ is also zero, that is, $p = 0$, then $\mathcal{R}e(\Lambda_k^*) = 1$ and $\mathcal{I}m(\Lambda_k^*) = 0$ in which case

$$\begin{aligned}\alpha^*(z) &= \Delta_S(z) - [\omega_k(z) - \omega_k^0(z)]\Delta_k^0(z) \\ &= \Delta_S(z) - \gamma_k[-(\cos\omega_k - z)z]\Delta_k^0(z) - \delta_k[\sin\omega_k z]\Delta_k^0(z).\end{aligned}$$

Therefore one can uniquely identify the unit root length restriction $\gamma_k = 0$ with the polynomial $-(\cos\omega_k - z)z\Delta_k^0(z)$ and, conditional on $\gamma_k = 0$, the phase shift restriction $\theta_k = 0$ with the polynomial $\sin\omega_k z\Delta_k^0(z)$, $k = 1, ..., S^*$. We comment further on the impact of these results for testing H_0 of (2.4) in the discussion of section 4.

For completeness, we detail the relationship between the general decomposition for $\alpha^*(z)$ given in Proposition 1 and the local expansion approach of, for example, ST1 and ST2. In order to do so we re-parameterise (3.1)-(3.3) as $\alpha_0 = 1 + \pi_0$, $\alpha_k = 1 + (\pi_{k,\alpha}/2)$, $\beta_k = \pi_{k,\beta}/2$, $k = 1, ..., S^*$, and, if S is even, $\alpha_{S/2} = 1 + \pi_{S/2}$. Hence, $\gamma_0 = \pi_0$, $\gamma_k = \pi_{k,\alpha} + o(\pi_{k,\alpha})$, $\delta_k = \pi_{k,\beta} + o(\pi_{k,\beta})$, $k = 1, ..., S^*$, and, if S is even, $\gamma_{S/2} = \pi_{S/2}$. We therefore obtain the following result.

Corollary 1 *(Local Expansion Characterisation Theorem.) The polynomial $\alpha^*(z) \equiv \alpha(z)\phi(z)$ may be expressed as*

$$\alpha^*(z) = \phi^*(z)\Delta_S(z) - \pi_0^* z\Delta_0^0(z) - \pi_{S/2}^* z\Delta_{S/2}^0(z)$$

$$-\sum_{k=1}^{S^*}\left(\pi_{k,\alpha}^*[-(\cos\omega_k - z)z] + \pi_{k,\beta}^*[\sin\omega_k z]\right)\Delta_k^0(z) + o(\{\pi_{k,\alpha}, \pi_{k,\beta}\}),$$

dropping the term $-\pi_{S/2}^* z\Delta_{S/2}^0(z)$ if S is odd, where

$$\pi_0^* \equiv -\phi(1)\pi_0, \pi_{S/2}^* \equiv \phi(-1)\pi_{S/2},$$

$$\begin{aligned}\pi_{k,\alpha}^* &\equiv \mathcal{R}e(\phi[\exp(-i\omega_k)])\pi_{k,\alpha} - \mathcal{I}m(\phi[\exp(-i\omega_k)])\pi_{k,\beta}, \\ \pi_{k,\beta}^* &\equiv \mathcal{R}e(\phi[\exp(-i\omega_k)])\pi_{k,\beta} + \mathcal{I}m(\phi[\exp(-i\omega_k)])\pi_{k,\alpha},\end{aligned}$$

$k = 1, ..., S^*$, and $\phi^*(z) \equiv 1 - \sum_{i=1}^{p}\phi_i^* z^i$ is a stationary polynomial.

As above, it is only in the case when the order of the AR polynomial $\phi(z)$ is zero, that is, $p = 0$, and, thus, $\mathcal{R}e(\phi[\exp(-i\omega_k)]) = 1$ and $\mathcal{I}m(\phi[\exp(-i\omega_k)]) = 0$, that in the local expansion of Corollary 3.1 one can uniquely identify the parameter $\pi_{k,\alpha}$ with the polynomial $-(\cos\omega_k - z)z\Delta_k^0(z)$ and the parameter $\pi_{k,\beta}$ with the polynomial $\sin\omega_k z\Delta_k^0(z)$, $k = 1, ..., S^*$; cf. ST1 and ST2.

4 Testing for Seasonal Unit Roots

Given the decomposition for $\alpha^*(z)$ detailed in Proposition 1, we substitute (3.8) into (2.3) to obtain the linear regression

$$\phi^*(L)\Delta_S x_{St+s} = \gamma_s^{**} + \delta_s^{**}(St+s) + \pi_0^* x_{0,St+s-1} + \pi_{S/2}^* x_{S/2,St+s-1}$$

$$+\sum_{k=1}^{S^*}\left(\pi_{k,\alpha}^* x_{k,St+s-1}^{\alpha} + \pi_{k,\beta}^* x_{k,St+s-1}^{\beta}\right) + \epsilon_{St+s}, \qquad (4.1)$$

dropping the term $\pi_{S/2}^* x_{S/2,St+s-1}$ if S is odd, $s = 1-S,...,0$, $t = 1,2,...$, with the transformed level variables $x_{0,St+s}$, $x_{S/2,St+s}$, $x_{k,St+s}^{\alpha}$ and $x_{k,St+s}^{\beta}$, $k = 1,...,S^*$, defined from (3.10) by

$$x_{0,St+s} = \sum_{j=0}^{S-1} x_{St+s-j}, x_{S/2,St+s} = \sum_{j=0}^{S-1}\cos[(j+1)\pi]x_{St+s-j},$$

$$x_{k,St+s}^{\alpha} = \sum_{j=0}^{S-1}\cos[(j+1)\omega_k]x_{St+s-j}, x_{k,St+s}^{\beta} = -\sum_{j=0}^{S-1}\sin[(j+1)\omega_k]x_{St+s-j}, \qquad (4.2)$$

$k = 1,...,S^*$. For quarterly, $S = 4$, and monthly, $S = 12$, data, the relevant transformations are given in ST2. In what follows we assume that the investigator has available sufficient $(p+S)$ pre-sample values of the lagged seasonal differences $\{\Delta_S x_{St+s}\}_{t\leq 0}$ to accommodate the pth order AR polynomial $\phi^*(z)$ and the seasonal difference operator Δ_S together with the observations x_{St+s}, $s = 1-S,...,0$, $t = 1,...,T$.

The auxiliary regression (4.1) provides the basis of a testing procedure for the presence or otherwise of unit roots at the kth seasonal frequency; that is, $H_{0,k}$ of (3.11) against $H_{1,k}^{\gamma} : \gamma_k < 0$, $k = 0,...,[S/2]$. In particular, in order to test whether or not a unit root is present at the kth seasonal frequency necessitates a test for the exclusion of the regressor $x_{0,St+s-1}$ if $k = 0$, $x_{S/2,St+s-1}$ if $k = S/2$, and both $x_{k,St+s-1}^{\alpha}$ and $x_{k,St+s-1}^{\beta}$ if $k = 1,...,S^*$, in (4.1).

In the quarterly context, $S = 4$, HEGY consider t-tests for the individual hypotheses $H_{0,0}^{\pi} : \pi_0^* = 0$, $H_{0,1}^{\pi_\alpha} : \pi_{1,\alpha}^* = 0$, $H_{0,1}^{\pi_\beta} : \pi_{1,\beta}^* = 0$ and $H_{0,2}^{\pi} : \pi_2^* = 0$ together with an F-test for $H_{0,1} = H_{0,1}^{\pi_\alpha} \cap H_{0,1}^{\pi_\beta}$, that is, $\pi_{1,\alpha}^* = \pi_{1,\beta}^* = 0$, in the auxiliary regression (4.1) with a common trend parameter $\delta_s = \delta$, $s = 1-S,...,0$. HEGY argue that a two-sided t-test of $H_{0,1}^{\pi_\beta} : \pi_{1,\beta}^* = 0$ is appropriate and that, if $H_{0,1}^{\pi_\beta}$ is accepted, one should continue with a one-sided t-test of $H_{0,1}^{\pi_\alpha} : \pi_{1,\alpha}^* = 0$ against $H_{1,1}^{\pi_\alpha} : \pi_{1,\alpha}^* < 0$. However, as detailed in Proposition 1, the coefficients $\pi_{1,\alpha}^*$ and $\pi_{1,\beta}^*$ are generally a linear composite of the length deviation parameter γ_1 and the parameter δ_1 which is a function of the phase shift parameter θ_1 but also involves γ_1 (unless $\gamma_1 = 0$); see (3.7). It is only in the presence of unit roots at *both* frequencies ω_0 and ω_2 *and* when the AR polynomial $\phi(z)$ of (2.2) is of order $p = 0$ that $\pi_{1,\alpha}^* = \gamma_1$ and $\pi_{1,\beta}^* = \delta_1$. These circumstances would suggest *firstly* conducting a one-sided t-test for $H_{0,1}^{\pi_\alpha} : \pi_{1,\alpha}^* = 0$

against $H_{1,1}^{\pi_\alpha} : \pi_{1,\alpha}^* < 0$ and *secondly*, if $H_{0,1}^{\pi_\alpha} : \pi_{1,\alpha}^* = 0$ is accepted in which case $\pi_{1,\beta}^* = -2\cos(\theta_k + \pi/2)$, conducting a two-sided t-test for $H_{0,1}^{\pi_\beta} : \pi_{1,\beta}^* = 0$ against $H_{1,1}^{\pi_\beta} : \pi_{1,\beta}^* \neq 0$.[1] However, in general, because in practice it is typically the case that $p > 0$ and there is usually no *a priori* basis for a belief that unit roots are present at all other frequencies, an appropriate test for a seasonal unit root at frequency ω_k should be an F-test for $H_{0,k} : \pi_{k,\alpha}^* = \pi_{k,\beta}^* = 0$, $k = 1, ..., S^*$, together with one-sided t-tests for $H_{0,0} : \pi_0^* = 0$ against $H_{1,0} : \pi_0^* < 0$ and, if S is even, $H_{0,S/2} : \pi_{S/2}^* = 0$ against $H_{1,S/2} : \pi_{S/2}^* < 0$, a recommendation supported by the limiting distribution theory in Section 5 and the Monte Carlo evidence of section 6. Note that the implicit alternative for the F-test for $H_{0,k} : \pi_{k,\alpha}^* = \pi_{k,\beta}^* = 0$ is $H_{1,k}^{\pi} : (\pi_{k,\alpha}^* \neq 0) \cup (\pi_{k,\beta}^* \neq 0)$ rather than $H_{1,k}^{\gamma} : \gamma_k < 0$, $k = 1, ..., S^*$.

In the following, we will denote by t_0 and $t_{S/2}$ the regression t-statistics for $H_{0,0} : \pi_0^* = 0$ and $H_{0,S/2} : \pi_{S/2}^* = 0$ respectively in (4.1), t_k^α and t_k^β the t-statistics for $H_{0,k}^{\pi_\alpha} : \pi_{k,\alpha}^* = 0$ and $H_{0,k}^{\pi_\beta} : \pi_{k,\beta}^* = 0$ respectively, $k = 1, ..., S^*$, F_k the F-statistic for $H_{0,k} = H_{0,k}^{\pi_\alpha} \cap H_{0,k}^{\pi_\beta}$, $k = 1, ..., S^*$; that is, the test statistics for the exclusion of $x_{0,St+s-1}$ if $k = 0$ (t_0), $x_{S/2,St+s-1}$ if $k = S/2$ ($t_{S/2}$), and $x_{k,St+s-1}^\alpha$ (t_k^α), $x_{k,St+s-1}^\beta$ (t_k^β), and *both* $x_{k,St+s-1}^\alpha$ *and* $x_{k,St+s-1}^\beta$ (F_k) if $k = 1, ..., S^*$, in (4.1). Furthermore, $F_{1...[S/2]}$ and $F_{0...[S/2]}$ are the F-statistics for the joint hypotheses $\cap_{k=1}^{[S/2]} H_{0,k}$ and $\cap_{k=0}^{[S/2]} H_{0,k}$ respectively. We will continue to adopt this notation to represent the corresponding t- and F- statistics for the special cases of (2.1), and hence of the auxiliary regression (4.1), considered at the end of section 2.

We conclude this section by demonstrating that tests based upon the statistics defined above from (4.1) are exact similar with respect to both the initial conditions $\{x_s\}_{s=1-S}^0$ and the seasonal drift parameters $\{\gamma_s^{**}\}_{s=1-S}^0$. For simplicity, but with no loss of generality, these results will be demonstrated under the overall null hypothesis, H_0 of (2.4).

As the time-trend parameters $\delta_s = 0$, $s = 1 - S, ..., 0$, under H_0 of (2.4), it follows from (2.6) that the seasonally de-meaned process $\{\tilde{x}_{St+s}\}$ may be written as

$$\tilde{x}_{St+s} = x_{St+s} - \bar{x}_s = \gamma_s^{**}(t - \bar{t}) + \sum_{v=1}^{t} u_{Sv+s} - T^{-1} \sum_{t=1}^{T} \sum_{v=1}^{t} u_{Sv+s}, \qquad (4.3)$$

which is invariant to the initial conditions $\{x_s\}_{s=1-S}^0$, where $\bar{t} \equiv T^{-1} \sum_{t=1}^T t$ and $\bar{x}_s = T^{-1} \sum_{t=1}^T x_{St+s}$, $s = 1 - S, ..., 0$, $t = 1, 2, ..., T$. From (4.3), we may correspondingly

[1] Moreover, as the maintained hypothesis $H_0 \cup H_1$ excludes the possibility of unit roots at other frequencies than ω_k, $k = 0, ..., [S/2]$, the latter test might be omitted.

express the seasonally de-meaned and seasonally de-trended process $\{\hat{x}_{St+s}\}$ as

$$\begin{aligned}
\hat{x}_{St+s} &= \tilde{x}_{St+s} - (t-\bar{t})\left(\sum_{t=1}^{T}(t-\bar{t})^2\right)^{-1}\sum_{t=1}^{T}(t-\bar{t})\tilde{x}_{St+s} \\
&= \sum_{v=1}^{t} u_{Sv+s} - T^{-1}\sum_{t=1}^{T}\sum_{v=1}^{t} u_{Sv+s} \\
&\quad -(t-\bar{t})\left(\sum_{t=1}^{T}(t-\bar{t})^2\right)^{-1}\sum_{t=1}^{T}(t-\bar{t})\left(\sum_{v=1}^{t} u_{Sv+s} - T^{-1}\sum_{t=1}^{T}\sum_{v=1}^{t} u_{Sv+s}\right),
\end{aligned} \tag{4.4}$$

which is invariant to the initial conditions $\{x_s\}_{s=1-S}^{0}$ *and* the seasonal drift parameters $\{\gamma_s^{**}\}_{s=1-S}^{0}$, $s = 1-S, ..., 0$, $t = 1, 2, ..., T$.

After regression on the seasonal intercepts and time-trends, the auxiliary regression equation (4.1) is consequently rendered from (4.4) as

$$\begin{aligned}
\hat{\Delta}_S x_{St+s} &= \sum_{i=1}^{p} \phi_i^* \hat{\Delta}_S x_{St+s-i} + \pi_0^* \hat{x}_{0,St+s-1} + \pi_{S/2}^* \hat{x}_{S/2,St+s-1} \\
&\quad + \sum_{k=1}^{S^*}\left(\pi_{k,\alpha}^* \hat{x}_{k,St+s-1}^{\alpha} + \pi_{k,\beta}^* \hat{x}_{k,St+s-1}^{\beta}\right) + \hat{\epsilon}_{St+s},
\end{aligned} \tag{4.5}$$

dropping the term $\pi_{S/2}^* \hat{x}_{S/2,St+s-1}$ if S is odd, $s = 1-S, ..., 0$, $t = 1, 2, ..., T$, where

$$\hat{\epsilon}_{St+s} = \tilde{\epsilon}_{Sv+s} - (t-\bar{t})\left(\sum_{t=1}^{T}(t-\bar{t})^2\right)^{-1}\sum_{t=1}^{T}(t-\bar{t})\tilde{\epsilon}_{Sv+s}$$

and the seasonally de-meaned error process $\tilde{\epsilon}_{St+s} \equiv \epsilon_{St+s} - T^{-1}\sum_{t=1}^{T}\epsilon_{St+s}$. In (4.5), $\hat{\Delta}_S x_{St+s}$ is $\Delta_S x_{St+s}$ seasonally de-meaned and seasonally de-trended, which is also invariant to the initial conditions $\{x_s\}_{s=1-S}^{0}$ *and* the seasonal drifts $\{\gamma_s^{**}\}_{s=1-S}^{0}$. We have defined the seasonally de-meaned and seasonally de-trended transformed variables from (4.2) as:

$$\hat{x}_{0,St+s} \equiv \sum_{j=0}^{S-1} \hat{x}_{St+s-j}, \hat{x}_{S/2,St+s} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\pi]\hat{x}_{St+s-j},$$

$$\hat{x}_{k,St+s}^{\alpha} \equiv \sum_{j=0}^{S-1} \cos[(j+1)\omega_k]\hat{x}_{St+s-j}, \hat{x}_{k,St+s}^{\beta} \equiv -\sum_{j=0}^{S-1} \sin[(j+1)\omega_k]\hat{x}_{St+s-j}, \tag{4.6}$$

$k = 1, ..., S^*$, $s = 1-S, ..., 0$, $t = 1, ..., T$. Consequently, tests based on regression t- and F-statistics from (4.1) will be exact similar with respect to both the initial conditions $\{x_s\}_{s=1-S}^{0}$ and the seasonal drift parameters $\{\gamma_s^{**}\}_{s=1-S}^{0}$. Using similar arguments (see, for example, Burridge and Taylor, 2004, pp.71-73, for the quarterly case) it is straightforward to show that these tests are also exact invariant to the seasonal drift parameters.

5 Asymptotic Distribution Theory

In the quarterly case, $S = 4$, HEGY and Engle, Granger, Hylleberg and Lee (1993) detail the limiting null distributions of the t-statistics together with the F-statistic F_1 for deterministic scenarios up to and including non-seasonal trend parameters $\delta_s = \delta$, $s = -3, ..., 0$, in the auxiliary regression (4.1), while GLN derive the limit distributions for the $F_{1...2}$ and $F_{0...2}$ statistics in these cases. ST1 generalise the quarterly results of HEGY and GLN to allow for the presence of differential seasonal trends, as in (4.1), and discuss the various deterministic scenarios given at the end of Section 1, also for quarterly data $S = 4$, while Taylor (1998) and BM deal with the monthly case, $S = 12$.

The limiting distributional results provided by the above authors have all been derived under the assumption that $p = 0$ in (2.2). In the quarterly case, Burridge and Taylor (2001) have extended the analysis to the case where $p > 0$ in (2.2). They demonstrate that provided (4.1), or the restricted cases (i)-(v) thereof discussed at the end of Section 2, contains at least p lags of $\Delta_4 x_{4t+s}$, then so the limiting distributions of the t_0, $t_{S/2}$, F_1, $F_{1...2}$ and $F_{0...2}$ statistics under H_0 of (2.4) coincide with those derived by the above authors, for $p = 0$. However, and contrary to what is claimed by the above authors, they demonstrate that the limiting null distributions of the harmonic frequency t-tests, t_1^{α} and t_1^{β}, are not of the form appropriate for $p = 0$, but that they depend on the parameters characterising $\phi(z)$. This even though (4.1) has been appropriately lag-augmented.

All of the above authors, including Burridge and Taylor (2001), derived their limiting null distribution theory under the assumption that $\alpha(z) = (1 - z^S)$; that is, under H_0 of (2.4) which imposes the unit root null hypothesis at both the zero and all of the seasonal frequencies; cf. (3.5). This is quite restrictive, particulary in the light of the discussion in Ghysels and Osborn (2001,p.90) who note that most empirical applications of seasonal unit root tests have led to rejections of the unit root hypothesis at at least one of the seasonal frequencies, implying the likely inappropriateness of the assumption that $\alpha(z) = (1 - z^S)$. Consequently, and in the light of our discussion in Sections 3 and 4, we now turn to deriving the limiting null distributions of the t- and F-statistics from (4.1) for an arbitrary seasonal aspect S in the more general case where it is not necessarily assumed to be the case that the unit root null hypothesis holds at each of the zero and seasonal frequencies. As in Burridge and Taylor (2001), we allow for the case where $p > 0$ in (2.2). Our main results are stated for the case where the auxiliary regression (4.1) contains seasonal intercepts and seasonal trends. The corresponding limiting null distributions under the special cases of (4.1) outlined at the end of section 2 are subsequently discussed in Remark 9.

In order to proceed, we first need to introduce some notation. First, let the Sth order polynomial $\alpha(z)$ of (2.1) be decomposed into $\alpha(z) = \bar{\alpha}(z)a(z)$, where all of the roots of $\bar{\alpha}(z) = 0$ (of $a(z) = 0$) lie on (outside) the unit circle. We then observe that (ignoring deterministic components for ease of exposition), x_{St+s} satisfies $\Delta_S x_{St+s} = [(1 - L^S)/\bar{\alpha}(L)]u^*_{St+s}$, where $a(L)u^*_{St+s} = u_{St+s}$. Next define the S-dimensional vector processes $\mathbf{X}_t \equiv [x_{St-(S-1)}, x_{St-(S-2)}, ..., x_{St-3}, x_{St-2},\ x_{St-1}, x_{St}]'$

and $\mathbf{U}_t^* \equiv [u_{St-(S-1)}^*, u_{St-(S-2)}^*, ..., u_{St-3}^*, u_{St-2}^*, \; u_{St-1}^*, u_{St}^*]'$, $t = 1, 2, ...$ Observe from Burridge and Taylor (2001) that

$$\mathbf{U}_t^* = \sum\nolimits_{j=0}^{\infty} \mathbf{\Psi}_j^* \mathbf{E}_{t-j}$$

where $\mathbf{E}_t \equiv [\epsilon_{St-(S-1)}, \epsilon_{St-(S-2)}, ..., \epsilon_{St-3}, \epsilon_{St-2}, \epsilon_{St-1}, \epsilon_{St}]'$ and the sequence of the $S \times S$ matrices:

$$\mathbf{\Psi}_0^* = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \psi_1 & 1 & 0 & 0 & \cdots & 0 \\ \psi_2 & \psi_1 & 1 & 0 & \cdots & 0 \\ \psi_3 & \psi_2 & \psi_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{S-1} & \psi_{S-2} & \psi_{S-3} & \psi_{S-4} & \cdots & 1 \end{bmatrix}$$

$$\mathbf{\Psi}_j^* = \begin{bmatrix} \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \psi_{jS-3} & \cdots & \psi_{jS-(S-1)} \\ \psi_{jS+1} & \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \cdots & \psi_{jS-(S-2)} \\ \psi_{jS+2} & \psi_{jS+1} & \psi_{jS} & \psi_{jS-1} & \cdots & \psi_{jS-(S-3)} \\ \psi_{jS+3} & \psi_{jS+2} & \psi_{jS+1} & \psi_{jS} & \cdots & \psi_{jS-(S-4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{jS+S-1} & \psi_{jS+S-2} & \psi_{jS+S-3} & \psi_{jS+S-4} & \cdots & \psi_{jS} \end{bmatrix}, \quad j = 1, 2, \cdots$$

where $\psi(z) \equiv 1 + \sum_{j=1}^{\infty} \psi_j z^j$ is the inverse of $\phi(z)\, a(z)$. Finally, let $\hat{\mathbf{X}}_t$ denote the de-meaned and de-trended counterpart of $\mathbf{X}_t$.

In Lemma 1 we now provide an invariance principle for $\hat{\mathbf{X}}_t$.

Lemma 1 *Let $\{x_{St+s}\}$ be generated according to (2.1)-(2.2) under the conditions stated in section 2. Then, as $T \to \infty$, and denoting weak convergence by "$\Rightarrow$"*

$$T^{-1/2} \hat{\mathbf{X}}_{[T\cdot]} \Rightarrow \sigma \mathbf{C} \mathbf{\Psi}^*(1)\, \hat{\mathbf{W}}(\cdot) \tag{5.1}$$

where $\hat{\mathbf{W}}(r) \equiv \mathbf{W}(r) - (4 - 6r)\int_0^1 \mathbf{W}(r) - (12r - 6)\int_0^1 s\mathbf{W}(s)ds$, $r \in [0, 1]$, is an $S \times 1$ vector (standard) de-meaned and de-trended Brownian motion process, with $\mathbf{W}$ a standard Brownian motion, $\mathbf{\Psi}^*(1) \equiv \sum_{j=1}^{\infty} \mathbf{\Psi}_j^*$, and $\mathbf{C}$ is a generic circulant matrix whose precise form depends on $\bar{\alpha}(z)$; see Remark 2 below.

Remark 2: The matrix $\mathbf{C}$ appearing in Lemma 1 is a circulant matrix (see, Davis, 1979, for details) whose rank is equal to the number of unit roots present in $\bar{\alpha}(z)$. This also coincides with the number of seasons minus the number of (linearly independent) co-integrating relationships that exist between them; cf. Franses (1994). For example, if there is a unit root only at frequency zero, such that $\bar{\alpha}(z) = (1 - z)$, then $\mathbf{C}$ is an $S \times S$ matrix of ones, $\mathbf{C} = \mathbf{C}_0 = \mathsf{Circ}[1, 1, 1, 1, \cdots, 1]$, while for a unit root only at the Nyquist frequency, $\bar{\alpha}(z) = (1 + z)$, $\mathbf{C} = \mathbf{C}_{S/2} = \mathsf{Circ}[1, -1, 1, -1, \cdots, -1]$. In both of these examples $\mathbf{C}$ has rank one, such that there will exist $S-1$ co-integrating relationships between the seasons. For the case of a complex pair of unit roots at frequency ω_k, $k=1,...,S^*$, such that $\bar{\alpha}(z)=(1-2\cos\omega_k z+z^2)$, we have that

$$\begin{aligned}\mathbf{C} &= \mathbf{C}_k = \mathsf{Circ}\left[\frac{\sin(\omega_k)}{\sin(\omega_k)}, \frac{\sin(2\omega_k)}{\sin(\omega_k)}, \frac{\sin(3\omega_k)}{\sin(\omega_k)}, \cdots, \frac{\sin((S-1)\omega_k)}{\sin(\omega_k)}\right] \\ &= \mathbf{C}_k^{\alpha} + \frac{\cos(\omega_k)}{\sin(\omega_k)}\mathbf{C}_k^{\beta}\end{aligned} \tag{5.2}$$

where:

$$\mathbf{C}_k^{\alpha} = \mathsf{Circ}\left[\cos(0\omega_k), \cos(\omega_k), \cos(2\omega_k), \cdots, \cos((S-1)\omega_k)\right]$$

and

$$\mathbf{C}_k^{\beta} = \mathsf{Circ}\left[\sin(0\omega_k), \sin((S-1)\omega_k), \sin((S-1)\omega_k), \cdots, \sin(\omega_k)\right],$$

which has rank two, and, hence, the number of co-integrating relationships between the seasons is $S-2$. For the case where H_0 of (2.4) holds, such that $\bar{\alpha}(z)=(1-z^S)$, we have that $\mathbf{C}=\mathbf{I}_S$ is of full rank and no co-integration exists between the seasons. Based on properties of the sums of circulant matrices (see, for example, Theorem 3.2.4 of Davis, 1979, and Theorem 3.1 of Gray, 2006, and the appendix for details) it is always possible to express $\mathbf{C}$ for any given $\bar{\alpha}(z)$ polynomial as the following weighted sum:

$$\mathbf{C} = c_0^{\bar{\alpha}}\mathbf{C}_0 + c_{S/2}^{\bar{\alpha}}\mathbf{C}_{S/2} + \sum_{k=1}^{S^*}\left(c_{k\alpha}^{\bar{\alpha}}\mathbf{C}_k^{\alpha} + c_{k\beta}^{\bar{\alpha}}\mathbf{C}_k^{\beta}\right) \tag{5.3}$$

(omitting the term in $\mathbf{C}_{S/2}$ when S is odd), where $c_0^{\bar{\alpha}}$, $c_{S/2}^{\bar{\alpha}}$, $c_{k\alpha}^{\bar{\alpha}}$ and $c_{k\beta}^{\bar{\alpha}}$, $k=1,...,S^*$, are scalars which are non-zero (zero) when the factors $(1-z)$, $(1+z)$ and $(1-2\cos\omega_k z+z^2)$, $k=1,...,S^*$, respectively, are present (not present) in $\bar{\alpha}(z)$. The values of these coefficients when they are non-zero depends upon the form of $\bar{\alpha}(z)$.[2] Finally, using the notation $\Psi^*(1)^j$ to denote the jth element in the first row of $\mathbf{\Psi}^*(1)$, observe that $\mathbf{\Psi}^*(1)$ is also a circulant matrix, since

$$\begin{aligned}\mathbf{\Psi}^*(1) &= \mathsf{Circ}\left[\Psi^*(1)^1, \Psi^*(1)^2, \Psi^*(1)^3, \cdots, \Psi^*(1)^S\right] \\ &= \mathsf{Circ}\left[1+\sum_{j=1}^{\infty}\psi_{jS}, \sum_{j=1}^{\infty}\psi_{jS-1}, \sum_{j=1}^{\infty}\psi_{jS-2}, \cdots, \sum_{j=1}^{\infty}\psi_{jS-(S-1)}\right].\end{aligned}$$

Some further results on circulant matrices are provided in the appendix. □

We are now in a position to state our main theorem, which establishes the asymptotic null distributions of the regression-based unit root statistics from (4.1).

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[2]For example, for the process (6.1) used in the Monte Carlo exercise in Section 6, where $\mathbf{C}=\mathsf{Circ}[1,0,0,0,1,-1]$, the following identity holds $\mathbf{C}=\frac{1}{6}\mathbf{C}_0+\frac{1}{2}\mathbf{C}_{S/2}+\frac{1}{3}\mathbf{C}_2^{\alpha}-\frac{1}{\sqrt{3}}\mathbf{C}_2^{\beta}$. As a second example consider the case where $\bar{\alpha}(z)=(1-z^S)$; here $\mathbf{C}=\mathbf{I}_S=\frac{1}{S}\mathbf{C}_0+\frac{1}{S}\mathbf{C}_{S/2}+\frac{2}{S}\sum_{k=1}^{S^*}\mathbf{C}_k^{\alpha}$, omitting the term in $\mathbf{C}_{S/2}$ where S is odd.

Theorem 1 *Let x_{St+s} be generated according to (2.1)-(2.2) under the conditions stated in Section 2, and let $\hat{w}_0(r)$, $\hat{w}_{S/2}(r)$ (S even), and $\hat{w}_k^\alpha(r)$ and $\hat{w}_k^\beta(r)$, $k = 1, ..., S^*$, denote S independent de-meaned and de-trended standard Brownian motions on $[0,1]$.*

(a) Under $H_{0,0}$, the t_0 statistic from (4.1) satisfies

$$t_0 \Rightarrow \frac{\int_0^1 \hat{w}_0(r)\, d\hat{w}_0(r)}{\sqrt{\int_0^1 \hat{w}_0(r)^2\, dr}} \equiv \eta_0 \tag{5.4}$$

(b) For S even, under $H_{0,S/2}$, the $t_{S/2}$ statistic from (4.1) satisfies

$$t_{S/2} \Rightarrow t_0 \Rightarrow \frac{\int_0^1 \hat{w}_{S/2}(r)\, d\hat{w}_{S/2}(r)}{\sqrt{\int_0^1 \hat{w}_{S/2}(r)^2\, dr}} \equiv \eta_{S/2}. \tag{5.5}$$

(c) Under $H_{0,k}$, the t_k^α, t_k^β and F_k, $k = 1, ..., S^*$, statistics from (4.1), satisfy

$$\begin{aligned} t_k^\alpha \;\Rightarrow\; & \frac{\left(c_{k\alpha}^{\bar{\alpha}} b_k - c_{k\beta}^{\bar{\alpha}} a_k\right)\left[\int_0^1 \hat{w}_k^\alpha(r)\, d\hat{w}_k^\alpha(r) + \int_0^1 \hat{w}_k^\beta(r)\, d\hat{w}_k^\beta(r)\right]}{\sqrt{\left(\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^\alpha(r)^2\, dr + \int_0^1 \hat{w}_k^\beta(r)^2\, dr\right)}} \\ & - \frac{\left(c_{k\alpha}^{\bar{\alpha}} a_k + c_{k\beta}^{\bar{\alpha}} b_k\right)\left[\int_0^1 \hat{w}_k^\beta(r)\, d\hat{w}_k^\alpha(r) - \int_0^1 \hat{w}_k^\alpha(r)\, d\hat{w}_k^\beta(r)\right]}{\sqrt{\left(\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^\alpha(r)^2\, dr + \int_0^1 \hat{w}_k^\beta(r)^2\, dr\right)}} \end{aligned} \tag{5.6}$$

$$\begin{aligned} t_k^\beta \;\Rightarrow\; & \frac{\left(c_{k\alpha}^{\bar{\alpha}} a_k + c_{k\beta}^{\bar{\alpha}} b_k\right)\left[\int_0^1 \hat{w}_k^\alpha(r)\, d\hat{w}_k^\alpha(r) + \int_0^1 \hat{w}_k^\beta(r)\, d\hat{w}_k^\beta(r)\right]}{\sqrt{\left(\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^\alpha(r)^2\, dr + \int_0^1 \hat{w}_k^\beta(r)^2\, dr\right)}} \\ & + \frac{\left(c_{k\alpha}^{\bar{\alpha}} b_k - c_{k\beta}^{\bar{\alpha}} a_k\right)\left[\int_0^1 \hat{w}_k^\beta(r)\, d\hat{w}_k^\alpha(r) - \int_0^1 \hat{w}_k^\alpha(r)\, d\hat{w}_k^\beta(r)\right]}{\sqrt{\left(\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^\alpha(r)^2\, dr + \int_0^1 \hat{w}_k^\beta(r)^2\, dr\right)}} \end{aligned} \tag{5.7}$$

$$F_k \;\Rightarrow\; \frac{\left[\int_0^1 \hat{w}_k^\alpha(r)\, d\hat{w}_k^\alpha(r) + \int_0^1 \hat{w}_k^\beta(r)\, d\hat{w}_k^\beta(r)\right]^2 + \left[\int_0^1 \hat{w}_k^\beta(r)\, d\hat{w}_k^\alpha(r) - \int_0^1 \hat{w}_k^\alpha(r)\, d\hat{w}_k^\beta(r)\right]^2}{2\left(\int_0^1 \hat{w}_k^\alpha(r)^2\, dr + \int_0^1 \hat{w}_k^\beta(r)^2\, dr\right)} \equiv \eta_k^2 \tag{5.8}$$

where, for $k = 1, ..., S^*$,

$$\begin{aligned} b_k &\equiv \sum_{j=0}^{S-1} \cos(j\omega_k)\, \Psi^*(1)^{j+1} = \mathcal{R}e(\psi[\exp(i\omega_k)]) \\ a_k &\equiv -\sum_{j=0}^{S-1} \sin(j\omega_k)\, \Psi^*(1)^{j+1} = \mathcal{I}m(\psi[\exp(i\omega_k)]). \end{aligned}$$

(d) Under $\cap_{k=1}^{[S/2]} H_{0,k}$*,* $F_{1\ldots[S/2]} \Rightarrow \frac{1}{S-1}\sum_{j=1}^{[S/2]}\eta_j^2$*. Moreover, under* $H_0 = \cap_{k=0}^{[S/2]} H_{0,k}$*,* $F_{0\ldots[S/2]} \Rightarrow \frac{1}{S}\sum_{k=0}^{[S/2]}\eta_k^2$*.*

Remark 3: Notice that in the absence of any stationary serial correlation in the process; that is, where $\phi(z)\,a(z) = 1$, we have that $b_k = 1$ and $a_k = 0$ and, hence, while the representations in (5.4), (5.5) and (5.8) remain unaltered, those for (5.6) and (5.7) simplify to

$$t_k^{\alpha} \Rightarrow \frac{c_{k\alpha}^{\bar{\alpha}}\left[\int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\alpha}(r) + \int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\beta}(r)\right] - c_{k\beta}^{\bar{\alpha}}\left[\int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\alpha}(r) - \int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\beta}(r)\right]}{\sqrt{\left((c_{k\alpha}^{\bar{\alpha}})^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right)\left(\int_0^1 \hat{w}_k^{\alpha}(r)^2\,dr + \int_0^1 \hat{w}_k^{\beta}(r)^2\,dr\right)}}$$

and

$$t_k^{\beta} \Rightarrow \frac{c_{k\beta}^{\bar{\alpha}}\left[\int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\alpha}(r) + \int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\beta}(r)\right] + c_{k\alpha}^{\bar{\alpha}}\left[\int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\alpha}(r) - \int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\beta}(r)\right]}{\sqrt{\left((c_{k\alpha}^{\bar{\alpha}})^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right)\left(\int_0^1 \hat{w}_k^{\alpha}(r)^2\,dr + \int_0^1 \hat{w}_k^{\beta}(r)^2\,dr\right)}}$$

respectively, which still depend on the parameters $c_{k\alpha}^{\bar{\alpha}}$ and $c_{k\beta}^{\bar{\alpha}}$ of Remark 2.

Remark 4: In the case where $\bar{\alpha}(z) = (1 - 2\cos\omega_k z + z^2)$ we have that $\mathbf{C} = \mathbf{C}_k^{\alpha} + \frac{\cos(\omega_k)}{\sin(\omega_k)}\mathbf{C}_k^{\beta}$, and, hence, $c_{k\alpha}^{\bar{\alpha}} = 1$ and $c_{k\beta}^{\bar{\alpha}} = \frac{\cos(\omega_k)}{\sin(\omega_k)}$. Consequently, the expressions in (5.6) and (5.7) simplify to

$$\begin{aligned} t_k^{\alpha} \;\Rightarrow\; & \frac{\left(b_k - \frac{\cos(\omega_k)}{\sin(\omega_k)}a_k\right)\left[\int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\alpha}(r) + \int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\beta}(r)\right]}{\sqrt{\left(1 + \left(\frac{\cos(\omega_k)}{\sin(\omega_k)}\right)^2\right)[b_k^2 + a_k^2]\left(\int_0^1 \hat{w}_k^{\alpha}(r)^2\,dr + \int_0^1 \hat{w}_k^{\beta}(r)^2\,dr\right)}} \\ & - \frac{\left(a_k + \frac{\cos(\omega_k)}{\sin(\omega_k)}b_k\right)\left[\int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\alpha}(r) - \int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\beta}(r)\right]}{\sqrt{\left(1 + \left(\frac{\cos(\omega_k)}{\sin(\omega_k)}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^{\alpha}(r)^2\,dr + \int_0^1 \hat{w}_k^{\beta}(r)^2\,dr\right)}} \end{aligned}$$

and

$$\begin{aligned} t_k^{\beta} \;\Rightarrow\; & \frac{\left(a_k + \frac{\cos(\omega_k)}{\sin(\omega_k)}b_k\right)\left[\int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\alpha}(r) + \int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\beta}(r)\right]}{\sqrt{\left(1 + \left(\frac{\cos(\omega_k)}{\sin(\omega_k)}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^{\alpha}(r)^2\,dr + \int_0^1 \hat{w}_k^{\beta}(r)^2\,dr\right)}} \\ & + \frac{\left(b_k - \frac{\cos(\omega_k)}{\sin(\omega_k)}a_k\right)\left[\int_0^1 \hat{w}_k^{\beta}(r)\,d\hat{w}_k^{\alpha}(r) - \int_0^1 \hat{w}_k^{\alpha}(r)\,d\hat{w}_k^{\beta}(r)\right]}{\sqrt{\left(1 + \left(\frac{\cos(\omega_k)}{\sin(\omega_k)}\right)^2\right)[a_k^2 + b_k^2]\left(\int_0^1 \hat{w}_k^{\alpha}(r)^2\,dr + \int_0^1 \hat{w}_k^{\beta}(r)^2\,dr\right)}} \end{aligned}$$

respectively. It is therefore evident that even in the absence of any stationary serial correlation in the process (so that $\phi(z)\,a(z) = 1$, and $b_k = 1$, $a_k = 0$), when $\bar{\alpha}(z) = (1 - 2\cos\omega_k z + z^2)$ the limiting null distributions of t_k^α and t_k^β are only pivotal for the case of $\omega_k = \pi/2$.

Remark 5: When $\bar{\alpha}(z) = (1 - z^S)$, we have that $\mathbf{C} = \mathbf{I}_S$ and, hence from footnote 2, that $c_{k\alpha}^{\bar{\alpha}} = \frac{2}{S}$ and $c_{k\beta}^{\bar{\alpha}} = 0$, $k = 1, ..., S^*$. It is therefore seen that the expressions in (5.6) and (5.7) reduce to the representations given in (3.2) and (3.3), respectively, of Theorem 3.1 of Rodrigues and Taylor (2004) with $c = 0$. Even here, it should be noted that these distributions will not in general be pivotal when $p > 0$. Some exceptions are discussed in Burridge and Taylor (2001).

Remark 6: The representations given in (5.4) and (5.5) are identical to the standard DF distribution for a regression with intercept and trend; cf. Fuller (1976, Table 8.5.2, p.373). Hence, asymptotically, the t-statistics t_0 and $t_{S/2}$ are independently and identically distributed under $H_{0,0} \cap H_{0,S/2}$

Remark 7: From the representations (5.6) and (5.7), it is clear that, under $H_{0,k} \cap H_{0,l}$, $k \neq l$, $k, l = 1, ..., S^*$, the pairs t_k^α and t_l^α, and t_k^β and t_l^β, possess independent but not, in general, identical limiting distributions, and are also asymptotically independent of t_0 and $t_{S/2}$ under $(H_{0,0} \cap H_{0,S/2}) \cap (H_{0,k} \cap H_{0,l})$, $k \neq l$, $k, l = 1, ..., S^*$. In contrast, under $H_{0,k} \cap H_{0,l}$, the limiting representations in (5.8) for the F_k and F_l statistics in (5.8) $k \neq l$, $k, l = 1, ..., S^*$, are both independent and also identical and, moreover, are identical to those given for the corresponding statistics in the quarterly and monthly contexts for the case of $p = 0$; see ST1 and Taylor (1998), respectively, and are asymptotically independent of t_0 and $t_{S/2}$ under $(H_{0,0} \cap H_{0,S/2}) \cap (H_{0,k} \cap H_{0,l})$, $k \neq l$, $k, l = 1, ..., S^*$.

Remark 8: The representations given for the limiting null distributions of the t_0, $t_{S/2}$ and F_k, $k = 1, ..., S^*$, statistics in (5.4), (5.5) and (5.8), respectively, and of the $F_{0...[S/2]}$ and $F_{1...[S/2]}$ statistics in part (d), do not depend on any nuisance parameters; that is, they take the same form irrespective of whether the unit root null holds at the other frequencies or not, and irrespective of whether $p = 0$ (serially uncorrelated shocks) or $p > 0$ (serially correlated shocks). In contrast, the corresponding representations for the t_k^α and t_k^β statistics, $k = 1, ..., S^*$, depend on two sets of nuisance parameters: a_k and b_k, relating to the stationary serial correlation in the process, and $c_{k\alpha}^{\bar{\alpha}}$ and $c_{k\beta}^{\bar{\alpha}}$, relating to those frequencies other than ω_k which admit unit roots. Consequently the use of tests for unit roots at the harmonic seasonal frequencies based the t_k^α and t_k^β, $k = 1, ..., S^*$, statistics cannot be recommended in practice.

Remark 9: In Theorem 1 we have assumed, *via* (2.1), that x_{St+s} of (2.1) contains seasonal indicators and seasonal time trends in its deterministic mean function and that, appropriate to this, (4.5) is constructed from seasonally de-meaned and seasonally de-trended data. However, the limiting representations are valid under the more restricted cases (i)-(v) discussed at the end of Section 2, provided the de-meaned and de-trended standard Brownian motions appearing in Theorem 1 are re-defined as appropriate to the deterministic scenario of interest within (2.1); cf. ST1 (Sections 4.1-4.5) and ST2 for a complete typology. Remarks 3-8 remain valid, with appropriate re-definitions where necessary, in each case. □

In the next section we investigate aspects of the finite sample size and power properties of these statistics via a series of Monte Carlo experiments for the case of $S = 6$.

6 Finite Sample Size and Power Properties

In this section we conduct a number of Monte Carlo simulation experiments to investigate the finite-sample size and power properties of the seasonal unit root test-statistics developed in section 4 when the process under investigation displays phase and/or length shifts at a particular seasonal frequency.

We will consider the test-statistics t_j, $j = 0, 3$ together with t_k^α, t_k^β and F_k, $k = 1, 2$, for the case of a time-series process $\{x_{6t+s}\}$ with seasonal aspect $S = 6$. In what follows we assume that the investigator has available the initial values of the level process, $x_{-5}, \dots, x_0$, together with the observations x_{6t+s}, $s = -5, \dots, 0$, $t = 1, \dots, T$. We shall investigate the effects on the above statistics resulting from movements in phase and/or length away from the unit root null hypothesis at frequency $\omega_1 \equiv \pi/3$, whilst maintaining the unit root null at all other frequencies $\omega_k \equiv \pi k/3$, $k = 0, 2, 3$. Specifically, the simulations computed in this section were based on the DGP:

$$(1 - L^2)(1 + L + L^2)(1 - 2r_1 \cos(\theta_1 + \pi/3)L + r_1^2 L^2)x_{6t+s} = \epsilon_{6t+s} \sim NID(0, 1),$$
$$t = 1, \dots, T,\ s = -5, \dots, 0, \qquad (6.1)$$

with $\epsilon_{6j+s} = x_{6j+s} = 0$, $j \leq 0$. As our Monte Carlo design, we vary the phase shift and length parameters θ_1 and r_1 respectively according to $\theta_1 \in \{0, \pm\pi/4, \pm 11\pi/36\}$, and $r_1 \in \{1.00 \times 1(\theta = 0), 0.99, 0.95, 0.80\}$, where $1(\cdot)$ is the indicator function. Under the overall null hypothesis, H_0 of (2.4), $\theta_1 = 0$ and $r_1 = 1$. Note the phases $\theta_1 + \omega_1$ specified in these alternatives, $\pi/12$, $\pi/36$, and $7\pi/12$, $23\pi/36$, are close to the frequencies $\omega_0 \equiv 0$ and $\omega_2 \equiv 2\pi/3$ respectively. All experiments were programmed using the RNDN function of GAUSS 3.1 on a Pentium 400Mhz micro-computer using $30,000$ replications for each experiment. Sample sizes $6T = 72, 150, 300$ and 600 are considered. All test-statistics were computed from a regression containing both seasonal intercepts and seasonal trend variables; see (4.1).

The DGP (6.1) allows us to investigate the power properties of the t_1^α, t_1^β and F_1 test-statistics when either $\theta_1 \neq 0$ and/or $r_1 \neq 1$, whilst simultaneously investigating the size properties of the remaining statistics. To the best of our knowledge, a study of this kind has not previously been conducted in the literature. Previous studies have looked at departures of the length of roots at the harmonic seasonal frequencies from unity but have always maintained a zero phase shift; see, for example, Beaulieu and Miron (1992, pp.41–42) for $S = 12$ and GLN (pp.431–434) for $S = 4$. Because the test-statistics which we consider in this section have not previously appeared in the literature, we have used Monte Carlo simulation to generate approximate finite-sample critical values for the above tests under H_0 of (2.4) for each of the sample sizes considered. We do not report these critical values here although they, and a general program which was used to run all the experiments in this section, are available from the authors on request. These simulated critical values were then used in computing the size and power properties of the tests as θ_1 and r_1 deviated from their null values as described above. For all tests we adopted the nominal level of 0.05.

Tables 1–4 report the results of our Monte Carlo experimental design for the sample sizes $6T = 72, 150, 300$ and 600, respectively. The top line of each of the tables corresponds to when the overall null hypothesis H_0 of (2.4) holds.

It is clear from the first panel of Tables 1–4 that, with a zero phase shift at the first harmonic frequency, $\theta_1 = 0$, as the length of the root at this frequency, r_1, decreases and as the sample size, $6T$, increases, so the powers of both the t_1^α and F_1 statistics increase towards unity. In contrast, the t_1^β statistic displays power less than size for all sample sizes; a similar pattern is seen in the simulation results reported in Beaulieu and Miron (1992). The t_0, t_3, t_2^α, t_2^β and F_2 tests all adhere well to their nominal 0.05 level, although there is some evidence of an increase for the t_2^β test.

This pattern changes, however, when we allow a phase shift at the first harmonic frequency, $\theta_1 \neq 0$. Here we see that both the t_1^β and F_1 tests are quite sensitive to movements of θ_1 from zero, even for small sample sizes, in both cases power increasing well above nominal size. An interesting picture arises for the t_1^α test whose power is well below nominal size when $r_1 = 0.99$ and $\theta_1 \neq 0$ for all sample sizes considered but rises as r_1 decreases and as the sample size grows. Turning to the tests at other frequencies, some very interesting patterns emerge here too. Firstly, depending on the values of θ_1 and r_1, the levels of the t_0 and t_3 tests are either under or above the nominal 0.05 level in the smaller sample sizes considered but, as the sample size increases, so these deviations quickly disappear, consonant with the results in Theorem 1. The same is also true of the F_2 statistic. To illustrate, the worst level distortions for the t_0 and F_2 tests occur for $6T = 72$ and $r_1 = 0.99$, the actual levels being 0.11 at $\theta_1 = -11\pi/36$ and 0.09 at $\theta_1 = 11\pi/36$ respectively. The same, however, cannot be said for the t_2^α and t_2^β test statistics which from a practical viewpoint appear to be highly unreliable for inference purposes. We can see that the level of the t_2^β test tends to exceed 0.05 for positive θ_1, increasing as θ_1 moves from $\pi/4$ to $11\pi/36$; that is, as $\theta_1 + \omega_1 = 7\pi/12$, $23\pi/36$ moves closer to the second harmonic frequency $\omega_2 = 2\pi/3$. For negative θ_1, the magnitude of θ_1 seems to have no effect on the magnitude of the size distortions, probably reflecting that $\theta_1 + \omega_1$ now takes values $\pi/12$ and $\pi/36$ close to the zero frequency $\omega_0 \equiv 0$. It is also apparent that, in the case of positive values of θ_1, there is a very strong interaction effect between r_1 and θ_1. For fixed values of r_1 and θ_1, the distortions seen in the level of the t_2^β statistic from the nominal 0.05 level appear to stabilise as sample size increases. This observation accords with (5.7) of Theorem 1 which says that for a particular (θ_1, r_1) combination, there is a finite shift in the limiting distribution of the t_2^β statistic. A similar observation may be made regarding the limiting distribution of the t_2^α statistic except that the level of the t_2^α statistic decreases below the nominal 0.05 level whereas that for the t_2^β statistic increases. Although the tests for $H_{0,2}$ based on the t_2^α and t_2^β statistics are neither exactly nor asymptotically similar under deviations from $H_{0,1}$, it is known from (5.8) of Theorem 1 that the test for $H_{0,2}$ based on the F_2 statistic does displays an asymptotic similarity property, which our numerical results suggest becomes apparent even in quite moderate sample sizes.

Although the above experiments are based on deviations of the phase and length of the root at the first seasonal harmonic frequency away from their null hypothesis values under $H_{0,1}$, we also investigated the effects of departures in phase and length from the null hypothesis $H_{0,2}$ at the second harmonic frequency, $\omega_2 \equiv 2\pi/3$, while maintaining seasonal unit roots at all other frequencies. These results were qualitatively no different from those observed above. In this case, the results pertaining to the t_0 and t_3 tests given above are interchanged. The t_2^α, t_2^β and F_2 tests display very similar patterns to those seen above for the corresponding t_1^α, t_1^β and F_1 tests. The t_1^α and t_1^β statistics show almost identical level distortions to those reported above for the t_2^α and t_2^β tests for movements of θ_1 away from zero and r_1 away from unity, except that the impact of the sign of θ_2 is reversed from that seen above for θ_1.

Consequently, and consonant with the predictions from the limiting distribution theory in section 5, in the presence of deviations from the seasonal unit root hypothesis at other frequencies, a test for a seasonal unit root at frequency ω_k based on the t-statistics t_k^α and t_k^β as described in HEGY or as above in section 4 is likely to be unreliable for inference purposes, whereas the use of the F-statistic F_k would appear to be more efficacious, $k = 1, ..., S^*$.

7 Conclusions

This paper has been concerned with providing regression-based test statistics for seasonal unit roots for a general seasonal aspect of the data which are similar both exactly and asymptotically with respect to initial values of the time series process and seasonal drift parameters. A general characterisation result is provided which clarifies precisely the null and alternative sub-hypotheses under test in the regression approach due to Hylleberg *et al.* (1990). Asymptotic distribution theory coupled with a set of Monte Carlo experiments indicates that a t-statistic approach, as advocated by HEGY in their original article, to testing for unit roots at the harmonic seasonal frequencies cannot be recommended, and here the use of a joint F-test based approach is appropriate.

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Mathematical Appendix

Due to the similarity properties of the statistics discussed in Section 4, we may set $x_s = \gamma_s^* = \delta_s^* = 0$, $s = 1-S, \ldots, 0$, in (2.1) with no loss of generality in what follows.

Proof of Proposition 1: Define $\omega_k(z) \equiv \delta_{k,\alpha}(z)\delta_{k,\beta}(z)$, where

$$\delta_{k,\alpha}(z) \equiv 1 - r_k \exp[i(\theta_k + \omega_k)]z, \delta_{k,\beta}(z) \equiv 1 - r_k \exp[-i(\theta_k + \omega_k)]z,$$

$k = 1, ..., S^*$, and, for frequencies ω_0 and $\omega_{S/2}$, $\delta_{0,\alpha}(z) \equiv 1 - r_0 z$, $\delta_{0,\beta}(z) \equiv 1$ and $\delta_{S/2,\alpha}(z) \equiv 1$, $\delta_{S/2,\beta}(z) \equiv 1 + r_{S/2}z$. Similarly, $\omega_k^0(z) \equiv \delta_{k,\alpha}^0(z)\delta_{k,\beta}^0(z)$, where

$$\delta_{k,\alpha}^0(z) \equiv 1 - \exp(i\omega_k)z, \delta_{k,\beta}^0(z) \equiv 1 - \exp(-i\omega_k)z,$$

$k = 1, ..., S^*$, and $\delta_{0,\alpha}^0(z) \equiv 1 - z$, $\delta_{0,\beta}^0(z) \equiv 1$ and $\delta_{S/2,\alpha}^0(z) \equiv 1$, $\delta_{S/2,\beta}^0(z) \equiv 1 + z$. Therefore, $\delta_{k,\alpha}^0[\exp(-i\omega_k)] = 0, \delta_{k,\beta}^0[\exp(i\omega_k)] = 0$, $k = 1, ..., S^*$. Now $\alpha(z) = \prod_{k=0}^{[S/2]} \omega_k(z) = \prod_{k=0}^{[S/2]} \delta_{k,\alpha}(z)\delta_{k,\beta}(z)$ and $\Delta_S(z) = \prod_{k=0}^{[S/2]} \omega_k^0(z) = \prod_{k=0}^{[S/2]} \delta_{k,\alpha}^0(z)\delta_{k,\beta}^0(z)$, where $\Delta_S(z) = 1 - z^S$. Define

$$\Delta_k(z) = -\prod_{j\neq k, j=0}^{[S/2]} \omega_j(z) = -\prod_{j\neq k, j=0}^{[S/2]} \delta_{j,\alpha}(z)\delta_{j,\beta}(z),$$

$$\Delta_k^0(z) = -\prod_{j\neq k, j=0}^{[S/2]} \omega_j^0(z) = -\prod_{j\neq k, j=0}^{[S/2]} \delta_{j,\alpha}^0(z)\delta_{j,\beta}^0(z).$$

Hence, $\Delta_k^0[\exp(-i\omega_j)] = \Delta_k^0[\exp(i\omega_j)] = 0$, $j \neq k$, and $\Delta_k^0[\exp(-i\omega_k)] \neq 0$, $\Delta_k^0[\exp(i\omega_k)] \neq 0$. Note that $\alpha(z) = -\omega_k(z)\Delta_k(z)$ and $\Delta_S(z) = -\omega_k^0(z)\Delta_k^0(z)$.

Consider the polynomial

$$\psi^*(z) \equiv \alpha^*(z) + \sum_{k=0}^{[S/2]} \left(\lambda_{k,\alpha}^*\delta_{k,\beta}^0(z)[\delta_{k,\alpha}(z) - \delta_{k,\alpha}^0(z)] + \lambda_{k,\beta}^*\delta_{k,\alpha}^0(z)[\delta_{k,\beta}(z) - \delta_{k,\beta}^0(z)]\right)\Delta_k^0(z),$$

where the complex conjugates

$$\lambda_{k,\alpha}^* \equiv -\frac{\alpha[\exp(-i\omega_k)]}{\delta_{k,\beta}^0[\exp(-i\omega_k)]\delta_{k,\alpha}[\exp(-i\omega_k)]\Delta_k^0[\exp(-i\omega_k)]}\phi[\exp(-i\omega_k)]$$

$$= \frac{\delta_{k,\beta}[\exp(-i\omega_k)]}{\delta_{k,\beta}^0[\exp(-i\omega_k)]}\frac{\Delta_k[\exp(-i\omega_k)]}{\Delta_k^0[\exp(-i\omega_k)]}\phi[\exp(-i\omega_k)] = \frac{\delta_{k,\beta}[\exp(-i\omega_k)]}{\delta_{k,\beta}^0[\exp(-i\omega_k)]}\Lambda_{k,\alpha}^*,$$

$$\lambda_{k,\beta}^* \equiv -\frac{\alpha[\exp(i\omega_k)]}{\delta_{k,\alpha}^0[\exp(i\omega_k)]\delta_{k,\beta}[\exp(i\omega_k)]\Delta_k^0[\exp(i\omega_k)]}\phi[\exp(i\omega_k)]$$

$$= \frac{\delta_{k,\alpha}[\exp(i\omega_k)]}{\delta^0_{k,\alpha}[\exp(i\omega_k)]} \frac{\Delta_k[\exp(i\omega_k)]}{\Delta^0_k[\exp(i\omega_k)]} \phi[\exp(i\omega_k)] = \frac{\delta_{k,\alpha}[\exp(i\omega_k)]}{\delta^0_{k,\alpha}[\exp(i\omega_k)]} \Lambda^*_{k,\beta},$$

in which we have defined

$$\Lambda^*_{k,\alpha} \equiv \frac{\Delta_k[\exp(-i\omega_k)]}{\Delta^0_k[\exp(-i\omega_k)]} \phi[\exp(-i\omega_k)], \Lambda^*_{k,\beta} \equiv \frac{\Delta_k[\exp(i\omega_k)]}{\Delta^0_k[\exp(i\omega_k)]} \phi[\exp(i\omega_k)].$$

Because of the stationarity assumption on the polynomial $\phi(z)$, $\phi[\exp(-i\omega_k)] \neq 0$ and $\phi[\exp(i\omega_k)] \neq 0$. Note that $\Lambda^*_{k,\alpha}$ and $\Lambda^*_{k,\beta}$ are complex conjugates and $\lambda^*_{k,\alpha}(= \Lambda^*_{k,\alpha}) = \lambda^*_{k,\beta}(= \Lambda^*_{k,\beta})$, $k = 0, S/2$, are real.

It is easily seen that $\exp(i\omega_k)$ and $\exp(-i\omega_k)$, $k = 0, ..., [S/2]$, are roots of the polynomial $\psi^*(z) = 0$ and, therefore, we may express

$$\alpha^*(z) = \phi^*(z)\Delta_S(z) - \sum_{k=0}^{[S/2]} \left(\lambda^*_{k,\alpha}\delta^0_{k,\beta}(z)[\delta_{k,\alpha}(z) - \delta^0_{k,\alpha}(z)] + \lambda^*_{k,\beta}\delta^0_{k,\alpha}(z)[\delta_{k,\beta}(z) - \delta^0_{k,\beta}(z)] \right) \Delta^0_k(z), \tag{A.1}$$

where $\phi^*(z)$ is a stationary pth order polynomial, $\phi^*(z) = 1 - \sum_{i=1}^p \phi^*_i z^i$, as $\alpha^*_0 = 1$.

For $k = 1, ..., S^*$, the second term in (A.1) involves the polynomial conjugate pair

$$\lambda^*_{k,\alpha}\delta^0_{k,\beta}(z)[\delta_{k,\alpha}(z) - \delta^0_{k,\alpha}(z)] = \delta^0_{k,\beta}(z) z \exp(i\omega_k)\delta_{k,\alpha}[\exp(-i\omega_k)] \frac{\delta_{k,\beta}[\exp(-i\omega_k)]}{\delta^0_{k,\beta}[\exp(-i\omega_k)]} \Lambda^*_{k,\alpha},$$

$$\lambda^*_{k,\beta}\delta^0_{k,\alpha}(z)[\delta_{k,\beta}(z) - \delta^0_{k,\beta}(z)] = \delta^0_{k,\alpha}(z) z \exp(-i\omega_k)\delta_{k,\beta}[\exp(i\omega_k)] \frac{\delta_{k,\alpha}[\exp(i\omega_k)]}{\delta^0_{k,\alpha}[\exp(i\omega_k)]} \Lambda^*_{k,\beta}, \tag{A.2}$$

noting $\delta_{k,\alpha}(z) - \delta^0_{k,\alpha}(z) = z \exp(i\omega_k)\delta_{k,\alpha}[\exp(-i\omega_k)]$ and $\delta_{k,\beta}(z) - \delta^0_{k,\beta}(z) = z \exp(-i\omega_k)\delta_{k,\beta}[\exp(i\omega_k)]$. Now,

$$\frac{\delta_{k,\beta}[\exp(-i\omega_k)]}{\delta^0_{k,\beta}[\exp(-i\omega_k)]} = \frac{\exp(i\omega_k) - r_k \exp(-i(\theta_k + \omega_k))}{2i \sin \omega_k},$$

$$\frac{\delta_{k,\alpha}[\exp(i\omega_k)]}{\delta^0_{k,\alpha}[\exp(i\omega_k)]} = -\frac{\exp(-i\omega_k) - r_k \exp(i(\theta_k + \omega_k))}{2i \sin \omega_k},$$

as

$$\delta_{k,\alpha}[\exp(i\omega_k)] = 1 - r_k \exp[i(\theta_k + 2\omega_k)], \delta_{k,\beta}[\exp(-i\omega_k)] = 1 - r_k \exp[-i(\theta_k + 2\omega_k)],$$

$$\delta^0_{k,\alpha}[\exp(i\omega_k)] = -2i \sin \omega_k \exp(i\omega_k), \delta^0_{k,\beta}[\exp(-i\omega_k)] = 2i \sin \omega_k \exp(-i\omega_k).$$

Hence, the polynomial conjugate pair in (A.2)

$$\delta^0_{k,\beta}(z) z \exp(i\omega_k)\delta_{k,\alpha}[\exp(-i\omega_k)] \frac{\delta_{k,\beta}[\exp(-i\omega_k)]}{\delta^0_{k,\beta}[\exp(-i\omega_k)]}$$

and

$$\delta^0_{k,\alpha}(z) z \exp(-i\omega_k)\delta_{k,\beta}[\exp(i\omega_k)] \frac{\delta_{k,\alpha}[\exp(i\omega_k)]}{\delta^0_{k,\alpha}[\exp(i\omega_k)]}$$

have real part

$$\frac{z}{2}\left(\gamma_k[-(\cos\omega_k - z)] + \delta_k[\sin\omega_k]\right)$$

and imaginary parts + and − times respectively

$$\frac{z}{2}\left(-\gamma_k[\sin\omega_k] + \delta_k[-(\cos\omega_k - z)]\right).$$

Therefore, from (A.1)

$$\begin{aligned}
\alpha^*(z) &= \phi^*(z)\Delta_S(z) - \Lambda^*_{0,\alpha}\left(-\gamma_0 z\right)\Delta^0_0(z) - \Lambda^*_{S/2,\beta}\left(\gamma_{S/2} z\right)\Delta^0_{S/2}(z) \\
&\quad - \sum_{k=1}^{S^*} \mathcal{R}e(\Lambda^*_{k,\alpha})\left(\gamma_k[-(\cos\omega_k - z)z] + \delta_k[\sin\omega_k z]\right)\Delta^0_k(z) \\
&\quad + \sum_{k=1}^{S^*} \mathcal{I}m(\Lambda^*_{k,\alpha})\left(-\gamma_k[\sin\omega_k z] + \delta_k[-(\cos\omega_k - z)z]\right)\Delta^0_k(z) \\
&= \phi^*(z)\Delta_S(z) - \Lambda^*_{0,\alpha}\left(-\gamma_0 z\right)\Delta^0_0(z) - \Lambda^*_{S/2,\beta}\left(\gamma_{S/2} z\right)\Delta^0_{S/2}(z) \\
&\quad - \sum_{k=1}^{S^*}[\mathcal{R}e(\Lambda^*_{k,\alpha})\gamma_k - \mathcal{I}m(\Lambda^*_{k,\alpha})\delta_k][-(\cos\omega_k - z)z]\Delta^0_k(z) \\
&\quad - \sum_{k=1}^{S^*}[\mathcal{R}e(\Lambda^*_{k,\alpha})\delta_k + \mathcal{I}m(\Lambda^*_{k,\alpha})\gamma_k][\sin\omega_k z]\Delta^0_k(z),
\end{aligned}$$

noting $\mathcal{R}e(\Lambda^*_{k,\alpha}) = \mathcal{R}e(\Lambda^*_{k,\beta})$ and $\mathcal{I}m(\Lambda^*_{k,\alpha}) = -\mathcal{I}m(\Lambda^*_{k,\beta})$, $k = 1, ..., S^*$. □

Proof of Lemma 1: It is straightforward to show that, as in Boswijk and Franses (1996) and Burridge and Taylor (2001), *inter alia*, the process $\{x_{St+s}\}$ admits the vector-of-seasons representation

$$(1 - B)\,\mathbf{X}_t = (\mathbf{\Theta}_0 + \mathbf{\Theta}_1 B)\,\mathbf{U}^*_t, \quad t = 1, ..., T, \tag{A.3}$$

$$\mathbf{U}^*_t = \sum\nolimits_{j=0}^{\infty} \mathbf{\Psi}^*_j \mathbf{E}_t \tag{A.4}$$

where B is the annual lag operator, such that $B^k\mathbf{X}_t \equiv \mathbf{X}_{t-k}$, $k = 0, \pm1, ...$, and the $S \times S$ matrices $\mathbf{\Theta}_0$ and $\mathbf{\Theta}_1$ are such that $\mathbf{\Theta}_0 + \mathbf{\Theta}_1 = \mathbf{C}$ and, hence, are again determined by the form of $\bar{\alpha}(z)$. To illustrate, for $\bar{\alpha}(z) = (1 - z)$,

$$\mathbf{\Theta}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \quad \mathbf{\Theta}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

while for $\bar{\alpha}(z) = (1 + z)$,

$$\mathbf{\Theta}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ -1 & 1 & -1 & 1 & \cdots & 1 \end{bmatrix} \quad \mathbf{\Theta}_1 = \begin{bmatrix} 0 & -1 & 1 & -1 & \cdots & -1 \\ 0 & 0 & -1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & -1 & \cdots & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and for $\bar{\alpha}(z) = (1 - 2\cos\omega_k z + z^2)$, $k = 1, ..., S^*$,

$$\mathbf{\Theta}_0 = \frac{1}{\sin(\omega_k)} \begin{bmatrix} \sin(\omega_k) & 0 & 0 & 0 & \cdots & 0 \\ \sin(2\omega_k) & \sin(\omega_k) & 0 & 0 & \cdots & 0 \\ \sin(3\omega_k) & \sin(2\omega_k) & \sin(\omega_k) & 0 & \cdots & 0 \\ \sin(4\omega_k) & \sin(3\omega_k) & \sin(2\omega_k) & \sin(\omega_k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin((S-1)\omega_k) & \sin((S-2)\omega_k) & \sin((S-3)\omega_k) & \sin((S-3)\omega_k) & \cdots & \sin(\omega_k) \end{bmatrix}$$

and

$$\mathbf{\Theta}_1 = \frac{1}{\sin(\omega_k)} \begin{bmatrix} 0 & \sin((S-1)\omega_k) & \sin((S-2)\omega_k) & \sin((S-3)\omega_k) & \cdots & \sin(2\omega_k) \\ 0 & 0 & \sin((S-1)\omega_k) & \sin((S-2)\omega_k) & \cdots & \sin(3\omega_k) \\ 0 & 0 & 0 & \sin((S-1)\omega_k) & \cdots & \sin(4\omega_k) \\ 0 & 0 & 0 & 0 & \cdots & \sin(5\omega_k) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sin(\omega_k) \end{bmatrix}.$$

It is then straightforward to establish, along the same lines as for the proof of Lemma 1 in Boswijk and Franses (1996), from (A.3)-(A.4) the result that $T^{-1/2}\mathbf{X}_{[T\cdot]} \Rightarrow \sigma\mathbf{C}\mathbf{\Psi}^*(1)\mathbf{W}(\cdot)$, where $\mathbf{W}(\cdot) \equiv (W_{1-S}(\cdot), W_{2-S}(\cdot), ..., W_0(\cdot))'$ is an $S \times 1$ standard Brownian motion process. The stated result then follows directly from an application of the continuous mapping theorem [CMT]. □

Before turning the proof of Theorem 1 it will prove instructive to first establish some properties of circulant matrices which will be used in the proof of Theorem 1.

Some Properties of Circulant matrices used in Remark 2 and Theorem 1

From Theorem 3.2.3 of Davis (1979), a generic circulant matrix $\mathbf{A} = \mathsf{Circ}[a_1, a_2, a_3, \cdots, a_S]$ of order $S \times S$ admits the following decomposition $\mathbf{A} = \mathbf{F}^*\mathbf{\Lambda}\mathbf{F}$ where $\mathbf{F}^*$ and $\mathbf{F}$ are the Fourier matrix of order S and its complex conjugate, respectively, (see section 2.5 of Davis, 1979, for details) and $\mathbf{\Lambda}$ is a diagonal matrix of the form $\mathbf{\Lambda} = diag[\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_S]$, where λ_j, $j = 1, 2, \ldots, S$ are the eigenvalues of $\mathbf{A}$. Now, from Theorem 3.2.2 of Davis (1979) these eigenvalues can be obtained using $\lambda_j = P_{\mathbf{A}}(\varpi^{j-1})$

where $P_{\mathbf{A}}(z) = \sum_{j=1}^{S} a_j z^{j-1}$ is the polynomial associated with the circulant matrix, $\varpi = \exp\left(\frac{2\pi i}{S}\right) = \cos\left(\frac{2\pi}{S}\right) + i\sin\left(\frac{2\pi}{S}\right)$, with $i = \sqrt{-1}$. From Theorem 3.2.4 of Davis (1979) and Theorem 3.1 of Gray (2006), we have the following properties of the sums and products of circulant matrices, where $\mathbf{B} = \mathsf{Circ}[b_1, b_2, b_3, \cdots, b_S]$ is a second circulant matrix with associated eigenvalues $\bar{\lambda}_j$, $j = 1, 2, \ldots, S$:

$$\begin{aligned} \mathbf{AB} &= \mathbf{F}^* diag[\lambda_1\bar{\lambda}_1, \lambda_2\bar{\lambda}_2, \cdots, \lambda_S\bar{\lambda}_S]\mathbf{F} \\ \mathbf{A}+\mathbf{B} &= \mathbf{F}^* diag[\lambda_1+\bar{\lambda}_1, \lambda_2+\bar{\lambda}_2, ..., \lambda_S+\bar{\lambda}_S]\mathbf{F} \\ c\mathbf{A} &= \mathbf{F}^* diag[c\lambda_1, c\lambda_2, \cdots, c\lambda_S]\mathbf{F}, \end{aligned}$$

where c is a scalar.

From the familiar Granger Representation Theorem (see Engle and Granger, 1987) we know that for the multivariate Wold representation in (A.3) the matrix $\mathbf{C} = (\mathbf{\Theta}_0 + \mathbf{\Theta}_1)$ has rank equal to the number of unit roots present in $\bar{\alpha}(z)$. Hence, the matrices $\mathbf{C}_0$ and $\mathbf{C}_{S/2}$ have rank one and the matrices $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$ have rank two. Consequently, only one of the eigenvalues of $\mathbf{C}_0$ and $\mathbf{C}_{S/2}$ is non-zero, and only two of the eigenvalues of $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$ are non-zero. Furthermore, using Theorem 3.1.1 of Fuller (1996), it is possible to establish the position and the value of the non-zero eigenvalues of matrices $\mathbf{C}_0$, $\mathbf{C}_{S/2}$, $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$, $k = 1, ..., S^*$. In particular, the non-zero eigenvalue of $\mathbf{C}_0$ and $\mathbf{C}_{S/2}$ is in both cases equal to S and has the first position and the $S/2-1$ position in the diagonal eigenmatrix, respectively. In the case of matrices $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$, the non-zero eigenvalues have the $j+1$ and $S-j+1$ positions, where j is such that $\omega_k = \frac{2\pi j}{S}$. The non-zero eigenvalues associated with $\mathbf{C}_k^{\alpha}$ are equal to $S/2$, and the non-zero eigenvalues associated with $\mathbf{C}_k^{\beta}$ are equal to $-S/2i$ in position $j+1$ and equal to $S/2i$ in position $S-j+1$. Based on the previous results relating to circulant matrices it is evident that the general circulant matrix $\mathbf{C}$ associated with $\bar{\alpha}(z)$ can always be decomposed as in (5.3).

We now state some important identities relating to products involving $\mathbf{C}_0$, $\mathbf{C}_{S/2}$, $\mathbf{C}_k^{\alpha}$, $\mathbf{C}_k^{\beta}$, $k = 1, ..., S^*$, and $\mathbf{C}$:

$$\begin{aligned} \mathbf{C}_j\mathbf{C}_j &= S\mathbf{C}_j,\ j = 0, S/2 \\ \mathbf{C}_k^{\alpha}\mathbf{C}_k^{\alpha} &= \frac{S}{2}\mathbf{C}_k^{\alpha},\ \ \mathbf{C}_k^{\alpha}\mathbf{C}_k^{\beta} = \frac{S}{2}\mathbf{C}_k^{\beta},\ \ \mathbf{C}_k^{\beta}\mathbf{C}_k^{\beta} = -\frac{S}{2}\mathbf{C}_k^{\alpha} \\ \mathbf{C}'\mathbf{C}_j &= \mathbf{C}_j\mathbf{C} = Sc_j^{\bar{\alpha}}\mathbf{C}_j,\ j = 0, S/2 \\ \mathbf{C}'\mathbf{C}_k^{\alpha} &= \frac{S}{2}\left(c_{k\alpha}^{\bar{\alpha}}\mathbf{C}_k^{\alpha} - c_{k\beta}^{\bar{\alpha}}\mathbf{C}_k^{\beta}\right),\ \ \mathbf{C}_k^{\alpha}\mathbf{C} = \frac{S}{2}\left(c_{k\alpha}^{\bar{\alpha}}\mathbf{C}_k^{\alpha} + c_{k\beta}^{\bar{\alpha}}\mathbf{C}_k^{\beta}\right) \\ \mathbf{C}'\mathbf{C}_k^{\beta} &= \frac{S}{2}\left(c_{k\alpha}^{\bar{\alpha}}\mathbf{C}_k^{\beta} + c_{k\beta}^{\bar{\alpha}}\mathbf{C}_k^{\alpha}\right),\ \ \mathbf{C}_k^{\beta}\mathbf{C} = \frac{S}{2}\left(c_{k\alpha}^{\bar{\alpha}}\mathbf{C}_k^{\beta} - c_{k\beta}^{\bar{\alpha}}\mathbf{C}_k^{\alpha}\right), \end{aligned} \tag{A.5}$$

where we have used the fact that $\mathbf{C}_0$, $\mathbf{C}_{S/2}$ and $\mathbf{C}_k^{\alpha}$, $k = 1, ..., S^*$, are symmetric and $(\mathbf{C}_k^{\beta})' = -\mathbf{C}_k^{\beta}$, $k = 1, ..., S^*$. Moreover, for products involving $\mathbf{\Psi}^*(1)$ and $\mathbf{C}_0$, $\mathbf{C}_{S/2}$, $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$, $k = 1, ..., S^*$, the following identities hold:

$$\begin{aligned}
\mathbf{\Psi}^*(1)'\mathbf{C}_0 &= \mathbf{C}_0\mathbf{\Psi}^*(1) = \psi(1)\mathbf{C}_0, \;\; \mathbf{\Psi}^*(1)'\mathbf{C}_{S/2} = \mathbf{C}_{S/2}\mathbf{\Psi}^*(1) = \psi(-1)\mathbf{C}_{S/2} \\
\mathbf{\Psi}^*(1)'\mathbf{C}_k^{\alpha} &= b_k\mathbf{C}_k^{\alpha} - a_k\mathbf{C}_k^{\beta}, \;\; \mathbf{C}_k^{\alpha}\mathbf{\Psi}^*(1) = b_k\mathbf{C}_k^{\alpha} + a_k\mathbf{C}_k^{\beta} \qquad \text{(A.6)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}_k^{\beta} &= a_k\mathbf{C}_k^{\alpha} + b_k\mathbf{C}_k^{\beta}, \;\; \mathbf{C}_k^{\beta}\mathbf{\Psi}^*(1) = -a_k\mathbf{C}_k^{\alpha} + b_k\mathbf{C}_k^{\beta}
\end{aligned}$$

where b_k and a_k are as defined in Theorem 1. Finally by using the identities in (A.5) and (A.6) we may establish the following identities which will be required in order to prove the results in Theorem 1:

$$\begin{aligned}
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_0 &= Sc_0^{\bar{\alpha}}\psi(1)\mathbf{C}_0 \qquad \text{(A.7)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_0\mathbf{C}\mathbf{\Psi}^*(1) &= [Sc_0^{\bar{\alpha}}\psi(1)]^2\mathbf{C}_0 \qquad \text{(A.8)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_{S/2} &= Sc_{S/2}^{\bar{\alpha}}\psi(-1)\mathbf{C}_{S/2} \qquad \text{(A.9)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_{S/2}\mathbf{C}\mathbf{\Psi}^*(1) &= [Sc_{S/2}^{\bar{\alpha}}\psi(-1)]^2\mathbf{C}_{S/2} \qquad \text{(A.10)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_k^{\alpha} &= \frac{S}{2}\left[\left(c_{k\alpha}^{\bar{\alpha}}b_k - c_{k\beta}^{\bar{\alpha}}a_k\right)\mathbf{C}_k^{\alpha} - \left(c_{k\alpha}^{\bar{\alpha}}a_k + c_{k\beta}^{\bar{\alpha}}b_k\right)\mathbf{C}_k^{\beta}\right] \qquad \text{(A.11)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_k^{\beta} &= \frac{S}{2}\left[\left(c_{k\alpha}^{\bar{\alpha}}a_k + c_{k\beta}^{\bar{\alpha}}b_k\right)\mathbf{C}_k^{\alpha} + \left(c_{k\alpha}^{\bar{\alpha}}b_k - c_{k\beta}^{\bar{\alpha}}a_k\right)\mathbf{C}_k^{\beta}\right] \qquad \text{(A.12)} \\
\mathbf{\Psi}^*(1)'\mathbf{C}'\mathbf{C}_k^{\alpha}\mathbf{C}\mathbf{\Psi}^*(1) &= \left(\frac{S}{2}\right)^2\left[\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right]\left(b_k^2 + a_k^2\right)\mathbf{C}_k^{\alpha}. \qquad \text{(A.13)}
\end{aligned}$$

Proof of Theorem 1: Note first that by Frisch-Waugh theorem the OLS estimators of π_0^*, $\pi_{S/2}^*$,$\pi_{k,\alpha}^*$, $\pi_{k,\beta}^*$, $k = 1, \ldots, S^*$, and ϕ_i^*, $i = 1, \ldots, p$, from (4.1) and (4.5) are identical. Moreover, (4.5) can be expressed in vector form as:

$$\hat{\mathbf{y}} = \left[\hat{\mathbf{Y}}_1|\hat{\mathbf{Y}}_2\right]\boldsymbol{\beta}_0 + \hat{\mathbf{u}}$$

where: $\hat{\mathbf{y}}$ is an $N \times 1$ vector with generic element $\hat{\Delta}_S x_{St+s}$; $\hat{\mathbf{Y}}_1$ and $\hat{\mathbf{Y}}_2$ are $N \times r$ and $N \times [(S-r)+p]$ matrices, where r is the number of unit roots present in $\bar{\alpha}(z)$, such that $\hat{\mathbf{Y}}_1$ collects together those non-stationary variables in (4.6) which are associated with the r unit roots present in $\bar{\alpha}(z)$, while $\hat{\mathbf{Y}}_2$ collects together the remaining (stationary) variables from (4.6) together with the p lags of $\hat{\Delta}_S x_{St+s}$; $\boldsymbol{\beta}_0 \equiv (\pi_0^*, \pi_{1,\alpha}^*, \pi_{1,\beta}^*, \ldots, \pi_{S^*,\alpha}^*, \pi_{S^*,\beta}^*, \pi_{S/2}^*, \phi_1^*, \ldots, \phi_p^*)'$, omitting $\pi_{S/2}^*$ when S is odd; finally $\hat{\mathbf{u}}$ is an $N \times 1$ vector with generic element $\hat{\epsilon}_{St+s}$. The scaled OLS estimator of $\boldsymbol{\beta}$ from (4.5) can then be written as

$$\mathbf{D}_T\hat{\boldsymbol{\beta}}_0 = \begin{bmatrix} N^{-2}\hat{\mathbf{Y}}_1'\hat{\mathbf{Y}}_1 & N^{-3/2}\hat{\mathbf{Y}}_1'\hat{\mathbf{Y}}_2 \\ N^{-3/2}\hat{\mathbf{Y}}_2'\hat{\mathbf{Y}}_1 & N^{-1}\hat{\mathbf{Y}}_2'\hat{\mathbf{Y}}_2 \end{bmatrix}^{-1} \times \begin{bmatrix} N^{-1}\hat{\mathbf{Y}}_1'\hat{\mathbf{y}} \\ N^{-1/2}\hat{\mathbf{Y}}_2'\hat{\mathbf{y}} \end{bmatrix}$$

where $\mathbf{D}_T$ is a diagonal scaling matrix of the form $\mathbf{D}_T = diag[N \times \mathbf{I}_r,\ N^{1/2} \times \mathbf{I}_{(S-r)+p}]$. Observe from the non-stationary nature of the elements of the matrix $\hat{\mathbf{Y}}_1$ and the stationarity of the elements of matrix $\hat{\mathbf{Y}}_2$, it follows straightforwardly that $N^{-3/2}\hat{\mathbf{Y}}_1'\hat{\mathbf{Y}}_2 \xrightarrow{p}$

0. Moreover, due to the (asymptotic) orthogonality of the non-stationary auxiliary variables in $\hat{\mathbf{Y}}_1$, the matrix $N^{-2}\hat{\mathbf{Y}}_1'\hat{\mathbf{Y}}_1$ weakly converges to a diagonal matrix the elements on the leading diagonal of which are well-defined random variables. We now establish the results in parts (a)-(d) in turn.

Proof of (a): Using the asymptotic orthogonality of the variables in $\hat{\mathbf{Y}}_1$, noted above, the t_0 statistic satisfies

$$t_0 = \frac{N^{-1}\hat{\mathbf{y}}_0'\mathbf{Q}\hat{\mathbf{y}}}{\sqrt{\hat{\sigma}^2 N^{-2}\hat{\mathbf{y}}_0'\mathbf{Q}\hat{\mathbf{y}}_0}} + o_p(1), \tag{A.14}$$

where $\hat{\mathbf{y}}_0$ is an $N \times 1$ vector with generic element $\hat{x}_{0,St+s-1}$, and $\mathbf{Q}$ is the $N \times N$ symmetric and idempotent matrix $\mathbf{Q} = \mathbf{I} - \hat{\mathbf{Y}}_2\left(\hat{\mathbf{Y}}_2'\hat{\mathbf{Y}}_2\right)^{-1}\hat{\mathbf{Y}}_2'$. Consider first the denominator of (A.14). It is straightforwardly to show that

$$\begin{aligned} N^{-2}\hat{\mathbf{y}}_0'\mathbf{Q}\hat{\mathbf{y}}_0 &= N^{-2}\hat{\mathbf{y}}_0'\hat{\mathbf{y}}_0 + o_p(1) \\ &= N^{-2}\sum\left(\hat{x}_{0,St+s-1}\right)^2 + o_p(1) \\ &= N^{-2}S\sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_0\hat{\mathbf{X}}_{t-1}\right) + o_p(1). \end{aligned}$$

Similarly, for the numerator

$$\begin{aligned} N^{-1}\hat{\mathbf{y}}_0'\mathbf{Q}\hat{\mathbf{y}} = N^{-1}\hat{\mathbf{y}}_0'\mathbf{Q}\hat{\mathbf{u}} &= N^{-1}\hat{\mathbf{y}}_0'\hat{\mathbf{u}} + o_p(1) \\ &= N^{-1}\sum \hat{x}_{0,St+s-1}\hat{\epsilon}_{St+s} + o_p(1) \\ &= N^{-1}\sum \hat{\mathbf{X}}_{t-1}'\mathbf{C}_0\hat{\mathbf{E}}_t + o_p(1), \end{aligned}$$

where $\hat{\mathbf{E}}_t$ is the de-meaned and de-trended counterpart of $\mathbf{E}_t$.

Using Lemma 1, (A.7), (A.8) and the CMT it follows that

$$\begin{aligned} N^{-2}S\sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_0\hat{\mathbf{X}}_{t-1}\right) &\Rightarrow \frac{\sigma^2}{S}\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{\Psi}^*(1)'\,\mathbf{C}'\mathbf{C}_0\mathbf{C}\mathbf{\Psi}^*(1)\,\hat{\mathbf{W}}(r)\,dr \\ &= \frac{\sigma^2}{S}[Sc_0^{\bar{\alpha}}\psi(1)]^2\int_0^1 \hat{\mathbf{W}}(r)'\,C_0\hat{\mathbf{W}}(r)\,dr \\ &= \sigma^2[Sc_0^{\bar{\alpha}}\psi(1)]^2\int_0^1 \hat{\mathbf{W}}^*(r)'\,\mathbf{C}_0\hat{\mathbf{W}}^*(r)\,dr \end{aligned}$$

and

$$\begin{aligned} N^{-1}\sum \hat{\mathbf{X}}_{t-1}'\mathbf{C}_0\hat{\mathbf{E}}_t &\Rightarrow \frac{\sigma^2}{S}\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{\Psi}^*(1)'\,\mathbf{C}'\mathbf{C}_0 d\hat{\mathbf{W}}(r) \\ &= \frac{\sigma^2}{S}c_0^{\bar{\alpha}}S\psi(1)\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{C}_0 d\hat{\mathbf{W}}(r) \\ &= \sigma^2 c_0^{\bar{\alpha}}S\psi(1)\int_0^1 \hat{\mathbf{W}}^*(r)'\,\mathbf{C}_0 d\hat{\mathbf{W}}^*(r), \end{aligned}$$

where $\hat{\mathbf{W}}^*(r) \equiv \frac{1}{\sqrt{S}}\hat{\mathbf{W}}(r)$. Hence, as $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$, and noting that $\mathbf{C}_0 = v_0 v_0'$ where $v_0' = (1,1,1,1,\ldots,1)$, it is easy to see that $\hat{w}_0(r) = v_0'\hat{\mathbf{W}}^*(r) = S^{-1/2}\sum_{j=1-S}^{0}\hat{W}_j(r)$, a standard de-meaned and de-trended Brownian motion, the result in (5.4) follows straightforwardly using the CMT.

Proof of (b): In similar fashion as was observed for t_0 in the proof of part (a), it can be shown that

$$\begin{aligned} t_{S/2} &= \frac{N^{-1}\hat{\mathbf{y}}_{S/2}'\mathbf{Q}\hat{\mathbf{y}}}{\sqrt{\hat{\sigma}^2 N^{-2}\hat{\mathbf{y}}_{S/2}'\mathbf{Q}\hat{\mathbf{y}}_{S/2}}} + o_p(1) \\ &= \frac{N^{-1}\sum_t \hat{\mathbf{X}}_{t-1}'\mathbf{C}_{S/2}\hat{\mathbf{E}}_t}{\sqrt{\hat{\sigma}^2 N^{-2} S \sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_{S/2}\hat{\mathbf{X}}_{t-1}\right)}} + o_p(1) \end{aligned}$$

where $\hat{\mathbf{y}}_{S/2}$ is an $N \times 1$ vector with generic element $\hat{x}_{S/2,St+s-1}$. Using Lemma 1, (A.9), (A.10) and the CMT we obtain that

$$\begin{aligned} N^{-2}S\sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_{S/2}\hat{\mathbf{X}}_{t-1}\right) &\Rightarrow \frac{\sigma^2}{S}\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{\Psi}^*(1)'\,\mathbf{C}'\mathbf{C}_{S/2}\mathbf{C}\mathbf{\Psi}^*(1)\,\hat{\mathbf{W}}(r)\,dr \\ &= \frac{\sigma^2}{S}[Sc_{S/2}^{\bar{\alpha}}\psi(-1)]^2\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{C}_{S/2}\hat{\mathbf{W}}(r)\,dr \\ &= \sigma^2[Sc_{S/2}^{\bar{\alpha}}\psi(-1)]^2\int_0^1 \hat{\mathbf{W}}^*(r)'\,\mathbf{C}_{S/2}\hat{\mathbf{W}}^*(r)\,dr \end{aligned}$$

and

$$\begin{aligned} N^{-1}\sum \hat{\mathbf{X}}_{t-1}'\mathbf{C}_{S/2}\hat{\mathbf{E}}_t &\Rightarrow \frac{\sigma^2}{S}\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{\Psi}^*(1)'\,\mathbf{C}'\mathbf{C}_{S/2}d\hat{\mathbf{W}}(r) \\ &= \frac{\sigma^2}{S}c_{S/2}^{\bar{\alpha}}S\psi(-1)\int_0^1 \hat{\mathbf{W}}(r)'\,\mathbf{C}_{S/2}d\hat{\mathbf{W}}(r) \\ &= \sigma^2 c_{S/2}^{\bar{\alpha}}S\psi(-1)\int_0^1 \hat{\mathbf{W}}^*(r)'\,\mathbf{C}_{S/2}d\hat{\mathbf{W}}^*(r) \end{aligned}$$

where $\hat{\mathbf{W}}^*(r) \equiv \frac{1}{\sqrt{S}}\hat{\mathbf{W}}(r)$. Hence, as $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ and noting that $\mathbf{C}_{S/2} = v_{S/2}v_{S/2}'$, where $v_{S/2}' = (-1,1,-1,1,...,1)$, it is easy to see that $\hat{w}_{S/2}(r) = v_{S/2}'\hat{\mathbf{W}}^*(r) = S^{-1/2}\sum_{j=1-S}^{0}(-1)^{-j}$ $\hat{W}_j(r)$, a standard de-meaned and de-trended Brownian motion independent of $\hat{w}_0$ from part (a), the result in (5.5) follows using the CMT.

Proof of (c): As in parts (a) and (b), it can be shown that

$$
\begin{aligned}
t_k^{\alpha} &= \frac{N^{-1}\hat{\mathbf{y}}_k^{\alpha\prime}\mathbf{Q}\hat{\mathbf{y}}}{\sqrt{\hat{\sigma}^2 N^{-2}\hat{\mathbf{y}}_k^{\alpha\prime}\mathbf{Q}\hat{\mathbf{y}}_k^{\alpha}}} + o_p(1) = \frac{N^{-1}\sum_t \hat{\mathbf{X}}_{t-1}'\mathbf{C}_k^{\alpha}\hat{\mathbf{E}}_t}{\sqrt{\hat{\sigma}^2 N^{-2} S \sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_k^{\alpha}\hat{\mathbf{X}}_{t-1}\right)}} + o_p\left(1\right) \\
t_k^{\beta} &= \frac{N^{-1}\hat{\mathbf{y}}_k^{\beta\prime}\mathbf{Q}\hat{\mathbf{y}}}{\sqrt{\hat{\sigma}^2 N^{-2}\hat{\mathbf{y}}_k^{\beta\prime}\mathbf{Q}\hat{\mathbf{y}}_k^{\beta}}} + o_p(1) = \frac{N^{-1}\sum_t \hat{\mathbf{X}}_{t-1}'\mathbf{C}_k^{\beta}\hat{\mathbf{E}}_t}{\sqrt{\hat{\sigma}^2 N^{-2} S \sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_k^{\alpha}\hat{\mathbf{X}}_{t-1}\right)}} + o_p\left(1\right)
\end{aligned}
$$

where $\hat{\mathbf{y}}_k^{\alpha}$ and $\hat{\mathbf{y}}_k^{\alpha}$ are $N \times 1$ vectors with generic elements $\hat{x}_{k,St+s-1}^{\alpha}$ and $\hat{x}_{k,St+s-1}^{\beta}$, respectively. Using Lemma 1, (A.11), (A.12), (A.13) and applications of the CMT we obtain the following results:

$$
\begin{aligned}
N^{-2}\frac{S}{2}\sum\left(\hat{\mathbf{X}}_{t-1}'\mathbf{C}_k^{\alpha}\hat{\mathbf{X}}_{t-1}\right) &\Rightarrow \frac{\sigma^2}{S^2}\left(\frac{S}{2}\right)\int_0^1 \hat{\mathbf{W}}\left(r\right)'\mathbf{\Psi}^*\left(1\right)'\mathbf{C}'\mathbf{C}_k^{\alpha}\mathbf{C}\mathbf{\Psi}^*\left(1\right)\hat{\mathbf{W}}\left(r\right)dr \\
&= \frac{\sigma^2}{S^2}\left(\frac{S}{2}\right)^3\left[\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right]\left(b_k^2 + a_k^2\right)\int_0^1 \hat{\mathbf{W}}\left(r\right)'\mathbf{C}_k^{\alpha}\hat{\mathbf{W}}\left(r\right)dr \\
&= \frac{\sigma^2}{S^2}\left(\frac{S}{2}\right)^4\left[\left(c_{k\alpha}^{\bar{\alpha}}\right)^2 + \left(c_{k\beta}^{\bar{\alpha}}\right)^2\right]\left(b_k^2 + a_k^2\right)\int_0^1 \hat{\mathbf{W}}^*\left(r\right)'\mathbf{C}_k^{\alpha}\hat{\mathbf{W}}^*\left(r\right)dr
\end{aligned}
$$

$$
\begin{aligned}
N^{-1}\sum\hat{\mathbf{X}}_{t-1}'\mathbf{C}_k^{\alpha}\hat{\mathbf{E}}_t &\Rightarrow \frac{\sigma^2}{S}\int_0^1 \hat{\mathbf{W}}\left(r\right)'\mathbf{\Psi}^*\left(1\right)'\mathbf{C}'\mathbf{C}_k^{\alpha}d\hat{\mathbf{W}}\left(r\right) \\
&= \frac{\sigma^2}{S}\left(\frac{S}{2}\right)\left(c_{k\alpha}^{\bar{\alpha}}b_k - c_{k\beta}^{\bar{\alpha}}a_k\right)\int_0^1 \hat{\mathbf{W}}\left(r\right)'\mathbf{C}_k^{\alpha}d\hat{\mathbf{W}}\left(r\right) \\
&\quad -\frac{\sigma^2}{S}\left(\frac{S}{2}\right)\left(c_{k\alpha}^{\bar{\alpha}}a_k + c_{k\beta}^{\bar{\alpha}}b_k\right)\int_0^1 \hat{\mathbf{W}}\left(r\right)'\mathbf{C}_k^{\beta}d\hat{\mathbf{W}}\left(r\right) \\
&= \frac{\sigma^2}{S}\left(\frac{S}{2}\right)^2\left(c_{k\alpha}^{\bar{\alpha}}b_k - c_{k\beta}^{\bar{\alpha}}a_k\right)\int_0^1 \hat{\mathbf{W}}^*\left(r\right)'\mathbf{C}_k^{\alpha}d\hat{\mathbf{W}}^*\left(r\right) \\
&\quad -\frac{\sigma^2}{S}\left(\frac{S}{2}\right)^2\left(c_{k\alpha}^{\bar{\alpha}}a_k + c_{k\beta}^{\bar{\alpha}}b_k\right)\int_0^1 \hat{\mathbf{W}}^*\left(r\right)'\mathbf{C}_k^{\beta}d\hat{\mathbf{W}}^*\left(r\right)
\end{aligned}
$$

and

$$
\begin{aligned}
N^{-1} \sum \hat{\mathbf{X}}_{t-1}' \mathbf{C}_k^{\beta} \hat{\mathbf{E}}_t &\Rightarrow \frac{\sigma^2}{S} \int_0^1 \hat{\mathbf{W}}(r)' \mathbf{\Psi}^*(1)' \mathbf{C}' \mathbf{C}_k^{\beta} d\hat{\mathbf{W}}(r) \\
&= \frac{\sigma^2}{S}\left(\frac{S}{2}\right)\left(c_{k\alpha}^{\bar{\alpha}} a_k + c_{k\beta}^{\bar{\alpha}} b_k\right) \int_0^1 \hat{\mathbf{W}}(r)' \mathbf{C}_k^{\alpha} d\hat{\mathbf{W}}(r) \\
&\quad + \frac{\sigma^2}{S}\left(\frac{S}{2}\right)\left(c_{k\alpha}^{\bar{\alpha}} b_k - c_{k\beta}^{\bar{\alpha}} a_k\right) \int_0^1 \hat{\mathbf{W}}(r)' \mathbf{C}_k^{\beta} d\hat{\mathbf{W}}(r) \\
&= \frac{\sigma^2}{S}\left(\frac{S}{2}\right)^2\left(c_{k\alpha}^{\bar{\alpha}} a_k + c_{k\beta}^{\bar{\alpha}} b_k\right) \int_0^1 \hat{\mathbf{W}}^*(r)' \mathbf{C}_k^{\alpha} d\hat{\mathbf{W}}^*(r) \\
&\quad + \frac{\sigma^2}{S}\left(\frac{S}{2}\right)^2\left(c_{k\alpha}^{\bar{\alpha}} b_k - c_{k\beta}^{\bar{\alpha}} a_k\right) \int_0^1 \hat{\mathbf{W}}^*(r)' \mathbf{C}_k^{\beta} d\hat{\mathbf{W}}^*(r)
\end{aligned}
$$

where $\hat{\mathbf{W}}^*(r) \equiv \frac{1}{\sqrt{S/2}} \hat{\mathbf{W}}(r)$. Noting that $\mathbf{C}_k^{\alpha} = v_k^{\alpha} v_k^{\alpha\prime}$ and $\mathbf{C}_k^{\beta} = v_k^{\alpha} v_k^{\beta\prime}$, where

$$
v_k^{\alpha\prime} = \begin{bmatrix} \cos(\omega_k[1-S]) & \cos(\omega_k[2-S]) & \cdots & \cos(\omega_k 0) \\ \sin(\omega_k[1-S]) & \sin(\omega_k[2-S]) & \cdots & \sin(\omega_k 0) \end{bmatrix}
$$

and

$$
v_k^{\beta\prime} = \begin{bmatrix} -\sin(\omega_k[1-S]) & -\sin(\omega_k[2-S]) & \cdots & -\sin(\omega_k 0) \\ \cos(\omega_k[1-S]) & \cos(\omega_k[2-S]) & \cdots & \cos(\omega_k 0) \end{bmatrix},
$$

we obtain that

$$
\begin{aligned}
v_k^{\alpha\prime} \hat{\mathbf{W}}^*(r) &= \begin{bmatrix} \hat{w}_k^{\alpha}(r) \\ \hat{w}_k^{\beta}(r) \end{bmatrix} = \begin{bmatrix} (S/2)^{-1/2} \sum_{j=1-S}^{0} \cos(j\omega_k) \hat{W}_j(r) \\ (S/2)^{-1/2} \sum_{j=1-S}^{0} \sin(j\omega_k) \hat{W}_j(r) \end{bmatrix} \\
v_k^{\beta\prime} \hat{\mathbf{W}}^*(r) &= \begin{bmatrix} -\hat{w}_k^{\beta}(r) \\ \hat{w}_k^{\alpha}(r) \end{bmatrix}
\end{aligned}
$$

where $\hat{w}_k^{\alpha}(r)$ and $\hat{w}_k^{\beta}(r)$ are independent standard de-meaned and de-trended Brownian motions, independent of $\hat{w}_0(r)$ and $\hat{w}_{S/2}(r)$ of parts (a) and (b), respectively. The results in (5.6) and (5.7) then follow straightforwardly after a little algebra using the fact that $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ and applications of the CMT. Finally using the fact that $F_k = \frac{1}{2}[(t_k^{\alpha})^2 + (t_k^{\beta})^2] + o_p(1)$, which follows from the asymptotic orthogonality of the elements of $\hat{\mathbf{Y}}_1$, the representation in (5.8) obtains from the results in (5.6) and (5.7) and the CMT, after a little algebra.

Proof of (d): The stated results follow trivially from the results in (5.4), (5.5), (5.8) and applications of the CMT, again using the asymptotic orthogonality of the elements of $\hat{\mathbf{Y}}_1$ in each case.

Table 1: Size and Power of Seasonal Unit Root Tests, $T = 12$

θ_1	r_1	t_0	t_1^{α}	t_1^{β}	F_1	t_2^{α}	t_2^{β}	F_2	t_3
0.00	1.00	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.99	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.95	0.05	0.08	0.04	0.08	0.05	0.05	0.05	0.05
	0.80	0.05	0.54	0.03	0.52	0.05	0.06	0.06	0.06
$\pi/4$	0.99	0.03	0.00	1.00	1.00	0.04	0.07	0.05	0.04
	0.95	0.03	0.01	1.00	1.00	0.04	0.10	0.05	0.04
	0.80	0.02	0.10	1.00	0.90	0.02	0.31	0.05	0.05
$-\pi/4$	0.99	0.05	0.00	1.00	1.00	0.03	0.07	0.04	0.04
	0.95	0.05	0.00	1.00	1.00	0.03	0.07	0.04	0.03
	0.80	0.05	0.11	1.00	1.00	0.03	0.07	0.04	0.04
$11\pi/36$	0.99	0.03	0.00	1.00	1.00	0.07	0.14	0.09	0.05
	0.95	0.03	0.01	1.00	1.00	0.04	0.31	0.07	0.05
	0.80	0.02	0.06	1.00	1.00	0.01	0.60	0.05	0.04
$-11\pi/36$	0.99	0.11	0.00	1.00	1.00	0.03	0.06	0.04	0.04
	0.95	0.09	0.00	1.00	1.00	0.03	0.07	0.04	0.04
	0.80	0.06	0.10	1.00	1.00	0.03	0.07	0.04	0.04

Table 2: Size and Power of Seasonal Unit Root Tests, $T = 35$

θ_1	r_1	t_0	t_1^α	t_1^β	F_1	t_2^α	t_2^β	F_2	t_3
0.00	1.00	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.99	0.05	0.06	0.05	0.06	0.05	0.05	0.05	0.05
	0.95	0.05	0.21	0.03	0.19	0.05	0.05	0.05	0.05
	0.80	0.05	1.00	0.02	1.00	0.05	0.06	0.05	0.05
$\pi/4$	0.99	0.04	0.00	1.00	1.00	0.04	0.06	0.05	0.04
	0.95	0.04	0.03	1.00	1.00	0.04	0.08	0.05	0.04
	0.80	0.04	0.68	1.00	1.00	0.02	0.29	0.05	0.05
$-\pi/4$	0.99	0.05	0.00	1.00	1.00	0.04	0.06	0.04	0.04
	0.95	0.05	0.01	1.00	1.00	0.04	0.06	0.04	0.04
	0.80	0.05	0.85	1.00	1.00	0.03	0.07	0.04	0.04
$11\pi/36$	0.99	0.04	0.00	1.00	1.00	0.05	0.09	0.06	0.04
	0.95	0.04	0.03	1.00	1.00	0.03	0.25	0.06	0.04
	0.80	0.03	0.52	1.00	1.00	0.01	0.60	0.05	0.04
$-11\pi/36$	0.99	0.07	0.00	1.00	1.00	0.04	0.06	0.04	0.04
	0.95	0.06	0.01	1.00	1.00	0.04	0.06	0.04	0.04
	0.80	0.06	0.83	1.00	1.00	0.04	0.06	0.04	0.04

Table 3: Size and Power of Seasonal Unit Root Tests, $T = 50$

θ_1	r_1	t_0	t_1^α	t_1^β	F_1	t_2^α	t_2^β	F_2	t_3
0.00	1.00	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.99	0.05	0.08	0.05	0.07	0.05	0.05	0.05	0.05
	0.95	0.05	0.69	0.03	0.63	0.05	0.05	0.05	0.05
	0.80	0.05	1.00	0.03	1.00	0.05	0.07	0.05	0.05
$\pi/4$	0.99	0.04	0.01	1.00	1.00	0.05	0.05	0.05	0.05
	0.95	0.04	0.17	1.00	1.00	0.04	0.08	0.05	0.05
	0.80	0.04	1.00	1.00	1.00	0.02	0.29	0.05	0.05
$-\pi/4$	0.99	0.05	0.00	1.00	1.00	0.04	0.05	0.05	0.05
	0.95	0.05	0.09	1.00	1.00	0.04	0.06	0.05	0.05
	0.80	0.05	1.00	1.00	1.00	0.04	0.07	0.05	0.05
$11\pi/36$	0.99	0.04	0.01	1.00	1.00	0.05	0.07	0.05	0.05
	0.95	0.05	0.12	1.00	1.00	0.03	0.23	0.05	0.05
	0.80	0.04	0.98	1.00	1.00	0.00	0.61	0.05	0.05
$-11\pi/36$	0.99	0.05	0.00	1.00	1.00	0.04	0.06	0.05	0.05
	0.95	0.05	0.09	1.00	1.00	0.04	0.06	0.05	0.05
	0.80	0.05	1.00	1.00	1.00	0.04	0.07	0.05	0.05

Table 4: Size and Power of Seasonal Unit Root Tests, $T = 100$

θ_1	r_1	t_0	t_1^α	t_1^β	F_1	t_2^α	t_2^β	F_2	t_3
0.00	1.00	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	0.99	0.05	0.15	0.04	0.14	0.05	0.05	0.05	0.05
	0.95	0.05	1.00	0.02	1.00	0.05	0.05	0.05	0.05
	0.80	0.05	1.00	0.04	1.00	0.05	0.07	0.05	0.05
$\pi/4$	0.99	0.05	0.00	1.00	1.00	0.05	0.05	0.05	0.05
	0.95	0.05	0.70	1.00	1.00	0.05	0.07	0.05	0.05
	0.80	0.05	1.00	1.00	1.00	0.03	0.28	0.05	0.05
$-\pi/4$	0.99	0.05	0.00	1.00	1.00	0.05	0.05	0.05	0.05
	0.95	0.05	0.82	1.00	1.00	0.05	0.05	0.05	0.05
	0.80	0.05	1.00	1.00	1.00	0.05	0.06	0.05	0.05
$11\pi/36$	0.99	0.05	0.02	1.00	1.00	0.05	0.06	0.05	0.05
	0.95	0.05	0.58	1.00	1.00	0.03	0.21	0.05	0.05
	0.80	0.05	1.00	1.00	1.00	0.01	0.60	0.05	0.05
$-11\pi/36$	0.99	0.05	0.00	1.00	1.00	0.05	0.05	0.05	0.05
	0.95	0.05	0.79	1.00	1.00	0.05	0.05	0.05	0.05
	0.80	0.05	1.00	1.00	1.00	0.05	0.06	0.05	0.05