CHAOS VS. PATIENCE IN A MACROECONOMIC MODEL OF CAPITAL ACCUMULATION: NEW APPLICATIONS OF A UNIFORM NEIGHBORHOOD TURNPIKE THEOREM

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Resumen: Presentamos nuevos resultados en relación con la fuerte incompatibilidad entre caos y paciencia en modelos macroeconómicos de acumulación de capital. Los resultados son aplicaciones no triviales y explícitas del teorema general probado en Guerrero-Luchtenberg (2000), en cuyo trabajo el concepto (teorema 2) 'el caos desaparece a medida que el factor de descuento tiende a uno', es formalizado y probado. Aquí, exponemos en detalle cómo dicho resultado se aplica a algunos importantes conceptos de caos no analizados anteriormente. Precisamente, mostraremos que, dada una familia de modelos de crecimiento óptimo, existe un valor del factor de descuento, digamos $\hat{\delta}$, tal que, para cualquier otro valor δ mayor que $\hat{\delta}$, cualquier tipo de caos es irrelevante.

Abstract: We present in this paper some new results on the strong incompatibility between chaos and patience in a macroeconomic model of capital accumulation. These results are explicit and non-trivial applications of the general theorem proven in Guerrero-Luchtenberg (2000), in which the statement (Theorem 2) 'Chaos vanishes as the discount factor tends to one', is formally presented. Here, we show precisely how this statement applies to some important indicators of chaos not analyzed before. Furthermore, we will show that, for a given family of optimal growth models, there is a bound on the discount factor, say $\hat{\delta}$, such that, for any δ larger than $\hat{\delta}$, any type of chaos is negligible.

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1. Introduction

Consider a deterministic, reduced form of optimal growth models with discounting, given by:

$$\sup \sum_{t=0}^{\infty} \delta^{t+1} u(x_t, x_{t+1})$$

$$\text{s.t. } (x_t, x_{t+1}) \in D \ \forall\, t \geq 0$$

$$x_0 \text{ given,}$$

where D is the feasible set, x_0 is the initial state, δ is the discount factor (a real number between zero and one) and u is the felicity function.[1]

It is well known that, in this type of models, chaos is precluded under strong concavity assumptions over the felicity function, if the felicity function is fixed (the standard turnpike theorems; see, especially, McKenzie (1986) and Sheinkman 1976). More precisely, if the discount factor is large enough, the optimal path of capital accumulation converges to the steady state, *given the felicity function.* On the other hand, it is also well known that chaos is possible in that type of models. See, for example, Boldrin and Montrucchio (1986) and, especially, Nishimura and Yano (1995) and Nishimura, Sorger and Yano (1994). Furthermore, in these last papers it is shown that there exist families of strictly concave felicity functions, such that for any value of the discount factor, there is a member of the family that displays chaos. These results highlight the necessity of an appropriate justification of the uniform comparative analysis used in empirical works, because it is impossible to find an upper bound on the discount factor in such a way that in all members of the family there is convergence to the steady state.[2]

Indeed, in applications, the set-up is typically a family of models instead of a single model, in such a way that the felicity functions are not fixed. Therefore, the standard turnpike theorems cannot be cited in order to ensure the existence of a single value of the discount factor such that there is convergence to the steady states for all the

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[1] Note that the model presented here is much more general than the Ramsey-Solow model of capital accumulation.

[2] See, for more studies on the presence of chaotic behavior in economics, the special issue of the *Journal of Economic Theory* (2001).

family. Hence, the typical dynamic comparative analysis, by means of the steady states, cannot be used without further justification.

To justify that type of comparative analysis, we may cite some works that show the existence of upper bounds for the discount factor in order for a given type of chaos to be possible, for instance, Mitra (1996) and (1998), Montrucchio and Sorger (1996), Nishimura and Yano (1995), and Sorger (1994), among others. Nevertheless, in all these studies, the type of chaos is fixed, and hence the upper bounds cannot preclude other types of chaos.

On the other hand, in order to find a global justification of the comparative analysis, we can appeal to the uniform turnpike theorem proven in Guerrero-Luchtenberg (2000), the theorem 3 in that paper. That theorem, nonetheless, is proven under strong assumptions of the type 'uniform strong concavity over the family' (assumption A'7 in that paper). Now, that assumption, however, is notable relaxed in the theorem 2 in Guerrero-Luchtenberg (2000), a result that can be interpreted as 'quasi uniform convergence to the steady states all over the family', and that can also be used to find a global justification of the comparative analysis. Nevertheless, that last theorem does not explicitly show how it can be applied to minimize specific concepts of chaos.[3] For this reason, in Guerrero-Luchtenberg (2000), the case of the ergodic chaos is treated in detail. Furthermore, in that paper it is suggested that the uniform neighborhood turnpike theorem can also be used to explicitly study other types of chaos.

The purpose of this study is, therefore, to show explicitly how theorem 2 in Guerrero-Luchtenberg (2000) is applied in order to minimize —in the sense expressed above— some well known and important indicators of chaos. Furthermore, we will show that, for a given family of optimal growth models, there is a bound on the discount factor $\tilde{\delta}$ such that, for any $\delta > \tilde{\delta}$, any type of chaos is negligible for all the family, providing a general justification for the comparative analysis used in empirical works.

The rest of the paper is as follows. In section 2, for the sake of completeness, we roughly introduce the model and the basic definitions, and we announce the theorem 2 given in Guerrero-Luchtenberg (2000). Section 3 presents some basic definitions about dynamical systems. In section 4, we formally present and prove the statement

[3] Informally, we understand 'minimize' in the sense that the chaos is irrelevant, in such a way that the typical dynamic comparative analysis can be applied (see the remark 1 bellow). The formal expression of this idea is the content of our theorem 2.

'chaos vanishes as the discount factor tends to one', our theorem 2; also, we prove the theorems 3 and 4, which are the applications of theorem 2 to some indicators of chaos presented in section 3. Finally, we conclude in section 5.

2. The Model

As our work is heavily based on Guerrero-Luchtenberg (2000) and McKenzie (1986), we refer to these studies for the details on the model and the proofs of the results used in this paper.

Take $D \subset \Re^n_+ \times \Re^n_+$ where $n \geq 1$, $u : D \rightarrow \Re$, and $\delta \in (0,1]$. The set D is the *technology*, the function u is the *felicity function* and δ is the *discount factor*. We say that a sequence $\{x_t\} \subset \Re^n_+$ is a *path* if $(x_t, x_{t+1}) \in D$, for all $t \in N$.[4] We define an *optimal path* from a capital stock $x \in \Re^n_+$, as a path $\{k_t\}$ such that: $k_0 = x$, and

$$\limsup_{T \to \infty} \sum_{t=0}^{T} \left[\delta^{t+1} u(x_t, x_{t+1}) - \delta^{t+1} u(k_t, k_{t+1})\right] \leq 0$$

for all paths $\{x_t\}$, such that $x_0 = x$.

A stationary optimal path $k_t = k$ for all $t \in N$ is called an *Optimal Steady State*, OSS.

All members of the family of optimal growth problems that we will define later are assumed to satisfy the assumptions A0-A" 7-A8 given in Guerrero-Luchtenberg (2000). It is important to notice that those assumptions are standard in optimal growth theory.

Notice that for any $\delta \in \left(\hat{\delta}, 1\right]$[5] and (u, D) satisfying A0-A5, there exists $(k^{u,\delta}, q^{u,\delta}) \in \Re^{2n}_+$, $q^{u,\delta} \neq 0$ and $(k^{u,\delta}, k^{u,\delta}) \in D$, such that the path $k_t^{u,\delta} = k^{u,\delta}$ for all $t \in N$, is an OSS supported by *full Weitzman prices*[6] of the form $p_t^{u,\delta} := \delta^t q^{u,\delta}$. Furthermore, for any $k_0 \in \Re^n_+$ there exists an optimal path $\left\{k_t^{u,\delta}\right\}$ from k_0. If $k_0 \in \Re^n_+ \setminus \{0\}$, any optimal path from k_0 is bounded and can be supported by full Weitzman prices $p_t^{u,\delta} := \delta^t q_t^{u,\delta}$.

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4 $N = \{0, 1, 2, ...\}$

5 For the definition of the number $\hat{\delta}$, see Guerrero-Luchtenberg (2000).

6 This is a standard concept in optimal growth theory. For a precise definition see Guerrero-Luchtenberg (2000).

Under A0-A"7-A8 it is possible to show that there is a well defined function $h : K \to K$, called the policy function, where K is a compact non-empty set of $\Re^n_+$. This function is continuous and satisfies that $\{k_t\}$ is an optimal path from x if and only if it satisfies that

$$k_{t+1} = h(k_t) \ \forall \ t \in N \text{ with } k_0 = x$$

For the sake of exposition, we will reproduce some notation and definitions given in Guerrero-Luchtenberg (2000). Once again, we refer to that paper for the details.

Let $\{k_t\}$ be an optimal path from $k_0 \in \Re^n_+ \backslash \{0\}$ and $\left\{\delta^t q_t^{u,\delta}\right\}$ be the corresponding supporting prices.

Let

$$Q_u(k_0, \left\{q_t^{u,\delta}\right\}) := \sup_{\delta \in (\hat{\delta},1];t \geq 0} \left\{\left|q^{u,\delta}\right|, \left|q_t^{u,\delta}\right|\right\}$$

Take any family as follows,

$$\tilde{U} = \{\tilde{u} : D \to \Re \mid (\tilde{u}, D) \text{ satisfying } A0 - A"7 - A8\} \qquad (1)$$

Now we define[7]

$$U(k_0, \left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\})$$

$$= \left\{u : D \to \Re \mid u = \frac{\tilde{u}}{Q_{\tilde{u}}(k_0, \left\{q_t^{\tilde{u},\delta}\right\})}, \tilde{u} \in \tilde{U}\right\} \qquad (2)$$

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[7] Notice that we write $U(k_0, \left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\})$ in order to emphasize the fact that the family depends on k_0, and

$$\left\{\left\{q_t^{\tilde{u},\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\}$$

provided that for a given $\tilde{u} \in \tilde{U}$ and k_0, the prices $\left\{q_t^{\tilde{u},\delta}\right\}$ are not necessarily unique.

In what follows we will consider any family as in (1) such that for any $k_0 \in \Re^n_+ \setminus \{0\}$ and for any optimal path from k_0 with $\left\{\delta^t q_t^{u,\delta}\right\}$ as a corresponding supporting prices for $\delta \in (\hat{\delta}, 1)$,

$$U(k_0, \left\{\left\{q_t^{u,\delta}\right\} \mid \tilde{u} \in \tilde{U}, \delta \in (\hat{\delta}, 1)\right\})$$

satisfies the concavity condition and is relatively compact. In this case, we say that $\tilde{U}$ *satisfies the concavity condition uniformly and that it is uniformly relatively compact.*

All our results are based on the following

THEOREM 1. *(A Uniform Neighborhood Turnpike Theorem) Take any* $\tilde{U}$ *as in (1) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Take also any* $k_0 \in \Re^n_+ \setminus \{0\}$, *and denote by* $\left\{k_t^{u,\delta}(k_0)\right\}$ *the optimal path from* $k_0 \in \Re^n_+ \setminus \{0\}$ *for a given* $u \in \tilde{U}$ *and* $\delta \in [0, 1]$. *Then, for any* $\varepsilon > 0$ *there exists* $N(\varepsilon)$ *and* $0 < \delta(\varepsilon) < 1$ *such that we have* $\left|k_t^{u,\delta}(k_0) - k^{u,\delta}\right| \le \varepsilon$ *for all* $t > N(\varepsilon)$, *for all* $\delta(\varepsilon) \le \delta < 1$, $u \in \tilde{U}$ *and* $k_0 \in \Re^n_+ \setminus \{0\}$.

PROOF. It follows at once from theorem 2 in Guerrero-Luchtenberg (2000). ∎

3. Dynamical Systems

In this section we will give some basic definitions about dynamical systems. Nevertheless, before we start with these definitions, we would like to make one more general comment about the concepts of chaos that we will consider. Most definitions are made for the sake of studying the long run behavior of optimal solutions and chaos is then defined by using concepts entailing, essentially, some kind of uncertainty about the 'final future' of the dynamical system under consideration. So the precise formulation of our basic results will be made following this basic idea, that is, that chaos vanishes if uncertainty vanishes, in the sense that in spite of any theoretical presence of some kind of chaos, the 'long run future' of the system is not uncertain: It will be possible to say that the system, in the long run, is so close to the OSS, that any theoretical comparative analysis made by means of the corresponding steady states is justified, which constitutes one of the main objectives of this study. The formal expression of this idea is precisely the content of the following theorem 2.

A point $y \in \Re^n_+$ is called an *ω – limit point of* $k_0 \in \Re^n_+$ if there is an optimal sequence $\{k_t\}$ from k_0 and a subsequence $\{k_{t_s}\}$ of $\{k_t\}$ such that $\lim_{s \to \infty} k_{t_s} = y$. Denote by $W(k_0)$ the set of all *ω – limit points of* k_0, called the *ω – limit set of* k_0.

A *discrete dynamical system* can be defined by a pair (K, h) where $K \subset \Re^n$ and h is a function from K to K. The set K is called the *state space.*

For any $x \in K$, define $h^0(x) = x$, and for any $k \geq 1$ $(k \in N)$

$$h^k(x) = h(h^{k-1}(x)).$$

We say that the sequence $\{h^t(x)\}$ *is generated by the iterations of h from x*, and that h^k *is the iteration of h up to order k*. Also, the sequence $\{h^t(x)\}$ is called the *orbit from x*. A point $x \in K$ is called *a periodic point* of h, if $\{h^t(x)\}$ is finite and $h^p(x) = x$ for some $p > 1$. The smallest such p is called *the period of x*. If there exists a periodic point of period k, then we say that the dynamical system (K, h) has *period-k cycles.*

For the sake of the exposition, we present the following definitions.

DEFINITION 1. *For any $\delta \in \left(\hat{\delta}, 1\right)$ and $u \in \tilde{U}$, let $W^{u,\delta}(k_0)$ denote the ω – limit set of k_0; take any family $\tilde{U}$ and define the function $f_{\tilde{U}} : \left(\hat{\delta}, 1\right) \to \Re \cup \{\infty\}$ given by*

$$f_{\tilde{U}}(\delta) = \sup_{k_0 \in \Re^n_+ \setminus \{0\}; u \in \tilde{U}} \left\{ \sup_{y \in W^{u,\delta}(k_0)} \left| y - k^{u,\delta} \right| \right\}$$

DEFINITION 2. *Take any $\tilde{U}$ as in (1) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Take, for a given $u \in \tilde{U}$ and δ, any two points x and y in $\Re^n_+$ and the corresponding w–limit sets, $W_{u,\delta}(x)$ and $W_{u,\delta}(y)$. Now take $d(W_{u,\delta}(x), W_{u,\delta}(y)) = \sup\{| z - v || z \in W_{u,\delta}(x), v \in W_{u,\delta}(y)\}$ (the maximum distance between any two possible w–limit points of x and y respectively). Define now $D(u, \delta) = \sup\{d(W_{u,\delta}(x), W_{u,\delta}(y)) \mid (x, y) \in \Re^{n2}_+\}$ (the maximum distance between any two possible w–limit points of the system (u, D, δ)) and*

$$D_{\tilde{U}}(\delta) = \sup\left\{ D(u,\delta) \,\middle|\, u \in \tilde{U} \right\}$$

(the maximum distance between any two possible w−limit points of the family of systems).

Now we can prove the main results of this paper.

4. The Main Results

4.1. *The General Statement 'Chaos Vanishes as the Discount Factor Tends to One'*

As we have commented above, the chaos vanishes if the uncertainty vanishes, in the sense that in spite of any theoretical presence of some kind of chaos, the 'long run future' of the system is not uncertain: It will be possible to say that the system, in the long run, is so close to the OSS that any theoretical comparative analysis made by means of the corresponding steady states is justified, which constitutes one of the main objectives of this study.

Formally:

THEOREM 2. *Take any $\tilde{U}$ as in (1) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Then,*

(i)

$$\lim_{\delta \to 1} f_{\tilde{U}}(\delta) = 0$$

and

(ii)

$$\lim_{\delta \to 1} f_{\tilde{U}}(\delta) = 0 \text{ if and only if } \lim_{\delta \to 1} D_{\tilde{U}}(\delta) = 0$$

PROOF. First, we prove (i). The proof is by contradiction. Suppose the theorem is false, then there exists a family $\tilde{U}$ satisfying the conditions of the theorem, such that $\lim_{\delta \to 1} f_{\tilde{U}}(\delta) \neq 0$. Then there is an $\varepsilon > 0$, a sequence $\{\delta_l\} \subset \left(\hat{\delta}, 1\right)$ such that $\delta_l \to 1$, and a sequence $k_0^l \in \Re_+^n \setminus \{0\}$ such that there exists $y_l \in W^{u_l,\delta_l}(k_0^l)$ such that

$$\left|y_l - k^{u_l,\delta_l}\right| > \varepsilon \ \forall\ l \text{ large enough}$$

for some sequence $\{u_l\} \subset \tilde{U}$.

Now, by construction of the set K, for any $k_0^l \in \Re_+^n \setminus \{0\}$, if $\{k_t(k_0^l)\}$ denotes the optimal path from k_0^l, then there exists an integer t_l such that $k_{t_l}(k_0^l)$ will belong to K.

Let $x_0^l \in K$ denote such a point, that is, we write

$$x_0^l := k_{t_l}(k_0^l) \ \forall\ l \in N$$

Notice that from the definition of ω−limit sets, one can prove that

$$W^{u_l,\delta_l}(x_0^l) = W^{u_l,\delta_l}(k_0^l)$$

Therefore, for all $l \in N$ we have that $y_l \in W^{u_l,\delta_l}(k_0^l)$, implies

$$y_l \in W^{u_l,\delta_l}(x_0^l)$$

then there exists a sequence $\left\{k_t^{u_l,\delta_l}\right\}$ from x_0^l and a sub-sequence $\left\{k_{t_k}^{u_l,\delta_l}\right\} \subset \left\{k_t^{u_l,\delta_l}\right\}$ such that

$$\lim_{k\to\infty} k_{t_k}^{u_l,\delta_l} = y_l.$$

Therefore,

$$\left|k_{t_k}^{u_l,\delta_l} - k^{u_l,\delta_l}\right| > \varepsilon \ \forall\ k \text{ and } l \text{ large enough,} \tag{3}$$

a contradiction with the theorem 1. This completes the proof of the part (i).

Now we prove (ii). First we prove the implication if $\lim\limits_{\delta\to 1} f_{\tilde{U}}(\delta) = 0$, then $\lim_{\delta\to 1} D_{\tilde{U}}(\delta) = 0$.

Take any pair (u, δ). Now, take any two possible initial states $(x, y) \in \Re_+^{n2}$, and any two possible w−limit points of x and y respectively, say $z(x) \in W_{u,\delta}(x)$ and $v(y) \in W_{u,\delta}(y)$. Recall that for $\delta > \hat{\delta}$, $k^{u,\delta}$ is well defined (the lemma 1 and the theorem 1). Hence, suppose that $\delta > \hat{\delta}$. Now we have

$$\mid z(x) - v(y) \mid \leq \mid z(x) - k^{u,\delta} \mid + \mid v(y) - k^{u,\delta} \mid .$$

Consequently,

$$D(u,\delta) \leq \sup_{y \in W_{u,\delta}(x)} \mid y - k^{u,\delta} \mid + \sup_{y \in W_{u,\delta}(y)} \mid y - k^{u,\delta} \mid .$$

Therefore,

$$D_{\tilde{U}}(\delta) \leq f_{\tilde{U}}(\delta) + f_{\tilde{U}}(\delta) = \frac{1}{2} f_{\tilde{U}}(\delta).$$

Hence, $\lim_{\delta \to 1} D_{\tilde{U}}(\delta) = 0$.

Finally, we prove the implication, if $\lim_{\delta \to 1} D_{\tilde{U}}(\delta) = 0$, then $\lim_{\delta \to 1} f_{\tilde{U}}(\delta) = 0$. Suppose the implication is false. In this case, $\lim_{\delta \to 1} D_{\tilde{U}}(\delta) = 0$, but $\lim_{\delta \to 1} f_{\tilde{U}}(\delta) \neq 0$. Now, if $\lim_{\delta \to 1} f_{\tilde{U}}(\delta) \neq 0$, we can reason as in part (i) and find, for a given $\varepsilon > 0$, a sequence

$$\{\delta_l\} \subset \left(\hat{\delta}, 1\right)$$

such that $\delta_l \to 1$ and optimal sequences

$$\left\{k_t^{u_l,\delta_l}\right\}$$

and subsequences

$$\left\{k_{t_k}^{u_l,\delta_l}\right\} \subset \left\{k_t^{u_l,\delta_l}\right\}$$

in such a way that (3) holds for some sequence $\{u_l\} \subset \tilde{U}$. Now for a fix l in (3), observe that

$$\left\{k_{t_k}^{u_l,\delta_l}\right\}$$

is a bounded sequence, and then there exists a point $z_l \in \Re_+^n$ that is the limit of a subsequence of

$$\left\{k_{t_k}^{u_l,\delta_l}\right\}$$

consequently, z_l is an ω-limit point of $\left\{k_t^{u_l,\delta_l}\right\}$ hence, $D(u_l, \delta_l) \geq \varepsilon$ for all l large enough, which would imply that $\lim_{\delta \to 1} D_{\tilde{U}}(\delta) \geq \varepsilon$, a contradiction. This ends the proof of the implication, if $\lim_{\delta \to 1} D_{\tilde{U}}(\delta) = 0$, then $\lim_{\delta \to 1} f_{\tilde{U}}(\delta) = 0$. ∎

REMARK 1. *Observe that theorem 2 constitutes an expression of our general statement 'chaos vanishes as the discount factor tends to one': Indeed, if point (i) in theorem 2 holds, given any arbitrary* $\varepsilon > 0$, *for* δ *large enough, all the* $w-$*limit points of all the systems will be inside a ball of radius* ε *about the corresponding steady states. In other words, but informally, we can say that no matter the type of chaos a family of systems displays, in the long run, the chaos is shut in an* $\varepsilon-$*ball (because all the* $w-$*limit points are shut in an* $e-$*ball about the steady states), if the discount factor is larger than a uniform* $\tilde{\delta}$, *in such a way that for all the family the chaos will be irrelevant, provided that* ε *is chosen small enough. This situation is quite analogous to the case in which there is convergence to the steady state, but the steady state is never reached.*

REMARK 2. *There is an important fact in the proof of the theorem 2 that has to be noticed: In the implication that if* $\lim_{\delta \to 1} D_{\tilde{U}}(\delta) = 0$ *then* $\lim\limits_{\delta \to 1} f_{\tilde{U}}(\delta) = 0$, *it is heavily used that all the optimal solutions are bounded. Without this property over the optimal solutions, our argument fails. It is our conjecture that the implication need not hold. The fact that all optimal solutions are bounded is also behind our intuitive interpretation of theorem 2 given in the remark 1. Indeed, take an optimal solution. Now, given that it is bounded, one can decompose the optimal path in convergent subsequences (convergent to some* $w-$*limit point), in the sense that the union of all those subsequences coincides with the original optimal path. Simply, take the family of the* $w-$*limit points of the optimal solution and then consider the corresponding family of subsequences converging to those* $w-$*limit points. Thus, given that the optimal path is bounded, there cannot remain a subsequence that is not convergent. Therefore, informally, one can say that the optimal solution, in the long run, is close close to some* $w-$ *limit point. Consequently, given that all the* $w-$*limit points are close to the steady state, the optimal solution itself, in the long run, is close to the steady state.*

Now we treat the case of the topological chaos.

4.2. *Topological Chaos*

We will say that a dynamical system (K, h) displays *topological chaos* or that is *topologically chaotic* if there exists a subset $\Sigma \subset K$ such that:

T1 Σ is uncountable,

T2 Σ does not contain any periodic point of h,

T3 For any $(x, y) \in \Sigma \times \Sigma$ such that $x \neq y$,

$$\liminf_{t \to \infty} \mid h^t(x) - h^t(y) \mid = 0$$

and

$$\limsup_{t \to \infty} \mid h^t(x) - h^t(y) \mid > 0,$$

T4 For any periodic point $y \in K$ and any $x \in \Sigma$,

$$\limsup_{t \to \infty} \mid h^t(x) - h^t(y) \mid > 0.$$

A set $\Sigma \subset K$ is called *a scrambled set* if it satisfies T1-T4.

Also, we say that *an optimal growth problem* (u, D, δ) *displays topological chaos* if the dynamical system (h, K) displays topological chaos, where h is the policy function of (u, D, δ).

Note that T3 and T4 are indeed a way to describe some type of uncertainty about final states, because it may be possible that two optimal paths from different points do not converge to the same point; further, they may not even converge to the same periodic point; also, no periodic point can be globally stable, again, a very undesirable fact regarding final states. Nevertheless, the relevance of this type of chaos depends on how "big" is the scrambled set in terms of probabilistic concepts. Indeed, it has been proven that the scrambled set may have zero Lebesgue measure (see Collet and Eckmann, 1986), in which case there is zero probability of choosing points satisfying T3 or T4. Notice that this may not imply that there is zero probability of observing topological chaos. Think of the case when the scrambled set is a global attractor.

Now we will show that the uncertainty implied by topological chaos vanishes as the discount factor tends to one. The intuition of the result is the following. As we have commented in the paragraph above, the concept of topological chaos entails the impossibility of certain predictions in the long run. Therefore, if we have a family such that for any value of the discount factor there is a member of the family such that the corresponding optimal growth problem displays topological chaos, the expression 'chaos vanishes as the discount factor tends to one', means that if the discount factor is large enough, the distance from any two possible final states is very close to zero, and then any uncertainty in the long run would be irrelevant. Formally:

THEOREM 3. *Take any $\tilde{U}$ as in (1) that satisfies the concavity condition uniformly and that is uniformly relatively compact. Suppose that for every $\delta \in (0,1)$ there exists $u_\delta \in \tilde{U}$ such that the optimal growth problem (u_δ, D, δ) displays topological chaos. Let h^{u_δ} denote the policy function of optimal growth problem (u_δ, D, δ). Let Σ^{h^u} denote the scrambled set of (h^{u_δ}, K).*

Let $C_{T3}(\delta)$

$$= \sup_{(x,y)\in\Sigma^{h^{u_\delta}}\times\Sigma^{h^{u_\delta}} \text{ such that } x\neq y} \left\{ \limsup_{t\to\infty} \left|(h^{u_\delta})^t(x) - (h^{u_\delta})^t(y)\right| \right\}$$

and

$C_{T4}(\delta)$ given by

$$\sup_{y\in K \text{ periodic point of } h^{u_\delta} \text{ and any } x\in\Sigma^{h^{u_\delta}}} \left\{ \limsup_{t\to\infty} \mid (h^{u_\delta})^t(x) \right.$$

$$\left. -(h^{u_\delta})^t(y) \mid \right\}.$$

Then we have

$$\lim_{\delta\to 1} C_{T3}(\delta) = 0 \tag{4}$$

and

$$\lim_{\delta\to 1} C_{T4}(\delta) = 0. \tag{5}$$

PROOF. It follows at once from corollary 1. Simply, if you assume the contrary, for a given $e > 0$, it will be possible to find, for any δ, $w-$limit points for which the distance between them is larger than ε, contradicting the part (ii) in theorem 2. ∎

Now we consider the concept of sensitive dependence on initial conditions.

4.3. *Sensitive Dependence on Initial Conditions*

We say that a dynamical system (K,h) has, for a given $\varepsilon > 0$, $\varepsilon -$ *sensitivity* or that it displays $e-$ *sensitive dependence on initial* *conditions* if there is a set $E \subset K$ of strictly positive Lebesgue measure such that, for every $y \in E$ and every neighborhood B of y, there exists a $z \in B$ and a $t \in N$ such that

$$\left|h^t(y) - h^t(z)\right| > \varepsilon$$

Intuitively, the general idea of this definition is that if $y \in E$, no matter how close to y we are studying the behavior of the system, there exists a point z that will be $\varepsilon-$separated sooner or later. So, if for some reason we are not able to distinguish between two points that are not $\varepsilon-$separated at the beginning of period of consideration, we will be able to distinguish between them after some time. Clearly, this is a way to describe some undesirable behavior of a dynamical system regarding final states, in the sense that minor changes in the initial conditions result, probably, in significant differences after some time.

We will say that *an optimal growth problem* (u, D, δ) *displays* $\varepsilon-$*sensitivity* if the dynamical system (h, K) displays $\varepsilon-$*sensitivity*, where h is the policy function of (u, D, δ).

Now we will prove that if there is a family $\tilde{U}$ as in (1) that satisfies the concavity condition uniformly and that is uniformly relatively compact, such that for every $\delta \in (0,1)$ there is a member u_δ of U and an $\varepsilon_{u_\delta} > 0$ such that (u, D, δ) displays $\varepsilon_{u_\delta}-$ sensitive dependence on initial conditions, then for every $\varepsilon > 0$, there is number $\delta(\varepsilon)$, such that if δ is larger than $\delta(\varepsilon)$, we have that ε_{u_δ} cannot be larger than ε. This means precisely that the $\varepsilon_{u_\delta}-$sensitive dependence on initial conditions is no longer relevant for a δ that is large enough. Formally:

THEOREM 4. *Take any U as in (1) that satisfies the concavity condition uniformly and that is uniformly relatively compact, such that for every $\delta \in (0,1)$ there exists a member $u_\delta \in \tilde{U}$ and an $\varepsilon_\delta > 0$ such that the optimal growth problem (u_δ, D, δ) displays ε_δ-sensitivity, then*

$$\lim_{\delta \to 1} \varepsilon_\delta = 0.$$

PROOF. Suppose that the corollary is false. In this case there is a family $\tilde{U}$, an $\varepsilon > 0$, a sequence $\{\delta_l\} \subset (0,1)$ a sequence $\{u_l\} \subset \tilde{U}$ and a sequence $\{\varepsilon_l\} \subset \Re_+$ such that $\delta_l \to 1$, $\varepsilon_l \geq \varepsilon$ for all l large enough and for all l, the dynamical system (h^l, K) displays ε_l- sensitivity, where h^l denotes the policy function of the optimal growth problem (u^l, D, δ_l).

Now, notice that if K is compact and h is continuous, then if (K, h) has *ε-sensitivity*, then for any $l \in N$ there exist $y_l \in E\backslash\{0\}$, $z_l \in K\backslash\{0\}$ and $t_l \in N$ with $t_l > l$, such that

$$| h^{t_l}(y_l) - h^{t_l}(z_l) | \geq \varepsilon$$

because given $\varepsilon > 0$, for any $l \in N$ fixed, there exists a $\alpha > 0$ such that $(y, z) \in K \times K$ and $| y - z | < \alpha$, implies $| h^t(y) - h^t(z) | \leq \varepsilon$ for all $t \leq l$

Consequently, for every $l \in N$ there exist $y_l \in E\backslash\{0\}$, $z_l \in K\backslash\{0\}$ and $t_l \in N$ with $t_l > l$, such that

$$\left|(h^l)^{t_l}(y_l) - (h^l)^{t_l}(z_l)\right| \geq \varepsilon_l \geq \varepsilon$$

Therefore,

$$\liminf_{l\to\infty} \left|(h^t)^{t_l}(y_l) - (h^l)^{t_l}(z_l)\right| \geq \varepsilon$$

and thus

$$\liminf_{l\to\infty} f_{\bar{U}}(\delta_l) \geq \varepsilon$$

a contradiction with theorem 2. Then the theorem 4 is proven. ∎

5. Conclusions

As commented in the introduction, in order to justify what we called 'the uniform comparative analysis' used in empirical works, neither the standard turnpike theorems nor the specific lower bound found in relation to some fixed type of chaos can be cited. In this paper, we show how the theorem 2 in Guerrero-Luchtenberg (2000) can be applied in order to minimize the topological chaos and the $\varepsilon-$sensitive dependence on the initial conditions, two concepts not analyzed in previous studies. Furthermore, we prove theorem 2, in which no special concept of chaos is considered. Therefore those results are an appropriate justification for the uniform comparative analysis. It rests then to study if another result of this type is possible under weaker conditions. Also, it would be interesting to find the necessary conditions for the theorem 2 to hold. Both questions are left for future research.

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