MARSHALLIAN MONEY, WELFARE, AND SIDE-PAYMENTS

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September 2009

COWLES FOUNDATION DISCUSSION PAPER NO. 1729

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http://cowles.econ.yale.edu/

Marshallian Money, Welfare, and Side-Payments

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April 17, 2009

Abstract: A link between a no-side-payment (NSP) market game and a side-payment (SP) market game can be established by introducing a sufficient amount of an ideal utility-money of constant marginal utility to all agents. At some point when there is "enough money" in the system, if it is "well distributed" the new game will be a SP game. This game can also be related to a pure NSP game where a set of default parameters have been introduced. These parameters play a role similar to the parameters specifying the interpersonal comparisons in the side-payment game. We study this game for the properties of the $\lambda-$core and consider both the conditions for the uniqueness of competitive equilibria and a new approach to the second welfare theorem. A discussion of the relationship between market games and strategic market games is also noted.

Keywords: λ-core, enough money, market games

JEL Code: C71, C72, D51, E4

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1 Introduction

This essay takes a first step in exploring the relationship between models of monetary exchange considered as no-side-payment or side-payment games in coalitional form. It also considers some basic problems in the relationship between games in strategic form and coalitional form. However, as there are several difficulties that need to be overcome, the approach adopted here is to note and label the important ones, but to confine this analysis to a limited scope.

The transfer of any commodity of value is a transfer of utility. In an advanced economy, specially a market economy, money provides a sophisticated and general way to transfer utility. Problems concerning fiat money have been dealt with elsewhere (see [16]) and are not considered further here. We confine our remarks to an ideal commodity money and to a credit system.

The ideal commodity money has a constant marginal utility for all, hence it enters all utility functions as a linear separable term. The linearity of utility in this commodity qualifies it as a Marshallian money. We also assume that the total endowment of this commodity is bounded and no one is able to hold it in negative quantities.

We consider a credit system as an alternative to the economy with Marshallian money. There is a relationship between a competitive equilibrium (CE in short) of an exchange economy viewed as the λ-core of an NSP game and a competitive equilibrium of an exchange economy with enough commodity money to become an SP game.

In terms of monetary theory, the credit balancing required in the NSP game can be interpreted as the outcome from the functioning of a perfect "inside money" system. In contrast the limiting core point in the SP game can be interpreted as the outcome from an economy with "outside money" only. Figure 2 shows the relationship between the two outcomes and the NSP and SP games for an example with three CEs. In each case the individual marginal utility of money and credit are in the same proportions for all.

In the game in strategic form that is related to the NSP game, it is as though there is a perfect clearinghouse that permits individuals to create their own credit; it then takes in all bids and offers and centralizes the calculation of market clearing prices that avoid default.[1] In contrast, in the game in strategic form associated with the SP game, prices are formed in a decentralized manner at individual trading posts.

[1]The association of the appropriate game in strategic form with the game in coalitional form requires a separate discussion which will be investigated in a separate paper.

If the underlying economy has multiple equilibria, the λ-core construction selects one of them. Figure 2 shows the possibility of three CEs: CE_λ, $\mathrm{CE}_{\lambda'}$, and $\mathrm{CE}_{\lambda''}$. But with the selection of a vector of λs, a single λ-core will be eventually selected as the traders' Marshallian money endowments are increased. The SP game associated with CE_λ has been constructed and C_S is the limit core in the associated SP game.

Our concern is with welfare economics and competitive equilibria as well as the core of an appropriately defined market game.

2 Two Basic Models

An exchange economy is an array $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i\in N}$, where X^i is consumer i's consumption set, $u^i : X^i \longrightarrow \Re$ his utility function, and $a^i \in X^i$ is his endowment. An *allocation* for $\mathcal{E}$ is a n-tuple of consumption bundles $x = (x^1, \cdots, x^n)$ with $x^i \in X^i$, for all i. Allocation x is *feasible* if $\sum_{i\in N} x^i \le \sum_{i\in N} a^i$. A *competitive equilibrium* (CE) for $\mathcal{E}$ is a pair $(\bar{x}, \bar{p})$ with an allocation $\bar{x}$ and a price vector $\bar{p}$ such that (i) for all $i \in$N, $\bar{x}^i$ maximizes u^i subject to budget constraint $\bar{p}\cdot x^i \le \bar{p}\cdot a^i$, $x^i \in X^i$; (ii) $\bar{x}$ is feasible.

2.1 Model 1: The Marshallian General Equilibrium Model

Let λ be a vector in $\Re^N_{++}$ and let α be a vector in $\Re^N_+$. We use the pair (λ, α) to generate another exchange economy $\mathcal{E}_{\lambda\alpha} = \{(X^i_\lambda, U^i_\lambda, a^i_\lambda)\}_{i\in N}$ from $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i\in N}$ by letting (i) $a^i_\lambda = (a^i, \alpha^i)$; (ii) $X^i_\lambda = \Re^m_+ \times X^i_{m+1}$ for some closed convex subset $X^i_{m+1} \subseteq \Re$ such that $\alpha^i \in X^i_{m+1}$; and (iii) $U^i_\lambda(x^i_\lambda) = u^i(x^i) + \lambda^i x^i_{m+1}$, for $x^i_\lambda = (x^i, x^i_{m+1}) \in X^i_\lambda$. In $\mathcal{E}_{\lambda\alpha}$, good $m+1$ yields a constant marginal utility for each consumer and hence, can be considered as a Marshallian money. Correspondingly, the vector α is considered as representing the initial distribution of the Marshallian money.

Notice that by redistributing the Marshallian money endowments, consumers can transfer their utilities at rates prescribed by the ratios of λ^i, $i \in N$. However, utility transfers are restricted by the nature of X^i_{m+1}, $i \in N$, as well as by the initial distribution α. Notice also that the utility functions $U^i_\lambda(x^i_\lambda) = u^i(x^i) + \lambda^i x^i_{m+1}$ in $\mathcal{E}_{\lambda\alpha}$ are equivalent to

$$\tilde{U}^i_\lambda(x^i_1, x^i_2, ...x^i_{m+1}) = \frac{1}{\lambda^i} u^i(x^i_1, x^i_2, ...x^i_m) + x^i_{m+1} \tag{1}$$

Thus, given $\lambda \in \Re^N_{++}$, provided that there is enough Marshallian money which is "well-distributed", the CE problem is reduced to a joint maximization of

$$\sum_{i=1}^{n} \left(\frac{1}{\lambda^i} u^i(x_1^i, x_2^i, ...x_m^i) + x_{m+1}^i \right). \tag{2}$$

Furthermore, for any given CE of $\mathcal{E}$, there exists an appropriately selected vector of parameters $\lambda = (\lambda^1, \lambda^2, \lambda^3, ..., \lambda^n)$ such that for all $\alpha \in \Re_+^N$, the given CE of $\mathcal{E}$ remains to be a CE for $\mathcal{E}_{\lambda\alpha}$ that does not involve any net trade in the Marshallian money. For example, this would be the case, if we let λ^i be the marginal utility of income of consumer i at the given CE.

As the Marshallian money is increased the Pareto surface of $\mathcal{E}_{\lambda\alpha}$ in utility space will be enlarged with a hyperplanar area appearing and growing on the surface. To see this more clearly, define a λ-*transfer segment* $H_{\lambda\alpha}$ by

$$H_{\lambda\alpha} = \{\Delta u \in \Re^N \mid \sum_{i \in N} \frac{1}{\lambda^i} \Delta u^i = \sum_{i \in N} \alpha^i, \Delta u^i \in \lambda^i X_{m+1}^i,\ i \in N\},$$

where $\lambda^i X_{m+1}^i$ denotes the multiplication of λ^i and X_{m+1}^i. This segment consists of vectors of consumers' utility changes achievable by redistributing the Marshallian money endowments. Given a feasible allocation x for economy $\mathcal{E}$, the set of utility allocations achievable from this allocation via the Marshallian money is

$$u(x) + H_{\lambda\alpha},$$

where $u(x)$ denotes the utility allocation $(u^1(x^1), \cdots, u^n(x^n))$.

The introduction of the increasing amounts of the Marshallian money has the effect of both raising and flattening the Pareto surface of the exchange economy. For the instance where there are only two consumers a simple diagram in Figure 1 indicates how the Pareto surface moves out and flattens

Figure 1: The flattening of the Pareto optimal surface.

Let $\bar{u}$ be a weighted welfare maximum with welfare weights $1/\lambda^1$ for consumer 1 and $1/\lambda^2$ for consumer 2. Then, given Marshallian money endowment distribution $\alpha = (\alpha^1, \alpha^2)$ with $\alpha^1, \alpha^2 > 0$, the Pareto surface of $\mathcal{E}_{\lambda\alpha}$ contains the line segment with end points $\bar{u} + (\lambda^1(\alpha^1 + \alpha^2), 0)$ and $\bar{u} + (0, \lambda^2(\alpha^1 + \alpha^2))$. To see this, consider any utility pair $(\bar{u}_1 + \lambda_1 \bar{x}_{m+1}^1, \bar{u}_2 + \lambda_2 \bar{x}_{m+1}^2) \in \bar{u} + H_{\lambda\alpha}$ for some redistribution $(\bar{x}_{m+1}^1, \bar{x}_{m+1}^2) \in X_{m+1}^1 \times X_{m+1}^2$ of the total Marhsallian money endowment $\alpha^1 + \alpha^2$.

If the utility pair were Pareto dominated, then there would be a utility pair $\tilde{u}$ achievable in $\mathcal{E}$ and a redistribution $(\tilde{x}^1_{m+1}, \tilde{x}^2_{m+1}) \in X^1_{m+1} \times X^2_{m+1}$ of the total Marshallian money endowment such that

$$\tilde{u}^1 + \lambda^1 \tilde{x}^1_{m+1} > \bar{u}^1 + \lambda^1 \bar{x}^1_{m+1} \text{ and } \tilde{u}^2 + \lambda^2 \tilde{x}^2_{m+1} > \bar{u}^2 + \lambda^2 \bar{x}^2_{m+1}$$

which would imply

$$\frac{1}{\lambda^1}\tilde{u}^1 + \frac{1}{\lambda^2}\tilde{u}^2 > \frac{1}{\lambda^1}\bar{u}^1 + \frac{1}{\lambda^2}\bar{u}^2.$$

This contradicts the fact that $\bar{u}$ is the weighted welfare maximum of $\mathcal{E}$ with welfare weights $w^1 = 1/\lambda^1$ and $w^2 = 1/\lambda^2$.

2.2 Fiat money, Default, and Uniqueness

The full details of the link between an explicit strategic market game and either the general equilibrium or the market game model are not presented here, they are dealt with in a subsequent essay. However, a way of imagining a playable game is that as soon as the individuals put up their goods for sale the clearing house extends a line of credit to each, but requires that after exchange and the netting of sales and purchases all accounts clear.

But if this is truly a game of strategy where individuals offer goods for sale, and bid to set prices, unless a perfect clearing house calculates matching prices, there is no guarantee that, in general, accounts will always balance. After credit settlement some individuals may have a residual credit and others may have a deficit. A rule to settle default must be supplied. A natural way to account for this situation is to introduce a default penalty. A simple form for such a penalty is a separable term of the form [2]

$$U^i_\mu(x^i_1, x^i_2, ..., x^i_m, x^i_{m+1}) = u^i(x^i) + \mu^i \min[x^i_{m+1}, 0]$$

where $x^i = (x^i_1, x^i_2, ..., x^i_m)$ and x^i_{m+1} is the difference between the monetary value of his endowment a^i and the monetary payment that i makes for purchasing commodity bundle x^i.

But if a vector of penalties $\mu = (\mu^1, \mu^2, ..., \mu^n)$ is specified, this is tantamount to selecting the marginal utilities of income as parameters, rather than as variables as

[2] The key feature is that at the point of default the marginal disutility of the penalty for defaulting is greater than the marginal utility of the gain from the extra increment of expenditure and beyond that point the penalty need only to be sufficiently harsh as to always discourage default. Any functional form that satisfies these conditions will suffice.

in the Slutsky or Hicks analysis. If the μ^i are selected so that $\mu^i = \lambda^i$ where λ^i is the Lagrangian multiplier associated with individual i's utility maximization problem at a CE of $\mathcal{E}$, this imposes on the underlying game extra constraints sufficient to guarantee this CE to be the unique one that survives these extra constraints (see [5]).

2.3 Model 2: The Game in Characteristic Function Form

Associated with Model 1 is a market game represented by a characteristic function without side-payment (an NSP game in short) as in Shubik [12] and Debreu and Scarf [2]. As the amount of Marshallian money is increased a class of NSP games is created that can be described as quasi-side-payment games. At one extreme, with no Marshallian money there is the pure NSP game and at the other extreme when there is "enough money, well-distributed' (to be specified below) the game becomes a pure side-payment game.

We are concerned with several explicit questions concerning the underlying NSP game and the associated class of quasi-SP games culminating in fully SP games under the appropriate conditions.

1. What happens to any multiple competitive equilibria as a Marshallian money is introduced?

2. Precisely what is meant by "enough money, well-distributed"?

3. What can we say about the structure of the core and its convergence in the process of replication when there is enough Marshallian money which is well distributed?

3 Enough Money, Well-Distributed, and Transferable Utility

3.1 Ideal gold or real u-money

The basic idea behind there being enough money in an economy that utilizes a commodity money is that whatever the transactions constraints may happen to be on any individual, there is sufficient money in the system that one can find a distribution of the money such that no individual is constrained by cash flow conditions.

The concept of enough money in an economy has always been peculiarly institutional, depending on custom of payment, details in the transactions technology and possible variations in velocity. We can, however, make precise the upper bound on the requirements for a money. The greatest amount of money required by an economy is given by the underlying sell-all condition, where all individuals are mandated by the rules of the game to sell all of their (non-monetary) assets, i.e. we require that all goods must pass through the market. Thus the quantity of money required at a CE with price vector p is

$$M = \sum_{i=1}^{n} \sum_{j=1}^{m} p_j a_j^i$$

but for this money to be well distributed we also require that the optimization for each individual is unconstrained. This requires that:

$$\alpha^i \geq \sum_{j=1}^{m} p_j a_j^i$$

3.2 Liquidity, Super-Liquidity, Fiat and Transferability

When we consider a game with a special "u-gold" where all non-monetary goods must go through the market, there are four zones that must be considered. They reflect the idea of liquidity: (i) there is not enough money in the economy; (ii) there is just enough money but it is not well-distributed; (iii) there is just enough money that is well-distributed; and (iv) Each individual has enough "u-gold" to buy all of the non-monetary assets in the economy.[3]

We note that the required condition that all goods (except u-gold) must go through the market brings in a subtle distinction in the concept of ownership and the definition of the value of a one-person coalition. The valuation of a coalition depends on the price at which the coalition's ownership claims are sold.

If there is not enough money in a one period strategic market game there is no interior solution and the shadow price of money is above its utilitarian worth. If there is enough money in a one period strategic market game but it is badly distributed there is no interior solution as trade for at least some individuals will be constrained by a cash flow constraint. This situation can be rectified by introducing a money market. The treatment of this possibility will be dealt with in a separate

[3]We note that the merchantlist fallacy that the unbounded accumulation of monetary gold represents an unbounded increase in the wealth of the society [7]. would not be a fallacy in a society with the existence of a gold with linearly separable utility in a sell-all economy. Enough money in a sell-all economy is precisely equal to the monetary value of all non-monetary goods.

paper establishing efficiency with $\rho = 0$. If there is enough money in a one period strategic market game and it is well distributed there is an interior solution.

With a separable linear utility for money and sell-all trading enough money calls for the wealth of all other goods to equal the amount of money. There is enough liquidity for all agents to purchase everything. Beyond that point the only other qualitative change brought on by extra liquidity is when each individual could purchase all goods. We might term this *super-liquidity* where with no prior knowledge of how the initial resources are distributed all individuals will have enough money to at least be able to purchase any ownership bundle assigned to them.[4]

4 Competitive Equilibrium and Second Welfare Theorem with Marshallian Money

Let $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i \in N}$ be an exchange economy. For all i, we assume A1: $X^i = \Re^m_+$; A2: u^i is continuous, concave, and strongly monotonically increasing; and A3: $u^i(a^i) > u^i(0)$. In addition, we also assume A4: $\sum_{i \in N} a^i_h > 0$ for $h = 1, 2, \cdots, m$. A2 implies that all CE price vectors must be strictly positive. By A3, for $i \in N$ there exists a commodity h such that $a^i_h > 0$. Consequently, A2 and A3 together imply $\bar{p} \cdot a^i > 0$ at each CE price vector $\bar{p}$. On the other hand, A 4 implies that for each commodity h, there exists a consumer i for whom $a^i_h > 0$.

Given price vector p, consumer i's utility maximization problem in economy $\mathcal{E}$ is:

$$\begin{array}{ll} \text{maximize} & u^i(x^i) \\ \text{subject to} & p \cdot a^i - p \cdot x^i \geq 0, \ x^i \in \Re^m_+. \end{array} \qquad \mathrm{UM}^i(p)$$

Let $\bar{p}$ be a price vector. Then, A2 and A3 imply that the problem $\mathrm{UM}^i(p)$ satisfies all the assumptions of the Saddle-Point Theorem (See [18], p. 75). It thus follows that $\bar{x}^i$ solves $\mathrm{UM}^i(\bar{p})$ if and only if there exists a Lagrangian multiplier $\bar{\xi}^i > 0$ such that [5]

$$u^i(x^i) + \bar{\xi}^i \bar{p} \cdot (a^i - x^i) \leq u^i(\bar{x}^i) + \bar{\xi}^i \bar{p} \cdot (a^i - \bar{x}^i) \leq u^i(\bar{x}^i) + \xi^i \bar{p} \cdot (a^i - \bar{x}^i) \tag{3}$$

for all $x^i \in \Re^m_+$ and for all $\xi^i \in \Re_+$.

[4] Going beyond an exchange economy, there is a more general question concerning the relationship between a no-side-payment and a related side-payment game. Is there a reasonable measure of "side-paymentness"?

[5] By the strong monotonicity of u^i, all Lagrangian multipliers must be positive.

Given a vector of welfare weights $w = (w^i)_{i\in N} \in \Re^N_+$ with $w \neq 0$, the weighted welfare maximization problem at w in economy $\mathcal{E}$ is:

$$\begin{array}{lll} \text{maximize} & \sum_{i\in N} w^i u^i(x^i) & \\ & & \text{WWM}(w) \\ \text{subject to} & \sum_{i\in N} a^i - \sum_{i\in N} x^i \geq 0,\ x^i \in \Re^m_+,\ i \in N. & \end{array}$$

A, A2, and A4 together imply that the problem WWM satisfies all the assumptions of the Saddle-Point theorem. Hence, $\bar{x} = (\bar{x}^i)_{i\in N}$ solves the problem WWM(w) if and only if there exists a vector $\bar{p} \in \Re^m_+$ of Lagrangian multipliers such that

$$\begin{aligned} & \textstyle\sum_{i\in N} w^i u^i(x^i) + \bar{p}\cdot \sum_{i\in N}(a^i - x^i) \\ & \leq \textstyle\sum_{i\in N} w^i u^i(\bar{x}^i) + \bar{p}\cdot \sum_{i\in N}(a^i - \bar{x}^i) \\ & \leq \textstyle\sum_{i\in N} w^i u^i(\bar{x}^i) + p\cdot \sum_{i\in N}(a^i - \bar{x}^i) \end{aligned} \tag{4}$$

for all $x^i \in \Re^m_+$, $i \in N$, and for all $p \in \Re^m_+$.

4.1 Competitive Equilibrium with Marshallian Money

With the presence of Marshallian money, we normalize the prices so as to make the price of the Marshallian money equal to 1. For $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_+$, let $\mathcal{E}_{\lambda\alpha} = \{(X^i_\lambda, U^i_\lambda, a^i_\lambda)\}_{i\in N}$ be the economy generated from $\mathcal{E}$ as in Section 2.1. Consumer i's utility maximization problem in $\mathcal{E}_{\lambda\alpha}$ at price vector $p_\lambda = (p, 1)$ with $p \in \Re^m_{++}$ is:

$$\begin{array}{lll} \text{maximize} & U^i_\lambda(x^i_\lambda) = u^i(x^i) + \lambda^i x^i_{m+1} & \\ & & \text{UM}^i(p,\lambda,\alpha) \\ \text{subject to} & p_\lambda \cdot a^i_\lambda - p_\lambda \cdot x^i_\lambda \geq 0,\ x^i_\lambda \in X^i_\lambda. & \end{array}$$

Let $\bar{p}_\lambda$ be a CE price vector for $\mathcal{E}_{\lambda\alpha}$. As before, with A1-A3 consumer i's utility maximization problem $\text{UM}^i(\bar{p},\lambda,\alpha)$ satisfies the conditions of the Saddle-Point Theorem. Consequently, $\bar{x}^i_\lambda = (\bar{x}^i, \bar{x}^i_{m+1})$ solves $\text{UM}^i(\bar{p},\lambda,\alpha)$ at if and only if there exists a Lagrangian multiplier $\bar{\xi}^i > 0$ such that

$$\begin{aligned} & u^i(x^i) + \lambda^i x^i_{m+1} + \bar{\xi}^i(\bar{p}\cdot a^i + \alpha^i - \bar{p}\cdot x^i - x^i_{m+1}) \\ & \leq u^i(\bar{x}^i) + \lambda^i \bar{x}^i_{m+1} + \bar{\xi}^i(\bar{p}\cdot a^i + \alpha^i - \bar{p}\cdot \bar{x}^i - \bar{x}^i_{m+1}) \\ & \leq u^i(\bar{x}^i) + \lambda^i \bar{x}^i_{m+1} + \xi^i(\bar{p}\cdot a^i + \alpha^i - \bar{p}\cdot \bar{x}^i - \bar{x}^i_{m+1}) \end{aligned} \tag{5}$$

holds for all $x^i \in \Re^m_+$, $x^i_{m+1} \in X^i_{m+1}$, and for all $\xi^i \in \Re_+$.

If no one uses the Marshallian money to finance consumption of the commodities in a CE of $\mathcal{E}_{\lambda\alpha}$, then it reduces to a CE for economy $\mathcal{E}$. Conversely, each CE, $(\bar{x}, \bar{p})$, for $\mathcal{E}$ can be embedded into a CE for $\mathcal{E}_{\lambda\alpha}$ with arbitrary distribution of the Marshallian money, but with each consumer i's marginal utility of the Marshallian money equal to the Lagrangian multiplier associated with problem $\mathrm{MU}^i(\bar{p})$. Notice also that with these marginal utilities of the Marshallian money, no net transfers in the Marshallian money are needed.

Remark 1: Let $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_+$ be given and let $(\bar{x}_\lambda, \bar{p}_\lambda)$ with $\bar{x}_\lambda = (\bar{x}, \bar{x}_{m+1})$ and $\bar{p}_\lambda = (\bar{p}, 1)$ be a CE for $\mathcal{E}_{\lambda\alpha}$. Then, $\bar{x}^i_\lambda = (\bar{x}^i, \bar{x}^i_{m+1})$ solves $\mathrm{UM}^i(\bar{p}, \lambda, \alpha)$ for all i;, hence, the bundle solves (5) for some Lagrangian multiplier $\bar{\xi}^i > 0$. In case when $\bar{\xi}^i = \lambda^i$, $x'^i_\lambda = (\bar{x}^i, x'^i_{m+1})$ also solves $\mathrm{UM}^i(\bar{p}, \lambda, \alpha')$ whenever $\alpha'^i - x'^i_{m+1} = \alpha^i - \bar{x}^i_{m+1}$. When $\bar{\xi}^i = \lambda^i$ for all $i \in N$, $\bar{x}$ is a weighted welfare maximum for $\mathcal{E}$ with $w^i = 1/\lambda^i$. Furthermore, $(\bar{x}'_\lambda, \bar{p}_\lambda)$ with $\bar{x}'_\lambda = (\bar{x}, \bar{x}'_{m+1})$ is a CE for $\mathcal{E}_{\lambda\alpha'}$ whenever $\alpha'^i - \bar{x}'^i_{m+1} = \alpha^i - \bar{x}^i_{m+1}$ for all $i \in N$. This implies that any utility allocation on the flat portion of the Pareto frontier can be achieved via a CE of $\mathcal{E}_{\lambda\alpha'}$ for a suitable redistribution $\alpha' \in \Re^N$ of the Marshallian money endowments prescribed by the allocation α.

4.2 An Alternative Formulation of the Second Welfare Theorem with Marshallian Money

From the first welfare theorem, each CE allocation is a welfare maximum of some weighted welfare function (e.g., with welfare weights equal to the inverses of the consumers' marginal utilities income associated with the CE allocation). The converse is, however, not necessarily true unless one can change the consumers' incomes as shown by the second welfare theorem. We want to know if the converse can hold with the introduction of a Marshallian money. Specifically, let $\bar{x}$ be a solution for WWM(w) with $w \in \Re^N_{++}$. We want to know how large α^i has to be in order for $\bar{x}$ to be a CE allocation for $\mathcal{E}_{\lambda\alpha}$ with $\lambda^i = 1/w^i$ for all i. The following theorem provides an answer.

Theorem 1 *Assume A1-A4. Let $w \in \Re^N_{++}$ be a vector of welfare weights, $\bar{x}$ a commodity allocation that solves WWM(w) with $w \in \Re^N_{++}$, and let $\bar{p} \in \Re^m_+$ be such that $(\bar{x}, \bar{p})$ is a saddle-point for WWM(w). Set (i) $\lambda^i = 1/w^i$; (ii) $\bar{x}^i_{m+1} = \max\{0, \bar{p} \cdot (a^i - \bar{x}^i)\}$; and (iii) $\alpha^i = \max\{0, \bar{p} \cdot (\bar{x}^i - a^i)\}$ for $i \in N$. Then, $(\bar{x}_\lambda, \bar{p}_\lambda) = ((\bar{x}^i, \bar{x}^i_{m+1})_{i \in N}, (\bar{p}, 1))$ is a CE for $\mathcal{E}_{\lambda\alpha}$.*

Proof: Since $(\bar{x}, \bar{p})$ satisfies (4) and since, by (i), $\lambda^i = 1/w^i$, taking $x^j = \bar{x}^j$ for $j \neq i$ we have from (4)

$$\begin{array}{l} u^i(x^i) + \lambda^i x^i_{m+1} + \lambda^i(\bar{p} \cdot a^i + \alpha^i - \bar{p} \cdot x^i - x^i_{m+1}) \\ \leq \\ u^i(\bar{x}^i) + \lambda^i \bar{x}^i_{m+1} + \lambda^i(\bar{p} \cdot a^i + \alpha^i - \bar{p} \cdot \bar{x}^i - \bar{x}^i_{m+1}) \end{array} \tag{6}$$

for all $x^i \in \Re^m_+$, $x^i_{m+1} \in \Re_+$. Next, by (ii) and (iii), $\alpha_i - \bar{x}^i_{m+1} = \bar{p} \cdot \bar{x}^i - \bar{p} \cdot a^i$. Hence, $\bar{p} \cdot a^i + \alpha^i - \bar{p} \cdot \bar{x}^i - \bar{x}^i_{m+1} = 0$. It follows

$$\begin{array}{l} u^i(\bar{x}^i) + \lambda^i \bar{x}^i_{m+1} + \lambda^i(\bar{p} \cdot a^i + \alpha^i - \bar{p} \cdot \bar{x}^i - \bar{x}^i_{m+1}) \\ \leq \\ u^i(\bar{x}^i) + \lambda^i \bar{x}^i_{m+1} + \xi^i(\bar{p} \cdot a^i + \alpha^i - \bar{p} \cdot \bar{x}^i - \bar{x}^i_{m+1}) \end{array} \tag{7}$$

for all $\xi^i \in \Re_+$. Together, (6) and (7) imply that the triplet $(\bar{x}^i, \bar{x}^i_{m+1}, \bar{\xi}^i)$ with $\bar{\xi}^i = \lambda^i$ satisfies (5). Consequently, by the Saddle-Point Theorem, $\bar{x}^i_\lambda = (\bar{x}^i, \bar{x}^i_{m+1})$ solves $\mathrm{UM}^i(\bar{p}, \lambda, \alpha)$.

To complete the proof, it only remains to show that the allocation $\bar{x}_\lambda$ is feasible. To this end, notice that by (ii) and (iii),

$$\sum_{i \in N} \bar{x}^i_{m+1} = \sum_{i : \bar{p} \cdot a^i > \bar{p} \cdot \bar{x}^i} \bar{p} \cdot (a^i - \bar{x}^i)$$

and

$$\sum_{i \in N} \alpha^i = \sum_{i : \bar{p} \cdot a^i < \bar{p} \cdot \bar{x}^i} \bar{p} \cdot (\bar{x}^i - a^i).$$

From these two equations, $\sum_{i \in N} \bar{x}^i_{m+1} = \sum_{i \in N} \alpha^i$ *if and only if* $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} a^i$. This last inequality holds because, as a weighted welfare maximum, $\bar{x}$ is feasible. ■

Theorem 1 provides an alternative formulation for the Second Fundamental Theorem of welfare economics. To understand this formulation, notice first that at welfare maximum $\bar{x}$ with welfare weights w, some consumers may violate budget constraints and others have budget surplus when evaluating endowments a^i and bundles $\bar{x}^i$ at shadow prices $\bar{p}$. Now add good $m+1$ to the economy and let it be measured in the unit of account as determined by the shadow prices $\bar{p}$, so that its price is 1. If each consumer i values this additional good at a constant rate equal to $\lambda^i = 1/w^i$, then the welfare maximum $\bar{x}$ becomes part of a CE by allowing those consumers whose budgets are violated at $(\bar{x}, \bar{p})$ to finance the extra expenses with good $m+1$, while allowing the others with budget surplus to purchase it at price equal to 1. It is as if we *subsidize* those who needs extra income with good $m+1$ and at the same time *increase* the others' utilities for possessing it.

Notice also that under the market implementation of $\bar{x}$ as a CE commodity allocation for $\mathcal{E}_{\lambda\alpha}$, consumers' marginal utilities of income are exactly the inverses of their welfare weights. The implementation does not require changes in the consumers' entire endowments. Rather, it requires the existence of an additional good, a Marshallian money, that yields a constant marginal utility equal to $\lambda^i = 1/w^i$ for all i.

4.3 Characterization of the CEs with Marshallian Money

Given $\lambda \in \Re^N_{++}$ and given $\alpha \in \Re^N_{+}$, how different are the CE commodity allocations for $\mathcal{E}_{\lambda\alpha}$ from those that solve WWM(w) with $w = (1/\lambda^1, \cdots, 1/\lambda^n)$ in $\mathcal{E}$? The following theorem provides an answer.

Theorem 2 *Assume A1-A4. Let* $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_{+}$, $(\bar{x}_\lambda, \bar{p}_\lambda) = ((\bar{x}^i, \bar{x}^i_{m+1})_{i\in N}, (\bar{p}, 1))$ *be a CE for* $\mathcal{E}_{\lambda\alpha}$*, and let* $\bar{\xi}^i$ *be a Lagrangian multiplier associated with* $UM^i(\bar{p}, \lambda, \alpha)$*. Then, (i)* $\bar{p}\cdot\bar{x}^i + \bar{x}^i_{m+1} = \bar{p}\cdot a^i + \alpha^i$*; (ii)* $\bar{\xi}^i = \lambda^i$ *when* $\bar{x}^i_{m+1} > 0$*; (iii)* $\bar{\xi}^i \geq \lambda^i$ *when* $\bar{x}^i_{m+1} = 0$*; (iv)* $\bar{x}$ *solves WWM(w) with* $w = (1/\bar{\xi}^1, \cdots, 1/\bar{\xi}^n)$ *in* $\mathcal{E}$*; and (v)* $\bar{x}$ *solves the following perturbed welfare maximization at welfare weights* $1/\lambda^i$ *in* $\mathcal{E}$*:*

$$\begin{array}{ll} \textit{maximize} & \sum_{i\in N} \frac{1}{\lambda^i}[u^i(x^i) - (\bar{\xi}^i - \lambda^i)\bar{p}\cdot x^i] \\ \textit{subject to} & \sum_{i\in N} x^i = \sum_{i\in N} a^i, \quad x^i \in \Re^m_+, \; i \in N. \end{array}$$

Before proving the theorem, a few comments are in order. Property (i) needs no comment. Property (ii) says that a consumer who keeps a positive amount of the Marshallian money has marginal utility of income equal to his marginal utility of the Marshallian money. This is necessary because the price of the Marshallian money is 1, so that the consumer can increase his utility by increasing his quantity of the Marshallian money when $\bar{\xi}^i < \lambda^i$ or by decreasing the quantity when $\bar{\xi}^i > \lambda^i$. Similiar reasoning can be given for property (iii). Property (iv) states that the commodity allocation $\bar{x}$ actually solves the weighted welfare maximization problem with $w = (1/\bar{\xi}^1, \cdots, 1/\bar{\xi}^n)$. In this maximization, less welfare weight is given to a consumer, as compared to the inverse of his marginal utility of the Marshallian money, whose budget would be violated were there no Marshallian money given to him. Property (v), on the other hand, states that the commodity allocation

$\bar{x}$ can be a weighted welfare maximum with welfare weights exactly equal to the inverses of the marginal utilities of the Marshallian money, provided the assignment of commodity bundle x^i to consumer i is penalized at a rate equal to $(\bar{\xi}^i - \lambda^i)$ per unit of total value $\bar{p} \cdot x^i$, for $i \in N$.

Proof: Since the triplet $(\bar{x}^i, \bar{x}^i_{m+1}, \bar{\xi}^i)$ satisfies (5), we have [6]

$$(\lambda^i - \bar{\xi}^i)(\bar{x}^i_{m+1} - x^i_{m+1}) \geq 0, \ \forall x^i_{m+1} \in \Re_+. \tag{8}$$

If $\bar{x}^i_{m+1} > 0$, then there are numbers $\tilde{x}^i_{m+1}, \hat{x}^i_{m+1} \in \Re$ such that $0 < \tilde{x}^i_{m+1} < \bar{x}^i_{m+1} < \hat{x}^i_{m+1}$. Thus, by (8), both $\lambda^i \geq \bar{\xi}^i$ and $\lambda^i \leq \bar{\xi}^i$ must hold. This establishes (ii). When $\bar{x}^i_{m+1} = 0$, (8) implies $(\lambda^i - \bar{\xi}^i)x^i_{m+1} \leq 0$ for all $x^i_{m+1} \geq 0$. Hence $\lambda^i \leq \bar{\xi}^i$. This establishes (iii). Since $\lambda^i > 0$, it follows from (ii) and (iii) that $\bar{\xi}^i > 0$. With $\bar{\xi}^i > 0$, the equality $\bar{p} \cdot x^i + \bar{x}^i_{m+1} = \bar{p} \cdot a^i + \alpha^i$ in (i) follows directly from the second inequality in (5).

To show (iv), notice that (ii) and (iii) together with the feasibility of the allocation $(\bar{x}^i_{m+1})_{i \in N}$ of the Marshallian money imply

$$\sum_{i \in N} \frac{\lambda^i}{\bar{\xi}^i} \bar{x}^i_{m+1} = \sum_{i \in N} \alpha^i. \tag{9}$$

On the other hand, since $X^i_{m+1} = \Re_+$ and since $\lambda^i \leq \bar{\xi}^i$, we have

$$\sum_{i \in N} \frac{\lambda^i}{\bar{\xi}^i} x^i_{m+1} \leq \sum_{i \in N} \alpha^i, \tag{10}$$

for all $(x^i_{m+1})_{i \in N} \in \Re^N_+$ such that $\sum_{i \in N} x^i_{m+1} = \sum_{i \in N} \alpha^i$. Dividing both sides of the first inequality in (5) by $\bar{\xi}^i$ for $i \in N$, property (iv) follows from (5), (9), and (10).

By similar reasoning,

$$\sum_{i \in N} \frac{\bar{\xi}^i}{\lambda^i} \bar{x}^i_{m+1} = \sum_{i \in N} \alpha^i, \tag{11}$$

and

$$\sum_{i \in N} \frac{\bar{\xi}^i}{\lambda^i} x^i_{m+1} \geq \sum_{i \in N} \alpha^i, \tag{12}$$

for all $(x^i_{m+1})_{i \in N} \in \Re^N_+$ such that $\sum_{i \in N} x^i_{m+1} = \sum_{i \in N} \alpha^i$. Now dividing both sides of the first inequality in (5) by λ^i for $i \in N$ and rearranging the terms, property (v) follows from (5), (11), and (12). ■

[6] Inequality (8) follows by letting $x^i = \bar{x}^i$ be the commodity bundle and x^i_{m+1} be arbitrary in the first inequality in (5).

4.4 The Uniqueness of the CE with Marshallian Money

In this section we show that the CEs for $\mathcal{E}_{\lambda\alpha}$ are unique if, among some other standard conditions, the Marshallian money endowment distribution satisfies a certain lower bound. To this end, we first establish a lemma that shows that when consumers' are endowed with large enough amounts of the Marshallian money, the marginal utilities of the money prescribed by λ coincide with the Lagrangian multipliers of the consumers in *all* the CEs of $\mathcal{E}_{\lambda\alpha}$.

For $i \in N$, let $\hat{X}^i$ be the set of consumption bundles in X^i each of which is determined by a feasible allocation of $\mathcal{E}$. That is, $\hat{X}^i$ is the projection of the set of feasible allocations of $\mathcal{E}$ onto consumer i's consumption set X^i. Standard results in general equilibrium analysis imply that $\hat{X}^i$ is a compact subset of X^i.

Lemma 1 *Assume $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i\in N}$ satisfies A1-A4. Then for any $\lambda = (\lambda^i)_{i\in N} \in \Re^N_{++}$ and for any $i \in N$, λ^i is the only Lagrangian multiplier for utility maximization problems $MU^i(\bar{p}, \lambda, \alpha)$ in any CE of $\mathcal{E}_{\lambda\alpha}$ for all $\alpha \in \Re^N_+$ such that*

$$\alpha^i > \max\left\{\frac{u^i(x^i) - u^i(a^i)}{\lambda^i} \mid x^i \in \hat{X}^i\right\}. \tag{13}$$

Proof: Let $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_+$ such that α satisfies (13) and let $(\bar{x}_\lambda, \bar{p}_\lambda)$ be a CE for $\mathcal{E}_{\lambda\alpha}$. Then,

$$u^i(\bar{x}_i) + \lambda^i \bar{x}^i_{m+1} \geq u^i(a^i) + \lambda^i \alpha^i$$

which is equivalent to

$$\bar{x}^i_{m+1} \geq \alpha^i - \frac{u^i(\bar{x}^i) - u^i(a^i)}{\lambda^i}.$$

This together with (13) implies $\bar{x}^i_{m+1} > 0$. Thus, by (ii) in Theorem 2, λ^i is the only Lagrangian multiplier for $\mathrm{MU}^i(\bar{p}, \lambda, \alpha)$. ■ .

We are now ready to establish conditions that guarantee the uniqueness of CE for $\mathcal{E}_{\lambda\alpha}$.

Theorem 3 *Assume $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i\in N}$ satisfies A1, A3, and A4. Assume further A2': u^i is continuous, strictly concave, and strongly monotonically increasing. Then, for any $\lambda \in \Re^m_{++}$, $\mathcal{E}_{\lambda\alpha} = \{(X^i_\lambda, U^i_\lambda, a^i_\lambda)\}_{i\in N}$ has the weighted welfare maximum with welfare weights $1/\lambda^i$ as the unique CE commodity allocation for initial distributions $\alpha \in \Re^N_+$ of the Marshallian money such that α^i satisfies (13) for all i.*

Proof: Let $\lambda \in \Re^N_{++}$ be given. Since A2′ implies A2, Lemma 1 implies that for all $\alpha \in \Re^N_{++}$ satisfying (13), the Lagrangian multiplier associated for consumer i's utility maximization problem $\mathrm{MU}^i(\bar{p}, \lambda, \alpha)$ is unique and equal to λ^i in any CE of $\mathcal{E}_{\lambda\alpha}$.

Fix $\alpha \in \Re^N_{++}$ satisfying (13) and let $(\bar{x}_\lambda, \bar{p}_\lambda) = ((\bar{x}^i, \bar{x}^i_{m+1})_{i\in N}, (\bar{p}, 1))$ be any CE of $\mathcal{E}_{\lambda\alpha}$. Then, by (5), $\bar{\xi}^i = \lambda^i$ implies

$$\frac{1}{\lambda^i} u^i(x^i) - \bar{p} \cdot x^i \leq \frac{1}{\lambda^i} u^i(\bar{x}^i) - \bar{p} \cdot \bar{x}^i, \tag{14}$$

for all $x^i \in X^i$ and for all $i \in N$. Consequently, for any feasible allocation $x = (x^i)_{i\in N}$ in $\mathcal{E}$, it follows from (14)

$$\sum_{i\in N} \frac{1}{\lambda^i} u^i(x^i) \leq \sum_{i\in N} \frac{1}{\lambda^i} u^i(\bar{x}^i).$$

This shows that $\bar{x}$ maximizes the weighted welfare function $\sum_{i\in N}(u^i(x^i)/\lambda^i)$. By A2′, the weighted welfare function is strictly concave; hence, it has a unique maximum point. This in turn implies that $\mathcal{E}_{\lambda\alpha}$ has a unique CE commodity allocation. ■

Given $\lambda \in \Re^N_{++}$, condition (13) can be considered as the condition of *enough money which is well-distributed* for the uniqueness of the CE in $\mathcal{E}_{\lambda\alpha}$. Theorem 3 confirms our earlier remark that when there is enough Marshallian money which is well-distributed, the CE problem in $\mathcal{E}_{\lambda\alpha}$ is reduced to a joint maximization of the total λ-transfer utility function in (2).

Remark 2: Let $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_{+}$ be such that α satisfies (13) and let $\bar{x}$ be the weighted welfare maximum with welfare weights equal to $1/\lambda_i$. Theorem 3 shows that $\bar{x}$ is the unique CE commodity allocation for $\mathcal{E}_{\lambda\alpha}$. Given any CE price vector $\bar{p}_\lambda = (\bar{p}, 1)$, the corresponding CE Marshallian money allocation is uniquely determined through budget constraints: $\bar{p} \cdot \bar{x}^i + \bar{x}^i_{m+1} = \bar{p} \cdot a^i + \alpha^i$ for all $i \in N$. By Lemma 1, λ^i is the Lagrangian multiplier for $MU^i(\lambda, \alpha)$ at $\bar{p}_\lambda$. Hence, by saddle-point characterization (5), $\lambda^i \bar{p} \in \partial u^i(\bar{x}^i)$. It follows that the CE Marshallian money allocation is also unique if, in addition, $\partial u^i(\bar{x}^i)$ has a unique element for all $i \in N$. This would be the case if the utility functions are differentiable. We end this section with a numerical illustration of the reduction from multiple CEs to a unique CE as the Marshallian money is increased.

Example 1: Consider the 2-person economy in [11]. There are two goods, 1 and 2, and two consumers, Ivan and John with endowments $a^I = (40, 0)$ for Ivan and

$a^J = (0, 50)$ for John. Ivan has utility function $u^I(x^I) = x_1^I + 100(1 - e^{-x_2^I/10})$ for $x^I \in \Re_+^2$ while John has utility function $u^J(x^J) = 110(1 - e^{-x_1^J/10}) + x_2^J$ for $x^J \in \Re_+^2$. There are three CEs in this economy as shown in [11].

Now consider $\lambda = (\lambda^I, \lambda^J) \in \Re_{++}^2$ and $\alpha = (\alpha^I, \alpha^J) \in \Re_{++}^2$ where [7]

$$\text{(i)} \quad \frac{11}{e^4} < \frac{\lambda^J}{\lambda^I} < 11.$$

Recall that in $\mathcal{E}_{\lambda\alpha}$, $U_\lambda^I(x_1^I, x_2^I, x_3^I) = x_1^I + 100(1 - e^{-x_2^I/10}) + \lambda^I x_3^I$ and $U_\lambda^J(x_1^J, x_2^J, x_3^J) = 110(1 - e^{-x_1^J/10}) + x_2^J + \lambda^J x_3^J$. It can be checked that without the non-negativity constraint on the Marshallian money, there is a unique CE in $\mathcal{E}_{\lambda\alpha}$ given by

$$\left\{ \begin{array}{l} \bar{p}_\lambda = (\frac{1}{\lambda^I}, \frac{1}{\lambda_J}, 1) \\ \bar{x}^I = (40 - 10\ln\frac{11\lambda^I}{\lambda^J}, 10\ln\frac{10\lambda^J}{\lambda^I}, \alpha^I + \frac{10}{\lambda^I}\ln\frac{11\lambda^I}{\lambda^J} - \frac{10}{\lambda^J}\ln\frac{10\lambda^J}{\lambda^I}) \\ \bar{x}_\lambda^J = (10\ln\frac{11\lambda^I}{\lambda^J}, 50 - 10\ln\frac{10\lambda^J}{\lambda^I}, \alpha^J + \frac{10}{\lambda^J}\ln\frac{10\lambda^J}{\lambda^I} - \frac{10}{\lambda^I}\ln\frac{11\lambda^I}{\lambda^J}). \end{array} \right. \tag{15}$$

Thus, $\mathcal{E}_{\lambda\alpha}$ has a unique CE whenever the Marshallian money allocation in (15) is non-negative, which holds if and only if

$$\text{(ii)} \;\; \alpha^I \geq \max\left\{0, \frac{10}{\lambda^J}\ln\frac{10\lambda^J}{\lambda^I} - \frac{10}{\lambda^I}\ln\frac{11\lambda^I}{\lambda^J}\right\}, \;\; \alpha^J \geq \max\left\{0, \frac{10}{\lambda^I}\ln\frac{11\lambda^I}{\lambda^J} - \frac{10}{\lambda^J}\ln\frac{10\lambda^J}{\lambda^I}\right\}.$$

Since there are three CEs without any Marshallian money, it follows that given $\lambda \in \Re_{++}^N$ satisfying (i), the number of CEs in $\mathcal{E}_{\lambda\alpha}$ reduces from three to one as α is increased to eventually satisfy (ii). Figure 2 provides an illustration.

Figure 2: The Pareto surface and transfer plane.

5 Salvage Values with Fiat and Marshallian Money

A fundamental difference between the economies with fiat or Marshallian money is that in the former the money does not provide a physical static store of value, in the later it does. Fiat money is supported by default laws and expectations. Its store of value features are in the dynamics (Bak, Norrelykke and Shubik [1]). In contrast Marshallian money is a direct store of value, reflected in the static model of equilibrium

[7] Simple calculation shows that these ratios are the rates of utility transfers implied by Pareto optimal allocations without any Mashallian money.

5.1 Clearinghouse credit, redistribution and salvage values

5.1.1 The selection of arbitrary weights: Competitive equilibria and individual endowments

In conventional general equilibrium theory initial resources for each individual and preferences are given. The Lagrangian multipliers λ^i associated with a CE are calculated. Above we also have considered a somewhat different problem. Suppose that the utility comparison weights μ^i (or penalties) are given, as are the total endowments a_j of each commodity j, but the individual endowments a_j^i are not given.

Each individual may be regarded as having a utility function of the form:

$$U_\mu^i(x_1^i, x_2^i, ...x_{m+1}^i) = u^i(x_1^i, x_2^i, ...x_m^i) + \mu^i x_{m+1}^i$$

where, with accounting credit only, the last term is a positive or negative amount of final credit.[8] We need to distinguish two cases: (1) Clearinghouse credit with individual endowments not given and (2) Clearinghouse credits with given individual endowments.

The distinction between an economy with full clearinghouse credit and one with "enough" fiat money well distributed must be made. The price level in an economy depends on volume of trade, velocity of transactions, and whether or not there is a physically defined specific amount of money and credit in the economy. The default conditions given by the μ^i place a lower bound on prices without default, as the lower the prices sink, the more attractive default becomes.

The upper bound on prices is defined by the amount of money and credit in the system and the transactions technology. There are two intuitively reasonable ways to construct a mathematically precise model, we can consider a well defined extreme case where all goods must be sold (sell-all); or a buy-sell model where individuals either buy or sell in each market. In the first instance trade will satisfy a cash-flow constraint of

$$\sum_{j=1}^{m} p_j x_j^i \leq a_{m+1}^i$$

In the second instance the cash flow constraint will be:

$$\sum_{j=1}^{m} p_j \max[x_j^i - a_j^i, 0] \leq a_{m+1}^i.$$

[8]The possibility of x_{m+1}^i being positive reflects expectations for its use in a future period. The negative reflects the severity of the default penalty. The dimensions of the penalty are utility/(unit of money).

The clear upper bound is provided by the sell-all model. The institutionally close model is where individuals can sell or buy more of their assets.

5.1.2 Clearinghouse credit with unknown individual endowments

A reasonable assumption is that at most all that is known is the aggregate amount of goods held in the economy. Thus if we wish to make sure that there is enough money around to be able to achieve efficient trade when only aggregates are given we need to consider what distribution of assets will generate the largest amount of trade. The sell-all model provides the extreme upper bound, but if the buy-sell model is considered the volume of trade will depend on the distribution of the assets and will be generically less than the sell-all.

5.2 Marshallian money, and redistribution

5.2.1 No money market

When the money is a physical commodity we may consider trade with or without a money market or banking system. If we specify that all payments must be made in u-gold and we have specified it as the numeraire with a price of $p_{m+1} = 1$ and there is no creation or introduction of outside money or credit via a banking system, international trade or a central bank then the quantity of money is well defined. The specification of the quantity of money combined with sell-all and a velocity of 1 is sufficient to specify a price level. Given the price level and the amount of trade we can calculate the amount of money for each individual required to achieve a CE. We require that

$$\sum_{j=1}^{m} p_j a_j^i \leq a_{m+1}^i$$

for all i.

5.2.2 A money market with enough money badly distributed

The meaning of enough money badly distributed is illustrated easily. If at price vector p

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_j a_j^i \leq \sum_{i=1}^{n} a_{m+1}^i,$$

but there is some i for whom

$$\sum_{j=1}^{m} p_j a_j^i > a_{m+1}^i,$$

then the economy as a whole has enough money yet individual i has cash shortage. Notice that this maldistribution can be resolved by introducing a money market with a rate of interest of $\rho = 0$.

6 The Core and Its Convergence with Marshallian Money

Given $(\lambda, \alpha) \in \Re_{++}^N \times \Re_+^N$, we now turn to modeling of the core with Marshallian money. For $S \subseteq N$, let $x(S, \lambda) = (x^i(S, \lambda))_{i \in S}$ denote a weighted welfare maximum for coalition S in $\mathcal{E}$ with welfare weights $1/\lambda^i$, $i \in S$.

Lemma 2 *Let $\mathcal{E} = \{X^i, u^i, a^i\}_{i \in N}$ be an economy satisfying A1-A4 and let $\lambda \in \Re_{++}^N$. Then, for any $\alpha \in \Re_+^N$ satisfying (13) and for any coalition $S \subseteq N$, its Pareto optimal (PO) and individually rational (IR) allocations in $\mathcal{E}_{\lambda\alpha}$ are composed of the commodity allocation $x(S, \lambda)$ and reallocations of Marshallian money endowments α^i, $i \in S$.*

Proof. Let $(\bar{y}^i, \bar{y}_{m+1}^i)_{i \in S}$ be a PO and IR allocation for coalition S. Suppose $(\bar{y}^i)_{i \in S} \neq (x^i(S, \lambda))_{i \in S}$. For each $i \in S$, define T^i by

$$T^i = \frac{u^i(\bar{y}^i) - u^i(x^i(S, \lambda))}{\lambda^i}. \tag{16}$$

Since $(\bar{y}^i, \bar{y}_{m+1}^i)_{i \in S}$ is IR, we have

$$u^i(\bar{y}^i) + \lambda^i \bar{y}_{m+1}^i \geq u^i(a^i) + \lambda^i \alpha^i, \ i \in N$$

which implies

$$\bar{y}_{m+1}^i \geq \alpha^i - \frac{u^i(\bar{y}^i) - u^i(a^i)}{\lambda^i}, \ i \in S. \tag{17}$$

By (16) and (17),

$$\bar{y}_{m+1}^i + T^i \geq \alpha^i - \frac{u^i(x^i(S, \lambda)) - u^i(a^i)}{\lambda^i}, \ i \in S. \tag{18}$$

It follows from (13) and (18) that $\bar{y}^i_{m+1} + T^i \geq 0$ for all $i \in S$.[9] Thus, $y^i_{m+1} + T^i \in X^i_{m+1} = \Re_+$ for all $i \in S$. Next, since $x(S,\lambda)$ is a welfare maximum for welfare function $\sum_{i\in S} u^i(x^i)/\lambda^i$, by (16),

$$\sum_{i\in S} T^i = \sum_{i\in S} \frac{1}{\lambda^i} u^i(\bar{y}^i) - \sum_{i\in S} \frac{1}{\lambda^i} u^i(x^i(S,\lambda)) < 0. \tag{19}$$

Now, let

$$\bar{x}^i_{m+1} = \bar{y}^i_{m+1} + T^i - \frac{\sum_{j\in S} T^j}{s},$$

where s is the number of traders in S. By (16) and (19),

$$u^i(x^i(S,\lambda)) + \lambda^i \bar{x}^i_{m+1} = u^i(\bar{y}^i) + \lambda^i \bar{y}^i_{m+1} - \lambda^i \frac{\sum_{j\in S} T^j}{s} > u^i(\bar{y}^i) + \lambda^i \bar{y}^i_{m+1},\ i \in S.$$

This contradicts the assumption that $(\bar{y}^i, \bar{y}^i_{m+1})_{i\in S}$ is PO for coalition S. ■

Lemma 2 implies that given $\lambda \in \Re^N_{++}$, condition (13) also provides a lower bound on the endowment of Marhsallian money for each trader that guarantees that the IR portion of the Pareto suffice is flat for each coalition.

Definition 1 *Given $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_+$, the λ-core of $\mathcal{E} = \{X^i, u^i, a^i\}_{i\in N}$ is the core of $\mathcal{E}_{\lambda\alpha}$.*

An allocation in the λ-core of $\mathcal{E}$ is not improvable by any coalition S, even if members of the coalition can transfer utilities via the Marshallian money at rates determined by λ^i, $i \in S$. A direct application of Lemma 2 establishes

Theorem 4 *Let $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_+$ be such that α satisfies (13). Then a utility vector u is in the λ-core of $\mathcal{E}$ if and only if there exists a distribution $(x^i_{m+1})_{i\in N}$ of α such that*

$$u^i = u^i(x^i(N,\lambda)) + \lambda^i x^i_{m+1}, i \in N \tag{20}$$

and for any coalition $S \subseteq N$

$$\sum_{i\in S} \frac{1}{\lambda^i} u^i \geq \sum_{i\in S} \frac{1}{\lambda^i} u^i(x^i(S,\lambda)) + \sum_{i\in S} \alpha^i. \tag{21}$$

[9]Notice that the set of commodity bundles trader $i \in S$ receives from feasible allocations for coalition S is no bigger than $\bar{X}^i$.

Proof. Let $(\lambda, \alpha) \in \Re^N_{++} \times \Re^N_+$ satisfying (13) and let utility vector u be in the λ-core. Then, u must be PO and IR. By Lemma 2, there exists a reallocation $(x^i_{m+1})_{i\in N}$ of the endowments α such that (20) is satisfied. For coalition S, if (21) is violated, then Lemma 2 implies that u is below the the PO and IR portion of the utility set of coalition S in $\mathcal{E}_{\lambda\alpha}$. This means that coalition S can improve upon u in $\mathcal{E}_{\lambda\alpha}$, which contradicts the assumption that u is in the core of $\mathcal{E}_{\lambda\alpha}$.

Conversely, the utility vector u in (20) is feasible for the grand coalition in $\mathcal{E}_{\lambda\alpha}$. On the other hand, by (21), no coalition can improve upon u in $\mathcal{E}_{\lambda\alpha}$. This concludes that u is in the core of $\mathcal{E}_{\lambda\alpha}$. ■

Given $\lambda \in \Re^N_{++}$, when the Marshallian money endowments satisfy condition (13), Theorem 4 shows that the core of $\mathcal{E}$ with Marshallian money (in utility space) coincides with the core obtained when utilities are unlimitedly transferable at rates determined by the marginal utilities of the Marshallian money. Consequently, condition (13) can be considered as a condition of enough money which is well-distributed for the market game of $\mathcal{E}_{\lambda\alpha}$ to be transferable.

A direct application of Theorems 3 and 4 together with the usual core convergence theorem establishes the following corollary.

Corollary 1 *Assume $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i\in N}$ satisfies A1, A3, and A4. Assume further A2′: u^i is continuous, strictly concave, and strongly monotonically increasing. Then, for any $\lambda \in \Re^m_{++}$, the λ-core of $\mathcal{E} = \{(X^i, u^i, a^i)\}_{i\in N}$ converges to the unique CE of $\mathcal{E}_{\lambda\alpha}$ in the process of replication, for all endowments $\alpha \in \Re^N_+$ satisfying (13) for all i.*

7 Concluding Remarks

7.1 Comments on C-games

A c-game is a game that is well represented by the characteristic function form. An example of a c-game is a game with orthogonal coalitions, i.e. as soon as a set of players S has decided to act together, what $N - S$ does is irrelevant (see [15]).

The link between the competitive equilibrium analysis and market games has been made utilizing the concept of a Walrasian Market Game, which differs from the Shapley and Shubik [9], [10] definition of a market game. This definition made use of Edgeworth's observation that a group S of traders could trade among themselves regardless of the actions of the complementary group $N - S$. The Walrasian Market Game has the extra condition requiring that all individuals must trade at a single price. Scarf [6] noted the difficulty in linking games in normal form with games in cooperative form without discussing any explicit role for money or markets. His approach was to consider the distinctions between the $\alpha - core$ and the $\beta - core$. Considerably earlier Viner [19] had described the influence of the linking of markets as a "pecuniary externality". Shubik [13] observed that Viner's comments could be interpreted in game theoretic terms as weakening the C-game property.

7.2 Comments on market games and strategic market games

In an explicit development of a noncooperative approach to studying the price system Shubik [14], Shapley and Shubik [11] and Shapley [8] developed a class of games in strategic form called strategic market games. In a separate paper we address the modeling problems in linking the normal form of the strategic market games to the cooperative market games. Here we sidestep the basic problem by making the assumption that we can adequately model the role of markets and money by modifying the conventional Edgeworth type of approach to market games, considering the coalition S to be orthogonal from $N - S$, although requiring all trading arrangements to take place at a single price. We call these games Walrasian Market Games. We further introduce the role of money by requiring that the optimization satisfy not only wealth constraints, but cash flow constraints. When these constraints are introduced we have Money Market Games. We justify this by arguing that as a reasonable first approximation the existence of the core will depend on large coalitions and these are almost orthogonal, i.e. if S is almost the size of N then $N - S$ scarcely influences S. Heuristically we can consider a coalition of size $n - 1$ and show that it generates at worst ($\max\min$) an inefficient imputation $\varepsilon(k)$ away from the CE where as the replication k grows it becomes arbitrarily close to the CE but does not attain it.

7.3 The Edgeworth, Walras, money, money market, and bank cores

The Edgeworth core provided a means to show the emergence of price system. The Walras core analysis takes an axiom of one price as given. It would seem that by Occam's razor adding such an axiom is unnecessary. However it serves as means to start to reconcile market games with strategic market games and to enable us to both maintain a high level of abstraction, while catching the essence of a monetary economy with cash flow constraints. This in turn leads to being able to define and analyze not only a money core but games involving a money market or a central bank. Here we have constrained the analysis to the simplest case of a side-payment money core.

7.4 Institutions and invariant properties

The general equilibrium analysis is presented at a high level of abstraction, but the cost of the assumptions made was that no process model was specified with the existence proof of the efficient price system. The strategic market game formulation offers a process model, and, if constrained to minimal institutional forms presents a reasonably parsimonious set of models. However as soon as the extensive form involves more than one or two players there is an enormous proliferation of special cases, many of which can be associated with the myriads of financial institutions and instruments which have evolved since the emergence of organized economies. The market game formulation is at a higher level of abstraction (there is a many to one mapping from exchange economies to market games). The Walrasian and cash flow modifications to the market game provide an opportunity to encompass the properties of the cash flow constraints and the role of money and credit without the institutional detail, but in such a manner that the connection to the strategic market games and their institutional richness can be made.

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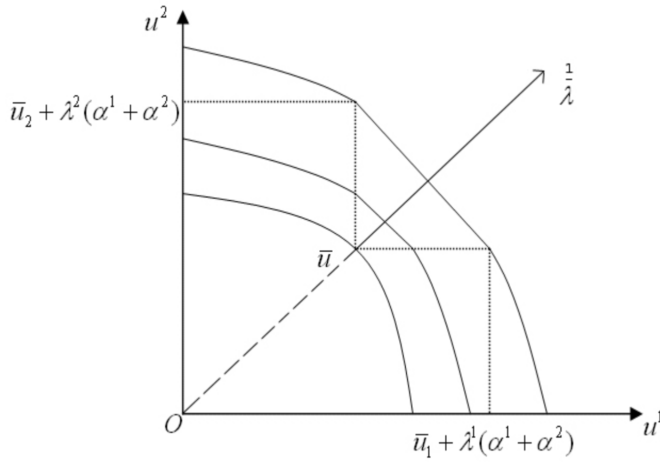

Figure 1: The Flattening of the Pareto Surface.

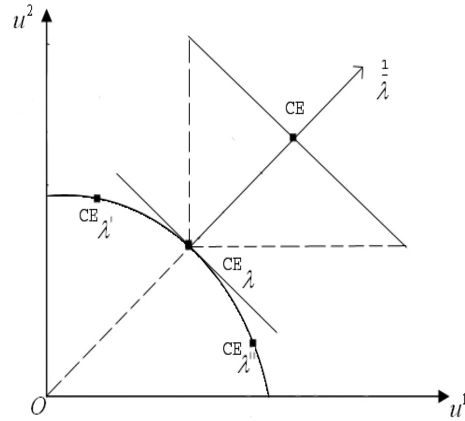

Figure 2

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