



When can you immunize a bond portfolio? ☆

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Abstract

This paper presents a condition equivalent to the existence of a Riskless Shadow Asset that guarantees a minimum return when the asset prices are convex functions of interest rates or other state variables. We apply this lemma to immunize default free and option free coupon bonds and reach three main conclusions. First, we give a solution to an old puzzle: why do simple duration matching portfolios work well in empirical studies of immunization even though they are derived in a model inconsistent with equilibrium and shifts on the term structure of interest rates are not parallel, as assumed? Second, we establish a clear distinction between the concepts of immunized and maxmin portfolios. Third, we develop a framework that includes the main results of this literature as special cases. Next, we present a new strategy of immunization that consists in matching duration and minimizing a new linear dispersion measure of immunization risk.

Keywords: Immunization; Maxmin portfolio; Weak immunization condition; Worst shock; Dispersion measures

1. Introduction

This paper develops a general framework to study bond portfolio immunization. This framework includes most models in this literature as special cases, and it allows three main contributions to the literature on immunization. First, the duration puzzle, presented in the literature years ago, is explained and resolved. Second, a clear distinction between the concepts of an immunized portfolio and a maxmin portfolio is made. Third, a theoretically sound new strategy for immunization is presented.

Denote the investor planning period by m , and let R be the return on a zero coupon bond that matures in m years. A portfolio is said to be immunized, if it guarantees a return R at m , regardless of changes in interest rates. All immunization strategies involve the use of a “duration measure”, or vector of duration measures, and all of these measures depend on the assumed shocks to interest rates, or on the factors that determine the term structure of interest rates.²

The immunization literature also presents a well-defined puzzle. In empirical immunization studies, Macaulay duration matching portfolios often work as well as more complex immunizing strategies, despite two strong criticisms. First, the Macaulay duration is derived from a model that implies arbitrage opportunities (Ingersoll and Skelton (1978)), and therefore, it is inconsistent with equilibrium (see also Cox et al., 1979). Second, shifts in the term structure of interest rates are far from parallel, as assumed. Although a solution to the first part of the puzzle is given, for instance, in Bierwag (1987), where it is shown that this duration measure may be also derived from an equilibrium process,³ the second problem is still unsolved.⁴

² For instance, Macaulay (1938) and Fisher and Weil (1971) assume additive shocks on the interest rates. Bierwag (1977), Bierwag et al. (1981), Khang (1979), Chambers et al. (1988), Prisman and Shores (1988), Prisman and Tian (1993), Paroush and Prisman (1997), and others assume other more general shocks. Cox et al. (1979), Brennan and Schwartz (1983), Nelson and Schaefer (1983) and others study immunization strategies in equilibrium models of the term structure. Finally, a more recent approach is to study empirically the factors that move the term structure as in Elton et al. (1990), Litterman and Scheinkman (1991), Ilmanen (1992), D’eclesia and Zenios (1994) and others.

³ We thank a referee for pointing out this reference.

⁴ Brennan and Schwartz (1983) and Schaefer (1992) have also attacked the duration puzzle from a more empirical point of view. They showed that in a two factor empirical model, one factor is similar to duration, and the relationship between both factors is approximately linear in duration for a coupon bonds example. This fact implies, among other things, that the portfolio hedging both empirical factors will be a matching duration portfolio. However, this answer is only partial. First, because it does not recognize that among matching duration portfolios there is a better portfolio, as Fong and Vasicek (1984) and this paper show theoretically, and Bierwag et al. (1993) and others show empirically. Furthermore, as these authors themselves recognize, this result depends on the empirical linear relationship between factors and duration.

Bierwag and Khang (1979) show that an immunized portfolio is a maxmin portfolio, i.e., it guarantees the highest return, under the assumption of parallel shocks on the interest rates. Later, Prisman (1986) extends this maxmin result to a more general context, and recently, Bowden (1997) points out that both concepts are not equivalent in more general models.

To study the immunization problem, in Section 2 of this paper, we assume three very simple and general hypotheses that are verified by most immunization models. These hypotheses allow us to prove Lemma 2.1 and, from this lemma, we establish three main results on immunization. First, Theorem 2.3 shows that there always exists a maxmin portfolio in an immunization model. Second, Theorem 2.6 yields a new condition equivalent to the existence of an immunized portfolio, the *weak immunization condition*, which does not require any duration measure. Third, Proposition 2.8 makes a distinction between the concepts of an immunized portfolio and a maxmin portfolio, showing that a maxmin portfolio is an immunized portfolio only if immunization is feasible. The concept of a maxmin portfolio includes and extends the concept of an immunized portfolio.

To explain the duration puzzle we need to introduce Lemma 2.1. Let us assume a set of feasible shocks K to the term structure of interest rates. This lemma says that there exists a portfolio that guarantees a return μ at m if and only if, for each feasible shock, there exists a bond, which depends on the shock, such that the return of the bond at m , when this shock occurs, is at least μ . This lemma invites a second look. There does not exist a portfolio that guarantees a return μ , if and only if, there exists a feasible shock k^* such that the last condition fails.

Lemma 2.1 has two main implications.

- First, there exists a portfolio that guarantees a return μ at m not only against the feasible set K , but maybe against a wider set of shocks. Therefore, we are hedged against a wider set of shocks. If μ is the highest value that may be guaranteed, then this portfolio is the *maxmin portfolio*.
- Second, the set of shocks K contains a subset of worst shocks such that the latter should be the only shocks to take into account in an immunization problem. The shock k^* , which makes the weak immunization condition to fail, belongs to the subset of worst shocks.

Ingersoll and Skelton (1978) have pointed out that the convexity hypotheses is the reason for a well-known arbitrage violation since the return of an immunized portfolio dominates the return of the zero coupon bond. Therefore, to solve this caveat under convexity assumptions, we have to assume that an immunized portfolio does not exist in the model.⁵

⁵ Prisman and Shores (1988) have a similar argument considering polynomial shocks since in this case immunization is not feasible either. This is the kind of arbitrage attributed to duration models and, of course, in order to prove that there are no arbitrage opportunities in the model, we should allow short positions. This is a very important point, but it is beyond the scope of this paper, and it is a line for future research.

To solve the second caveat, changes in interest rates are far from parallel, we return to Lemma 2.1. Divide the analysis to two steps. In the first one, consider additive shocks, so that, a matching duration portfolio guarantees a return R . In the second step incorporate a wider set of additive and non-additive shocks, which includes worst shocks k^* , such that an immunized portfolio does not exist. How does the matching duration portfolio work against this wider set of shocks? Of course, one cannot guarantee a return R , but from Lemma 2.1, if the effect of these worst shocks k^* are small, or if the probability that they will happen is also very small, then the matching duration portfolio should behave in a manner close to that of an immunized portfolio.⁶

In Section 3 of the paper we introduce the set of worst shocks in models where immunization is not feasible. One can set an upper bound on the possible loss on a non-immunized bond portfolio through Theorem 3.2. This theorem includes remarkable immunization strategies, such as those derived in Fong and Vasicek (1984), Chambers et al. (1988), Prisman and Shores (1988) and others. This upper bound is obtained applying the concept of Gateaux differential, which has also been used by Bowden (1997) to show how to compute the worst shocks.

In Section 4 of the paper we introduce a new set of shocks, and from the theory developed in the previous sections, we present a new strategy of immunization. We argue that this new set of shocks is more reasonable than any of the previous set of shocks considered in the literature, and it is specially related to the shocks introduced by Fong and Vasicek (1984). This new strategy consists in matching duration and in minimizing a new linear dispersion measure, the \tilde{N} measure. With an example, we also show that this strategy can include a maturity matching bond, and therefore, we can explain the empirical results of Bierwag et al. (1993).

Furthermore, we will show the tie between Theorem 2.6 and the strategy of minimizing the \tilde{N} dispersion measure. By minimizing the \tilde{N} dispersion measure we are minimizing the effect of the worst shock, among the shocks previously discussed, which causes the weak immunization condition to fail.

2. The riskless shadow asset and the weak immunization condition

Let $[0, T]$ be the time interval with $t=0$ the present moment, and let m be the investor planning period, $0 < m < T$. We model the problem as if only one

⁶ At the end of this paper we will introduce a set of shocks from which we will propose a new strategy of immunization that includes shocks far from parallel. This set of shocks contains a shock such that the weak immunization condition fails, and therefore, the model will be consistent with equilibrium. For this set of shocks, we will show with an example a specific duration matching portfolio performing close to an immunized portfolio. We interpret this example as a confirmation of the ideas previously discussed and a proof that we have given a solution to the puzzle.

shock occurs at $t=0$. Consequently, if many shocks are going to take place between $t = 0$ and $t = m$, we should rebalance the portfolio and hedge against the shock occurring at that time.

Let us consider n default-free and option-free bonds with maturities less than or equal to T , and with prices P_1, P_2, \dots, P_n , respectively. We will let K be the set of admissible shocks over the term structure of instantaneous forward interest rates, and therefore, K will be a subset of the vector space of real valued functions defined on $[0, T]$. If the elements of K are only constant functions, we will be working with additive shocks like Fisher and Weil (1971). If these elements are polynomials we will have polynomial shocks like the ones considered by Chambers et al. (1988) or Prisman and Shores (1988) amongst others, and if these elements are continuously differentiable functions we are under the hypotheses of Fong and Vasicek (1984). Clearly, K may be a very general set of shocks.

Consider n real valued functionals:

$$V_i : K \rightarrow R, \quad i = 1, 2, \dots, n,$$

such that $V_i(k)$ (where $k \in K$ is any admissible shock) is the i th bond value at time m , which includes the coupons paid before m ,⁷ if the shock k takes place.

We will assume the following three hypotheses.

Hypothesis 1 (H1). K is a convex set.

Hypothesis 2 (H2). V_i is a convex functional for $i = 1, 2, \dots, n$.

Hypothesis 3 (H3). $V_i(k) > 0$ for $i = 1, 2, \dots, n$ and for any $k \in K$.

H1 is a regularity hypothesis. We assume that the value of any bond verifies H2 since convexity holds in most models on immunization. Some discussion about this hypotheses, however, will be presented at the end of this section. Finally, H3 is a necessary condition, if we do not want arbitrage to exist, since a bond only pays positive amounts.

Although we are working in an abstract and general context, concrete representations by basis (for instance: polynomials, or Fourier series) of the function space where K is included would be possible. It would allow translating conditions H1, H2 and H3 to conditions on the coefficients of the linear combinations. Moreover, one could substitute these coefficients for the variable k in K .

⁷ As usual, the coupons paid before m will be reinvested by purchasing the considered n bonds or the new bonds appearing in the market

If $C > 0$ represents the total amount of investment, and the vector $q = (q_1, q_2, \dots, q_n)$ gives us the number of units q_i of the i th bond that the investor has bought, then the portfolios satisfying the following constraints:

$$\sum_{i=1}^n q_i P_i = C, \quad q_i \geq 0, \quad i = 1, \dots, n, \quad (1)$$

will be called feasible portfolios. The functional

$$V(q, k) = \sum_{i=1}^n q_i V_i(k) \quad (2)$$

gives us the value at time m of portfolio q if the shock k takes place. Obviously, V is a convex functional in the k variable since Eq. (1) guarantees that it is a non-negative linear combination of convex functionals.⁸

Let us introduce the guaranteed value by portfolio q which will be

$$\bar{V}(q) = \text{Inf}\{V(q, k); k \in K\}, \quad (3)$$

that is, the infimum of all possible values (at m) of portfolio q depending on the shocks $k \in K$.

The following lemma forms the central result, which is the basis of the rest of the paper.

Lemma 2.1. *Let $\mu_0 \geq 0$. Then, there is a feasible portfolio q^* such that*

$$\frac{\bar{V}(q^*)}{C} \geq \mu_0$$

if and only if for every admissible shock $k \in K$ there is at least a bond i th (which depends on k) such that

$$\frac{V_i(k)}{P_i} \geq \mu_0.$$

Proof. See Appendix A. \square

As can be seen, Lemma 2.1 establishes a necessary and sufficient condition to guarantee the existence of a Riskless Shadow Asset⁹ such that its return is at least μ_0 . This number is said to be a Riskless Shadow Return. The lemma is proved under the techniques of convex analysis, which were also applied to immunization theory by Prisman (1986).

⁸ In order to avoid the loss of the convexity property, short positions are excluded from the analysis. With short positions the functional $V(q, k)$ is not a convex functional of k , and therefore, the separation theorems do not apply and we cannot prove Lemma 2.1.

⁹ We have taken this name from Ingersoll (1987, p.48).

We should be interested in the greatest values for which the lemma is verified. For instance, if immunization is feasible, μ_0 would be any number between 0 and the return on the zero coupon bond with m years to maturity.

A portfolio is called maxmin if it guarantees as much as possible. To formally introduce this concept we will consider the optimization program

$$\left. \begin{array}{l} \text{Max } \bar{V}(q) \\ q \text{ subject to (1).} \end{array} \right\} \quad (\text{P1})$$

Definition 2.2. A feasible portfolio q^* is maxmin if it solves program (P1).

The first interesting consequence which can be derived from Lemma 2.1 is that under hypotheses H1, H2 and H3 one can always find a maxmin portfolio.

Theorem 2.3. *Program (P1) has a solution, i.e., there always exists a maxmin portfolio.*

Proof. See Appendix A. \square

The above theorem gives us the highest value of μ_0 for which it is possible to find a Riskless Shadow Asset. This value is given by $\mu_0^* = \bar{V}(q^*)/C$, where q^* is a maxmin portfolio.

We are now in a position to answer the question posed by the title of this paper. Let us denote R as the return of the zero coupon bond with m years to maturity that we observe in the initial term structure of the interest rates. Then, we will say that RC is the promised amount. A portfolio q^* is said to be immunized when it guarantees at least RC . Formally we have the following definition.

Definition 2.4. A feasible portfolio q^* is immunized if $\bar{V}(q^*) \geq RC$.

Let us now introduce the “weak immunization condition”.

Definition 2.5. We will say that the set of admissible shocks K and the n considered bonds verify the weak immunization condition if for any shock $k \in K$ there exists at least one bond, the i th (which depends on k), such that

$$\frac{V_i(k)}{P_i} \geq R.$$

We could interpret this concept as follows. Let us consider an investor interested in an immunized portfolio, i.e., a portfolio which guarantees the promised amount RC . If our investor knew the real future shock k then he or she would buy the bond which does not lose value, i.e., a bond such that

$V_i(k) \geq RP_i$. If the investor can find this bond for any feasible shock, the weak immunization condition is achieved.

The following results show that the weak immunization condition is necessary and sufficient to guarantee the existence of an immunized portfolio. This condition does not need any kind of duration measure, and we will show that it is very easy to apply in practice.

Theorem 2.6. *The weak immunization condition is necessary and sufficient to guarantee the existence of an immunized portfolio.*

Proof. It is an immediate consequence of Lemma 2.1. taking $\mu_0 = R$. \square

The latter theorem has another interpretation. “Immunization is not possible, if and only if, there is an admissible shock for which all the bonds lose value at m , i.e., do not reach the value RP_i , $i = 1, 2, \dots, n$, at m ”.

Theorem 2.3 and the latter theorem have interesting consequences on the K set of admissible shocks. Let us assume for example that in addition to the n considered bonds, the zero coupon bond with m maturity is available on the market. Consider that its present price is P and let 1 be its value at maturity. Ingersoll and Skelton (1978) showed that in convex models of immunization, an immunized portfolio, i.e. a portfolio q such that $\bar{V}(q)/C \geq 1/P$, cannot be found because otherwise the investor can buy it and sell the zero coupon bond. Clearly, this would be an arbitrage. On the other hand, in Theorem 2.3, we have just proved that the highest riskless shadow asset does exist, and the hypotheses under which it has been proved are general enough to ensure that they will always hold in practice. Therefore, since this arbitrage cannot be accepted, there must be at least a shock k^* in the set K such that

$$\frac{V_i(k^*)}{P_i} < \frac{1}{P} \quad i = 1, 2, \dots, n.$$

The latter condition rules out riskless and profitable arbitrage and therefore gives a solution to the first caveat of the puzzle.

Let us denote by μ_0^* , $\mu_0^* < R$, the return of the maxmin portfolio for a given set of shocks K . Lemma 2.1 is saying that we can guarantee the return μ_0^* not only against K , but against any convex set that contains the set K and *for which the lemma's hypotheses are verified*. Therefore, we are immunizing or hedging against a wider set of shocks than K , although we do not realize it. ¹⁰

¹⁰ An example of this result could be Bierwag (1987), where it is shown that a one to one correspondence between a duration measure and a stochastic process does not obtain. It is possible then for several stochastic processes (or sets of shocks, K) to imply the same immunizing durations. Combining the sets; and permitting convex combinations of them would then expand set K .

This fact has important implications in order to solve the second caveat of the puzzle. It would be now necessary to show that μ_0^* is close to R , and that the portfolio that guarantees μ_0^* is close to a duration matching portfolio. We will do that at the end of paper, where the discussion about the puzzle will be continued.

Once we know that a maxmin portfolio always exists and the conditions under which there exists an immunized portfolio, we can analyze the relationship between both concepts. We require an additional hypothesis in the model, which will be maintained until the end of this paper.

Hypothesis 4 (H4). The set K contains the zero shock (denoted by $k=0$), and there exists a number $R > 0$ such that $V_i(0) = RP_i$ for $i = 1, 2, \dots, n$.

Hypothesis H4 is verified in most immunization models, and if investors believe that forward rates will be the future actual rates, then H4 means that any shock to the term structure is the difference between futures rates and forward rates.

Let us first verify the following relationship.

Proposition 2.7. *The following expressions hold for any feasible portfolio q .*

$$0 \leq \bar{V}(q) \leq RC.$$

Proof. The first inequality follows from H3. Furthermore

$$\begin{aligned} \bar{V}(q) &= \text{Inf}\{V(q, k); k \in K\} \leq V(q, 0) = \sum_{i=1}^n q_i V_i(0) \\ &= \sum_{i=1}^n q_i RP_i = R \sum_{i=1}^n q_i P_i = RC. \quad \square \end{aligned}$$

Again one can see that RC is the promised return. The exact relationship between an immunized and a maxmin portfolio can now be stated.

Proposition 2.8. *If q^* is an immunized portfolio, then it is maxmin.*

Proof. Definition 2.4 and Proposition 2.7 imply $\bar{V}(q^*) = RC$, and by applying Proposition 2.7 to any feasible portfolio q we have

$$\bar{V}(q) \leq RC = \bar{V}(q^*)$$

and therefore q^* solves (P1). \square

The above result has been proved in an extraordinarily simple way and in a very general context because of the apparent power of the introduced notation.

This was initially established by Bierwag and Khang (1979) in a model where the shocks are additives. The model of Bierwag and Khang (1979), where an immunization strategy always exists, was extended by Prisman (1986), who relaxed hypothesis H4, allowing $V_i(0) \leq RP_i$ for $i = 1, 2, \dots, n$, to take into account a bond market with tax. However, because we focus on a model where an immunized portfolio does not necessarily exist and a maxmin portfolio always exists, we prefer not to relax hypothesis H4 and separate the tax effects on immunization in the model. Nevertheless, Propositions 2.7 and 2.8 would still be true in our model with hypothesis H4 relaxed, and the distortions observed by Prisman (1986) would also be present in such a model.

There are other assets, whose price is a convex functional that we are not considering (for instance options on bonds) and for which the Lemma 2.1 is also verified. However, they do not verify the hypothesis H4. Anyway, Propositions 2.7 and 2.8 still hold if we relax this hypothesis and impose $V_i(0) \leq RP_i$ for $i = 1, 2, \dots, n$. This would allow the main results of this second section to be extended to more general models that incorporate bonds, and some derivative securities on the bonds.

Let us point out that the converse of Proposition 2.8 is generally false. In fact, the maxmin portfolio (i.e. a portfolio that maximizes the guaranteed amount at time m) always exists but it will be seen later that the weak immunization condition is not always satisfied, and therefore an immunized portfolio does not exist. Moreover, it is well known that in the literature one can find many models where immunization is not possible. In any case, it can be easily proved in our general context that if an immunized portfolio does exist, then immunized and maxmin portfolios are equivalent. The weak immunization condition may be interpreted as a necessary and sufficient condition to guarantee that R is the highest riskless shadow return. When this condition fails, this riskless shadow return is smaller than R .

Bierwag and Khang (1979), working with additive shocks, proved that a bond with a duration greater than m increases (decreases) its value at m if there is a negative (positive) shock. The converse holds for bonds of a shorter duration. Therefore, if we have both bonds, we are under the hypotheses of Theorem 2.6, and so we can conclude (as Bierwag and Khang showed under other arguments) that an immunized portfolio does exist because the weak immunization condition holds.

Prisman and Shores (1988) proved that the model for polynomial shocks proposed by Chambers et al. (1988) has no solution if the polynomials have a degree equal or greater than one. Thus, immunization against polynomial shock is not possible. Their proof is based on the fact that it is not possible to match a duration vector without short sales. However, we will offer a simple but very different proof. An example of an admissible polynomial shock is given by $k^*(t) = \lambda(t - m)$ where λ is any positive number. Since $k^*(t) < 0$ if $t < m$ and $k^*(t) > 0$ if $t > m$, the instantaneous forward interest rates are going

to decrease from $t = 0$ to $t = m$ and they are going to increase for $t > m$. Thus, the coupons we have to re-invest (the coupons paid before m) will lose value at m and so will the ones we have to discount (the ones paid later than m). In this situation, only the zero coupon bond with maturity m would not lose value at m , but if this bond were not on the market, immunization would not be possible because the weak immunization condition fails.

3. The set of worst shocks. An upper bound to the loss of a non-immunized portfolio

This section is devoted to an analysis of the *set of worst shocks*, which contains a shock such that the weak immunization condition fails. It seems clear that in an immunization context, the worst shocks in the set of feasible shocks are the only ones to be worried about. Furthermore, these shocks can help to approximate the maxmin portfolio and its guaranteed return μ_0^* .

In general, any hedging model has to explain which are the worst states that can arise. The other states would be irrelevant. Fisher and Weil (1971) and Bierwag and Khang (1979) have already worked with the concept of worst shock, since for an immunized bond portfolio, the worst shock is the null shock.

Given this concept and that a convex function is bound from below by its tangent, one can derive very simply the upper bound to the possible losses from a non-immunized portfolio. Results has been obtained in this manner by many authors including the outstanding immunization strategies of Fong and Vasicek (1984) and Prisman and Shores (1988).

Definition 3.1. We will say that a set k_1, k_2, \dots, k_h , of feasible shocks is a set of worst shocks, if given any shock $k \in K$ there exist h real numbers, which depend on k , $\lambda_1(k), \lambda_2(k), \dots, \lambda_h(k)$, such that

$$\sum_{j=1}^h \lambda_j(k) k_j \in K,$$

$$V_i(k) \geq V_i \left(\sum_{j=1}^h \lambda_j(k) k_j \right), \quad i = 1, 2, \dots, n.$$

This concept simply means that the value at m of the n considered bonds is always bound from below by their values by considering linear combinations of elements in the set of worst shocks.

From now on, we take into account the following additional assumption.

Hypothesis 5 (H5). The set K of admissible shocks is a subset of a normed space X whose elements are real valued functions over the interval $[0, T]$. The zero shock is interior to the set K . The functionals V_i , $i = 1, 2, \dots, n$, are Gateaux differentiable with respect to their k variable in an open set containing the zero shock.

The concepts of normed space and Gateaux differential may be found for example in Luenberger (1969). The hypothesis of being V_i Gateaux differentiable may be more easily written if we consider shocks k which depend on $p + 1$ parameters (for instance, polynomial shocks with p degree). If this dependence is linear, then it means that V_i is differentiable with respect to the parameters.¹¹ As is well known, a convex function is always bound from below by its tangent plane. This is also true for convex functionals in normed spaces (see Luenberger, 1969) and we can apply this fact to obtain some properties for functional V .

Let q be a feasible portfolio and let v_j , $j = 1, 2, \dots, h$ be the value of the Gateaux differential of $V(q, k)$ with respect to its variable k , evaluated at $k = 0$, and applied over k_j (see Luenberger, 1969). Then we have the following result.

Theorem 3.2. *Under assumptions H1 H5 the following inequality holds for every feasible q and every feasible $k \in K$:*

$$\frac{V(q, k) - RC}{RC} \geq \sum_{j=1}^h \frac{v_j}{RC} \lambda_j, \quad (4)$$

where $\lambda_1, \lambda_2, \dots, \lambda_h$ are the values given by Definition 3.1, where the dependence of k has been omitted.

Proof. It obviously follows from Definition 3.1 and expression (2) that

$$V(q, k) \geq V\left(q, \sum_{j=1}^h \lambda_j k_j\right).$$

Since V is convex in its second variable, it is bound from below by its differential.

$$V\left(q, \sum_{j=1}^h \lambda_j k_j\right) \geq V(q, 0) + \sum_{j=1}^h v_j \lambda_j.$$

And the result obviously follows from the equality $V(q, 0) = RC$. \square

¹¹ Gateaux differentiable is also assumed in Bowden (1997).

Definition 3.1 and Theorem 3.2 can be used to explain some of the results from the immunization literature.

First, consider the shocks with the derivative bound by $\lambda > 0$ of Fong and Vasicek (1984). That is, $dk(t)/dt \leq \lambda$ for every $t \in [0, T]$. It is easily shown that the weak immunization condition fails because there exists a shock $\lambda(t - m)$ such that every bond loses value when this shock takes place. Furthermore, it is immediately deduced from the Taylor formula that

$$k(t) \geq k(m) + \lambda(t - m) \quad \text{if } t \leq m,$$

$$k(t) \leq k(m) + \lambda(t - m) \quad \text{if } t > m.$$

Since the latter inequalities show that the shock $k(m) + \lambda(t - m)$ has a more negative effect on all the coupons than the shock $k(t)$, we have

$$V_i(k) \geq V_i(k(m) + \lambda(t - m)).$$

Then, Definition 3.1 holds, $\{1, \lambda(t - m)\}$ is a set of worst shocks, and their linear combinations are given by the lines $k^*(t) = \lambda_0 + \lambda(t - m)$ where λ_0 is any real number. Now, by applying Theorem 3.2, one may easily prove the upper bound of Fong and Vasicek ¹² which contains the M^2 measure.

Considering polynomial shocks with a degree not greater than p , the upper bound obtained in Prisman and Shores (1988) may be also deduced from Proposition 4.2. In this case, a set of worst shocks is given by the polynomials $\{1, t, t^2, \dots, t^p\}$.

Prisman and Shores (1988) and Bierwag et al. (1993) showed that the upper bound of the Fong and Vasicek (1984) model, and of a polynomial shocks of degree one model are related because both upper bounds depend on the first and second duration measures. Lacey and Nawhalka (1993) showed empirically that hedging against both factors of a degree 1 polynomial turns out an immunized portfolio. It can be observed that although the shocks of Fong and Vasicek (1984) are much more general than a polynomial of degree one with a bound derivative, their worst shocks are just given by a polynomial of degree one. This shows why both upper bounds are related and points out that the worst shocks are the only relevant shocks in this immunization context. Otherwise, Lemma 2.1 says that when we are immunizing against a degree one polynomial shock with bound derivative, in fact, we are hedging against a wider convex set of shocks that are all the shocks with bound derivative introduced by Fong and Vasicek (1984).

¹² The results of Fong and Vasicek (1984) were extended in Montrucchio and Peccati (1991) for non differentiable shocks, but shocks with bounded Dini's derivative. See also Shiu (1987).

4. Working with bound shocks. A new dispersion measure

The previous analysis suggests a new methodology for immunizing a bond portfolio. In previously developed immunization strategies, a particular duration strategy pursued depend on the set of shocks considered. If the set of shocks is changed, then the strategies for achieving immunization may also change.

As Fong and Vasicek (1984) argue, the factor models i.e., Fisher and Weil (1971), Bierwag (1977), Khang (1979) and others, or the equilibrium models, i.e., Cox et al. (1978), Brennan and Schwartz (1983), Nelson and Schaefer (1983) and others, are too constrained because they only permit hedging against the shocks which are compatible with the model.

The polynomial multifactorial models of Chambers et al. (1988) and Prisman and Shores (1988) seem appropriate because a polynomial can approximate whatever continuous function. However, these models are too unconstrained. A simple polynomial of degree one contains a shock given by $k^*(t) = \lambda(t - m)$, which is unreal if $\lambda \rightarrow \infty$ or $|t - m|$ is big. If λ is bound, then we are in the Fong and Vasicek (1984) situation, but the second problem still holds.

The empirical factor models, such as Litterman and Scheinkman (1991), would also be appropriate. Nevertheless, they require no short selling constraints, and they depend on the fact that the estimated factors are stationary.

The latter section has shown that, in a general context, if an immunized portfolio does not exist, one may look for the set of worst shocks in order to obtain an upper bound on possible capital losses. Let us consider the case were the set of admissible shocks is the set of bound and integrable functions defined on $[0, T]$ and let us prove that there is not an immunized portfolio.

Since we are working with bound and integrable functions $k(t)$, an admissible shock is given by

$$k(t) = \begin{cases} -\lambda_1 & \text{if } t < m, \\ \lambda_1 & \text{if } t \geq m, \end{cases} \quad (5)$$

where λ_1 is arbitrary but positive. This shock is negative before m and positive after m , and assuming that the zero coupon bond with m maturity is not on the market, the weak immunization condition does not hold, and therefore, an immunized bond portfolio does not exist.

Let us consider a constant $\lambda > 0$ and let us assume that the set of admissible shocks consists of the set of bound and integrable functions:

$$K = \{k(t); |k(t_1) - k(t_2)| \leq \lambda, 0 \leq t_1 \leq t_2 \leq T\}. \quad (6)$$

In this situation, the set

$$k_0(t) = 1, \\ k_1(t) = \begin{cases} -\frac{\lambda}{2} & \text{if } t < m, \\ \frac{\lambda}{2} & \text{if } t \geq m, \end{cases}$$

is a set of worst shocks in the sense given by Definition 3.1 since given any $k(t)$ we have that

$$\begin{aligned} k(t) &\geq \lambda_0 - \frac{\lambda}{2} & \text{if } t \leq m, \\ k(t) &\leq \lambda_0 + \frac{\lambda}{2} & \text{if } t > m, \end{aligned} \quad (7)$$

being

$$\lambda_0 = \frac{\sup\{k(t); t \in [0, T]\} + \inf\{k(t); t \in [0, T]\}}{2}. \quad (8)$$

Following the usual assumptions, let portfolio q pays a continuous coupon $c(t) \geq 0$, $0 \leq t \leq T$, and assume the balloon payment at maturity is zero. If $g(t)$, $0 \leq t \leq T$, represents the instantaneous forward interest rates and $k(t)$ is a shock on $g(t)$, then the q portfolio value at m is given by

$$V(q, k) = \int_0^T c(t) \exp \left[\int_t^m (g(s) + k(s)) ds \right] dt. \quad (9)$$

Denoting the return between 0 and m by

$$R = \exp \left[\int_0^m g(s) ds \right] \quad (10)$$

and the coupon's present value by

$$c(t, 0) = c(t) \exp \left[- \int_0^t g(s) ds \right] \quad (11)$$

we have

$$V(q, k) = R \int_0^T c(t, 0) \exp \left[\int_t^m k(s) ds \right] dt. \quad (12)$$

The differential of functional V with respect to its variable k evaluated at the zero shock and applied over the shock k (i.e., the derivative of functional V evaluated on the zero shock and in the direction given by shock k) will be given as

$$R \int_0^T c(t, 0) \left(\int_t^m k(s) ds \right) dt. \quad (13)$$

Theorem 4.1. *For any feasible portfolio q and for any admissible shock k the following inequality holds:*

$$\frac{V(q, k) - RC}{RC} \geq \lambda_0(m - D) - \frac{\lambda}{2} \tilde{N}, \quad (14)$$

where λ_0 is given by Eq. (8), D is the Macaulay duration of the q portfolio and \tilde{N} is the dispersion measure given by

$$\tilde{N} = \int_0^T \frac{c(t, 0)}{C} |t - m| dt. \quad (15)$$

Proof. See Appendix A. \square

It immediately follows that for a portfolio q with duration equal to m , we have the following upper bound on the possible losses after a shock on the forward interest rates

$$\frac{V(q, k) - RC}{RC} \geq -\frac{\lambda}{2} \tilde{N}. \quad (16)$$

The development in (14)–(16) shows a new strategy for immunizing a bond portfolio, i.e., picking the portfolio with the minimum \tilde{N} measure from the duration matching portfolios. By buying a duration matching portfolio, the investor is immunized against additive shocks, which have an important percentage of the total changes on the interest rates, as shown empirically by Litterman and Scheinkman (1991) and others.

We can also note the tie between this strategy and Theorem 2.6; by minimizing the \tilde{N} measure, we are minimizing the effect due to the shock, for which the weak immunization condition does not hold. The effect of this shock on the coupon paid in t is given by $(\lambda/2) |t - m|$, and such effect is weighed by $c(t, 0)$. We have a parallel situation for the M^2 measure of Fong and Vasicek (1984) and their worst shocks.

If we work with differentiable shocks with a bound derivative by a parameter λ , then we can follow the strategy proposed by Fong and Vasicek (1984). But we think there are three important reasons to work with the previously proposed bound and integrable shocks, which depend on parameter λ .

First, the bound shocks have a theoretical argument in their favor with respect to the Fong and Vasicek shocks. In the case of bound shocks, the parameter λ can be understood as a volatility measure, as how much the shocks

on the forward instantaneous interest rates can differ between two time dates. This parameter can be estimated. On the other hand, the Fong and Vasicek shocks parameter, a derivative, has a more complex economic meaning and it is more difficult to estimate.

Second, shocks with a bound derivative are also bound, but the converse is false. Shocks with small variations could have a very big derivative. Then, we have that bound shocks include most of the Fong and Vasicek shocks, but, once again, the converse is false.

Third, the worst shocks in the Fong and Vasicek situation are unreal because it entails very big values when t is far from m . In the bound case the worst shocks on the Term Structure of Interest Rates are given by

$$\frac{1}{t} \int_0^t \begin{cases} \lambda_0 - \frac{\lambda}{2} & \text{if } s \leq m \\ \lambda_0 + \frac{\lambda}{2} & \text{if } s > m \end{cases} ds = \begin{cases} \lambda_0 - \frac{\lambda}{2} & \text{if } t \leq m, \\ \lambda_0 + \frac{\lambda}{2} - \lambda \frac{m}{t} & \text{if } t > m, \end{cases} \quad (17)$$

and in the Fong and Vasicek situation

$$\frac{1}{t} \int_0^t (\lambda_0 + \lambda(s - m)) ds = \lambda_0 + \lambda \left(\frac{t}{2} - m \right). \quad (18)$$

We can also observe for both sets that the worst shocks are twists of the term structure of interest rates at the term $2m$.¹³

In short, the bound shock which depends on a parameter λ can be considered as the sum of two components. A parallel shift of all interest rates and a second change such that the interest rates move in a band of width λ . This parameter λ can represent a volatility measure. These two components seem an appropriate scenario to describe the changes in the interest rates in this immunization context.

To throw more light on the developed strategy, we now present a simple example to see the portfolios that minimize both dispersion measures. We will take an investor planning period of five years, $m = 5$, and we will assume a flat term structure on the interest rates, $r = 10\%$, to make it easier.

Let us consider the set of coupon bonds presented in Table 1. The first column in Table 1 is the bond number, the second one is its maturity, the third is the coupon (as a percentage), the fourth is the coupon periodicity (in months), the fifth is the bond duration (in years), the sixth is its M^2 measure, and the last one is its \tilde{N} measure.

In Table 2 we give the duration matching portfolios. The first column is the portfolio number, the second is the first bond in the portfolio, the third is the second bond in the portfolio, the fourth is the first bond percentage and the last

¹³ Now, one can refer to almost any empirical paper on the factors that move the term structure of interest rates, as in Litterman and Scheinkman (1991), to see if in fact these twists are in term $2m$, or if they are not.

Table 1
Set of bonds

Bond number	Maturity (years)	coupon (%)	coupon (monthly)	Duration (years)	M^2	N
1	4	10	12	3.48232	3.27130	1.51767
2	5	10	12	4.16101	2.59822	0.83898
3	6	10	12	4.77597	3.22315	1.45880
4	7	10	12	5.33285	4.90932	2.02011
5	8	10	12	5.83689	7.45157	2.52818
6	9	10	12	6.29287	10.6728	2.98782

Table 2
Matching duration portfolios

Number	1st bond	2nd bond	% (1st)	M^2	N
1	3	4	59.77	3.901	1.6846132
2	3	5	78.88	4.116	1.6846172
3	2	4	28.40	4.252	1.6846198
4	3	6	85.23	4.323	1.6846211
5	2	5	49.93	5.027	1.6846342
6	2	6	60.64	5.775	1.6846482
7	1	4	17.98	4.614	1.9297429
8	1	5	35.54	5.965	2.1690124
9	1	6	46.00	7.268	2.3115433

columns are their M^2 and \tilde{N} dispersion measures. The portfolios are arranged according to their \tilde{N} measure.

We can observe that the portfolio 1, which is a bullet portfolio, minimizes both dispersion measures, and therefore, it would be a very interesting portfolio. However, portfolios 3, 5 and 6 also minimize the \tilde{N} measure. They include a maturity matching bond, and they are not a bullet portfolio.

The proposed strategy can be empirically tested and would explain both empirical results of Bierwag et al. (1993) and others. First, they point out that the best strategy for immunization consists of matching duration but including a maturity matching bond. This maturity matching bond would be in the duration matching portfolio that minimizes the \tilde{N} measure. Second, they also show that the latter strategy and a bullet portfolio perform very close. Both strategies would minimize the \tilde{N} dispersion measure.

It is worthwhile to point out that, in general, the bond shocks include the shocks described by the empirical factors that move the term structure, as in Litterman and Scheinkman (1991). Furthermore, they also include the GAP management approach that consists of dividing the yield curve into sections and shifting independently to each one of the sections.¹⁴ However, we have shown that it is only necessary to take into account the worst shocks.

¹⁴ Hull (1993, pp. 103, 408).

Let us get back to the puzzle. It is necessary to show that a matching duration portfolio performs as an immunized portfolio in a model that includes shocks far from parallel. Let us take the set of bonds previously described, and let us suppose that the bound shocks are an adequate framework that describes the shocks on the interest rates. Let us assume that our volatility measure is not very high, for example $\lambda = 2\%$. Then the interest rates can suffer a additive movement and further they can move in any way in a 2% wide band. If we now use the bound developed in Theorem 3.2, then the six first portfolios guarantee a 98.32% of the promised value at m . This corresponds to a continuous composed yield of 9.66% against the 10.00% promised. If furthermore this shock has very small probability, then the yield could be closer to the $R = 10.00\%$ promised.

Finally, let us point out that the proposed set of shocks and the developed dispersion measure are a natural consequence of the results obtained in Sections 2 and 3. However, these sections prove important results that should be taken into account.

First, the Riskless Shadow Asset or the maxmin portfolio does exist and it is an interesting task to obtain it. Furthermore, the weak immunization condition easily shows why total immunization is not possible in most models. But this condition also shows that immunization against shocks (which could be non additive) in very general convex sets may be possible if there are appropriate bonds on the market. Consequently, the results of section two will allow one to obtain upper bounds for possible capital losses if the real shock on the forward interest rate is not in the considered convex set.

When we immunize against additive shocks and minimize the \tilde{N} measure, we are choosing one possible way amongst many others that should be analyzed. To minimize the \tilde{N} measure, we have taken into account many considerations about the possible shocks on the interest rates, but another analysis would be welcome.

5. Conclusions

This paper presents in a very general framework a condition equivalent to the existence of a *Riskless Shadow Asset* that guarantees a minimum return when the assets prices are convex functions of interest rates or other state variables. We show that the *weak immunization condition* is equivalent to the existence of an immunized portfolio. Furthermore, this new condition does not require any duration measure and is easy to use in practice.

To introduce the weak immunization condition we have applied convex analysis methods, which also allow us to prove the existence of *maxmin portfolios* in these models. We have distinguished between the concepts of a maxmin

and an immunized portfolio, showing that both concepts are equivalent only if immunization is feasible. Therefore, the concept of maxmin portfolio is more general and includes immunized portfolios.

The existence of maxmin portfolios also allows us to solve some of the problems originated by immunization models. For instance, the existence of maxmin portfolios, in models where immunization is not feasible, should not be inconsistent with equilibrium since it does not create the arbitrage opportunities attributed to immunization models.

Studying maxmin portfolios in immunization is a very important choice, but it is not the only one. We show that the weak immunization condition leads also to the *set of worst shocks*, whose existence also allows us to reconcile immunization theory with equilibrium. They are the shocks that affect more negatively each bond return and, consequently, produce the greater risk on bond portfolios. By means of this concept, we give an upper bound on the possible loss on a non-immunized bond portfolio that includes outstanding immunization strategies previously proposed in the literature.

Both ideas, maxmin portfolios and the set of worst shocks, are complementary and, probably, related. This should be an important research topic for the future.

Finally, we have introduced a new set of shocks. These shocks depend on a parameter λ that can be understood as a volatility parameter for interest rates. This set of shocks includes most sets previously studied in the literature and seems to be a more reasonable set to describe changes in interest rates. These changes can be considered as the sum of two components: a parallel shift in interest rates and a second change where the interest rates move within a band of width λ .

From these shocks, and taking into account the theory developed, a new *linear dispersion measure* \tilde{N} is introduced, and a new strategy to minimize the immunization risk is proposed; to match duration and to minimize the \tilde{N} measure. Following this strategy, the investor has an upper bound for a possible loss on the portfolio. With an example, we also show that this portfolio can include a *maturity matching bond*, and therefore, we can explain the empirical results of Bierwag et al. (1993). In addition, this example shows that this portfolio guarantees a return close to an immunized portfolio and, thus, it could be a solution to the *duration puzzle*.

Furthermore, we show the tie between the theorems of this paper and the strategy of minimizing the \tilde{N} measure. By minimizing the \tilde{N} measure, we are minimizing the effect of the worst shock (among the bounded shocks previously discussed), which causes the weak immunization to fail.

The set of shocks considered and the measure are developed bearing in mind the weak immunization condition, but it is not the only application, since the theory developed is quite general and the upper bound of Theorem 3.2 should allow us to analyze many other situations.

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Appendix A

Proof of Lemma 1. Let us assume the existence of portfolio q^* . Then

$$V(q^*, k) \geq \mu_0 C$$

for any admissible shock k . From Eqs. (1) and (2)

$$\sum_{i=1}^n q_i^* V_i(k) \geq \sum_{i=1}^n q_i^* \mu_0 P_i$$

for any k . Since the terms in both sides of last inequality are non-negative, this is only possible if at least for one i th we have

$$V_i(k) \geq \mu_0 P_i.$$

Conversely, let us consider that the given condition holds and let us prove the existence of q^* portfolio.

The following set is obviously convex in R^n

$$A = \{(\alpha_1, \alpha_2, \dots, \alpha_n); \alpha_j \leq \mu_0 P_j, j = 1, 2, \dots, n\}.$$

Also consider the set

$$B = \{(\beta_1, \beta_2, \dots, \beta_n); \exists k \in K \text{ with } \beta_j \geq V_j(k), j = 1, 2, \dots, n\}.$$

Let us prove that B is a convex set. In fact, if $(\beta_1, \beta_2, \dots, \beta_n)$ and $(\beta'_1, \beta'_2, \dots, \beta'_n)$ are in B , we can find two shocks k and k' in K such that

$$\beta_j \geq V_j(k), \quad \beta'_j \geq V_j(k'), \quad j = 1, 2, \dots, n.$$

Since K is a convex set, given τ with $0 \leq \tau \leq 1$, $\tau k + (1 - \tau)k' \in K$ and being V_j a convex functional for any j , we have that

$$\begin{aligned} \tau\beta_j + (1 - \tau)\beta'_j &\geq \tau V_j(k) + (1 - \tau)V_j(k') \geq V_j(\tau k + (1 - \tau)k'), \\ j &= 1, 2, \dots, n, \end{aligned}$$

$$\tau(\beta_1, \beta_2, \dots, \beta_n) + (1 - \tau)(\beta'_1, \beta'_2, \dots, \beta'_n) \in B.$$

We will prove now that there are no points in A^0 (interior of A) and B simultaneously. In fact, if $(\alpha_1, \alpha_2, \dots, \alpha_n)$ were in both A^0 and B , then $\alpha_j < \mu_0 P_j$, $j = 1, 2, \dots, n$ and we could find a shock k such that

$$\alpha_j \geq V_j(k), \quad j = 1, 2, \dots, n.$$

Therefore

$$\mu_0 P_j > \alpha_j \geq V_j(k), \quad j = 1, 2, \dots, n,$$

and it is a contradiction with the assumptions.

The separation theorems (see Luenberger, 1969) show that we can find n real numbers q'_1, q'_2, \dots, q'_n such that q'_i is not zero for at least one i th and

$$\sum_{j=1}^n q'_j \alpha_j \leq \sum_{j=1}^n q'_j \beta_j$$

if $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is in A and $(\beta_1, \beta_2, \dots, \beta_n)$ is in B . In particular, taking $\alpha_j = \mu_0 P_j$ and $\beta_j = V_j(k) + r_j$, $j = 1, 2, \dots, n$, where k is any admissible shock and r_j is any non-negative number,

$$\mu_0 \sum_{j=1}^n q'_j P_j \leq \sum_{j=1}^n q'_j (V_j(k) + r_j). \quad (\text{A.1})$$

We have $q'_i \geq 0$ because if we had $q'_i < 0$ then the right side in last inequality would tend to $-\infty$ if r_i tends to infinite and this is not compatible with the inequality. Analogously $q'_2 \geq 0, \dots, q'_n \geq 0$. Since at least one q'_i is not zero,

$$S = \sum_{j=1}^n q'_j P_j > 0$$

and then, taking $q_j^* = (C/S)q'_j$, $j = 1, 2, \dots, n$, we have that $(q_1^*, q_2^*, \dots, q_n^*)$ verifies Eq. (1) and from Eqs. (2) and (A.1) (with $r_j = 0$ for any j)

$$\mu_0 C \leq V(q^*, k)$$

for any shock k . \square

Proof of Theorem 2.3. Let us consider the following real valued functional over the admissible shocks:

$$U(k) = \text{Max} \left\{ \frac{V_1(k)}{P_1}, \frac{V_2(k)}{P_2}, \dots, \frac{V_n(k)}{P_n} \right\} \quad \text{for } k \in K.$$

Define

$$\mu_0^* = \text{Inf} \{ U(k); k \in K \},$$

then, for any shock k we have $U(k) \geq \mu_0^*$ and then there exists a i th bond (which depends on k) such that

$$\frac{V_i(k)}{P_i} \geq \mu_0^*.$$

Lemma 2.1 shows that we can find a portfolio q^* such that $V(q^*, k) \geq \mu_0^* C$ for any $k \in K$ and then

$$\bar{V}(q^*) = \text{Inf}\{V(q^*, k); k \in K\} \geq \mu_0^* C.$$

We will have proved that q^* is a solution of P1 if we show that $\bar{V}(q) \leq \mu_0^* C$ for any portfolio $q = (q_1, q_2, \dots, q_n)$ subject to Eq. (1).

Clearly, for any feasible shock k we have

$$\bar{V}(q) \leq V(q, k) = \sum_{i=1}^n q_i V_i(k) = \sum_{i=1}^n q_i P_i \frac{V_i(k)}{P_i} \leq U(k) \sum_{i=1}^n q_i P_i = CU(k).$$

Therefore

$$\bar{V}(q) \leq C \text{Inf}\{U(k); k \in K\} = C\mu_0^*. \quad \square$$

Proof of Theorem 4.1. Since $\{k_0(t), k_1(t)\}$ is a set of worst shocks, it follows from Eq. (7) and Proposition 3.2 that

$$\frac{V(q, k) - RC}{RC} \geq \lambda_0 \frac{v_0}{RC} + \frac{v_1}{RC}, \quad (\text{A.2})$$

where λ_0 is given by Eq. (8) and $v_i = \lim_{h \rightarrow 0} (V(q, hk_i) - V(q, 0)/h)$, $i = 0, 1$.

From Eq. (13)

$$v_0 = R \int_0^T c(t, 0)(m - t) dt = RC(m - D), \quad (\text{A.3})$$

$$v_1 = R \int_0^T c(t, 0) \left[-\frac{\lambda}{2} |m - t| \right] dt = -RC \frac{\lambda}{2} \tilde{N}, \quad (\text{A.4})$$

and Eq. (14) trivially follows. \square

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