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**The Analytics of New Keynesian  
Phillips Curves**

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# The Analytics of New Keynesian Phillips Curves

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## Abstract

This paper introduces the reader into the apparatus behind the popular New Keynesian Phillips (NKPC) curve. It derives several log-linear versions of this curve and recursive formulations of the Calvo-Yun price staggering model that is behind this curve. These formulations can be used for higher-order approximations of the NKPC or for implementations that use other non-linear solution techniques, as, e.g., projection methods.

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# 1 Basic Framework

## 1.1 Production Functions

There is a continuum of firms indexed by  $j \in [0, 1]$ . The demand function of firm  $j$  is

$$Y_{jt} = \left( \frac{P_{jt}}{P_t} \right)^{-\epsilon} Y_t, \quad \epsilon > 1, \quad (1.1)$$

where  $P_{jt}$ ,  $P_t$ , and  $Y_t$  denote the firm's price, the aggregate price level, and aggregate output, respectively. The price index is given by

$$P_t = \left( \int_0^1 P_{jt}^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}. \quad (1.2)$$

The demand function (1.1) derives from minimizing the costs to purchase the bundle  $P_t Y_t = \int_0^1 P_{jt} Y_{jt} dj$ , where

$$Y_t = \left( \int_0^1 Y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}. \quad (1.3)$$

The production function is either

$$Y_{jt} = Z_t N_{jt}^{1-\alpha} \quad \alpha \in (0, 1] \quad (1.4)$$

or

$$Y_{jt} = Z_t N_{jt}^{1-\alpha} K_{jt}^{\alpha} \quad \alpha \in (0, 1). \quad (1.5)$$

In the first case (considered for instance by Galí et al. (2001)) labor  $N_{jt}$  is the single factor of production. In the second case (considered for instance by Heer and Maußner (2009) or Christiano et al. (2005)) capital services  $K_{jt}$  are an additional factor of production.  $Z_t$  is a productivity shock common to all firms. Cost minimization at the given real wage  $w_t$  implies

$$w_t = (1 - \alpha) g_{jt} Z_t N_{jt}^{-\alpha} \quad (1.6)$$

in the case of production function (1.4) and

$$w_t = g_{jt} (1 - \alpha) Z_t (K_{jt}/N_{jt})^{\alpha}, \quad (1.7a)$$

$$r_t = g_{jt} \alpha Z_t (K_{jt}/N_{jt})^{\alpha-1} \quad (1.7b)$$

in the case of (1.5).

There is an important difference between the two settings. In the second case the first-order conditions (1.7) ensure that all firms choose the same capital-labor ratio  $\kappa_t := K_{jt}/N_{jt} = K_t/N_t$ . Hence, all firms have the same marginal costs  $g_t = g_{jt} \forall j \in [0, 1]$ . This does not hold in the case of production function (1.4) unless  $\alpha = 1$ .

## 1.2 Aggregation

Aggregate output in this economy is given by (1.3). However, this will not allow us to define output in terms of sums of factor inputs. Yun (1996) proposes to use a second price index  $\tilde{P}_t$  defined by

$$\tilde{P}_t = \left( \int_0^1 P_{jt}^{-\epsilon} dj \right)^{-\frac{1}{\epsilon}} \quad (1.8)$$

so that

$$\tilde{Y}_t := \int_0^1 Y_{jt} dj \stackrel{(1.1)}{=} \left( \frac{P_{jt} P_t}{Y_t} \right) = \left( \frac{P_t}{\tilde{P}_t} \right)^\epsilon Y_t. \quad (1.9)$$

In the case of production function (1.5) this allows us to relate  $Y_t$  to aggregate labor input  $N_t = \int_0^1 N_{jt} dj$  and aggregate capital input  $K_t = \int_0^1 K_{jt} dj$ : since  $k_{jt} = (K_{jt}/N_{jt}) = k_t$  for all firms, we get

$$\tilde{Y}_t = \int_0^1 Y_{jt} dj = \int_0^1 Z_t N_{jt} k_t^\alpha dj = Z_t N_t k_t^\alpha = Z_t N_t^{1-\alpha} K_t^\alpha.$$

In addition, we can rewrite equation (1.7) in terms of aggregate variables:

$$w_t = g_t(1 - \alpha) Z_t N_t^{-\alpha} K_t^\alpha, \quad (1.10a)$$

$$r_t = g_t \alpha Z_t N_t^{1-\alpha} K_t^{\alpha-1}. \quad (1.10b)$$

It is not possible to follow the same procedure in the case of production function (1.4). Since

$$\tilde{Y}_t = \int_0^1 Y_{jt} dj = \int_0^1 Z_t N_{jt}^{1-\alpha} dj \neq Z_t N_t^{1-\alpha},$$

we define

$$\tilde{N}_t^{1-\alpha} = \int_0^1 N_{jt}^{1-\alpha} dj \quad (1.11)$$

so that

$$\tilde{Y}_t = Z_t \tilde{N}_t^{1-\alpha}. \quad (1.12)$$

Accordingly, we define aggregate marginal costs  $\tilde{g}$  by

$$w_t = \tilde{g}_t(1 - \alpha) Z_t \tilde{N}_t^{-\alpha}. \quad (1.13)$$

This allows us to relate the marginal costs of firm  $j$  to our measure of average marginal costs  $\tilde{g}_t$ : from (1.6) and (1.13):

$$\frac{g_{jt}}{\tilde{g}_t} \left( \frac{N_{jt}}{\tilde{N}_t} \right)^{-\alpha}.$$

Using (1.1) and the aggregate production function (1.12) to substitute for  $N_{jt}/\tilde{N}_t$  we can write:

$$g_{jt} = \tilde{g}_t \left( \frac{P_{jt}}{P_t} \right)^{\frac{-\alpha\epsilon}{1-\alpha}} \left( \frac{Y_t}{\tilde{Y}_t} \right)^{\frac{\alpha}{1-\alpha}}. \quad (1.14)$$

### 1.3 Price Setting

In each period  $(1 - \varphi)$  of the firms are allowed to set their relative price  $P_{jt}/P_t$  optimally. Henceforth we use the index  $A$  to refer to these firms. The remaining fraction of firms, indexed by  $N$ , adjusts their price according to a rule of thumb. We consider two rules. The first rule implies a forward-looking Phillips curve. We assume,

$$P_{Nt+1} = \pi P_{Nt}, \quad \pi_t := \frac{P_t}{P_{t-1}}, \quad (1.15a)$$

where  $\pi_t$  is the inflation factor (1 plus the rate of inflation) and  $\pi$  its value in a non-stochastic stationary equilibrium. Note that with zero inflation (i.e.  $\pi = 1$ ) these firms do not change their nominal price. The second rule (used in Christiano et al. (2005) and Walsh (2005)) accounts for the backward-looking element in the Phillips curve. It posits

$$P_{Nt+1} = \pi_t P_{Nt}. \quad (1.15b)$$

Since  $1 - \varphi$  firms choose  $P_{jt} = P_{At}$  and the remaining fraction sets  $P_{jt} = P_{Nt}$ , the formula for the price index (1.2) implies

$$P_t^{1-\epsilon} = (1 - \varphi)P_{At}^{1-\epsilon} + \varphi P_{Nt}^{1-\epsilon}. \quad (1.16)$$

In the case of the first rule of thumb this implies

$$P_t^{1-\epsilon} = (1 - \varphi)P_{At}^{1-\epsilon} + \varphi(\pi P_{Nt-1})^{1-\epsilon}. \quad (1.17a)$$

For the second rule we get

$$P_t^{1-\epsilon} = (1 - \varphi)P_{At}^{1-\epsilon} + \varphi(\pi_{t-1}P_{Nt-1})^{1-\epsilon}. \quad (1.17b)$$

Since  $P_{Nt-1}$  is itself an index of the prices of those firms that adjusted their price in  $t - 2$  optimally and those firms that obeyed to a rule of thumb,

$$P_{Nt-1}^{1-\epsilon} = (1 - \varphi)P_{At-1}^{1-\epsilon} + \varphi(\pi_{t-2}P_{Nt-2})^{1-\epsilon}, \quad (1.18)$$

we can derive a recursive formulation for the price index. I demonstrate this for the updating scheme (1.15b):

$$\begin{aligned}
P_t^{1-\epsilon} &= \varphi P_{At}^{1-\epsilon} + \varphi(\pi_{t-1} P_{Nt-1})^{1-\epsilon}, \\
&\stackrel{(1.18)}{=} \varphi P_{At}^{1-\epsilon} + \varphi(1-\varphi)(\pi_{t-1} P_{At-1})^{1-\epsilon} + \varphi^2(\pi_{t-1}\pi_{t-2} P_{Nt-2})^{1-\epsilon}, \\
&\stackrel{(1.18)}{=} \varphi P_{At}^{1-\epsilon} + \varphi(1-\varphi)(\pi_{t-1} P_{At-1})^{1-\epsilon} + \varphi^2(1-\varphi)(\pi_{t-1}\pi_{t-2} P_{At-2})^{1-\epsilon} + \dots
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\pi_{t-1} P_{t-1})^{1-\epsilon} &= \varphi(\pi_{t-1} P_{At-1})^{1-\epsilon} + \varphi^2(1-\varphi)(\pi_{t-1}\pi_{t-2} P_{At-2})^{1-\epsilon} \\
&\quad + \varphi^3(1-\varphi)(\pi_{t-1}\pi_{t-2}\pi_{t-3} P_{At-3})^{1-\epsilon} + \dots,
\end{aligned}$$

and, thus,

$$P_t^{1-\epsilon} = (1-\varphi)P_{At}^{1-\epsilon} + \varphi(\pi_{t-1} P_{t-1})^{1-\epsilon}. \quad (1.19a)$$

Similarly, we can derive a recursive formulation of equation (1.17a):

$$P_t^{1-\epsilon} = (1-\varphi)P_{At}^{1-\epsilon} + \varphi(\pi P_{t-1})^{1-\epsilon}. \quad (1.19b)$$

In the case of rule (1.15a) this implies the following relation between the relative price of firms that optimally adjust their price and the inflation factor :

$$1 = (1-\varphi)(P_{At}/P_t)^{1-\epsilon} + \varphi(\pi/\pi_t)^{1-\epsilon}. \quad (1.20a)$$

In the case of rule (1.15b) this relation is

$$1 = (1-\varphi)(P_{At}/P_t)^{1-\epsilon} + \varphi(\pi_{t-1}/\pi_t)^{1-\epsilon}. \quad (1.20b)$$

The same line of reasoning applied to

$$q_t := \left( \frac{\tilde{P}_t}{P_t} \right)^{-\epsilon} = \int_0^1 \left( \frac{P_{jt}}{P_t} \right)^{-\epsilon} dj = (1-\varphi)(P_{At}/P_t) + \varphi(P_{Nt}/P_t)$$

yields:

$$q_t = (1-\varphi)(P_{At}/P_t)^{-\epsilon} + \varphi(\pi_t/\pi)^\epsilon q_{t-1}, \quad (1.21a)$$

and

$$q_t = (1-\varphi)(P_{At}/P_t)^{-\epsilon} + \varphi(\pi_t/\pi_{t-1})^\epsilon q_{t-1}, \quad (1.21b)$$

respectively.

Given  $q_t$  we can write the aggregate resource constraint as

$$Y_t = \frac{1}{q_t} Z_t N_t^{1-\alpha} K_t^\alpha \quad (1.22)$$

if the production function is given by (1.5) or as

$$Y_t = \frac{1}{q_t} Z_t \tilde{N}_t^{1-\alpha}, \quad (1.23)$$

if the production function is given by (1.4). Note that our measure of aggregate labor input is related to  $N_t = \int_0^1 N_{jt} dj$ ,  $N_{At}$  and  $N_{Nt}$  via:

$$\begin{aligned} \tilde{N}_t^{1-\alpha} &= (1 - \varphi) N_{At}^{1-\alpha} + \varphi N_{Nt}^{1-\alpha}, \\ N_{Nt} &= \frac{N_t - (1 - \varphi) N_{At}}{\varphi}. \end{aligned} \quad (1.24)$$

## 2 The Optimal Price

### 2.1 Preliminaries

Now consider a firm in period  $t$  that is allowed to set its price optimally at  $P_{At}$ . As long as the firm will not be able to optimize again, its price in period  $t + s$ ,  $s = 1, 2, \dots$  is related to  $P_{At}$  according to

$$P_{jt+s} = \pi^s P_{At}, \quad (2.1a)$$

$$P_{jt+s} = \prod_{i=1}^s \pi_{t+i-1} P_{At}, \quad (2.1b)$$

where the first equation holds for rule (1.15a) and the second equation rests on rule (1.15b). Note that the aggregate price level can be written as

$$P_{t+s} = \prod_{i=1}^s \pi_{t+i} P_t. \quad (2.2)$$

and, thus, the relative price is either given by

$$\frac{P_{jt+s}}{P_{t+s}} = \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \frac{P_{At}}{P_t}, \quad (2.3a)$$

or by

$$\frac{P_{jt+s}}{P_{t+s}} = \frac{\pi_t}{\pi_{t+s}} \frac{P_{At}}{P_t}. \quad (2.3b)$$



## 2.2 First-Order Conditions

The profit per unit of output in terms of the aggregate price level equals

$$\begin{aligned} G_{jt+s} &= \frac{P_{jt+1}}{P_{t+s}} Y_{jt+s} - C(Y_{jt+s}), \\ &= \left( \frac{P_{jt+1}}{P_{t+1}} \right)^{1-\epsilon} Y_t - C \left( \left( \frac{P_{jt+1}}{P_{t+1}} \right)^{-\epsilon} Y_t \right), \end{aligned} \quad (2.4)$$

where  $C(\cdot)$  is the cost function with derivative  $c'(\cdot) = g_{jt+s}$ . Differentiating this function with respect to  $P_{At}/P_t$  yields:

$$\begin{aligned} \frac{\partial G_{jt+s}}{\partial P_{At}/P_t} &= \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} Y_{jt+s} - \epsilon \left( \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \frac{P_{At}}{P_t} - g_{jt+s} \right) \frac{Y_{jt+s}}{P_{At}/P_t}, \\ &= \frac{1}{P_{At}/P_t} \left( (1-\epsilon) \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \frac{P_{At}}{P_t} + \epsilon g_{jt+s} \right) Y_{jt+s}, \\ &= \frac{1-\epsilon}{P_{At}/P_t} \left( \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \frac{P_{At}}{P_t} - \frac{\epsilon}{\epsilon-1} g_{jt+s} \right) Y_{jt+s}, \end{aligned} \quad (2.5a)$$

if  $P_{jt+s}/P_{t+s}$  is given by (2.3a) and

$$\begin{aligned} \frac{\partial G_{jt+s}}{\partial P_{At}/P_t} &= \frac{\pi_t}{\pi_{t+s}} Y_{jt+s} - \epsilon \left( \frac{\pi_t P_{At}}{\pi_{t+s} P_t} - g_{jt+s} \right) \frac{Y_{jt+s}}{P_{At}/P_t}, \\ &= \frac{1}{P_{At}/P_t} \left( (1-\epsilon) \frac{\pi_t P_{At}}{\pi_{t+s} P_t} + \epsilon g_{jt+s} \right) Y_{jt+s}, \\ &= \frac{1-\epsilon}{P_{At}/P_t} \left( \frac{\pi_t P_{At}}{\pi_{t+s} P_t} - \frac{\epsilon}{\epsilon-1} g_{jt+s} \right) Y_{jt+s}. \end{aligned} \quad (2.5b)$$

if  $P_{jt+s}/P_{t+s}$  equals (2.3b). The firm chooses  $P_{At}/P_t$  to maximize the discounted stream of profits:

$$\max_{P_{At}/P_t} \mathbb{E}_t \sum_{s=0}^{\infty} \varphi^s \varphi_{t+s} G_{jt+s}, \quad (2.6)$$

where  $\varphi_t$  denotes the discount factor. The first order condition for this problem is:

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} \varphi^s \varphi_{t+s} \frac{\partial G_{jt+s}}{\partial P_{At}/P_t}. \quad (2.7)$$

Since the common non-stochastic term  $\frac{1-\epsilon}{P_{At}/P_t}$  in (2.5a) and (2.5b) can be canceled in (2.7) we obtain

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} \varphi^s \varphi_{t+s} \left( \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \frac{P_{At}}{P_t} - \frac{\epsilon}{\epsilon-1} g_{jt+s} \right) Y_{jt+s}, \quad (2.8a)$$

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} \varphi^s \varphi_{t+s} \left( \frac{\pi_t P_{At}}{\pi_{t+s} P_t} - \frac{\epsilon}{\epsilon-1} g_{jt+s} \right) Y_{jt+s}. \quad (2.8b)$$

The household's Euler equation implies

$$\varphi_{t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda_t}. \quad (2.9)$$

for the stochastic discount factor. This allows us to simplify equations (2.8) further:

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \lambda_{t+s} Y_{jt+s} \left( \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \frac{P_{At}}{P_t} - \frac{\epsilon}{\epsilon-1} g_{jt+s} \right), \quad (2.10a)$$

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \lambda_{t+s} Y_{jt+s} \left( \frac{\pi_t P_{At}}{\pi_{t+s} P_t} - \frac{\epsilon}{\epsilon-1} g_{jt+s} \right), \quad (2.10b)$$

where we canceled  $\lambda_t$  (a non-stochastic variable from the point of view of period  $t$ ).

### 2.3 Recursive Formulation of the First-Order Conditions

It is convenient to replace the infinite sums in the first-order conditions (2.10a) and (2.10b) (see Schmitt-Grohe and Uribe (2004)). Consider condition (2.10a). If marginal costs are equal across firms, it can be rewritten as

$$\begin{aligned} \frac{P_{At}}{P_t} &= \frac{\mu \Gamma_{1t}}{\Gamma_{2t}}, \quad \mu := \frac{\epsilon}{\epsilon-1}, \\ \Gamma_{1t} &:= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \left( \frac{\pi^s P_{At}}{\prod_{i=1}^s \pi_{t+i} P_t} \right)^{-\epsilon} Y_{t+1} g_{t+s} \lambda_{t+s}, \\ \Gamma_{2t} &:= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} \left( \frac{\pi^s}{\prod_{i=1}^s \pi_{t+i}} \right)^{1-\epsilon} Y_{t+s} \lambda_{t+s}. \end{aligned} \quad (2.11)$$

Since

$$\begin{aligned} \Gamma_{1t} = \mathbb{E}_t \left\{ \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t g_t + (\beta\varphi) \left( \frac{\pi P_{At}}{\pi_{t+1} P_t} \right)^{-\epsilon} Y_{t+1} \lambda_{t+1} g_{t+1} \right. \\ \left. + (\beta\varphi)^2 \left( \frac{\pi^2 P_{At}}{\pi_{t+1} \pi_{t+2} P_t} \right)^{-\epsilon} Y_{t+2} \lambda_{t+2} g_{t+2} + \dots \right\}, \end{aligned} \quad (2.12)$$

we get

$$\begin{aligned} \Gamma_{1t+1} = \mathbb{E}_{t+1} \left\{ \left( \frac{P_{At+1}}{P_{t+1}} \right)^{-\epsilon} Y_{t+1} \lambda_{t+1} g_{t+1} + (\beta\varphi) \left( \frac{\pi P_{At+1}}{\pi_{t+2} P_{t+1}} \right)^{-\epsilon} Y_{t+2} \lambda_{t+2} g_{t+2} \right. \\ \left. + (\beta\varphi)^2 \left( \frac{\pi^2 P_{At+1}}{\pi_{t+2} \pi_{t+3} P_{t+1}} \right)^{-\epsilon} Y_{t+3} \lambda_{t+3} g_{t+3} + \dots \right\}. \end{aligned}$$

From the point of view of period  $t + 1$  all variables dated  $t + 1$  and earlier are non-stochastic and can be post-multiplied the expectations operator  $\mathbb{E}_{t+1}$ . Thus,

$$\begin{aligned} \beta\varphi \left( \frac{\pi(P_{At}/P_t)}{\pi_{t+1}(P_{At+1}/P_{t+1})} \right)^{-\epsilon} \Gamma_{1t+1} &= \mathbb{E}_{t+1} \left\{ (\beta\varphi) \left( \frac{\pi P_{At}}{\pi_{t+1} P_t} \right)^{-\epsilon} Y_{t+1} \lambda_{t+1} g_{t+1} \right. \\ &\quad \left. + (\beta\varphi)^2 \left( \frac{\pi^2 P_{At}}{\pi_{t+1} \pi_{t+2} P_t} \right)^{-\epsilon} Y_{t+2} \lambda_{t+2} g_{t+2} + \dots \right\} \end{aligned}$$

By the law of iterated expectations,  $\mathbb{E}_t \mathbb{E}_{t+1} \{\cdot\} = \mathbb{E}_t \{\cdot\}$  so that the right-hand side of (2.12) equals:

$$\Gamma_{1t} = \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t g_t + \beta\varphi \mathbb{E}_t \left( \frac{\pi(P_{At}/P_t)}{\pi_{t+1}(P_{At+1}/P_{t+1})} \right)^{-\epsilon} \Gamma_{1t+1}. \quad (2.13)$$

In the same way, we can derive a recursive definition of  $\Gamma_{2t}$ :

$$\Gamma_{2t} = \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t + \beta\varphi \mathbb{E}_t \left( \frac{P_{At}/P_t}{P_{At+1}/P_{t+1}} \right)^{-\epsilon} \left( \frac{\pi}{\pi_{t+1}} \right)^{1-\epsilon} \Gamma_{2t+1}. \quad (2.14)$$

Similarly, we can derive a recursive formulation of the first-order condition (2.10b):

$$\begin{aligned} \frac{P_{At}}{P_t} &= \frac{\mu \Gamma_{1t}}{\Gamma_{2t}}, \quad \mu := \frac{\epsilon}{\epsilon - 1}, \\ \Gamma_{1t} &:= \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t g_t + (\beta\varphi) \mathbb{E}_t \left( \frac{\pi_t(P_{At}/P_t)}{\pi_{t+1}(P_{At+1}/P_{t+1})} \right)^{-\epsilon} \Gamma_{1t+1}, \\ \Gamma_{2t} &:= \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t + \beta\varphi \mathbb{E}_t \left( \frac{P_{At}/P_t}{P_{At+1}/P_{t+1}} \right)^{-\epsilon} \left( \frac{\pi_t}{\pi_{t+1}} \right)^{1-\epsilon} \Gamma_{2t+1}. \end{aligned} \quad (2.15)$$

In the case of the production function (1.4) we substitute for  $g_{jt+s}$  from equation (1.14) and obtain from (2.10a):

$$\begin{aligned} \frac{P_{At}}{P_t} &= \frac{\mu \Gamma_{1t}}{\Gamma_{2t}}, \quad \mu := \frac{\epsilon}{\epsilon - 1}, \\ \Gamma_{1t} &:= \left( \frac{P_{At}}{P_t} \right)^{\frac{-\epsilon}{1-\alpha}} Y_t^{\frac{1}{1-\alpha}} \tilde{Y}_t^{\frac{-\alpha}{1-\alpha}} \lambda_t \tilde{g}_t + (\beta\varphi) \mathbb{E}_t \left( \frac{\pi(P_{At}/P_t)}{\pi_{t+1}(P_{At+1}/P_{t+1})} \right)^{\frac{-\epsilon}{1-\alpha}} \Gamma_{1t+1}, \\ \Gamma_{2t} &:= \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t + \beta\varphi \mathbb{E}_t \left( \frac{P_{At}/P_t}{P_{At+1}/P_{t+1}} \right)^{-\epsilon} \left( \frac{\pi}{\pi_{t+1}} \right)^{1-\epsilon} \Gamma_{2t+1}. \end{aligned} \quad (2.16)$$

and from (2.10b):

$$\begin{aligned} \frac{P_{At}}{P_t} &= \frac{\mu \Gamma_{1t}}{\Gamma_{2t}}, \quad \mu := \frac{\epsilon}{\epsilon - 1}, \\ \Gamma_{1t} &:= \left( \frac{P_{At}}{P_t} \right)^{\frac{-\epsilon}{1-\alpha}} Y_t^{\frac{1}{1-\alpha}} \tilde{Y}_t^{\frac{-\alpha}{1-\alpha}} \lambda_t \tilde{g}_t + (\beta\varphi) \mathbb{E}_t \left( \frac{\pi_t(P_{At}/P_t)}{\pi_{t+1}(P_{At+1}/P_{t+1})} \right)^{\frac{-\epsilon}{1-\alpha}} \Gamma_{1t+1}, \\ \Gamma_{2t} &:= \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t + \beta\varphi \mathbb{E}_t \left( \frac{P_{At}/P_t}{P_{At+1}/P_{t+1}} \right)^{-\epsilon} \left( \frac{\pi_t}{\pi_{t+1}} \right)^{1-\epsilon} \Gamma_{2t+1}. \end{aligned} \quad (2.17)$$

### 3 Log-Linear Equations

#### 3.1 The Stationary Solution

Consider the non-stochastic equilibrium with constant inflation factor  $\pi$ . Equations (1.20a) and (1.20b) imply  $P_{At}/P_t = 1$ . In addition,  $P_{Nt}/P_t = 1$  (see (1.15a) and (1.15b)). Thus all firms produce the same amount  $Y_{jt} = Y$  at the same marginal costs  $g_{jt} = g$ . In this case, both equation (2.10a) and (2.10b) imply

$$g = \frac{\epsilon - 1}{\epsilon}. \quad (3.1)$$

To embed any of our models of sticky prices into a linearized model we can linearize  $P_{At}/P_t = \mu\Gamma_{1t}/\Gamma_{2t}$  together with the respective recursive formulations. It has, however, become common practice to linearize (2.10a) or (2.10b) directly to get a Phillips curve equation that relates the current rate of inflation to expected future inflation, past inflation, and a measure of cost pressure.

#### 3.2 First Steps

When we linearize equation (2.10a) and (2.10b) at the stationary solution we can disregard the terms involving  $\hat{\lambda}_{t+s}$  and  $\hat{Y}_{jt+s}$  since these terms are multiplied by the term in square brackets that vanishes at the stationary solution. Let  $\hat{p}_t := \widehat{P_{At}/P_t}$ . Then the log-linear version of (2.10a) can be written as

$$\begin{aligned} 0 &= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \lambda Y \left[ \hat{p}_t - \sum_{i=1}^s \hat{\pi}_{t+i} - \underbrace{\frac{\epsilon}{\epsilon-1} g}_{=1} \hat{g}_{jt+s} \right], \\ \underbrace{\lambda Y \sum_{s=0}^{\infty} (\beta\varphi)^s}_{=1/(1-\beta\varphi)} \hat{p}_t &= \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \lambda Y \left[ \sum_{i=1}^s \hat{\pi}_{t+i} + \hat{g}_{jt+s} \right], \\ \hat{p}_t &= (1 - \beta\varphi) \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \left[ \sum_{i=1}^s \hat{\pi}_{t+i} + \hat{g}_{jt+s} \right]. \end{aligned} \quad (3.2a)$$

Similarly, we obtain the log-linear version of (2.10b):

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \lambda Y \left[ \hat{p}_t + \hat{\pi}_t - \hat{\pi}_{t+s} - \underbrace{\frac{\epsilon}{\epsilon-1} g \hat{g}_{jt+s}}_{=1} \right],$$

$$\lambda Y \underbrace{\sum_{s=0}^{\infty} (\beta\varphi)^s}_{=1/(1-\beta\varphi)} [\hat{p}_t + \hat{\pi}_t] = \lambda Y \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s [\hat{\pi}_{t+s} + \hat{g}_{jt+s}],$$

$$\hat{p}_t + \hat{\pi}_t = (1 - \beta\varphi) \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s [\hat{\pi}_{t+s} + \hat{g}_{jt+s}]. \quad (3.2b)$$

Log-linearizing equation (1.20a) at  $P_{At}/P_t = 1$  yields

$$\hat{p}_t = \frac{\varphi}{1-\varphi} \hat{\pi}_t. \quad (3.3a)$$

Furthermore

$$\widehat{(P_{Nt}/P_t)} = -\hat{\pi}_t, \quad (3.3b)$$

since  $P_{Nt}/P_t = \pi/\pi_t$  in the case of (1.15a). If non-optimizers change their price according to the rule of thumb in equation (1.15b) the relation between  $\hat{p}_t$  and the rate of inflation is given by

$$\hat{p}_t = \frac{\varphi}{1-\varphi} (\hat{\pi}_t - \hat{\pi}_{t-1}), \quad (3.3c)$$

and for  $P_{Nt}/P_t$  we obtain

$$\widehat{(P_{Nt}/P_t)} = \hat{\pi}_{t-1} - \hat{\pi}_t. \quad (3.3d)$$

Given these relations the log-linear version of both (1.21a) and (1.21b) reduce to

$$\hat{q}_t = \varphi \hat{q}_{t-1}. \quad (3.4)$$

Since we are free to choose the initial condition, it will be convenient to set  $\hat{q}_{t-1} = 0$  so that we can disregard this variable and can work with the log-linearized aggregate production function (1.5) and the respective market clearing conditions (1.10). This is also possible in the case of the production function (1.4), since log-linearizing (1.24) implies  $\hat{\tilde{N}}_t = \hat{N}_t$ . This demonstrates that the common practice not to distinguish between  $Y_t$  and  $N_t$  on the one hand side and  $\tilde{Y}_t$  and  $\tilde{N}_t$  on the other is valid in a linearized model. However, it is not justified to do so if higher order approximations of the model's equilibrium conditions are used. In this case one has to resort to the recursive formulations presented in the previous sections.

### 3.3 Forward Looking Phillips Curves

First, we consider the case where the marginal costs do not differ between optimizing and non-optimizing firms so that  $\hat{g}_{jt+s} = \hat{g}_{t+s} \forall j \in [0, 1]$ . From the point of view of period  $t + 1$  equation (3.2a) can be written as

$$\hat{p}_{t+1} = (1 - \beta\varphi)\mathbb{E}_{t+1} \sum_{s=0}^{\infty} (\beta\varphi)^s \left[ \sum_{i=1}^s \hat{\pi}_{t+i+1} + \hat{g}_{t+s+1} \right].$$

Taking expectations as of period  $t$  on both sides and noting that (by the law of iterated expectations)  $\mathbb{E}_t(\cdot) = \mathbb{E}_t\mathbb{E}_{t+1}(\cdot)$  provides

$$\mathbb{E}_t\hat{p}_{t+1} = (1 - \beta\varphi)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s \left[ \sum_{i=1}^s \hat{\pi}_{t+i+1} + \hat{g}_{t+s+1} \right].$$

Therefore,

$$\begin{aligned} \hat{p}_t - \beta\varphi\mathbb{E}_t\hat{p}_{t+1} &= \mathbb{E}_t \left\{ (1 - \beta\varphi) \left[ \hat{g}_t + (\beta\varphi)\hat{g}_{t+1} + (\beta\varphi)^2\hat{g}_{t+2} + \dots \right] \right. \\ &\quad - (1 - \beta\varphi) \left[ (\beta\varphi)\hat{g}_{t+1} + (\beta\varphi)^2\hat{g}_{t+2} + \dots \right] \\ &\quad + (1 - \beta\varphi) \left[ \beta\varphi\hat{\pi}_{t+1} + (\beta\varphi)^2(\hat{\pi}_{t+1} + \hat{\pi}_{t+2}) \right. \\ &\quad \left. + (\beta\varphi)^3(\hat{\pi}_{t+1} + \hat{\pi}_{t+2} + \hat{\pi}_{t+3}) + \dots \right] \\ &\quad \left. - (1 - \beta\varphi) \left[ (\beta\varphi)^2\hat{\pi}_{t+2} + (\beta\varphi)^3(\hat{\pi}_{t+2} + \hat{\pi}_{t+3}) + \dots \right] \right\} \\ &= \mathbb{E}_t \left\{ (1 - \beta\varphi)\hat{g}_t + \left[ (1 - \beta\varphi)\beta\varphi\hat{\pi}_{t+1} \left( 1 + \beta\varphi + (\beta\varphi)^2 + \dots \right) \right] \right\} \\ &= \mathbb{E}_t \left\{ (1 - \beta\varphi)\hat{g}_t + \beta\varphi\hat{\pi}_{t+1} \right\} \end{aligned} \tag{3.5}$$

Using equation (3.3a) to substitute for  $\hat{p}_t$  and  $\mathbb{E}_t\hat{p}_{t+1}$  in equation (3.5) we obtain

$$\frac{\varphi}{1 - \varphi}\hat{\pi}_t = (1 - \beta\varphi)\hat{g}_t + \beta\varphi \left( \frac{\varphi}{1 - \varphi} + 1 \right) \mathbb{E}_t\hat{\pi}_{t+1}$$

or

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \frac{(1 - \varphi)(1 - \beta\varphi)}{\varphi}\hat{g}_t. \tag{3.6}$$

This is the New Keynesian Phillips curve that appears in a substantial number of papers.

In case of  $g_{At} \neq g_{Nt}$  we use equations (1.14) and (2.3a) to eliminate  $\hat{g}_{jt+s}$  from (3.2a). Since (for ease of writing, I use  $\hat{g}_t \equiv \hat{g}_t$  in the following paragraphs)

$$\hat{g}_{jt+s} = \hat{g}_{t+s} - \underbrace{\frac{\alpha\epsilon}{1 - \alpha}}_{=:A} \left( \hat{p}_t - \sum_{i=1}^s \hat{\pi}_{t+i} \right),$$

we get

$$\begin{aligned}
\hat{p}_t - \beta\varphi\mathbb{E}_t\hat{p}_{t+1} &= (1 - \beta\varphi)\mathbb{E}_t\left\{\hat{g}_t - A\hat{p}_t + \beta\varphi\left(\hat{g}_{t+1} - A(\hat{p}_t - \hat{\pi}_{t+1})\right)\right. \\
&\quad + (\beta\varphi)^2\left(\hat{g}_{t+2} - A(\hat{p}_t - \hat{\pi}_{t+1} - \hat{\pi}_{t+2})\right) + \dots \\
&\quad - (\beta\varphi)\left(\hat{g}_{t+1} - A\hat{p}_{t+1}\right) - (\beta\varphi)^2\left(\hat{g}_{t+2} - A(\hat{p}_{t+1} - \hat{\pi}_{t+2})\right) \\
&\quad \left. - (\beta\varphi)^3\left(\hat{g}_{t+3} - A(\hat{p}_{t+1} - \hat{\pi}_{t+2} - \hat{\pi}_{t+3})\right) - \dots\right\} + \beta\varphi\mathbb{E}_t\hat{\pi}_{t+1}, \\
&= (1 - \beta\varphi)\mathbb{E}_t\left\{\hat{g}_t - A\underbrace{(1 + \beta\varphi + (\beta\varphi) + ^2 + \dots)}_{=1/(1-\beta\varphi)}\hat{p}_t\right. \\
&\quad + A\beta\varphi\underbrace{(1 + \beta\varphi + (\beta\varphi) + ^2 + \dots)}_{1/(1-\beta\varphi)}\hat{p}_{t+1} \\
&\quad \left. + A\beta\varphi\underbrace{(1 + \beta\varphi + (\beta\varphi) + ^2 + \dots)}_{1/(1-\beta\varphi)}\hat{\pi}_{t+1}\right\} + \beta\varphi\mathbb{E}_t\hat{\pi}_{t+1}.
\end{aligned}$$

Rearranging terms yields

$$\hat{p}_t(1 + A) = (1 - \beta\varphi)\hat{g}_t + \beta\varphi(1 + A)\mathbb{E}_t\hat{p}_{t+1} + \beta\varphi(1 + A)\mathbb{E}_t\hat{\pi}_{t+1}. \quad (3.7)$$

Using equation (3.3a) to substitute for  $\hat{p}_t$  finally delivers

$$\hat{\pi}_t = \frac{(1 - \varphi)(1 - \beta\varphi)(1 - \alpha)}{\varphi[1 + \alpha(\epsilon - 1)]}\hat{g}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}. \quad (3.8)$$

This is the forward looking Phillips curve that appears in Galí et al. (2001) and Sbordone (2002).

### 3.4 Forward and Backward Looking Phillips Curves

Assume  $\hat{g}_{j+t+s} = \hat{g}_{t+s}\forall j \in [0, 1]$ . Proceeding as in the previous section, equation (3.2b) implies

$$\mathbb{E}_t[\hat{p}_{t+1} + \hat{\pi}_{t+1}] = (1 - \beta\varphi)E_t \sum_{s=0}^{\infty} (\beta\varphi)^s [\hat{\pi}_{t+s+1} + \hat{g}_{t+s+1}].$$

Thus,

$$\begin{aligned}
\hat{p}_t + \hat{\pi}_t - \beta\varphi\mathbb{E}_t[\hat{p}_{t+1} + \hat{\pi}_{t+1}] &= \\
\hat{g}_t + \hat{\pi}_t + \beta\varphi[\hat{g}_{t+1} + \hat{\pi}_{t+1}] + (\beta\varphi)^2[\hat{g}_{t+2} + \hat{\pi}_{t+2}] + (\beta\varphi)^3[\hat{g}_{t+3} + \hat{\pi}_{t+3}] \dots \\
- \beta\varphi[\hat{g}_t + \hat{\pi}_t] - (\beta\varphi)^2[\hat{g}_{t+1} + \hat{\pi}_{t+1}] - (\beta\varphi)^3[\hat{g}_{t+2} + \hat{\pi}_{t+2}] - \dots \\
- \beta\varphi[\hat{g}_{t+1} + \hat{\pi}_{t+1}] - (\beta\varphi)^2[\hat{g}_{t+2} + \hat{\pi}_{t+2}] - (\beta\varphi)^3[\hat{g}_{t+3} + \hat{\pi}_{t+3}] - \dots \\
+ (\beta\varphi)^2[\hat{g}_{t+1} + \hat{\pi}_{t+1}] + (\beta\varphi)^3[\hat{g}_{t+2} + \hat{\pi}_{t+2}] + (\beta\varphi)^4[\hat{g}_{t+3} + \hat{\pi}_{t+3}] + \dots \\
&= (1 - \beta\varphi)[\hat{g}_t + \hat{\pi}_t]. \quad (3.9)
\end{aligned}$$

Rearranging yields:

$$\hat{p}_t - \beta\varphi\mathbb{E}_t\hat{p}_{t+1} = (1 - \beta\varphi)\hat{g}_t + \beta\varphi\mathbb{E}_t[\hat{\pi}_{t+1} - \hat{\pi}_t]. \quad (3.10)$$

Substituting from equation (3.3c) for  $\hat{p}_t$  and  $\mathbb{E}_t\hat{p}_{t+1}$  delivers

$$\frac{\varphi}{1 - \varphi}[\hat{\pi}_t - \hat{\pi}_{t-1}] - \frac{\beta\varphi^2}{1 - \varphi}\mathbb{E}_t[\hat{\pi}_{t+1} - \hat{\pi}_t] = (1 - \beta\varphi)\hat{g}_t + \beta\varphi\mathbb{E}_t[\hat{\pi}_{t+1} - \hat{\pi}_t].$$

Collecting terms yields the forward and backward looking Phillips curve that appears in Christiano et al. (2005) and in Walsh (2005):

$$\hat{\pi}_t = \frac{1}{1 + \beta}\hat{\pi}_{t-1} + \frac{\beta}{1 + \beta}\mathbb{E}_t\hat{\pi}_{t+1} + \frac{(1 - \varphi)(1 - \beta\varphi)}{(1 + \beta)\varphi}\hat{g}_t. \quad (3.11)$$

Note that there is an alternative way to write equation (3.2b). Since its rhs equals:

$$\begin{aligned} & (1 - \beta\varphi)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s [\hat{\pi}_{t+s} + \hat{g}_{t+s}] = \\ & \hat{\pi}_t + \hat{g}_t + \mathbb{E}_t \left\{ (\beta\varphi) [\hat{\pi}_{t+1} + \hat{g}_{t+1}] + (\beta\varphi)^2 [\hat{\pi}_{t+2} + \hat{g}_{t+2}] + \dots \right. \\ & \left. - (\beta\varphi) [\hat{\pi}_t + \hat{g}_t] - (\beta\varphi)^2 [\hat{\pi}_{t+1} + \hat{g}_{t+1}] - (\beta\varphi)^3 [\hat{\pi}_{t+2} + \hat{g}_{t+2}] - \dots \right\} \\ & = \hat{\pi}_t + \hat{g}_t + \mathbb{E}_t \sum_{s=1}^{\infty} (\beta\varphi)^s [\hat{\pi}_{t+s} - \hat{\pi}_{t+s-1} + \hat{g}_{t+s} - \hat{g}_{t+s-1}] \end{aligned}$$

we can also write

$$\hat{p}_t = \hat{g}_t + \mathbb{E}_t \sum_{s=1}^{\infty} (\beta\varphi)^s [\hat{\pi}_{t+s} - \hat{\pi}_{t+s-1} + \hat{g}_{t+s} - \hat{g}_{t+s-1}]. \quad (3.12)$$

In the case where marginal costs differ between optimizing and non-optimizing firms we obtain (see also (3.9))

$$\begin{aligned} & \hat{p}_t + \hat{\pi}_t - \beta\varphi\mathbb{E}_t(\hat{p}_{t+1} + \hat{\pi}_{t+1}) = (1 - \beta\varphi) \\ & \times \mathbb{E}_t \left\{ \hat{g}_t - A\hat{p}_t + \beta\varphi \left( \hat{g}_{t+1} - A(\hat{p}_t - \hat{\pi}_{t+1}) + (\beta\varphi)^2 \left( \hat{g}_{t+2} - A(\hat{p}_t - \hat{\pi}_{t+1} - \hat{\pi}_{t+2}) \right) \right) + \dots \right. \\ & \left. - (\beta\varphi) \left( \hat{g}_{t+1} - A\hat{p}_{t+1} \right) - (\beta\varphi)^2 \left( \hat{g}_{t+2} - A(\hat{p}_{t+1} - \hat{\pi}_{t+2}) \right) - \dots \right. \\ & \left. + \hat{\pi}_t + \beta\varphi\hat{\pi}_{t+1} + (\beta\varphi)^2\hat{\pi}_{t+2} + \dots - \beta\varphi\hat{\pi}_{t+1} - (\beta\varphi)^2\hat{\pi}_{t+2} - \dots \right\} \\ & = (1 - \beta\varphi)(\hat{g}_t + \hat{\pi}_t) - A\hat{p}_t + \beta\varphi A\mathbb{E}_t(\hat{p}_{t+1} + \hat{\pi}_{t+1}). \end{aligned}$$



Replacing  $\hat{p}_t$  and  $\hat{p}_{t+1}$  yields after a modest amount of algebra the final solution:

$$\hat{\pi}_t = \frac{(1-\varphi)(1-\beta\varphi)}{\varphi B} \hat{g}_t + \frac{1+A}{B} \hat{\pi}_{t-1} + \frac{\beta(1+A)}{B} \hat{\pi}_{t+1},$$

where:  $A := \frac{\epsilon\alpha}{1-\alpha}$ ,  $B := (1+A)(1+\beta\varphi) + \beta(1-\varphi)$ . (3.13)

Galí et al. (2001) use a different assumption about backward looking behavior. They also assume that a fraction  $1-\varphi$  of firms adjusts their price according to (1.15a). Yet, among those firms that receive the signal to choose their price optimally only the fraction  $1-\omega$  does so. These firms set their relative price according to the first-order condition (2.10a). We use  $P_{At}^f$  to refer to their optimal nominal price. The remaining  $\omega(1-\varphi)$  backward looking firms update their price according to

$$P_{At}^b = \pi_{t-1} P_{At-1}, \tag{3.14}$$

where

$$P_{At} := \left[ (1-\omega)(P_{At}^f)^{1-\epsilon} + \omega(P_{At}^b)^{1-\epsilon} \right]^{1/(1-\epsilon)}. \tag{3.15}$$

is the average of the prices of those firm that truly optimize and the prices of those firms that adopt a backward looking update formula. The overall price level is still given by equation (1.17a).

The index formula (3.15) implies

$$\widehat{(P_{At}/P_t)} = (1-\omega)\hat{p}_t + \omega\widehat{(P_{At}^b/P_t)}, \tag{3.16}$$

where we continue to use the symbol  $\hat{p}_t$  for the percentage deviation of the optimal relative price of optimizing firms from its non-stochastic stationary value of unity. From (3.14) we obtain

$$\frac{P_{At}^b}{P_t} = \frac{\pi_{t-1} P_{At-1}}{P_t} = \frac{\pi_{t-1} P_{At-1}}{\pi_t P_{t-1}}$$

implying

$$\widehat{(P_{At}^b/P_t)} = \hat{\pi}_{t-1} - \hat{\pi}_t + \widehat{(P_{At-1}/P_{t-1})}.$$

Since (note that now  $\widehat{(P_{At}/P_t)}$  plays the role of  $\hat{p}_t$  in (3.3a))

$$\widehat{(P_{At}/P_t)} = \frac{\varphi}{1-\varphi} \hat{\pi}_t \tag{3.17}$$

this yields

$$\widehat{(P_{At}^b/P_t)} = \frac{1}{1-\varphi} \hat{\pi}_{t-1} - \hat{\pi}_t. \tag{3.18}$$

Substituting (3.17) and (3.18) into (3.16) we obtain a new relation between  $\hat{p}_t$  and the current and lagged rate of inflation:

$$\hat{p}_t = \frac{\varphi + \omega(1 - \varphi)}{(1 - \varphi)(1 - \omega)} \hat{\pi}_t - \frac{\omega}{(1 - \varphi)(1 - \omega)} \hat{\pi}_{t-1}. \quad (3.19)$$

Since equation (3.7) still gives log-linear approximation to the first-order condition (2.10a) we find the final solution after substitution for  $\hat{p}_t$  from (3.19). This yields

$$\begin{aligned} \hat{\pi}_t &= \frac{(1 - \omega)(1 - \varphi)(1 - \beta\varphi)(1 - \alpha)}{\xi[1 + \alpha(\epsilon - 1)]} \hat{g}_t + \frac{\omega}{\xi} \hat{\pi}_{t-1} + \frac{\beta\varphi}{\xi} \mathbb{E}_t \hat{\pi}_{t+1}, \\ \xi &:= \varphi + \omega(1 - \varphi(1 - \beta)). \end{aligned} \quad (3.20)$$

This is the hybrid Phillips curve equation from Galí et al. (2001). It nests several models:  $\omega = 0$  implies the purely forward looking Phillips curve (3.8),  $\omega=0$  and  $\alpha = 0$  imply the standard solution in (3.6).

## 4 Example

In order to see how the apparatus presented in the previous sections can be integrated into a model, I consider a simple New Keynesian macro model taken from Walsh (2003), Section 5.4.

### 4.1 The Model

**Households.** The representative household consumes a basket of goods

$$C_t = \left( \int_0^1 C_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon \geq 1 \quad (4.1)$$

with prices  $P_{jt}$ . Minimizing the costs  $P_t C_t = \int_0^1 P_{jt} C_{jt} dj$  of obtaining a given quantity  $C_t$  of this basket provides his demand for good  $j$ :

$$C_{jt} = \left( \frac{P_{jt}}{P_t} \right)^{-\epsilon} C_t, \quad (4.2)$$

where  $P_t$  is the price index defined in (1.2).

In this economy there are two stores of value: nominal money balances  $M_t$  and nominal bonds  $B_t$ , both issued by the government. Bonds pay a nominal interest  $q_t - 1$  which is determined at the end of period  $t - 1$  and, thus, a state variable. The household receives nominal wages  $W_t$  and real profits  $\Pi_t$  from firms and real

transfers  $T_t$  from the government. His period-to-period budget constraint in units of the final good  $Y_t$  is:

$$\frac{M_{t+1} + B_{t+1}}{P_t} = \frac{M_t}{P_t} + q_t \frac{B_t}{P_t} + \frac{W_t}{P_t} N_t + Z_t + \Pi_t. \quad (4.3)$$

The household maximizes

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[ \frac{C_{t+s}^{1-\eta}}{1-\eta} + \frac{\gamma_1}{1-\chi} \left( \frac{M_{t+s+1}}{P_{t+s}} \right)^{1-\chi} + \frac{\gamma_2}{1+\theta} N_{t+s}^{1+\theta} \right], \quad \beta \in (0, 1), \gamma_1, \gamma_2, \eta\theta \geq 0$$

subject to (4.3) and given initial levels of  $M_t$  and  $B_t$ .

The first-order conditions of this problem are:

$$\lambda_t = C_t^{-\eta}, \quad (4.4a)$$

$$N_t^\theta = \frac{1}{\gamma_2} \lambda_t w_t, \quad (4.4b)$$

$$\lambda_t = \beta q_{t+1} \mathbb{E}_t \frac{\lambda_{t+1}}{\pi_{t+1}}, \quad (4.4c)$$

$$\frac{\lambda_t}{P_t} = \gamma_1 \left( \frac{M_{t+1}}{P_t} \right)^{-\chi} \frac{1}{P_t} + \beta \mathbb{E}_t \frac{\lambda_{t+1}}{P_{t+1}}, \quad (4.4d)$$

where  $w_t := W_t/P_t$  denotes the real wage rate.

**Government.** The government's budget constraint is

$$T_t + \frac{M_{t+1} - M_t + B_{t+1} - B_t}{P_t} = (q_t - 1) \frac{B_t}{P_t}, \quad (4.5)$$

and we assume that each sequence of transfers, interest rates and money balances satisfies the no Ponzi game condition:

$$M_t + B_t = - \sum_{s=0}^{\infty} \frac{P_{t+s} T_{t+s} - (q_{t+s} - 1) M_{t+s}}{\prod_{i=0}^s q_{t+i}}.$$

In this example we consider a simple Taylor rule for the nominal interest rate. Let  $q > 1$  denote the desired rate, then

$$\frac{q_{t+1}}{q} = \left( \frac{\pi_t}{\pi} \right)^\delta e^{\nu_t}, \quad \delta > 1, \quad (4.6a)$$

$$\nu_t = \rho_\nu \nu_{t-1} + \nu_t, \quad \nu_t \sim N(0, \sigma_\nu^2), \rho_\nu \in (0, 1). \quad (4.6b)$$

**Firms.** Each of the  $j \in [0, 1]$  goods is produced by one firm according to the production function (1.4). The fraction  $\varphi$  of firms that is not allowed to set their optimal price use the rule (1.15a) to update their nominal price.

## 4.2 Dynamics

In a temporary equilibrium of this economy the goods, the labor market, and the money market clear. For given  $(x_t := s_{t-1}, \Gamma_{1t}, \Gamma_{2t}, \text{ and } \lambda_t)$  the 12 equations

$$C_t^{-\eta} = \lambda_t, \quad (4.7a)$$

$$C_t = Y_t, \quad (4.7b)$$

$$N_t^\theta = \frac{1}{\gamma_2} \lambda_t w_t, \quad (4.7c)$$

$$w_t = (1 - \alpha) \tilde{g}_t Z_t \tilde{N}_t^{-\alpha}, \quad (4.7d)$$

$$\tilde{Y}_t = Z_t \tilde{N}_t^{1-\alpha}, \quad (4.7e)$$

$$\tilde{Y}_t = s_t Y_t, \quad (4.7f)$$

$$1 = (1 - \varphi) (P_{At} P_t)^{1-\epsilon} - \varphi \left( \frac{\pi}{\pi_t} \right)^{1-\epsilon}, \quad (4.7g)$$

$$\left( \frac{P_{At}}{P_t} \right) = \frac{\mu \Gamma_{1t}}{\Gamma_{2t}}, \quad \mu := \frac{\epsilon_t}{\epsilon_t - 1}, \quad (4.7h)$$

$$\frac{q_{t+1}}{q} = \left( \frac{\pi_t}{\pi} \right)^\delta e^{v_t} \quad (4.7i)$$

$$\tilde{N}_t = (1 - \varphi) N_{At}^{1-\alpha} + \varphi \left( \frac{N_t}{\varphi} - \frac{1 - \varphi}{\varphi} N_{At}^{\frac{1}{1-\alpha}} \right)^{1-\alpha}, \quad (4.7j)$$

$$N_{At}^{1-\alpha} = \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} \frac{Y_t}{Z_t}, \quad (4.7k)$$

$$s_t = (1 - \varphi) \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} + \varphi \left( \frac{\pi_t}{\pi} \right)^\epsilon s_{t-1}, \quad (4.7l)$$

determine the 12 variables  $Y_t, C_t, N_t, \tilde{Y}_t, \tilde{N}_t, p_t := P_{At}/P_t, w_t, \tilde{g}_t, \pi_t, q_{t+1}, s_t,$  and  $N_{At}$ . The model's dynamics govern the next four equations:

$$x_{t+1} = s_t, \quad (4.8a)$$

$$\lambda_{t+1} = \beta q_{t+1} \mathbb{E}_t \frac{\lambda_{t+1}}{\pi_{t+1}}, \quad (4.8b)$$

$$\Gamma_{1t} := \left( \frac{P_{At}}{P_t} \right)^{\frac{-\epsilon}{1-\alpha}} Y_t^{\frac{1}{1-\alpha}} \tilde{Y}_t^{\frac{-\alpha}{1-\alpha}} \lambda_t \tilde{g}_t + (\beta \varphi) \mathbb{E}_t \left( \frac{\pi(P_{At}/P_t)}{\pi_{t+1}(P_{At+1}/P_{t+1})} \right)^{\frac{-\epsilon}{1-\alpha}} \Gamma_{1t+1}, \quad (4.8c)$$

$$\Gamma_{2t} := \left( \frac{P_{At}}{P_t} \right)^{-\epsilon} Y_t \lambda_t + \beta \varphi \mathbb{E}_t \left( \frac{P_{At}/P_t}{P_{At+1}/P_{t+1}} \right)^{-\epsilon} \left( \frac{\pi}{\pi_{t+1}} \right)^{1-\epsilon} \Gamma_{2t+1}. \quad (4.8d)$$

In order to solve this model via linear or quadratic feed back rules we must compute the stationary equilibrium of the deterministic counterpart of the model. This is obtained from (4.7) and (4.8) by ignoring the expectations operator, setting  $Z_t \equiv 1$  and  $v_t = 0$  for all  $t$ , and dropping the time indices. This delivers  $P_A/P = 1$ ,

$$s = 1, \tilde{g} = \epsilon/(\epsilon - 1), Y = \tilde{Y}, N = \tilde{N},$$

$$N = \left( \frac{1 - \alpha \epsilon - 1}{\gamma_2 \epsilon} \right)^{\frac{1}{\alpha + \eta(1 - \alpha) + \theta}},$$

$$Y = N^\alpha = C, \text{ and } \lambda = C^{-\eta}.$$

**Figure 4.1:** Impulse Responses to an Interest Rate Shock

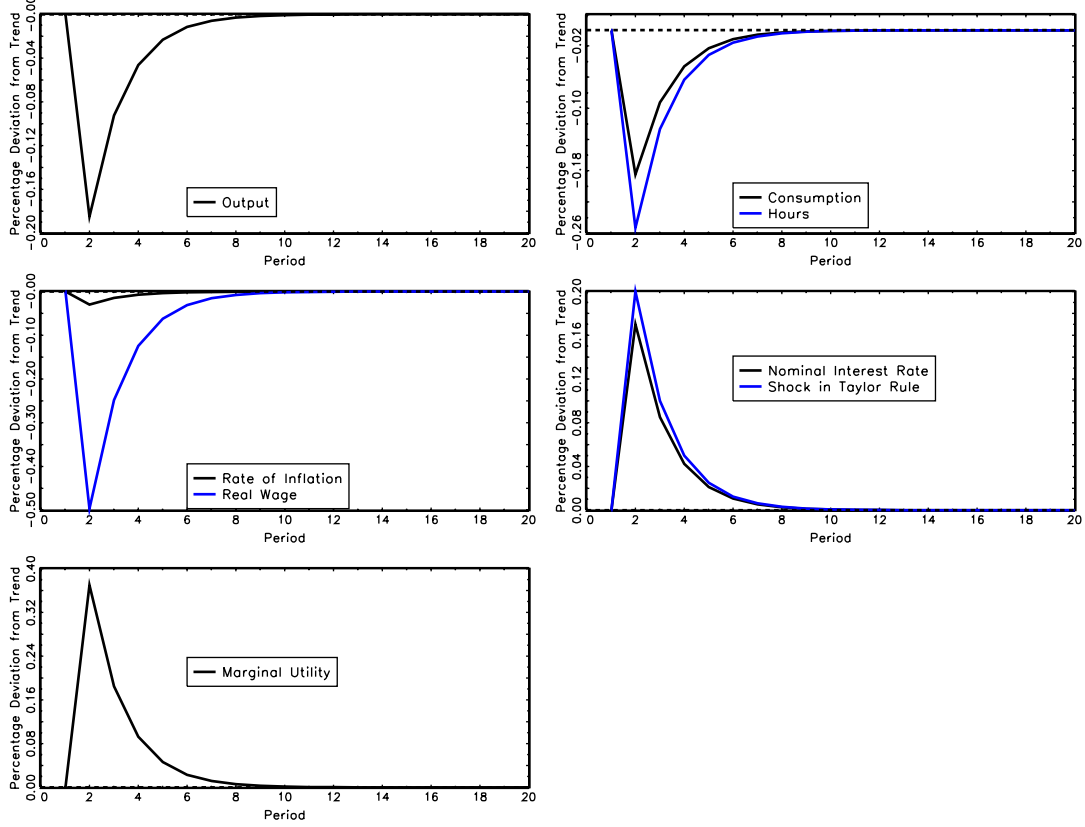


Figure 4.1 displays the response of the model to a one time shock of size  $\sigma_\nu$  in the interest rate equation computed with the program `NKPC_1.g`. The parameters are  $\alpha = 0.27$ ,  $\beta = \text{beta} = 0.994$ ,  $\delta = 1.01$ ,  $\epsilon = 6$ ,  $\eta = 2$ ,  $\varphi = 0.75$ ,  $\theta = 0.5$ ,  $\rho_\nu = 0.50$ , and  $q = \pi/\beta$  with  $\pi = 1.0167$ .

## 5 References

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