FUNDAÇÃO GETULIO VARGAS

EPGE

Escola de Pós-Graduação em Economia

Ensaios Econômicos

Escola de Pós-Graduação em Economia da Fundação Getulio Vargas

N° 715 ISSN 0104-8910

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Rodrigo Jardim Raad

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URL: http://hdl.handle.net/10438/7914

Jardim Raad, Rodrigo
Exogenous Expectations on Endogenous Uncertainty: Recursive Equilibrium and Survival/ Rodrigo Jardim Raad - Rio de Janeiro : FGV,EPGE, 2011
38p. - (Ensaios Econômicos; 715)

Inclui bibliografia.

CDD-330

Exogenous Expectations on Endogenous Uncertainty: Recursive Equilibrium and Survival*

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Abstract

This paper analyses general equilibrium models with finite heterogeneous agents having exogenous expectations on endogenous uncertainty. It is shown that there exists a recursive equilibrium with the state space consisting of the past aggregate portfolio distribution and the current state of the nature and that it implements the sequential equilibrium. We establish conditions under which the recursive equilibrium is continuous. Moreover, we use the continuous recursive relation of the aggregate variables to prove that if the economy has two types of agents, the one who commits persistent mistakes on the expectation rules of the future endogenous variables is driven out of the market by the others with correct anticipations of the variables, that is, the rational expectations agents.

Keywords: Recursive Equilibrium, Endogenous and Exogenous Uncertainty, Survival

*I thank to Victor Filipe Martins-da-Rocha for comments and suggestions.
†Escola de Pós Graduação em Economia, Fundação Getulio Vargas

1 Introduction

Several models in economics study recursive equilibrium (RE) (which is a relation between the equilibrium over consecutive periods) assuming that agents have rational expectations hypothesis as presented in Lucas Jr (1978), Mehra & Prescott (1980), Coleman (1991), Stokey et al. (1989) and Ljungqvist & Sargent (2000) for example. For exchange economies with homogeneous infinitely lived agents, Lucas Jr (1978) has shown the existence of RE with a state space that only contains exogenous variables. The analysis of the equilibrium in most models with heterogeneous agents and complete markets can be reduced to the Lucas Tree Model because it can be showed that any such equilibrium is the equilibrium of an appropriately chosen representative agent, so it will display the properties derived in Lucas' analysis. With incomplete markets and heterogeneous agents, however, it is well known that equilibrium allocations are not typically efficient, ruling out the possibility of a representative agent. More recently, Mirman et al. (2008) provide general results of existence and convergence for a large class of infinite horizon economies with capital and incomplete markets, using lattice programming and order theoretic fixed-point theory. For pure exchange economies with a finite number of infinitely lived agents and incomplete markets, Duffie et al. (1994) have shown the existence of RE with a compact state space that includes exogenous variables and the endogenous variables consumption, asset prices and portfolio holdings. A recursive approach to the model studied in Magill & Quinzii (1994) is given in Kubler & Schmedders (2002). In this work, they give counter-examples to the existence of RE for reduced state spaces and show that uniqueness of the sequential equilibrium is a sufficient condition for existence of RE. Takeoka (2006) also provides a recursive approach to economies based on the model without rational expectations presented in Grandmont (1977) which defines the temporary equilibrium concept where trading takes place sequentially over time and where each agent makes decisions at every date in the light of his expectations about his future environment. Agents anticipate the endogenous variables using exogenous rules which are described by "expectation functions" of his information on the present and past states of the economy. Following Grandmont's framework, Takeoka (2006) examines the existence of stationary processes of temporary equilibrium in an OLG model, where there are finitely many commodities and consumers in each period and the state space is taken as the set of all payoff-relevant variables.

In models such that markets reopen sequentially and the (sequential) equilibrium is defined for each period of time, the existence of a transition function defined in a reduced set of variables and determining the equilibrium over consecutive periods provides a tool to compute it and to study the dynamics of the state variable evolution. This is the case of economies in which the sequential equilibrium is implemented by some RE and consequently inherits the main properties satisfied by it. The existence of recursive equilibrium with a reduced state space can also be viewed as a defense of the "common and correct expectations" which is a concept given in Radner (1972) that requires traders to associate the same future prices to the same future exogenous events, but does not require them to agree on the (subjective) probabilities associated with those events. If the sequential equilibrium is implemented by a recursive equilibrium with a reduced state space, then prices can be anticipated correctly using the recursive structure and hence agents only need to anticipate a relative simple price transition function instead of all future contingent equilibrium prices itself as assumed in most classical general equilibrium models. The existence of recursive equilibrium can also be used to justify the correct anticipation of the endogenous variables of the economy using a market selection argument. Indeed, in an economy with a continuous recursive equilibrium and two types of agents, we prove that the one who commits persistent mistakes on the rules of anticipation of the future endogenous variables is driven out of the market by the others with correct expectation functions of the variables, that is, the rational expectations agents.

One contribution of this paper is to show existence of RE in a model such as Grandmont (1977) where it is assumed that agents have exogenous expectations on endogenous variables, but contrary to Grandmont (1977), we do not assume that agents are myopic. Following Svensson (1981), we define endogenous uncertainty as the inaccuracy in anticipation of some future aggregate endogenous variables and, consequently, depending on agents choices.[1] The state space is composed of past mean portfolio distribution and the current state of the nature and does not include all pay-off relevant variables as in Takeoka (2006). In the case of rational expectations and heterogeneous agents, Kubler & Schmedders (2002) have shown examples in which RE prices must depend on portfolio distribution and conclude that the minimal state space necessarily contains the aggregate portfolio distribution of the economy. Since we prove that the state space contains only the past aggregate portfolio distribution and the current state of the nature, it is also clear here that the RE has a minimal state space by the same reasons given in Kubler & Schmedders (2002). The intuition for this fact is that in economies with risk aversion heterogeneity for example, the reallocation of the asset shares from one agent to another with greater risk aversion would typically require a new set of equilibrium prices.

Finally, under some conditions on the primitives of the model, we prove that we can find a RE in an economy having at least one agent with correct price expectation function. If the recursive equilibrium is continuous then agents with this knowledge dominate the market when trading with the one who commits persistent mistakes on the rules of anticipation of future endogenous variables.

The paper is structured as follows. In Section 2 we set out the model. In Section 3 we define the equilibrium concept and exhibit some results. In section 4 we show survival results. In Section 5 we define an economy with one agent displaying the Price Perfect Foresight ability and show similar results as those given in Section 3. Conclusions are

[1] For example, uncertainty about the next period prices.

given in Section 6.

2 The model

2.1 Definitions

Suppose that there exist finite agent types in the economy denoted by the set[2] $I = \{1, ..., I\}$ and such that each type $i \in I$ has a continuum of agents trading in a competitive environment. Time is indexed by t in the set $\mathbb{N} = \{1, 2, ...\}$ for current periods and r for future periods. In this model there exists exogenous uncertainty, in the sense of being independent of agents' actions. Each agent knows the whole set of possible states of the nature and trade contingent claims. Let S be a topological space containing all states of the nature and Σ its Borelians. Denote by (S_t, Σ_t) a copy of (S, Σ) for all $t \in \mathbb{N}$. Exogenous uncertainty is described by the streams $s^t = (s_1, ..., s_t) \in S_1 \times \cdots \times S_t = S^t$ for all $t \in \mathbb{N}$.

There are one good and a finite set[3] H of long lived assets in net supply equal to one and with dividends measurable bounded functions $\hat{d} : S \to \mathbb{R}^H_{++}$. Denote by $\Theta^i \subset \mathbb{R}^H_+$ for all $i \in I$ the convex set where asset choices are defined and $C^i \subset \mathbb{R}_+$ be the convex set where agent i's consumption is chosen. Observe that we are not allowing for short-sales. Define the symbol without upper index as the Cartesian product. For example write $C = \prod_{i \in I} C^i$.

Denote by $Q = \{(q^c, q^a) \in \mathbb{R}_+ \times \mathbb{R}^H_+ : q^c + \sum_{h \in H} q^a_h = 1\}$ the set where the prices are defined and write $Q^\circ = Q \cap \mathbb{R}^{H+1}_{++}$. The symbol $q = (q^c, q^a) \in Q$ stands for the consumption and asset prices respectively.

Write $\overline{\Theta} = \{\bar{\theta} \in \Theta : \sum_{i \in I} \bar{\theta}^i_h = 1 \text{ for all } h \in H\}$. An element $\bar{\theta} \in \overline{\Theta}$ stands for the mean aggregate asset choice of the agents.

Let $Y = \overline{\Theta} \times S$ be the space of state variables endowed with the product topology. Write $\mathscr{Y}$ the Borelians of Y and $(Y_t, \mathscr{Y}_t)$ a copy of $(Y, \mathscr{Y})$ for all $t \in \mathbb{N}$. The set Y_t contains the variables on which the beliefs will be defined. Write the set of all functions $\hat{q} : Y \to Q$ by $\widehat{Q}$ and the set of all functions $\hat{q} : Y \to Q^\circ$ by $\widehat{Q}^\circ$.

Every Cartesian product of topological spaces is endowed with the product topology. In particular, $\widehat{Q}$ is endowed with the pointwise convergence topology, which is equivalent to the product topology.

The instantaneous utility is a bounded real valued function $u^i : \mathbb{R}_+ \to \mathbb{R}$ continuously differentiable on $\mathbb{R}_{++}$, strictly concave, strictly increasing for all $i \in I$, satisfying $u^i(0) = 0$ and $\lim_{c^i \to 0} \partial u^i(c^i) = \infty$ where the symbol $\partial u^i(c^i)$ stands for the derivative of u^i evaluated at the point c^i.

[2]We index the cardinality with the same symbol for convenience.
[3]We write $H = \{1, ..., H\}$.

2.2 Agents' characteristics

In this model we consider endogenous uncertainty, which consists basically in the uncertainty about the correct relation between prices and exogenous events[4] and consequently can not be separated from individuals decisions. If there is a market only in the present and none in the future, the endogenous uncertainty about prices does not matter and each agent only needs to observe the present equilibrium price to choose his actions. If some agent believes that a market will open in the future, his current choices depend on the price in the future. Since the price depends on what other agents decide to choose, and since this agent is uncertain about other agents' characteristics and hence what they will trade in the future, it is reasonable to conceive of him being uncertain about the price in the future.

We suppose that agents anticipate (possibly without accuracy) next period prices using a continuous price expectation function as in Grandmont (1977), depending on the current prices and contingent on the next period exogenous shock and aggregate mean asset share allocation that will be chosen in the current period. The price expectation can be viewed as a continuous function $\check{q}^i : Q \to C(Y, Q^\circ)$ from Q to the space of all continuous functions[5] from Y to Q° with the following rule: given a price q_τ, then $\check{q}^i(q_\tau)$ yields the agent i's next period anticipated price contingent to all next period realization of $y_\tau = (\bar{\theta}_\tau, s_{\tau+1}) \in \overline{\Theta} \times S = Y$ for $\tau \in \mathbb{N}$. It is convenient to denote the price expectation function $\check{q}^i$ by the function $\tilde{q}^i : Y \times Q \to Q^\circ$ defined by $\tilde{q}^i(y, q) = \check{q}^i(q)(y)$ for all $(y, q) \in Y \times Q$.

At current period t, given the observed variables $\bar{\theta}_{t-1}$, s_t and q_t, the beliefs about r-forward realization of state variables are given by the probability $\mu^i_r(y_t, q_t)$ where $\mu^i_r : Y \times Q \to \text{Prob}(Y^r)$ is a continuous[6] kernel for $r \in \mathbb{N}$. We suppose that these beliefs are $(\lambda^i, \tilde{q}^i)$-*predictive* in the context of Blackwell & Dubins (1962) with continuous probability transition rules $\lambda^i : Y \times Q \to \text{Prob}(Y)$. Rigorously, we assume that the measure μ^i_r satisfies[7]

$$\mu^i_r(y, q)(A_1, ..., A_r) = \int_{A_1} \cdots \int_{A_r} \lambda^i(y_{r-1}, q^i_{r-1}(y^{r-1}), dy_r) \cdots \lambda^i(y, q, dy_1)$$

for each rectangle $A_1 \times ... \times A_r$ where $\{q^i_r(y^r)\}_{r \geq 0}$ is the sequence of prices with $q^i_0 = q$ and recursively $q^i_r(y^r) = \tilde{q}^i(y_r, q^i_{r-1}(y^{r-1}))$ for $r \in \mathbb{N}$. Notice that we assume, to simplify notation, that agents use only one period backward to estimate future variables of the economy.

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[4]The model of Radner (1972) does not take into account this uncertainty because it is assumed that agents have *common (and correct) expectations*.

[5]The set $C(Y, Q^\circ)$ is endowed with the sup metric.

[6]As a function from $Y \times Q$ to $\text{Prob}(Y^r)$ with $\text{Prob}(Y^r)$ endowed with the *weak** topology.

[7]See Stokey et al. (1989) Chapters 8 and 9 for details about the construction of a probability measure based on the composition of probability transition rules and results about expectations over this measure. We use the Caratheodory Extension Theorem to define the measure over the entire product space.

Remark 2.1. We could suppose in this paper that agents choose non degenerated probability distributions on prices without altering the conclusions and results. In this case the probability transition rules provide the probabilities for the next period state variable y and prices, that is, $\tilde{\lambda}^i : Y \times Q \to \mathrm{Prob}(Y \times Q)$. Observe that in General Equilibrium models where agents perfectly anticipate future contingent prices, the beliefs on such prices can be viewed as degenerated probabilities. Rigorously, here we have that $\tilde{\lambda}^i(y,q) = \lambda^i(y,q) \otimes \mathrm{dirac}(\tilde{q}^i(y,q))$.

Since agents do not perfectly anticipate the future state variables which contain endogenous variables, they must make plans at each period t contingent to all possible future trajectories of these variables. Moreover, the optimal plans may be different over time, that is, we may not have intertemporal consistency in this model.

Definition 2.1. *A plan* $(\boldsymbol{c}^i_r, \boldsymbol{\theta}^i_r)_{r \geq 0}$ *is defined as the current period choice* $(\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0) \in C^i \times \Theta^i$ *and the streams* $(\boldsymbol{c}^i_r, \boldsymbol{\theta}^i_r)_{r \in \mathbb{N}}$ *of measurable maps* $\boldsymbol{c}^i_r : Y^r \to C^i$ *and* $\boldsymbol{\theta}^i_r : Y^r \to \Theta^i$ *representing future plans.*

It is convenient to write the agents' price forecasts as in the definition below.

Definition 2.2. *The agent i's future price forecasting stream* $\boldsymbol{q}^i_r : Y^r \times Q \to Q^\circ$ *is defined by*[8] $\boldsymbol{q}^i_0(q) = q$ *and recursively*

$$\boldsymbol{q}^i_r(y^r, q) = \tilde{q}^i(y_r, \boldsymbol{q}^i_{r-1}(y^{r-1}, q)) \text{ for } r \in \mathbb{N}.$$

Notice that in the current period agents observe q and use the $r-1$ periods forward anticipated price $\boldsymbol{q}^i_{r-1}(y^{r-1}, q)$ to anticipate the r periods forward price $\boldsymbol{q}^i_r(y^r, q)$ for each $r \in \mathbb{N}$.

Definition 2.3. *Let* $B^i : \Theta^i \times S \times Q \to C^i \times \Theta^i$ *be defined as*

$$B^i(\theta^i_-, s, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : q^c c^i + q^a \theta^i \leq (q^a + q^c \hat{d}(s))\theta^i_-\}.$$

A plan $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ *is feasible from* (θ^i_-, s, q) *if* $(\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0) \in B^i(\theta^i_-, s, q)$ *and*

$$(\boldsymbol{c}^i_r(y^r), \boldsymbol{\theta}^i_r(y^r)) \in B^i(\boldsymbol{\theta}^i_{r-1}(y^{r-1}), s_r, \boldsymbol{q}^i_r(y^r, q)) \text{ for all } y^r \in Y^r$$

where $y_r = (\bar{\theta}_{r-1}, s_r)$ *and* $\boldsymbol{q}^i_r : Y^r \times Q \to Q$ *is given by Definition 2.2.*

Denote by $\boldsymbol{F}^i(\theta^i_-, s, q)$ *the set of all feasible plans from* (θ^i_-, s, q).

Observe that one may have $\boldsymbol{\theta}^i_r(y^r) \neq \bar{\theta}^i_r$, that is, agents can plan asset purchases different from the realization of the mean aggregate asset share with respect to their own type[9] at period r.

[8] The function $\boldsymbol{q}^i_0$ is defined on Q.

[9] Recall that in this model there is a continuum of identical agents for each type.

Remark 2.2. The results given in Section 3 keep unaltered if we suppose that agents have contingent endowments of the good $e^i : S \to \mathbb{R}_+$, that is, the budget set is defined for each (θ^i_-, s, q) as

$$B^i(\theta^i_-, s, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : q^c c^i + q^a \theta^i \leq (q^a + q^c \hat{d}(s))\theta^i_- + q^c e^i(s)\}.$$

Now we can define the expected utility.

Definition 2.4. *Let $\boldsymbol{C}^i$ be the set of all sequence of measurable functions $\{\boldsymbol{c}^i_r\}_{r\geq 0}$ with $\boldsymbol{c}^i_0 \in C^i$ constant and $\boldsymbol{c}^i_r : Y^r \to C^i$ for $r \in \mathbb{N}$. We define agent i's expected utility $\boldsymbol{U}^i : \boldsymbol{C}^i \times Y \times Q \to \mathbb{R}$ of a contingent consumption $\boldsymbol{c}^i \in \boldsymbol{C}^i$ given the state y and the price q by the integral:*

$$\boldsymbol{U}^i(\boldsymbol{c}^i, y, q) = u^i(\boldsymbol{c}^i_0) + \sum_{r\in\mathbb{N}} \int_{Y^r} \beta^r u^i(\boldsymbol{c}^i_r(y^r))\mu^i_r(y, q, dy^r).$$

Definition 2.5. *Define the value function $v^i : \Theta^i \times Y \times Q \to \mathbb{R}$ by:*

$$v^i(\theta^i_-, y, q) = \sup\{\boldsymbol{U}^i(\boldsymbol{c}^i, y, q) : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\theta^i_-, s, q)\} \tag{1}$$

and the optimal correspondence $\widehat{\boldsymbol{F}}^i \subset \boldsymbol{F}^i$ by:

$$\widehat{\boldsymbol{F}}^i(\theta^i_-, y, q) = \operatorname{argmax}\{\boldsymbol{U}^i(\boldsymbol{c}^i, y, q) : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\theta^i_-, s, q)\}.$$

Although the demand defined below is independent of time, it yields the current choice at period t given some past and current observed variables. This approach allows us to write the problem recursively and hence to describe it in a more tractable manner as we will show in the next section.

Definition 2.6 (Agents'demand)**.** *We define agent i's demand for good and asset by:*

$$\delta^i(\theta^i_-, y, q) = \{(\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0) \in C^i \times \Theta^i : (\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \widehat{\boldsymbol{F}}^i(\theta^i_-, y, q)\}.$$

3 Sequential and recursive equilibrium

In this section we present the sequential and recursive equilibrium concepts. Moreover, we prove existence and characterize the connection between the sequential and recursive equilibrium.

Definition 3.1. *A sequential equilibrium with initial asset holdings $\boldsymbol{\theta}_0 \in \overline{\Theta}$ is a measurable family of contingent prices $\{\boldsymbol{q}_t : S^t \to Q^\circ\}_{t\in\mathbb{N}}$, contingent consumption allocations $\{\boldsymbol{c}_t : S^t \to C\}_{t\in\mathbb{N}}$ and contingent portfolio allocations $\{\boldsymbol{\theta}_t : S^t \to \Theta\}_{t\in\mathbb{N}}$ satisfying for all $s^t \in S^t$:*

1. *optimality: for every* i

$$(\boldsymbol{c}_t^i(s^t), \boldsymbol{\theta}_t^i(s^t)) \in \delta^i(\boldsymbol{\theta}_{t-1}^i(s^{t-1}), (\boldsymbol{\theta}_{t-1}(s^{t-1}), s_t), \boldsymbol{q}_t(s^t));$$

2. *asset markets clear:* $\sum_{i\in I} \boldsymbol{\theta}_t^i(s^t) = \mathbf{1} \in \mathbb{R}^H$*;*

3. *good markets clear:* $\sum_{i\in I} \boldsymbol{c}_t^i(s^t) = \mathbf{1} \cdot \hat{d}(s_t)$*.*

We introduce now the concept of recursive equilibrium and show in the appendix that it implements the sequential equilibrium of the economy. A well known result states that there exists a bounded continuous value function $v^i : \Theta^i \times Y \times Q^\circ \to \mathbb{R}$ satisfying the Bellman Equation:

$$v^i(\theta_-^i, y, q) = \sup \left\{ u^i(c^i) + \beta \int_Y v^i(\theta^i, y', \tilde{q}^i(y', q))\lambda^i(y, q, dy') \right\} \tag{2}$$

over all $(c^i, \theta^i) \in B^i(\theta_-^i, s, q)$ where $y = (\bar{\theta}, s)$. Indeed, consider the operator[10] $T^i : \mathbb{V} \to \mathbb{V}$, defined by

$$T^i(v^i)(\theta_-^i, y, q) = \sup \left\{ u^i(c^i) + \beta \int_Y v^i(\theta^i, y', \tilde{q}^i(y', q))\lambda^i(y, q, dy') \right\} \tag{3}$$

over all $(c^i, \theta^i) \in B^i(\theta_-^i, s, q)$. Clearly,[11] T satisfies the Blackwell's sufficient conditions for a contraction and hence has a fixed point. See Stokey et al. (1989) for further details. We stand out the argmax of agent i's Bellman Equation (2) in the following definition.

Definition 3.2. *Define the agent i's consumption and portfolio policy correspondence $\tilde{x}^i : \Theta^i \times Y \times Q^\circ \to C^i \times \Theta^i$ with $\tilde{x}^i = \tilde{c}^i \times \tilde{\theta}^i$ as*

$$\tilde{x}^i(\theta_-^i, y, q) = \operatorname{argmax} \left\{ u^i(c^i) + \beta \int_Y v^i(\theta^i, y', \tilde{q}^i(y', q))\lambda^i(y, q, dy') \right\}$$

over all $(c^i, \theta^i) \in B^i(\theta_-^i, s, q)$.

Remark 3.1. Observe that the value function $v^i(\,\cdot\,, y, q)$ is concave for all $(y, q) \in Y \times Q$ since the subset of all value functions $v^i \in \mathbb{V}$ such that $v^i(\,\cdot\,, y, q)$ is concave for all $(y, q) \in Y \times Q$ is a nonempty and closed subset of $\mathbb{V}$. Indeed, Lemma 7.2 assures the stability of this subspace under T and hence the fixed point must belong to it. Therefore, the projection of the policy correspondence into the consumption coordinate $\tilde{c}^i : \Theta^i \times Y \times Q^\circ \to C^i$ is actually a function. Moreover, if $C^i = \mathbb{R}_+$, Lemma 7.2 assures

[10] Define $\mathbb{V}$ the (Banach) space of all bounded continuous functions $v^i : \Theta^i \times Y \times Q^\circ \to \mathbb{R}_+$ endowed with the sup norm.

[11] This operator is well defined using the Berge Maximum Theorem, Lemmas 7.1 and 7.3 in the appendix because λ^i is continuous as well as B^i for $W^i = \{(\theta^i \hat{d}(s), \theta^i) : \theta^i \in \Theta^i \text{ and } s \in S\}$. Observe that $W^i \times Q^\circ \subset A^i$ where A^i is defined in Appendix (item 6) and that in Lemma 7.1 $Z = \Theta^i \times Q$ and $f(y, \theta^i, q) = v^i(\theta^i, y, \tilde{q}^i(y, q))$.

that $v^i(\cdot, y, q)$ is is strictly increasing for each $(y, q) \in Y \times Q$. Note also that we can allow u^i unbounded if we assume that C^i is compact.

Definition 3.3. *We say that the economy has a recursive equilibrium if there exist functions $\hat{c}^i : Y \to C^i$, $\hat{\theta}^i : Y \to \Theta^i$ for all $i \in I$ and $\hat{q} : Y \to Q^\circ$ satisfying for each $y = (\bar{\theta}, s) \in Y$*

1. *optimality: $(\hat{c}^i(y), \hat{\theta}^i(y)) \in \tilde{x}^i(\bar{\theta}^i, y, \hat{q}(y))$ for all $i \in I$;*
2. *asset market clearing: $\sum_{i \in I} \hat{\theta}^i(y) = \mathbf{1} \in \mathbb{R}^H$;*
3. *consumption market clearing: $\sum_{i \in I} \hat{c}^i(y) = \mathbf{1} \cdot \hat{d}(s)$.*

Definition 3.4. *We say that the functions $\hat{c} : Y \to C$, $\hat{\theta} : Y \to \Theta$ and $\hat{q} : Y \to Q$ implement the process $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t \in \mathbb{N}}$ starting from $\boldsymbol{\theta}_0 \in \overline{\Theta}$ if for each $(s_t)_{t \in \mathbb{N}}$*

$$\boldsymbol{q}_1(s_1) = \hat{q}(\boldsymbol{\theta}_0, s_1), \quad \boldsymbol{\theta}_1^i(s_1) = \hat{\theta}^i(\boldsymbol{\theta}_0, s_1), \quad \boldsymbol{c}^i(s_1) = \hat{c}^i(\boldsymbol{\theta}_0, s_1)$$

and recursively for $t \geq 2$

$$\boldsymbol{c}_t^i(s^t) = \hat{c}^i(\boldsymbol{\theta}_{t-1}(s^{t-1}), s_t) \qquad \boldsymbol{\theta}_t^i(s^t) = \hat{\theta}^i(\boldsymbol{\theta}_{t-1}(s^{t-1}), s_t) \tag{4}$$

for all $i \in I$ and

$$\boldsymbol{q}_t(s^t) = \hat{q}(\boldsymbol{\theta}_{t-1}(s^{t-1}), s_t). \tag{5}$$

The next result assures that a recursive equilibrium can actually be used to construct a sequential equilibrium.

Theorem 3.5. *If $(\hat{c}, \hat{\theta}, \hat{q})$ is a recursive equilibrium then its implemented process $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t \in \mathbb{N}}$ starting from $\boldsymbol{\theta}_0 \in \overline{\Theta}$ is a sequential equilibrium of the economy with initial asset holdings $\boldsymbol{\theta}_0 \in \overline{\Theta}$.*

Proof: See Theorem 7.8 in the appendix.

□

The theorem below assures the existence of a recursive equilibrium and the next proposition yields a sufficient condition under which it is continuous.

Theorem 3.6. *Suppose that $C^i = \mathbb{R}_+$ and $\Theta^i = \mathbb{R}_+^H$. Then there exists a recursive equilibrium for the economy $\mathcal{E} = \{u, \hat{d}, \lambda^i, \tilde{q}^i\}_{i \in I}$.*

Proof: See Theorem 7.7 in the appendix.

□

A similar result for the following proposition can be found in Kubler & Schmedders (2002).

Proposition 3.7. *Suppose that there exists a unique recursive equilibrium* $(\hat{c}, \hat{\theta}, \hat{q})$ *for the economy* $\mathcal{E} = \{u, \hat{d}, \lambda^i, \tilde{q}^i\}_{i \in I}$. *Then it is continuous.*

Proof: See Proposition 7.6 in the appendix.

□

4 Survival

In this section we prove a survival result based on some conditions on the beliefs and price expectations.

4.1 Euler equations

For simplification, at present section and in sections 4.2 and 4.3, we restrict our attention to an economy with only one state of nature, that is, with only endogenous uncertainty and consequently deterministic dividends. Moreover, suppose that the economy has only one asset and $C^i \times \Theta^i = \mathbb{R}_+^{H+1}$. In this case we write $Y = \overline{\Theta}$ and $y = \bar{\theta}$. One reason for this restriction is the application of a fixed point existence theorem for continuous functions defined on $[0, 1] \subset \mathbb{R}_+$ in the proof of Theorem 4.5. Indeed, we need that the policy correspondence must actually be a continuous function. In the section 4.4 we address the model with S a convex or finite space.

In the economy with one asset and one state of nature, the budget correspondence $B^i : \Theta^i \times Q^\circ \to C^i \times \Theta^i$ becomes

$$B^i(\theta^i_-, q) = \{(c^i, \theta^i) \in C^i \times \Theta^i : q^c c^i + q^a \theta^i \leq (q^a + q^c \hat{d})\theta^i_-\}.$$

Notation 4.1. Write, for $q \in Q^\circ$ the price $p = q^a/q^c$ of the asset in units of the good. Denote by $\partial_k f(x_1, x_2, ..., x_n)$ the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to the k-th coordinate evaluated at $(x_1, x_2, ..., x_n)$ when $n > 1$ and $\partial f(x)$ when $n = 1$. Define the price expectation function in units of the good $\tilde{p}^i : \overline{\Theta} \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{p}^i(\bar{\theta}, p) = \frac{\tilde{q}^{i,a}(\bar{\theta}, q)}{\tilde{q}^{i,c}(\bar{\theta}, q)} \quad \text{where} \quad q = \left(\frac{1}{1+p}, \frac{p}{1+p}\right).$$

For each $(\bar{\theta}, q) \in \overline{\Theta} \times Q$ the function $v^i(\,\cdot\,, \bar{\theta}, q)$ satisfies all assumptions of Benveniste and Scheinkman Theorem[12] which assures its differentiability at positive asset endowments whenever the argmax of the Bellman Equation is interior.[13] Since we do not have labor income in this model and the utility function satisfies the Inada conditions, Lemma 7.9 in appendix shows that portfolio optimal choices must be positive (and hence interior) if asset endowment is positive.

[12]See Benveniste & Scheinkman (1979).

[13]This argmax is always nonempty since $B^i(\theta^i, s, q)$ is compact for all $(\theta^i_-, s, q) \in \Theta^i \times S \times Q^\circ$.

Let $\tilde{\theta}^i : \Theta^i \times \overline{\Theta} \times Q^\circ \to \Theta^i$ and $\tilde{c}^i : \Theta^i \times \overline{\Theta} \times Q^\circ \to C^i$ be the policy functions[14] as in Definition 3.2 and observe that for $q \in Q^\circ$ and $\theta^i_- \in \Theta^i$:

$$\tilde{c}^i(\theta^i_-, \bar{\theta}, q) = -p\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) + (p + \hat{d})\theta^i_- \tag{6}$$

where we recall that $p = q^a/q^c \in \mathbb{R}_{++}$. Fix $(\theta^i_-, \bar{\theta}, q) \in \Theta^i \times \overline{\Theta} \times Q^\circ$ with $\theta^i_- > 0$. Applying the Benveniste Scheinkman Theorem we conclude that

$$\partial_1 v^i(\theta^i_-, \bar{\theta}, q) = \partial u^i(\tilde{c}^i(\theta^i_-, \bar{\theta}, q))(p + \hat{d}) \tag{7}$$

for all $\theta^i_- \in \Theta^i$ with $\theta^i_- > 0$, $\bar{\theta} \in \overline{\Theta}$ and $q \in Q^\circ$. Replacing c^i by $-p\theta^i + (p + \hat{d})\theta^i_-$ in the right hand side of (2), differentiating with respect to θ^i and evaluating at $\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q)$ we have the following F.O.C.:

$$-p\partial u^i(\tilde{c}^i(\theta^i_-, \bar{\theta}, q)) + \beta \int_{\overline{\Theta}} \partial_1 v^i[\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q), \bar{\theta}', \tilde{q}^i(\bar{\theta}', q)]\lambda^i(\bar{\theta}, q, d\bar{\theta}') = 0.$$

Using equation (7) then agent i's Euler Equation is given by

$$p\partial u^i(\tilde{c}^i(\theta^i_-, \bar{\theta}, q)) = \beta \int_{\overline{\Theta}} \partial u^i[\tilde{c}^i_+(\theta^i_-, \bar{\theta}, \bar{\theta}', q)](\tilde{p}^i(\bar{\theta}', p) + \hat{d})\lambda^i(\bar{\theta}, q, d\bar{\theta}') \tag{8}$$

for all $(\theta^i_-, \bar{\theta}, q) \in \Theta^i \times \overline{\Theta} \times Q^\circ$ with $\theta^i_- > 0$ where

$$\tilde{c}^i_+(\theta^i_-, \bar{\theta}, \bar{\theta}', q) = \tilde{c}^i(\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q), \bar{\theta}', \tilde{q}^i(\bar{\theta}', q)).$$

The example below exhibits an environment with a continuous unique recursive equilibrium.

Example 4.1. Suppose that the price expectation functions $\tilde{q}^i : \overline{\Theta} \times Q \to Q$ are given by

$$\tilde{q}^i(\bar{\theta}, q) = (1/(1 + \tilde{p}^i(\bar{\theta}, q)), \tilde{p}^i(\bar{\theta}, q)/(1 + \tilde{p}^i(\bar{\theta}, q)))$$

where[15] $\tilde{p}^i(\bar{\theta}, q) = p\gamma^i - \hat{d}$ with[16] $\gamma^i < \beta^{-1/\alpha}$ for a given $0 < \alpha < 1$. The instantaneous utility function is $u(c) = c^\alpha/\alpha$. We claim that the policy function $\tilde{\theta}^i : \Theta^i \times \overline{\Theta} \times Q \to \Theta^i$

[14] In this case the correspondences are actually functions because for each $(\theta^i_-, q) \in \Theta^i \times Q^\circ$ if $(c^i, \theta^i) \in B^i(\theta^i_-, q)$ and $(c^i, \tilde{\theta}^i) \in B^i(\theta^i_-, q)$ with binding inequalities then $\theta^i = \tilde{\theta}^i$. The local non satiation implies that the budget inequalities must bind in the optimum.

[15] Note that even considering the beliefs on future prices independent of the variable $\bar{\theta}$, the recursive equilibrium price $\hat{q}$ may depend on this variable.

[16] Under these beliefs, the ratio of expected next period payoff $p_{s+1} + d$ and the current period asset price p_s in units of the good is given by γ^i. More precisely, we define $\tilde{p}^i(\bar{\theta}, q) = \max\{p\gamma^i - \hat{d}, 0\}$ but since in the equilibrium $\tilde{p}^i(\bar{\theta}, \hat{q}(\bar{\theta})) > 0$ then this not relevant for the example.

is given by $\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) = \nu^i(p+\hat{d})\theta^i_-/p$ where $\nu^i = (\beta\gamma^i)^{1/(1-\alpha)}/\gamma^i < 1$. Indeed,

$$\begin{aligned} \tilde{c}^i(\theta^i_-, \bar{\theta}, q) &= -p\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) + (p+\hat{d})\theta^i_- \\ &= (1-\nu^i)(p+\hat{d})\theta^i_- \end{aligned}$$

and

$$\begin{aligned} \tilde{c}^i(\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q), \bar{\theta}', \tilde{p}^i(\bar{\theta}, q)) &= (1-\nu^i)(\tilde{p}^i(\bar{\theta}, q) + \hat{d})\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) \\ &= (1-\nu^i)p\gamma^i\nu^i(p+\hat{d})\theta^i_-/p \\ &= (1-\nu^i)(\beta\gamma^i)^{1/(1-\alpha)}(p+\hat{d})\theta^i_-. \end{aligned}$$

Therefore,

$$\tilde{c}^i(\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q), \bar{\theta}', \tilde{p}^i(\bar{\theta}, q)) = (\beta\gamma^i)^{1/(1-\alpha)}\tilde{c}^i(\theta^i_-, \bar{\theta}', q)$$

and hence Euler equation (8) holds since $\tilde{c}^i, \tilde{\theta}^i$ and $\tilde{p}^i$ do not depend on $\bar{\theta}$, $\partial u(x) = x^{\alpha-1}$ and $\gamma^i = (\tilde{p}^i(\bar{\theta}, q) + \hat{d})/p$.

To find the equilibrium price $\hat{q}$, notice that if $\hat{p}(\bar{\theta}) = \hat{d}\nu\bar{\theta}/(1-\nu\bar{\theta})$ where $\nu\bar{\theta} = \sum_{i\in I}\nu^i\bar{\theta}^i$, then

$$(1+\hat{d}/\hat{p}(\bar{\theta}))\sum_{i\in I}\nu^i\bar{\theta}^i = 1 \text{ for all } \bar{\theta} \in \overline{\Theta}$$

and hence $\hat{q}$ is the continuous recursive equilibrium price and it is unique.

Notice that we can suppose γ^i depending on $(\bar{\theta}, q)$. In this case if there exists a function $\nu^i : \overline{\Theta} \to (0,1)$ such that

$$\beta\int_{\overline{\Theta}}(1-\nu^i(\bar{\theta}'))^{\alpha-1}\gamma^i(\bar{\theta}', q)^\alpha\lambda(\bar{\theta}, q, d\bar{\theta}') = \left(\frac{1-\nu^i(\bar{\theta})}{\nu^i(\bar{\theta})}\right)^{\alpha-1}$$

then the asset demand will be given by $\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) = \nu^i(\bar{\theta})(p+\hat{d})\theta^i_-/p$.

4.2 Perfect foresight

We provide hereafter the definition of an equilibrium where agents eventually anticipate correctly future prices and aggregate portfolio transitions. To simplify we can omit the variable s^t on the sequential equilibrium without ambiguity because $S = \{1\}$.

Definition 4.1. *We say that an agent k is eventually Perfect Foresight with respect to the equilibrium $\{(\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t)\}_{t\in\mathbb{N}}$ if there exists $T \in \mathbb{N}$ such that the updating rule λ^k and the price expectation $\hat{q}^k$ satisfy*

1. $\lambda^k(\boldsymbol{\theta}_{t-1}, \boldsymbol{q}_t) = \mathrm{dirac}(\boldsymbol{\theta}_t)$
2. $\hat{q}^k(\boldsymbol{\theta}_t, \boldsymbol{q}_t) = \boldsymbol{q}_{t+1}$

for all $t \geq T$.

Remark 4.1. Observe that nothing is imposed on $\hat{q}^k$ and λ^k outside the equilibrium path $\{\boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$. Raad (2011) has shown existence of sequential equilibrium with at least one agent satisfying the properties of Definition 4.1. However, it is not clear that this sequential equilibrium can be implemented by some recursive equilibrium even if all expectations are independent of time.

Notice that for a sequential equilibrium implemented by a recursive equilibrium, a sufficient condition under which one agent satisfies the properties of Definition 4.1 is the eventually correct anticipation of the price and portfolio transitions $\hat{q}$ and $\hat{\theta}$ respectively in the sense that there exists $T \in \mathbb{N}$ such that $\lambda^k(\bar{\theta}, \boldsymbol{q}_t) = \mathrm{dirac}(\hat{\theta}(\bar{\theta}))$ and $\hat{q}^k(\bar{\theta}, \boldsymbol{q}_t) = \hat{q}(\bar{\theta})$ for all $\bar{\theta} \in \overline{\Theta}$ and $t \geq T$. Indeed, observing $\boldsymbol{\theta}_{t-1}$ and the price $\boldsymbol{q}_t$, agent k expects next period asset distribution $\lambda^k(\boldsymbol{\theta}_{t-1}, \boldsymbol{q}_t) = \mathrm{dirac}(\hat{\theta}(\boldsymbol{\theta}_{t-1})) = \mathrm{dirac}(\boldsymbol{\theta}_t)$ and next period price $\hat{q}^k(\boldsymbol{\theta}_t, \boldsymbol{q}_t) = \hat{q}(\boldsymbol{\theta}_t) = \boldsymbol{q}_{t+1}$ because transitions and prices follow the recursive relations (4) and (5) respectively. Let T as in Definition 4.1 and the previous and current period information data of the economy $(\boldsymbol{\theta}_{T-1}, \boldsymbol{q}_T)$. Using Definition 2.2 and that $\hat{q}^k$ satisfies item *2.* of Definition 4.1, the expected price at period $T+1$ becomes $\boldsymbol{q}_1^k(\boldsymbol{\theta}_T, \boldsymbol{q}_T) = \hat{q}^k(\boldsymbol{\theta}_T, \boldsymbol{q}_T) = \boldsymbol{q}_{T+1}$ and recursively

$$\begin{aligned} \boldsymbol{q}_r^k((\boldsymbol{\theta}_T, ..., \boldsymbol{\theta}_{T+r-1}), \boldsymbol{q}_T) &= \hat{q}^k[\boldsymbol{\theta}_{T+r-1}, \boldsymbol{q}_{r-1}^k((\boldsymbol{\theta}_T, ..., \boldsymbol{\theta}_{T+r-2}), \boldsymbol{q}_T)] \\ &= \boldsymbol{q}_{T+r} \text{ for } r \in \mathbb{N}. \end{aligned} \tag{9}$$

Moreover, since λ^k satisfies *1.* of Definition 4.1 and $\boldsymbol{q}_r^k$ satisfies (9), we have that $\mu_r^k(\boldsymbol{\theta}_{T-1}, \boldsymbol{q}_T) = \mathrm{dirac}((\boldsymbol{\theta}_T, ..., \boldsymbol{\theta}_{T+r-1}))$ for $r \in \mathbb{N}$. This characterizes completely the eventual intertemporal consistency of the eventually Perfect Foresight agents.

Since we did not show conditions on the premises of this economy assuring the existence of a continuous recursive equilibrium and one agent with rational expectations, we exhibit the example below that guarantees the existence of at least one economy with these properties.

Example 4.2. Let $\alpha \in (0,1)$ and consider[17] $R : \mathrm{Int}\,\overline{\Theta} \to \mathbb{R}_+$ defined by $R(\bar{\theta}) = \beta^{-1}(\bar{\theta}^k)^{\alpha-1} > 1$. Write the asset price in units of the good by the function $\hat{p} : \overline{\Theta} \to \mathbb{R}_+$ defined as $\hat{p}(\bar{\theta}) = \hat{d}/(R(\bar{\theta}) - 1)$. The normalized price $\hat{q} : \overline{\Theta} \to Q$ is given by $\hat{q}(\bar{\theta}) = ((1 - \hat{p}(\bar{\theta}))^{-1}, \hat{p}(\bar{\theta})(1 - \hat{p}(\bar{\theta}))^{-1})$. The recursive transition function $\hat{\theta} : \overline{\Theta} \to \Theta$ is given by $\hat{\theta}(\bar{\theta}) := (\hat{\theta}^j(\bar{\theta}), \hat{\theta}^k(\bar{\theta})) = (1 - (\bar{\theta}^k)^\alpha, (\bar{\theta}^k)^\alpha)$. Clearly,[18] for $u(c) = \ln(c)$ these functions satisfy agent k's Euler equation (8) evaluated at $\theta_-^k = \bar{\theta}^k$ with $\tilde{p}^k$ and λ^k defined by $\tilde{p}^k(\bar{\theta}, p) = \hat{p}(\bar{\theta})$ and $\lambda^k(\bar{\theta}, q, d\bar{\theta}') = \mathrm{dirac}(\hat{\theta}(\bar{\theta}))$ for all $\bar{\theta} \in \overline{\Theta}$. Indeed, equation (6) implies that

[17] This function plays the role of the gross asset return in units of the good, that is, $R(\bar{\theta}) = 1 + \hat{d}/\hat{p}(\bar{\theta})$. The fact that R is defined on $\mathrm{Int}\,\overline{\Theta}$ does not matter for our analysis since the initial asset endowments are strictly positive and the demand function are interior by construction.

[18] The condition $\lim_{c \to 0^+} \ln(c) = -\infty$ implies that the argmax of the Bellman Equation must be interior for positive prices.

$$\begin{aligned}\tilde{c}^k(\bar{\theta}^k, \bar{\theta}, \hat{q}(\bar{\theta})) &= \hat{c}^k(\bar{\theta}) \\ &= \hat{p}(\bar{\theta})(R(\bar{\theta})\bar{\theta}^k - \hat{\theta}^k(\bar{\theta})) \\ &= \hat{p}(\bar{\theta})\beta^{-1}(1-\beta)(\bar{\theta}^k)^\alpha\end{aligned}$$

and

$$\begin{aligned}\tilde{c}^k(\tilde{\theta}^k(\bar{\theta}^k, \bar{\theta}, \hat{q}(\bar{\theta})), \hat{\theta}(\bar{\theta}), \tilde{q}^k(\hat{\theta}(\bar{\theta}), \hat{q}(\bar{\theta}))) &= \tilde{c}^k(\hat{\theta}^k(\bar{\theta}), \hat{\theta}(\bar{\theta}), \hat{q}(\hat{\theta}(\bar{\theta}))) \\ &= \hat{c}^k(\hat{\theta}(\bar{\theta})) \\ &= \hat{p}(\hat{\theta}(\bar{\theta}))(R(\hat{\theta}(\bar{\theta}))\hat{\theta}^k(\bar{\theta}) - \hat{\theta}^k(\hat{\theta}(\bar{\theta}))) \\ &= \hat{p}(\hat{\theta}(\bar{\theta}))\beta^{-1}(1-\beta)(\hat{\theta}^k(\bar{\theta}))^\alpha.\end{aligned}$$

Thus $\beta^{-1}(\hat{\theta}^k(\bar{\theta}))^{\alpha-1} = R(\hat{\theta}(\bar{\theta}))$ implies that $(\hat{\theta}^k(\bar{\theta}))^\alpha = \beta R(\hat{\theta}(\bar{\theta}))(\bar{\theta}^k)^\alpha$ and hence, using that $R(\hat{\theta}(\bar{\theta})) = (\hat{p}(\hat{\theta}(\bar{\theta})) + \hat{d})/\hat{p}(\hat{\theta}(\bar{\theta}))$ we get

$$\hat{p}(\bar{\theta})\hat{p}(\hat{\theta}(\bar{\theta}))\beta^{-1}(1-\beta)(\hat{\theta}^k(\bar{\theta}))^\alpha = \beta(\hat{p}(\hat{\theta}(\bar{\theta})) + \hat{d})\hat{p}(\bar{\theta})\beta^{-1}(1-\beta)(\bar{\theta}^k)^\alpha.$$

Therefore, since $\partial u(c) = 1/c$ we conclude that

$$\hat{p}(\bar{\theta})\tilde{c}^k(\tilde{\theta}^k(\bar{\theta}^k, \bar{\theta}, \hat{q}(\bar{\theta})), \hat{\theta}(\bar{\theta}), \tilde{q}^k(\hat{\theta}(\bar{\theta}), \hat{q}(\bar{\theta}))) = \beta(\hat{p}(\hat{\theta}(\bar{\theta})) + \hat{d})\tilde{c}^k(\bar{\theta}^k, \bar{\theta}, \hat{q}(\bar{\theta}))$$

which is the Euler equation (8) rearranging the terms.

To exhibit the characteristics of the agent j demand, let $\tilde{c}^k : \Theta^k \times \overline{\Theta} \times Q \to C^k$ and $\tilde{\theta}^k : \Theta^k \times \overline{\Theta} \times Q \to \Theta^k$ be the policy functions of agent k. Define $\tilde{c}^j : \Theta^j \times \overline{\Theta} \times Q \to C^j$ and $\tilde{\theta}^j : \Theta^j \times \overline{\Theta} \times Q \to \Theta^j$ by $\tilde{c}^j(\theta^j_-, \bar{\theta}, q) = \hat{d} - \tilde{c}^k(1-\theta^j_-, \bar{\theta}, q)$ and $\tilde{\theta}^j(\theta^j_-, \bar{\theta}, q) = 1 - \tilde{\theta}^k(1-\theta^j_-, \bar{\theta}, q) = 1 - (1-\theta^j_-)^\alpha < 1$. Therefore,

$$\tilde{c}^j(\bar{\theta}^j, \bar{\theta}, q) = -p\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, q) + (p+\hat{d})\bar{\theta}^j = -p(1-(1-\bar{\theta}^j)^\alpha) + (p+\hat{d})\bar{\theta}^j.$$

Keeping all other agent j characteristics identical to agent k, it is easy to see that one can find a price expectation function $\tilde{q}^j : \overline{\Theta} \times Q \to Q$ such that

$$\tilde{c}^j(\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, q), \hat{\theta}(\bar{\theta}), \tilde{q}^j(\hat{\theta}(\bar{\theta}), q)) = \beta\tilde{c}^j(\bar{\theta}^j, \bar{\theta}, q)(\tilde{p}^j(\hat{\theta}(\bar{\theta}), p) + \hat{d})/p \tag{10}$$

for $q = \hat{q}(\bar{\theta})$ where we recall that $p = q^a/q^c$. Indeed, the left hand side of (10) is equal to

$$-\tilde{p}^j(\hat{\theta}(\bar{\theta}), q)(1-(1-\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, q))^\alpha) + (p^j(\hat{\theta}(\bar{\theta}), q) + \hat{d})\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, q).$$

Since equation (10) is linear in $\tilde{p}^j$ and $\hat{\theta}$ is invertible, we can solve it for $\tilde{p}^j$. Observe that for $\boldsymbol{\theta}_0 > 0$ if we define recursively $\boldsymbol{\theta}_t = \hat{\theta}(\boldsymbol{\theta}_{t-1})$ for all $t \in \mathbb{N}$ then $\lim_{t\to\infty} \boldsymbol{\theta}_t = (0,1)$. Moreover, defining recursively $\boldsymbol{q}_t = \hat{q}(\boldsymbol{\theta}_{t-1})$ for all $t \in \mathbb{N}$ then $\lim_{t\to\infty} \boldsymbol{q}_t = \beta\hat{d}/(1-\beta)$.

The following proposition shows that if some agent k satisfies Assumption 4.1 with respect to a (convergent) sequential equilibrium, then the limit price must be equal to

$\beta\hat{d}/(1-\beta) = \beta\hat{d} + \beta^2\hat{d} + \cdots$, that is, the present value (in units of the good) over all subsequent periods of a constant flow $\hat{d}$ with discounted rate β.

Proposition 4.2. *Let $\{(\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t)\}_{t\in\mathbb{N}}$ be a convergent sequential equilibrium such that there exists some agent k satisfying Assumption 4.1. If $(\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t)$ converges to $(\boldsymbol{c}_*, \boldsymbol{\theta}_*, \boldsymbol{q}_*)$ as $t \to \infty$ with $\boldsymbol{c}_*^k > 0$ and $\boldsymbol{q}_* > 0$ then $\boldsymbol{q}_*^a/\boldsymbol{q}_*^c = \beta\hat{d}/(1-\beta)$.*

Proof: Let $T > 0$ be given as in Assumption 4.1. Then writing $\boldsymbol{p}_t = \boldsymbol{q}_t^a/\boldsymbol{q}_t^c$ and $\boldsymbol{p}_* = \boldsymbol{q}_*^a/\boldsymbol{q}_*^c$, the Euler Equation (8) for the agent k evaluated on the equilibrium path for each $t \geq T$ becomes

$$\boldsymbol{p}_t \partial u^k(\tilde{c}^k(\boldsymbol{\theta}_{t-1}^k, \boldsymbol{\theta}_{t-1}, \boldsymbol{q}_t)) = \beta \partial u^k(\tilde{c}^k(\boldsymbol{\theta}_t^k, \boldsymbol{\theta}_t, \boldsymbol{q}_{t+1}))(\boldsymbol{p}_{t+1} + \hat{d}) \tag{11}$$

since $\tilde{\theta}^k(\boldsymbol{\theta}_{t-1}^k, \boldsymbol{\theta}_{t-1}, \boldsymbol{q}_t) = \boldsymbol{\theta}_t^k$ and $\tilde{c}^k(\boldsymbol{\theta}_{t-1}^k, \boldsymbol{\theta}_{t-1}, \boldsymbol{q}_t) = \boldsymbol{c}_t^k$ for all $t \in \mathbb{N}$ by Lemma 7.10 in appendix. Passing to the limit Equation (11) as $t \to \infty$ then

$$\boldsymbol{p}_* \partial u^k(\tilde{c}^k(\boldsymbol{\theta}_*^k, \boldsymbol{\theta}_*, \boldsymbol{q}_*)) = \beta \partial u^k(\tilde{c}^k(\boldsymbol{\theta}_*^k, \boldsymbol{\theta}_*, \boldsymbol{q}_*))(\boldsymbol{p}_* + \hat{d})$$

and consequently, $\boldsymbol{p}_* = \beta\hat{d}/(1-\beta)$ since $\boldsymbol{c}_*^k > 0$ and

$$\boldsymbol{c}_*^k = \lim_{t\to\infty} \boldsymbol{c}_t^k = \lim_{t\to\infty} \tilde{c}^k(\boldsymbol{\theta}_{t-1}^k, \boldsymbol{\theta}_{t-1}, \boldsymbol{q}_t) = \tilde{c}^k(\boldsymbol{\theta}_*^k, \boldsymbol{\theta}_*, \boldsymbol{q}_*).$$

□

4.3 Inaccurate expectations

In this section we analyse survival for equilibria with trade. In the case of no-trade equilibrium, trivially, all agents with positive initial asset endowment survive. Roughly speaking, we show that if the economy has a continuous recursive equilibrium and two types then agents satisfying Assumption 4.1 below, which states that they have incentive to trade in the optimum for every possible realization of the state variable, are dominated in the market, that is, have zero consumption level in the long run. In particular, if some agent k is eventually Perfect Foresight and the other agent j has price expectations bounded away from the limit price $\boldsymbol{p}_* = \beta\hat{d}/(1-\beta)$ given in Proposition 4.2, then agent j is driven out of the market.

Notation 4.2. Write $I = K \cup J$ where J is the set of agents satisfying Assumption 4.1 below and K is the set of agents with eventually correct expectations as in Definition 4.1.

Assumption 4.1. The asset policy function $\tilde{\theta}^j$ satisfies $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, q) \neq \bar{\theta}^j$ for all $(\bar{\theta}, q) \in \overline{\Theta} \times Q$ with $\bar{\theta}^j > 0$.

Lemma 4.3. *Assumption 4.1 holds if and only if the price expectation* $\tilde{p}^j$ *and the beliefs* λ^j *satisfy for each* $\bar{\theta} \in \overline{\Theta}$ *with* $\bar{\theta}^j > 0$ *and* $p = q^a/q^c \in \mathbb{R}_+$

$$p\partial u^j(\hat{d}\bar{\theta}^j) \neq \beta \int_{\overline{\Theta}} \partial u^j[\tilde{c}^j(\bar{\theta}^j, \bar{\theta}', \hat{q}^j(\bar{\theta}', q))](\tilde{p}^j(\bar{\theta}', p) + \hat{d})\lambda^j(\bar{\theta}, d\bar{\theta}')$$

where $\tilde{c}^j : \Theta^j \times \overline{\Theta} \times Q \to C^j$ *is the argmax of agent j's Bellman Equation (2).*

Proof: This is a direct consequence of the Euler Equation (8) evaluated on $\bar{\theta}$ such that $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, q) = \bar{\theta}^j$.

□

Lemma 4.4 below gives some conditions on the beliefs such that the demand of agent j satisfies Assumption 4.1. Under these beliefs, aggregate portfolio distribution is an i.i.d process and the current asset price (in units of the good) is not the rate β discounted value of the next period expected pay off. We know from the recursive relations given in Definition 3.4 that current aggregate portfolio distribution is correlated with the past aggregate portfolio distribution in the sequential equilibrium implemented by a recursive equilibrium. Therefore there is a kind of inaccuracy on these beliefs because they are not specifying correctly the relation that describes the transition of the variables.

Lemma 4.4. *Suppose there exists a probability* $\nu^j \in \text{Prob}(\overline{\Theta})$ *such that* $\lambda^j(\bar{\theta}) = \nu^j$ *for all* $\bar{\theta} \in \overline{\Theta}$ *and the price expectation function* $\tilde{p}^j$ *satisfies*

$$\beta \left(\hat{d} + \int_{\overline{\Theta}} \tilde{p}^j(\bar{\theta}', p)\nu^j(d\bar{\theta}') \right) \neq p \text{ for all } p \in \mathbb{R}_+. \tag{12}$$

Then Assumption 4.1 holds.

Proof: The Bellman operator of agent j becomes for $\bar{\theta}^j > 0$

$$T^j(v^j)(\theta^j_-, \bar{\theta}, q) = \sup \left\{ u^j(c^j) + \beta \int_{\overline{\Theta}} v^j(\theta^j, \bar{\theta}', \tilde{q}^j(\bar{\theta}', q))\nu^j(d\bar{\theta}') \right\}$$

over all $(c^j, \theta^j) \in B^j(\theta^j_-, q)$. Observe that if we consider the closed[19] set $\mathbb{V}' \subset \mathbb{V}$ of all value functions constant on $\bar{\theta}$ then $\mathbb{V}'$ is invariant under T^j. Therefore the value function v^j is constant[20] on the variable $\bar{\theta}$. Suppose now that Assumption 4.1 does not hold. Then there exists $(\bar{\theta}, \bar{q}) \in \overline{\Theta} \times Q$ such that $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, \bar{q}) = \bar{\theta}^j$ and $\tilde{c}^j(\bar{\theta}^j, \bar{\theta}, \bar{q}) = \hat{d}\bar{\theta}^j$. Since the policy function $\tilde{\theta}^j$ is constant on $\bar{\theta}$ we have $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}', \bar{q}) = \bar{\theta}^j$ for all $\bar{\theta}' \in \overline{\Theta}$ and hence

$$\tilde{c}^j(\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, \bar{q}), \bar{\theta}', \tilde{q}^j(\bar{\theta}', \bar{q})) = -\tilde{p}^j(\bar{\theta}', \bar{p})\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}', \bar{q}) + (\tilde{p}^j(\bar{\theta}', \bar{p}) + \hat{d})\bar{\theta}^j = \hat{d}\bar{\theta}^j$$

[19]For the sup norm topology.
[20]Because it is a fixed point of T^j.

where we recall that $\bar{p} = \bar{q}^a/\bar{q}^c$. Therefore the Euler Equation (8) becomes[21]

$$\bar{p} = \beta \int_{\overline{\Theta}} (\tilde{p}^j(\bar{\theta}', \bar{p}) + \hat{d})\nu^j(d\bar{\theta}')$$

which contradicts (12) for $p = \bar{p}$.

□

Notice that an example of expectation function $\tilde{p}^j$ under which equation (12) holds is $\tilde{p}^j(\bar{\theta}, p) = p/\beta$, that is, agent j believes that asset price measured in units of the good increases at a gross rate $1/\beta$ over any two consecutive periods. Another example for which the equation (12) holds is the one such that $\tilde{p}^j$ satisfies $\tilde{p}^j(\bar{\theta}, p) > p/\beta - \hat{d}$ for all $(\bar{\theta}, p) \in \overline{\Theta} \times \mathbb{R}_+$. Notice that if $\bar{p} = \beta\hat{d}/(1-\beta)$ then $\tilde{p}^j(\bar{\theta}, \bar{p}) > \bar{p}/\beta - \hat{d} = \beta\hat{d}/(1-\beta) = \beta\hat{d} + \beta^2\hat{d} + \cdots$, that is, asset price expected on the next period when the current price is $\beta d/(1-\beta)$ is greater than the present value (in units of the good) over all subsequent periods of a constant flow $\hat{d}$ with discounted rate β. In economies in which some agent is eventually Perfect Foresight and survives, Proposition 4.2 assures that asset price in a convergent sequential equilibrium (in units of the good) is $\beta\hat{d}/(1-\beta)$ in the long run. Therefore, agent j price expectation function is bounded away from the correct limit price in the long run.[22]

In the proof of next theorem, we use the continuous recursive relations given in the previous section and a fixed point result for continuous functions defined on the compact interval $[0, 1] \subset \mathbb{R}_+$.

Theorem 4.5. *Suppose that $I = \{j, k\}$ with agent j satisfying Assumption 4.1. Let $(\hat{c}, \hat{\theta}, \hat{q})$ be a continuous recursive equilibrium and $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ be the sequential equilibrium implemented by it and starting from $\boldsymbol{\theta}_0 > 0$. If $\hat{q}$ is continuous then the sequential equilibrium converges and agent j is dominated*[23] *in the market.*

Proof: Let $(\hat{c}, \hat{\theta}, \hat{q})$ be the recursive equilibrium and $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ the sequential equilibrium implemented by it and starting from $\boldsymbol{\theta}_0 > 0$. Let $\tilde{c}^i : \Theta^i \times \overline{\Theta} \times Q^\circ \to C^i$ and $\tilde{\theta}^i : \Theta^i \times \overline{\Theta} \times Q^\circ \to \Theta^i$ be the argmax of the Bellman Equation (2). We recall that[24] $\hat{\theta}(\bar{\theta}) = (\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, \hat{q}(\bar{\theta})), \tilde{\theta}^k(\bar{\theta}^k, \bar{\theta}, \hat{q}(\bar{\theta})))$ for all $\bar{\theta} \in \overline{\Theta}$.

Assumption 4.1 assures that $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, \hat{q}(\bar{\theta})) \neq \bar{\theta}^j$ for all $\bar{\theta} \in \overline{\Theta}$ with $\bar{\theta}^j > 0$. Moreover, the continuity of $\tilde{\theta}^j$ and $\hat{q}$ implies that either $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, \hat{q}(\bar{\theta})) > \bar{\theta}^j$ for all $\bar{\theta} \in \overline{\Theta}$ with $\bar{\theta}^j > 0$ or $\tilde{\theta}^j(\bar{\theta}^j, \bar{\theta}, \hat{q}(\bar{\theta})) < \bar{\theta}^j$ for all $\bar{\theta} \in \overline{\Theta}$ with $\bar{\theta}^j > 0$. Using that the sequential equilibrium is implemented by the recursive equilibrium, $\boldsymbol{\theta}_0^j > 0$ and

$$\boldsymbol{\theta}_t^j = \hat{\theta}^j(\boldsymbol{\theta}_{t-1}) = \tilde{\theta}^j(\boldsymbol{\theta}_{t-1}^j, \boldsymbol{\theta}_{t-1}, \hat{q}(\boldsymbol{\theta}_{t-1})) \text{ for all } t \in \mathbb{N}, \tag{13}$$

[21] Observe that $\partial u^i(c^i) \neq 0$ for all $c^i \in C^i$.

[22] Notice that the beliefs of agent j do not take into account the possibility of no trade equilibrium.

[23] That is, $\lim_{t\to\infty} \boldsymbol{\theta}_t^j = 0$

[24] Recall that the policy correspondence must actually be a function since there is only one asset.

we conclude that the sequence $\{\boldsymbol{\theta}_t^j\}_{t\in\mathbb{N}}$ is monotone and hence converges to $\boldsymbol{\theta}_*^j > 0$. Moreover the sequence $\boldsymbol{\theta}_t = (\boldsymbol{\theta}_t^j, \boldsymbol{\theta}_t^k) = (\boldsymbol{\theta}_t^j, 1-\boldsymbol{\theta}_t^j)$ converges to $\boldsymbol{\theta}_* := (\boldsymbol{\theta}_*^j, 1-\boldsymbol{\theta}_*^j)$ implying that $\boldsymbol{q}_t = \hat{q}(\boldsymbol{\theta}_{t-1})$ converges to $\boldsymbol{q}_* := \hat{q}(\boldsymbol{\theta}_*)$ as $t\to\infty$. Passing to the limit equation (13) we get $\boldsymbol{\theta}_*^j = \tilde{\theta}^j(\boldsymbol{\theta}_*^j, \boldsymbol{\theta}_*, \hat{q}(\boldsymbol{\theta}_*))$ which contradicts Assumption 4.1 for $q = \boldsymbol{q}_*$ and $\bar{\theta} = \boldsymbol{\theta}_*$ if $\boldsymbol{\theta}_*^j > 0$. Therefore $\boldsymbol{\theta}_*^j = 0$ and hence agent j is dominated in the financial market.

□

Remark 4.2. Notice that if all agents satisfy the condition of Assumption 4.1 then there does not exist a recursive equilibrium with a continuous price.

4.4 The case S convex or finite

Theorem 4.5 holds, under some conditions which will be specified, if we suppose $S \subset \mathbb{R}^n$ convex or finite and that the exogenous uncertainty is governed by a stochastic process $\{\hat{s}_t : \Omega \to S\}_{t\in\mathbb{N}}$ defined on a probability space $(\Omega, \Sigma, \mathbb{P})$. Assumption 4.1 in this case becomes

Assumption 4.2. The asset policy function $\tilde{\theta}^j$ satisfies $\tilde{\theta}^j(\bar{\theta}^j, y, q) \neq \bar{\theta}^j$ for all $(y,q) \in Y \times Q$ with $\bar{\theta}^j > 0$ where $y = (\bar{\theta}, s)$.

Thus we have the following theorem

Theorem 4.6. *Suppose that $I = J \cup \{k\}$ with each agent $j \in J$ satisfying Assumption 4.2. Let $(\hat{c}, \hat{\theta}, \hat{q})$ be a continuous recursive equilibrium and $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ be the sequential equilibrium implemented by it and starting from $\boldsymbol{\theta}_0 > 0$. If S is convex and $\hat{s}_t$ converges in distribution, or S is finite and $\hat{s}_t$ converges almost everywhere, then each agent $j \in J$ is dominated[25] in the market.*

Proof: Let $(\hat{c}, \hat{\theta}, \hat{q})$ be the recursive equilibrium and $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ the sequential equilibrium implemented by it and starting from $\boldsymbol{\theta}_0 > 0$. Let $\tilde{c}^i : \Theta^i \times Y \times Q^\circ \to C^i$ and $\tilde{\theta}^i : \Theta^i \times Y \times Q^\circ \to \Theta^i$ be the argmax of the Bellman Equation (2).

Suppose that S is convex and $\{\hat{s}_t\}_{t\in\mathbb{N}}$ converges in distribution to $\hat{s}_*$. Assumption 4.1 assures that $\tilde{\theta}^j(\bar{\theta}^j, y, \hat{q}(y)) \neq \bar{\theta}^j$ for all $y \in Y$ with $\bar{\theta}^j > 0$. Using the same arguments given in Theorem 4.5, we conclude that the stochastic process $\{\hat{\boldsymbol{\theta}}_t : \Omega \to S\}_{t\in\mathbb{N}}$ defined recursively by $\hat{\boldsymbol{\theta}}_t^i(\omega) = \tilde{\theta}^i(\hat{\boldsymbol{\theta}}_{t-1}^i(\omega), \hat{\boldsymbol{y}}_t(\omega), \hat{q}(\hat{\boldsymbol{y}}_t(\omega)))$ for all $i \in I$ and the stochastic process $\{\hat{\boldsymbol{q}}_t = \hat{q}(\hat{\boldsymbol{y}}_t)\}_{t\in\mathbb{N}}$ where $\hat{\boldsymbol{y}}_t(\omega) = (\hat{\boldsymbol{\theta}}_{t-1}(\omega), \hat{s}_t(\omega))$, converge in distribution to $\hat{\boldsymbol{\theta}}_*$ and $\hat{\boldsymbol{q}}_*$ respectively as $t \to \infty$. Suppose that $\hat{\boldsymbol{\theta}}_*(\omega') > 0$ for some $\omega' \in \Omega$. Since $\hat{\boldsymbol{\theta}}_t^j(\omega) = \hat{\theta}^j(\hat{\boldsymbol{y}}_t(\omega)) = \tilde{\theta}^j(\hat{\boldsymbol{\theta}}_{t-1}^j(\omega), \hat{\boldsymbol{y}}_t(\omega), \hat{q}(\hat{\boldsymbol{y}}_t(\omega)))$ for each $\omega \in \Omega$, passing to the limit in distribution we get $\hat{\boldsymbol{\theta}}_*^j(\omega') = \tilde{\theta}^j(\hat{\boldsymbol{\theta}}_*^j(\omega'), \hat{\boldsymbol{y}}_*(\omega'), \hat{q}(\hat{\boldsymbol{y}}_*(\omega')))$ where $\hat{\boldsymbol{y}}_* : \Omega \to Y$ is defined by $\hat{\boldsymbol{y}}_*(\omega) = (\hat{\boldsymbol{\theta}}_*(\omega), \hat{s}_*(\omega))$. This contradicts Assumption 4.1 for $q = \hat{\boldsymbol{q}}_*(\omega')$ and $y = (\hat{\boldsymbol{\theta}}_*(\omega'), \hat{s}_*(\omega'))$. Therefore, $\hat{\boldsymbol{\theta}}_*^j(\omega) = 0$ for each $\omega \in \Omega$.

[25] That is $\boldsymbol{\theta}_t^j(\boldsymbol{\theta}_{t-1}(\hat{s}^{t-1}(\omega)), \hat{s}_t(\omega))$ converges to zero for almost all $\omega \in \Omega$.

Suppose that S is finite. Since $\{\hat{s}_t\}_{t\in\mathbb{N}}$ converges almost everywhere, taking $\epsilon > 0$ sufficiently small, one can find a set $A \in \Sigma$ with $\mathbb{P}(A) = 1$ satisfying the following property: for each fixed $\omega \in A$, there exists t_ω such that $\hat{s}_t(\omega) = s' \in S$ for $t \geq t_\omega$. Moreover, since $\tilde{\theta}^j(\bar{\theta}^j, (\bar{\theta}, s'), \hat{q}(\bar{\theta}, s')) \neq \bar{\theta}^j$ for all $\bar{\theta} \in \overline{\Theta}$ with $\bar{\theta}^j > 0$ we can apply again the contradiction arguments above to conclude the theorem.

□

5 Price perfect foresight

Many existence theorems and counter examples can be found in the literature when agents have Rational Expectations. Lucas Jr (1978) proves existence of recursive equilibrium with homogeneous agents. Kubler & Schmedders (2002) give examples of non existence of recursive equilibrium in models with heterogeneous agents, short lived assets and restriction of non Ponzi Schemes. Coleman (1991) shows existence of recursive equilibrium for models with homogeneous agents, production and income tax. Krebs (2004) shows non existence of recursive equilibrium in compact state spaces and incomplete markets such that borrowing credit constrains never bind and Braido (2008) proves existence of an ergodic Markov equilibrium[26] for a class of economies with incomplete markets, default and without the usual utility penalties as in Dubey et al. (2005). In the previous sections we showed existence of recursive equilibrium in economies where agents have exogenous expectations on endogenous and exogenous variables. The objective of this section is to address the existence of recursive equilibrium when agents display some ability to anticipate some (but not all) endogenous variables in the economy. In this approach, for the sequential equilibrium implemented by a recursive equilibrium, agents may not have *common and correct expectations*[27] which requires traders to associate the same future prices to the same future exogenous events, but does not require them to agree on the (subjective) probabilities associated with those events. More precisely, here agents may correctly anticipate the relation between prices and the state variables but not necessarily anticipate with accuracy the transition of the mean aggregate portfolio of the economy. Clearly, in an equilibrium implemented by the recursive relation (5) agents who do not anticipate the transition of the mean aggregate portfolio of the economy may not have *common and correct expectations.* The existence of a recursive equilibrium with the state space $\overline{\Theta} \times S$ and heterogeneous agents having *common and correct expectations* in the sequential equilibrium implemented by it is an open question. Consequently, the existence of a recursive equilibrium with the same state space and agents having or not the ability to anticipate all endogenous uncertainty of the economy is also an open question. Nevertheless, we show under some conditions on the primitives of the model that it is

[26]The ergodic Markov equilibrium is recursive and the state space contains all aggregate variables of the economy. See Duffie et al. (1994) for further details.

[27]This concept is defined in Radner (1972).

possible to find a recursive equilibrium with the state space $\overline{\Theta} \times S$ and such that at least one type anticipates correctly the price expectation function. We say that agents have *Price Perfect Foresight* or PPF when they anticipate correctly the relation between price and state variables. We say that agents have *Exogenous Expectations* or EE when they use an expectation function to anticipate (maybe incorrectly) prices. Moreover, we index these agents in the sets K and J respectively and write $I = K \cup J$.

In this section we define the PPF agents' characteristics and the sequential equilibrium analogously to definitions given in Sections 2 and 3 respectively. The concept of recursive equilibrium is modified to incorporate the PPF agents. Under some conditions on the primitives of the economy, we prove its existence and that it implements the sequential equilibrium. In the existence proof we show that the expectation function of agents with PPF, which was exogenous in the economy defined on the previous sections, is determined endogenously and coincides with the recursive equilibrium price.

The difference of definitions given in Section 2 and definitions of this section for the PPF agents is that we suppose here the state space endowed with the σ-algebra $\mathscr{Y}$ of all subsets of Y and denote it by Y again to simplify the notation. The PPF agents' plans are given as in Definition 2.1 because they may not anticipate correctly the portfolio aggregate transition $\hat{\theta}$ of the economy and hence choose plans contingent to all streams $y^r \in Y^r$ for $r \in \mathbb{N}$ as the EE agents. The (exogenous) beliefs $\mu_r^k : Y \to \text{Prob}(Y^r)$ are generated by the probability transition rules $\lambda_k : Y \to \text{Prob}(Y)$ and do not depend on the current observed price in this case. The feasible plans, expected utility[28] and demand correspondence of the agents with PPF are given as in Section 2. For agents with Exogenous Expectations, all definitions given in Section 2 are the same.

The next definition specifies an equilibrium of this sequential economy and clarifies how agents with PPF anticipate correctly the contingent prices.

Definition 5.1. *A PPF sequential equilibrium with initial asset holdings $\boldsymbol{\theta}_0 \in \overline{\Theta}$ is a measurable family of contingent prices $\{\boldsymbol{q}_t : S^t \to Q^\circ\}_{t\in\mathbb{N}}$, contingent consumption allocations $\{\boldsymbol{c}_t : S^t \to C\}_{t\in\mathbb{N}}$ and contingent portfolio allocations $\{\boldsymbol{\theta}_t : S^t \to \Theta\}_{t\in\mathbb{N}}$ satisfying for all $s^t \in S^t$:*

1. *optimality: for every $i \in I$,*

$$(\boldsymbol{c}_t^i(s^t), \boldsymbol{\theta}_t^i(s^t)) \in \delta^i(\boldsymbol{\theta}_{t-1}^i(s^{t-1}), (\boldsymbol{\theta}_{t-1}(s^{t-1}), s_t), \boldsymbol{q}_t(s^t));$$

2. *agent k's price consistency: $\hat{q}^k((\boldsymbol{\theta}_t(s^t), s_{t+1}), \boldsymbol{q}_t(s^t)) = \boldsymbol{q}_{t+1}(s^{t+1})$;*
3. *asset markets clear: $\sum_{i\in I} \boldsymbol{\theta}^i(s^t) = \mathbf{1} \in \mathbb{R}^H$;*
4. *good markets clear: $\sum_{i\in I} \boldsymbol{c}^i(s^t) = \mathbf{1} \cdot \hat{d}(s_t)$.*

[28] Notice that the utility function in Definition 2.4 does not depend on current prices because $\mu_r^k : Y \to Y^r$.

Remark 5.1. Condition *2* means that agents with Price Perfect Foresight have an expectation function which yields exactly the next period price given the current equilibrium price and the next period equilibrium state variable. Observe that we only impose this consistency on prices expectations over the equilibrium path $\{\boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$.

The definition of the recursive equilibrium follows by a way analogous to the one given in Section 3, that is, we show existence of the value function satisfying the Bellman Equation and after we define the recursive equilibrium. In the recursive approach we consider price expectation consistency imposed at all state variables for agents with PPF.

Notation 5.1. Write

1. $Z = \{(\bar{\theta}, s) \in \overline{\Theta} \times S : \sum_{k\in K} \bar{\theta}^k_h \neq 1 \text{ for all } h \in H\} \subset Y$;
2. $\widehat{Q}' = \{\hat{q} : Y \to Q : \hat{q}(y) \in Q^\circ \text{ for all } y \in Z\} \supset \widehat{Q}^\circ$ endowed with the topology of pointwise convergence.
3. $\mathbb{V}$ the Banach space of all bounded functions $v^k : \Theta^k \times Y \times \widehat{Q}' \to \mathbb{R}$ with $v^k(\,\cdot\,, y, \,\cdot\,)$ continuous for each fixed $y \in Z$ and endowed with the sup norm.

Since the space $\widehat{Q}$ endowed with the product topology is compact but not metrizable, we need the assumption below to assure the continuity of the integration over stochastic kernels λ^k given by Lemma 7.11. This assumption assures that agent k's beliefs of next period state variables yield zero probability to the set of mean aggregate portfolios for which each agent j has zero portfolio endowment.[29]

Assumption 5.1. For each fixed $y \in Y$ there exists a countable set $W \subset Z$ such that $\lambda^k(y)(W) = 1$.

Remark 5.2. Is not clear that the results of this section hold if there are only non identical PPF agents satisfying Assumption 5.1 because in the proof of existence we use that there always exists one agent j with positive wealth to assure positivity of equilibrium prices.

We claim that there exists a function $v^k_{pf} : \Theta^k \times Y \times \widehat{Q}' \to \mathbb{R}$ with $\hat{v}^k_{pf}(\,\cdot\,, y, \,\cdot\,)$ continuous for each $y \in Z$ and satisfying the Bellman Equation

$$v^k_{pf}(\theta^k_-, y, \hat{q}) = \sup\left\{u^k(c^k) + \beta \int_Y v^k_{pf}(\theta^k, y', \hat{q})\lambda^k(y, dy')\right\} \tag{14}$$

over all $(c^k, \theta^k) \in B^k(\theta^k_-, s, \hat{q}(y))$. Indeed let $T^k : \mathbb{V} \to \mathbb{V}$ be the operator defined by

$$T^k(v^k_{pf})(\theta^k_-, y, \hat{q}) = \sup\left\{u^k(c^k) + \beta \int_Y v^k_{pf}(\theta^k, y', \hat{q})\lambda^k(y, dy')\right\} \tag{15}$$

over all $(c^k, \theta^k) \in B^k(\theta^k_-, s, \hat{q}(y))$. To see that T^k is well defined, notice that for each fixed $y = (\bar{\theta}, s) \in Z$, the correspondence $(\theta^k_-, \hat{q}) \to B^k(\theta^k_-, s, \hat{q}(y))$ defined on $\Theta^k \times \widehat{Q}'$

[29] If we consider that some agent j has positive good endowment at each period then he may choose a current positive portfolio even if the current asset endowment is null in some period.

is continuous as a composition of continuous correspondences by Lemma 7.3 because[30] $((\theta^k_- \hat{d}(s), \theta^k_-), \hat{q}(y)) \in A^k$ for all $\hat{q} \in \widehat{Q}'$ where A^k is defined in the appendix. Moreover the objective function is continuous on c^k, θ^k and $\hat{q}$ for each fixed $y \in Y$ by Lemma 7.11. Applying the Berge Maximum Theorem, we conclude that if $v^k_{pf} \in \mathbb{V}$, that is, $v^k_{pf}(\,\cdot\,, y, \,\cdot\,)$ is continuous for each $y \in Z$, then $T(v^k_{pf})(\,\cdot\,, y, \,\cdot\,)$ is continuous for each $y \in Z$. Clearly, T^k satisfies the Blackwell's sufficient conditions for a contraction and hence has a fixed point. Notice that $v^k_{pf}(\,\cdot\,, y, \hat{q})$ is strictly increasing for each $(y, \hat{q}) \in Z \times \widehat{Q}'$ if $C^k = \mathbb{R}_+$.

Definition 5.2. *Define the agent k's consumption and portfolio policy correspondence*[31] $\tilde{x}^k_{pf} : \Theta^k \times Y \times \widehat{Q}' \to C^k \times \Theta^k$ *with* $\tilde{x}^k_{pf} = \tilde{c}^k \times \tilde{\theta}^k$ *as*

$$\tilde{x}^k_{pf}(\theta^k_-, y, \hat{q}) = \text{argmax}\left\{u^k(c^k) + \beta \int_Y v^k_{pf}(\theta^k, y', \hat{q})\lambda^k(y, dy')\right\}$$

over all $(c^k, \theta^k) \in B^k(\theta^k_-, s, \hat{q}(y))$.

Definition 5.3. *We say that the economy has a PPF recursive equilibrium if there exist functions* $\hat{c}^i : Y \to C^i$, $\hat{\theta}^i : Y \to \Theta^i$ *for all* $i \in I$ *and* $\hat{q} : Y \to Q$ *satisfying for each* $y = (\bar{\theta}, s) \in Y$

1. *EE's optimality:* $(\hat{c}^j(y), \hat{\theta}^j(y)) \in \tilde{x}^j(\bar{\theta}^j, y, \hat{q}(y))$;
2. *PPF's optimality:* $(\hat{c}^k(y), \hat{\theta}^k(y)) \in \tilde{x}^k_{pf}(\bar{\theta}^j, y, \hat{q})$;
3. *asset market clearing:* $\sum_{i\in I} \hat{\theta}^i(y) = \mathbf{1} \in \mathbb{R}^H$;
4. *consumption market clearing:* $\sum_{i\in I} \hat{c}^i(y) = \mathbf{1} \cdot \hat{d}(s)$.

Under Assumption 5.1 the following results are similar to the ones given in previous sections.

Theorem 5.4. *Under Assumption 5.1, there exists a PPF recursive equilibrium.*

Proof: See Theorem 7.12 in Appendix.

□

Theorem 5.5. *If* $(\hat{c}, \hat{\theta}, \hat{q})$ *is a PPF recursive equilibrium then its implemented process* $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t\in\mathbb{N}}$ *starting from* $\boldsymbol{\theta}_0 \in \overline{\Theta}$ *is a PPF sequential equilibrium of the economy with initial asset holdings* $\boldsymbol{\theta}_0 \in \overline{\Theta}$.

Proof: The proof is analogous[32] to Theorem 3.5 replacing the price expectation function $\hat{q}^k$ by the price recursive equilibrium $\hat{q}$.

□

[30] Take $W^k = \{(\theta^k \hat{d}(s), \theta^k) : \theta^k \in \Theta^k\}$ in this lemma.
[31] This correspondence may be empty.
[32] In the arguments of Theorem 3.5 we use the choice axiom to select the measurable plans in the optimum since any function defined on Y^r is measurable.

6 Conclusion

Existence of recursive equilibrium is a propriety of economies where trade takes place sequentially over time and where each agent makes decisions at every date in the light of his (possibly incorrect) expectations about his future environment. Moreover there exists a recursive equilibrium even if some type anticipate correctly the function which specifies the recursive relation between the sequential price of equilibrium and state variables. The uniqueness of the recursive equilibrium assures the continuity. The continuity of the recursive equilibrium allows us to conclude that if one agent has price expectation functions bounded away from the discounted cash flow of future dividends and the other agent is eventually Perfect Foresight, then the first agent has zero asset endowment[33] in the long run.

7 Appendix

7.1 Results related to Section 3

For the sake of completeness we enunciate the lemma below. A similar result can be found in Grandmont (1972).

Lemma 7.1. *Let $(Y, \mathscr{Y})$ be a compact metric space with $\mathscr{Y}$ its Borelians and Z a metric space. Consider a bounded continuous*[34] *$f : Y \times Z \to \mathbb{R}_+$ and the continuous kernel*[35] *$\nu : Z \to \text{Prob}(Y)$. Then the function $g : Z \to \mathbb{R}_+$ defined by $g(z) = \int_Y f(y', z)\nu(z, dy')$ is continuous.*

Proof: Fix $z \in Z$ and let $z_n \to z$ as $n \to \infty$. Using that ν and $f(\,\cdot\,, z)$ are continuous, we can take n' such that $n \geq n'$ implies

$$\left| \int_Y f(y, z)\nu(z_n, dy) - \int_Y f(y, z)\nu(z, dy) \right| < \epsilon/2.$$

Write $Z' = \{z_1, z_2, ...\} \cup \{z\}$. The continuity of f and the compactness of $Y \times Z'$ allow us to conclude that f is uniformly continuous on $Y \times Z'$, and hence, we find[36] an $n'' \in \mathbb{N}$ such that

$$|f(y, z_n) - f(y, z)| < \epsilon/2 \text{ for all } y \in Y \text{ and } n \geq n''$$

[33] And hence have zero consumption if there is no income at each period.

[34] On the product topology of $Y \times Z$ induced by the metric $d_{Y\times Z}((y, z), (y', z')) = d_Y(y, y') + d_Z(z, z')$. Notice that $Y \times Z'$ is compact on this topology for each compact subset $Z' \subset Z$.

[35] The space Prob(Y) is endowed with the weak topology.

[36] Notice that $d_Z(z', z'') < \epsilon$ implies $d_{Y\times Z}((y, z'), (y, z'')) < \epsilon$.

Choose $n_0 = \max\{n', n''\}$. Then $n \geq n_0$ implies

$$\begin{aligned} |g(z_n) - g(z)| &\leq \left| \int_Y f(y, z_n) - f(y, z)\nu(z_n, dy) \right| \\ &+ \left| \int_Y f(y, z)\nu(z_n, dy) - \int_Y f(y, z)\nu(z, dy) \right| \\ &\leq \int_Y |f(y, z_n) - f(y, z)|\nu(z_n, dy) \\ &+ \left| \int_Y f(y, z)\nu(z_n, dy) - \int_Y f(y, z)\nu(z, dy) \right| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

□

Lemma 7.2. *Let v be the fixed point of the operator T given by (3). Then for every fixed $(y, q) \in Y \times Q^\circ$, $v(\,\cdot\,, y, q)$ is concave. Moreover, if $C^i = \mathbb{R}_+$ then v^i is strictly increasing in the first coordinate.*

Proof: A similar proof is found in Stokey et al. (1989). First notice that if a contraction is invariant under a nonempty closed subspace F then the fixed point belongs to F. It is easy to see that the set of concave and increasing functions on some fixed coordinate is closed and nonempty. To prove that it is invariant by T, let v be a concave function, $\alpha > 0$, $(c^i, \theta^i) \in \tilde{x}^i(\theta^i_-, y, q)$, $(\check{c}^i, \check{\theta}^i) \in \tilde{x}^i(\check{\theta}^i_-, y, q)$, $c^i_\alpha = \alpha c^i + (1 - \alpha)\check{c}^i$ and $\theta^i_\alpha = \alpha\theta^i + (1 - \alpha)\check{\theta}^i$ where $\tilde{x}^i$ is given by Definition 3.2. Then $(c^i_\alpha, \theta^i_\alpha) \in B^i(\theta^i_{\alpha-}, q)$ where $\theta^i_{\alpha-} = \alpha\theta^i_- + (1-\alpha)\check{\theta}^i_-$ and

$$\begin{aligned} [T(v)](\theta^i_{\alpha-}, q) &= \sup \left\{ u^i(c^i) + \beta \int_Y v(\theta^i, q)\lambda(y, q, dy) \right\} \\ &\geq u^i(c^i_\alpha) + \beta \int_Y v(\theta^i_\alpha, q)\lambda(y, q, dy) \\ &\geq \alpha[T(v)](\theta^i, q) + (1 - \alpha)[T(v)](\check{\theta}^i, q). \end{aligned}$$

The first inequality holds because the sup is taken over $B^i(\theta^i_{\alpha-}, q)$ and the second holds because u^i and v are concave by hypothesis. To show that $v(\,\cdot\,, y, q)$ is strictly increasing, we use that the function $\theta^i_- \mapsto (q^a + q^c\hat{d}(s))\theta^i_-$ is strictly increasing in θ^i_{h-} for each $h \in H$ since $q \in Q^\circ$ and $C^i = \mathbb{R}_+$. Indeed, an increasing in θ^i_{h-} allows agent i to increase current consumption keeping it feasible.

□

Notation 7.1. Write for $L \in \mathbb{N}$:

1. $Q = \{q \in \mathbb{R}^L_+ : \sum_l q_l = 1\}$ and $Q^\circ = Q \cap \mathbb{R}^L_{++}$;

2. $X^i \subset \mathbb{R}^L_+$ nonempty convex for all $i \in I$;

3. $W^i \subset \mathbb{R}^L_+$ convex bounded with $W^i \cap \mathbb{R}^L_{++} \neq \emptyset$ for all $i \in I$;

4. S a compact metric space endowed with the σ-algebra of the Borelians;

5. $Y \subset \mathbb{R}^L_+ \times S$ nonempty compact with $\mathscr{Y}$ its σ-algebra;

6. $A^i = \{(w^i, q) \in W^i \times Q : q \in Q^\circ \quad \text{or} \quad qw^i > 0\}$;

7. $\widehat{Q}$ the set of all functions[37] $\hat{q} : Y \to Q$ and $\widehat{X}^i$ the set of all functions $\hat{x}^i : Y \to X^i$ both endowed with the τ weak topology of pointwise convergence.[38]

Lemma 7.3. *Suppose that $X^i \subset \mathbb{R}^L_+$ is a convex set with $0 \in \operatorname{Int} X^i$. Let $B^i : \mathbb{R}^L_+ \times Q \to X^i$ be the budget correspondence defined by*

$$B^i(w^i, q) = \{x^i \in X^i : qx^i \leq qw^i\}.$$

Then B^i is continuous on A^i when X^i is compact and on $\mathbb{R}^L_+ \times Q^\circ$ when $X^i = \mathbb{R}^L_+$.

Proof: Suppose that X^i is compact and convex. The upper hemicontinuity follows from the fact that B^i has closed graph and compact range space. To show the lower hemicontinuity, let $(w^i_n, q_n) \in A^i$ converging to $(\bar{w}^i, \bar{q}) \in A^i$ as $n \to \infty$ and $\bar{x}^i \in B^i(\bar{w}^i, \bar{q})$.

Suppose first that $\bar{q}\bar{w}^i > 0$. Then there exists an open set[39] O of A^i containing $(\bar{w}^i, \bar{q})$ such that $qw^i > 0$ for all $(w^i, q) \in O$. Let $\operatorname{Int} B^i : O \to X^i$ be the correspondence defined by $\operatorname{Int} B^i(w^i, q) = \{x^i \in X^i : qx^i < qw^i\}$. Since $0 \in X^i$, $\operatorname{Int} B^i$ is nonempty on the set O and X^i is convex, then[40] $B^i(w^i, q) = \operatorname{cl}[\operatorname{Int} B^i(w^i, q)]$ for all $(w^i, q) \in O$. Clearly, $\operatorname{Int} B^i$ has open graph. Therefore, using that an open graph correspondence is lower hemicontinuous and that the closure of a lower hemicontinuous correspondence is lower hemicontinuous, we conclude that B^i is lower hemicontinuous on O and hence there exists an $N \subset \mathbb{N}$ and a sequence[41] $x^i_n \in B^i(w^i_n, q_n)$ for each $n \in N$ such that $x^i_n \to \bar{x}^i$ as $n \to \infty$.

If $\bar{q}\bar{w}^i = 0$ then $(\bar{w}^i, \bar{q}) \in A^i$ implies that $\bar{q} \in Q^\circ$ and $\bar{x}^i = 0$. Since $\bar{q}_1 > 0$ and $0^L \in \operatorname{Int} X^i$, there exists $N \subset \mathbb{N}$ such that $q_{1n} > 0$ and $(q_n w^i_n / q_{1n}, 0^{L-1}) \in X^i$ for $n \in N$. Choose the sequence $x^i_{1n} = q_n w^i_n / q_{1n}$ and $x^i_{ln} = 0$ for $l > 1$ and $n \in N$. Then $x^i_{1n} = q_n w^i_n / q_{1n} \to \bar{q}\bar{w}^i / \bar{q}_1 = 0$ and hence $x^i_n \to \bar{x}^i = 0$ as $n \to \infty$. Moreover, by construction, $x^i_n \in B^i(w^i_n, q_n)$ for each $n \in N$.

In the case of $X^i = \mathbb{R}^L_+$ and $Z^i = \mathbb{R}^L_+$ the lower hemicontinuity is clear by the arguments above. For the upper hemicontinuity, consider $(z^i_n, q_n) \to (z^i, q) \in Z^i \times Q^\circ$ and $\{x^i_n\}_{n \in \mathbb{N}}$ with $x^i_n \in B^i(z^i_n, q_n)$ for each $n \in \mathbb{N}$. Since $q \in Q^\circ$, the set $\{q_n\}_{n \in \mathbb{N}}$ is eventually

[37] Notice that $\widehat{Q}$ is compact by the Tychonoff Product Theorem.

[38] Recall that this topology is equivalent to the product topology.

[39] Recall that we are using the relative topology.

[40] To see the inclusion $B^i(w^i, q) \subset \operatorname{cl}[\operatorname{Int} B^i(w^i, q)]$, given $x^i \in B^i(w^i, q)$ notice that if we choose $\tilde{x}^i \in \operatorname{Int} B^i(w^i, q)$ then $x^i_n := (1 - 1/n)x^i + \tilde{x}^i/n \in \operatorname{Int} B^i(w^i, q)$ and $x^i_n \to x^i$. Thus $x^i \in \operatorname{cl}[\operatorname{Int} B^i(w^i, q)]$.

[41] The set N is chosen such that $(w^i_n, q_n) \in O$ for each $n \in N$.

bounded away from zero. Moreover, since $z^i_n \to z^i$ then the set $\{z^i_n\}_{n\in\mathbb{N}}$ is bounded. Thus $\{x^i_n\}_{n\in\mathbb{N}}$ is bounded and hence has a subsequence convergent to $x^i \in B^i(z^i, q)$.

□

Theorem 7.4. *Suppose that $W^i \subset \mathbb{R}^L_{++}$. Let $V^i : \mathbb{R}^L_+ \times Y \times Q \to \mathbb{R}$ and $\hat{w}^i : Y \to W^i$ be bounded functions with $V^i(\,\cdot\,, y,\,\cdot\,)$ continuous for each $y \in Y$. Suppose that $V^i(\,\cdot\,, y, q)$ is concave and strictly increasing for all $(y, q) \in Y \times Q$. Then there exist functions $\hat{x}^i : Y \to \mathbb{R}^L_+$ for all $i \in I$ and $\hat{q} : Y \to Q$ such that for all $y \in Y$*

$$\hat{x}^i(y) \in \operatorname{argmax}\left\{V^i(x^i, y, \hat{q}(y)) : x^i \in \mathbb{R}^L_+ \text{ and } \hat{q}(y)x^i \le \hat{q}(y)\hat{w}^i(y)\right\}$$

and $\sum_{i\in I} \hat{x}^i(y) = \sum_{i\in I} \hat{w}^i(y)$.

Proof: The notation without upper index stands for the Cartesian product. Let $\pi^i : \mathbb{R}^{LI}_+ \to \mathbb{R}^L_+$ the projection on the ith-coordinate and write X^i a compact convex set containing the set[42]

$$\pi^i\left(\left\{x \in \mathbb{R}^{LI}_+ : \sum_{i\in I} x^i \le \sum_{i\in I} \sup\{\hat{w}^i(y) : y \in Y\}\right\}\right) \tag{16}$$

in its interior relative to $\mathbb{R}^L_+$. Moreover, let $\widehat{X}^i$ be the compact convex set of all functions $\hat{x}^i : Y \to X^i$ for all $i \in I$ endowed with the τ topology of pointwise convergence.

Define the correspondence $B^i : A^i \to X^i$ for all $i \in I$ by

$$B^i(w^i, q) = \{x^i \in X^i : \; qx^i \le qw^i\}.$$

Define $\widehat{B}^i : Y \times \widehat{Q} \to X^i$ by $\widehat{B}^i(y, \hat{q}) = B^i(\hat{w}^i(y), \hat{q}(y))$ for all $(y, \hat{q}) \in Y \times \widehat{Q}$. Clearly, $\widehat{B}^i$ is well defined since $W^i \subset \mathbb{R}_{++}$ implies $(\hat{w}^i(y), \hat{q}(y)) \in A^i$ for all $(y, \hat{q}) \in Y \times \widehat{Q}$. To see that $\widehat{B}^i(y,\,\cdot\,)$ is continuous for each fixed $y \in Y$ we use that the composition of continuous correspondences is continuous.[43] Indeed, fixing $y \in Y$, we have that $\widehat{B}^i(y, \hat{q}) = B^i(\hat{w}^i(y), \pi_y(\hat{q}))$ for all $\hat{q} \in \widehat{Q}$ where $\pi_y : \widehat{Q} \to Q$ is the τ continuous projection defined by $\pi_y(\hat{q}) = \hat{q}(y)$ for all $\hat{q} \in \widehat{Q}$. Analogously, for each fixed $y \in Y$ the map $(x^i, \hat{q}) \mapsto V^i(x^i, y, \hat{q}(y))$ is continuous because $\widehat{Q}$ is endowed with the τ topology. Therefore, using the Berge Maximum Theorem, the correspondence $\zeta^i : Y \times \widehat{Q} \to X^i$ defined by

$$\zeta^i(y, \hat{q}) = \operatorname{argmax}\left\{V^i(x^i, y, \hat{q}(y)) : x^i \in \widehat{B}^i(y, \hat{q})\right\} \tag{17}$$

is upper hemicontinuous on $\hat{q}$ for each $y \in Y$ fixed and hence has closed graph because X^i is a compact Hausdorff space and[44] $\zeta^i(y,\,\cdot\,)$ is closed valued.

[42] For a set $Z^i \subset \mathbb{R}^L_+$, write $\sup Z^i = (\sup Z^i_l)_{l\le L} \in \mathbb{R}^L_+$ where $Z^i_l \subset \mathbb{R}_+$ is the projection of Z^i into the l-th coordinate. Observe also that $0 \in X$.

[43] See Aliprantis & Border (1999) sec 17.4.

[44] To conclude that the correspondence $\zeta^i(y,\,\cdot\,)$ is closed valued and has closed graph, we use the Berge

Let $\delta^i : \widehat{Q} \to \widehat{X}^i$ be the demand correspondence given by

$$\delta^i(\hat{q}) = \left\{ \hat{x}^i \in \widehat{X}^i : \hat{x}^i(y) \in \zeta^i(y, \hat{q}) \text{ for all } y \in Y \right\}. \tag{18}$$

First we show that δ^i has closed graph. To see this, take the nets $\hat{q}_\alpha \in \widehat{Q}$ and $\hat{x}^i_\alpha \in \widehat{X}^i$ for $\alpha \in D$ converging to $\hat{q} \in \widehat{Q}$ and $\hat{x}^i \in \widehat{X}^i$ respectively, with $\hat{x}^i_\alpha \in \delta^i(\hat{q}_\alpha)$ for $\alpha \in D$. Fix $y \in Y$. Since $\zeta^i(y, \cdot)$ has closed graph and $\hat{x}^i_\alpha(y) \in \zeta^i(y, \hat{q}_\alpha)$ then $\hat{x}^i(y) \in \zeta^i(y, \hat{q})$. This is the same to state that $\hat{x}^i \in \delta^i(\hat{q})$ because y was chosen arbitrary. Notice that δ^i is convex valued because V^i is concave on x^i.

Let $\hat{z} : X \times W \to \mathbb{R}^L$ be the excess of demand function defined by

$$\hat{z}(x, w) = \sum_{i \in I} x^i - \sum_{i \in I} w^i \tag{19}$$

and the upper hemicontinuous[45] correspondence $\Delta' : X \times W \to Q$ defined by

$$\Delta'(x, w) = \operatorname{argmax} \{ q\hat{z}(x, w) : q \in Q \}.$$

We define the correspondence $\Delta : \widehat{X} \to \widehat{Q}$ by

$$\Delta(\hat{x}) = \{ \hat{q} \in \widehat{Q} : \hat{q}(y) \in \Delta'(\hat{x}(y), \hat{w}(y)) \text{ for all } y \in Y. \}$$

We claim that the correspondence Δ has closed graph. Indeed, consider the net $\{(\hat{q}_\alpha, \hat{x}_\alpha)\}_{\alpha \in D}$ converging to $(\hat{q}, \hat{x})$ and such that $\hat{q}_\alpha \in \Delta(\hat{x}_\alpha)$ for $\alpha \in D$. Using the definition of Δ we conclude that $\hat{q}_\alpha(y) \in \Delta'(\hat{x}_\alpha(y), \hat{w}(y))$ for all $y \in Y$. The upper hemicontinuity[46] of Δ' allows us to conclude that $\hat{q}(y) \in \Delta'(\hat{x}(y), \hat{w}(y))$ for all $y \in Y$, that is, $\hat{q} \in \Delta(\hat{x})$. Trivially, Δ is convex valued.

Let $T : \widehat{X} \times \widehat{Q} \to \widehat{X} \times \widehat{Q}$ be the convex valued correspondence defined by:

$$T(\hat{x}, \hat{q}) = \prod_{i \in I} \delta^i(\hat{q}) \times \Delta(\hat{x}).$$

Since $\widehat{X} \times \widehat{Q}$ is a nonempty compact convex space endowed with a locally convex Hausdorff topology and T has closed graph, we can apply the Kakutani-Fan-Gliksberg Fixed Point Theorem[47] to conclude that T has a fixed point, say, $(\hat{x}, \hat{q})$.

To show the market clearing conditions notice that

$$\hat{x}^i(y) \in B^i(\hat{w}^i(y), \hat{q}(y)) \text{ for all } y \in Y$$

Maximum Theorem and the Closed Graph Theorem. For further details, see Aliprantis & Border (1999) Theorems 17.11 and 17.31.

[45] Using the Berge Maximum Theorem and that every constant correspondence is continuous.

[46] And the closed graph property of Δ'.

[47] See Aliprantis & Border (1999) Theorem 17.55.

and if we add over $i \in I$ the budget restrictions we have

$$\hat{q}(y)\hat{z}(\hat{x}(y), \hat{w}(y)) \leq 0 \text{ for all } y \in Y. \tag{20}$$

Suppose that $\hat{z}_l(\hat{x}(y'), \hat{w}(y')) > 0$ for some $y' \in Y$. Then $\hat{q} \in \Delta(\hat{x})$ implies that $\hat{q}(y') \in \operatorname{argmax}\{q\hat{z}(\hat{x}(y'), \hat{w}(y')) : q \in Q\}$ and choosing $q' \in Q$ such that $q'_l = 1$ and $q'_k = 0$ if $k \neq l$ we have

$$0 < \hat{z}_l(\hat{x}(y'), \hat{w}(y')) = q'\hat{z}(\hat{x}(y'), \hat{w}(y')) \leq \hat{q}(y')\hat{z}(\hat{x}(y'), \hat{w}(y'))$$

which is a contradiction to (20) for $y = y'$ because $q' \in Q$. We have thus proved that $\hat{z}(\hat{x}(y'), \hat{w}(y')) \leq 0$.

We claim that $\hat{q}_l(y) \in \mathbb{R}^L_{++}$. Assume by way of contradiction that $\hat{q}_l(y) = 0$ for some $l \leq L$. Since $\hat{z}(\hat{x}(y), \hat{w}(y)) \leq 0$ we have that $\hat{x}^i_l(y)$ is in the interior of X^i_l relative to $\mathbb{R}_+$ and hence cannot be optimal given that the price $\hat{q}_l(y)$ is zero.[48] Therefore, we must have $\hat{q}(y) > 0$ for all $y \in Y$. Moreover, the local non satiation property[49] and the fact that all allocations are interior allows us to conclude that all budget restrictions are binding and hence

$$\hat{q}(y)\hat{z}(\hat{x}(y), \hat{w}(y)) = 0 \text{ for all } y \in Y. \tag{21}$$

Since $\hat{z} \leq 0$, and the prices are positive, the relation (21) implies the market clearing conditions.

Finally we have to prove that the equilibrium $(\hat{x}, \hat{q})$ is optimal for $\{V^i\}_{i \in I}$ in the set $\mathbb{R}^{LI}_+$. Define the budget correspondence $\widetilde{B}^i : W^i \times Q \to \mathbb{R}^L_+$ with the same inequalities of $B^i : W^i \times Q \to X^i$. Suppose that, for some $y \in Y$, $\hat{x}^i(y)$ is not optimal on the set $\widetilde{B}^i(\hat{w}^i(y), \hat{q}(y))$. Then there exists $x^i \in \widetilde{B}^i(\hat{w}^i(y), \hat{q}(y))$ such that $V^i(x^i, y, \hat{q}(y)) > V^i(\hat{x}^i(y), y, \hat{q}(y))$. Since the market clearing conditions imply that $\hat{x}^i(y) \in \operatorname{Int} X^i$, there exists a number $\alpha > 0$ such that

$$x^i_\alpha := \alpha x^i + (1 - \alpha)\hat{x}^i(y) \in B^i(\hat{w}^i(y), \hat{q}(y)).$$

Therefore

$$V^i(x^i_\alpha, y, \hat{q}(y)) \geq \alpha V^i(x^i, y, \hat{q}(y)) + (1 - \alpha)V^i(\hat{x}^i(y), y, \hat{q}(y)) > V^i(\hat{x}^i(y), y, \hat{q}(y))$$

which is a contradiction. Thus the equilibrium found above is optimal on the set R^{LI}_+.

□

Theorem 7.5. *Let $\{V^i, \hat{w}^i\}_{i \in I}$ satisfying all assumptions of Theorem 7.4 except that $W^i \subset \mathbb{R}_{++}$. Suppose that for each $y \in Y$ and $l \in L$ there exists $j \in I$ such that $\hat{w}^j_l(y) > 0$.*

[48] Recall that $V^i(\,\cdot\,, y, \hat{q}(y))$ is strictly increasing.

[49] That is, the fact that V^i is strictly increasing on the first coordinate for all $i \in I$.

Then there exist functions $\hat{x}^i : Y \to \mathbb{R}^L_+$ for all $i \in I$ and $\hat{q} : Y \to Q$ such that for all $y \in Y$

$$\hat{x}^i(y) \in \operatorname{argmax}\left\{V^i(x^i, y, \hat{q}(y)) : x^i \in \mathbb{R}^L_+ \text{ and } \hat{q}(y)x^i \le \hat{q}(y)\hat{w}^i(y)\right\}$$

and $\sum_{i\in I} \hat{x}^i(y) = \sum_{i\in I} \hat{w}^i(y)$.

Proof: Let $\bar{w}^i \in W^i \cap \mathbb{R}^L_{++}$ and apply Theorem 7.4 using the bounded endowment functions $\hat{w}^i_n : Y \to W^i \cap \mathbb{R}^L_{++}$ defined by $\hat{w}^i_n(y) = (1 - 1/n)\hat{w}^i(y) + \bar{w}^i/n \in W^i \cap \mathbb{R}^L_{++}$. Write X^i a compact set containing the set given in (16) with $\hat{w}^i$ replaced by $\hat{w}^i_n$ for $n \in \mathbb{N}$. Therefore by Theorem 7.4, there exists a recursive equilibrium $(\hat{x}'_n, \hat{q}'_n) \in \widehat{X} \times \widehat{Q}$ for all $n \in \mathbb{N}$. Since $\widehat{X} \times \widehat{Q} \times [0,1]$ is compact we can choose a subnet $(\hat{x}_\alpha, \hat{q}_\alpha, \epsilon_\alpha)_{\alpha\in D}$ of the (net) sequence $(\hat{x}_n, \hat{q}_n, 1/n)_{n\in\mathbb{N}}$ converging[50] to $(\hat{x}, \hat{q}, 0) \in \widehat{X} \times \widehat{Q} \times [0,1]$. Clearly, $\hat{x}$ satisfies the market clearing conditions because $\sum_{i\in I} \hat{x}^i_\alpha(y) = \sum_{i\in I}((1 - \epsilon_\alpha)\hat{w}^i(y) + \epsilon_\alpha \bar{w}^i)$ for all $\alpha \in D$ and all $y \in Y$. Let $\{V^i\}_{i\in I}$ be the value functions and $\{B^i\}_{i\in I}$ be the budget correspondences defined in Theorem 7.4 for $\{X_i\}_{i\in I}$. Define $\tilde{\zeta}^i : Y \times Q \times [0,1] \to X^i$ by

$$\tilde{\zeta}^i(y, q, \epsilon) = \operatorname{argmax}\left\{V^i(x^i, y, q) : x^i \in B^i((1 - \epsilon)\hat{w}^i(y) + \epsilon\bar{w}^i, q)\right\}.$$

By Lemma 7.3, the correspondence B^i is continuous on the set A^i. Therefore, for each fixed $y \in Y$ we can apply the Berge Maximum Theorem for the set $D^i_y = \{(q, \epsilon) \in Q\times[0,1] : ((1-\epsilon)\hat{w}^i(y)+\epsilon\bar{w}^i, q) \in A^i\}$ to conclude that $\tilde{\zeta}^i(y, \cdot, \cdot)$ is upper hemicontinuous on D^i_y for all $i \in I$. Fix $y \in Y$. Since $\hat{q}(y) \in Q$ then there exists $l \le L$ such that $\hat{q}_l(y) > 0$. Moreover, by hypothesis, there exists $j \in I$ such that $\hat{w}^j_l(y) > 0$. The conditions $\hat{w}^j_l(y) > 0$ and $\hat{q}_l(y) > 0$ assure that $(\hat{q}(y), 0) \in D^j_y$. Therefore, using that $\hat{x}^j_\alpha(y) \in \tilde{\zeta}^j(y, \hat{q}_\alpha(y), \epsilon_\alpha)$ for all $\alpha \in D$ then $\hat{x}^j(y) \in \tilde{\zeta}^j(y, \hat{q}(y), 0)$ which implies that $\hat{x}^j(y)$ is optimal. Therefore we conclude, by local non satiation, that $\hat{q}(y)$ is positive since $\hat{x}(y)$ satisfies the market clearing conditions and hence $\hat{x}^j(y)$ belongs to the interior of X^j. Moreover, $\hat{q}(y) \in Q^\circ$ implies $(\hat{q}(y), 0) \in D^i_y$ for all $i \in I$ and hence $\hat{x}^i(y) \in \tilde{\zeta}^i(y, \hat{q}(y), 0)$ because $\hat{x}^i_\alpha(y) \in \tilde{\zeta}^i(y, \hat{q}_\alpha(y), \epsilon_\alpha)$ for all $\alpha \in D$, that is, $\hat{x}^i(y)$ is optimal for each $i \in I$. Since y was chosen arbitrary, this is the same to state that $\hat{x}(y)$ and $\hat{q}(y)$ constitutes a recursive equilibrium for the configuration $\epsilon = 0$, that is, in the economy with the initial endowment functions $\{\hat{w}^i\}_{i\in I}$.[51]

□

Proposition 7.6. *Suppose that $\{V^i, \hat{w}^i\}$ are continuous with $W^i \subset \mathbb{R}_+$, the recursive equilibrium $(\hat{x}, \hat{q})$ is unique*[52] *and for each $y \in Y$ and $l \le L$ there exists $j \in I$ such that $\hat{w}^j_l(y) > 0$. Then $(\hat{x}, \hat{q})$ is continuous.*

[50] Since $(1/n)_{n\in\mathbb{N}}$ converges to zero then $\{\epsilon_\alpha\}_{\alpha\in D}$ also converges to zero.

[51] Notice that the proof that $\hat{x}^i$ is optimal considering $X^i = \mathbb{R}^L_+$ is identical to the one found at the end of Theorem 7.4.

[52] That is, for each $y \in Y$ there exist only one $(x, q) \in X \times Q$ satisfying the optimality and market clearing conditions.

Proof: Take $y_n = (\bar{\theta}_n, s_n)_{n\in\mathbb{N}}$ such that $y_n \to y = (\bar{\theta}, s) \in Y$ as $n \to \infty$ and X^i the compact set containing the set given in (16) in its interior. By Lemma 7.3, the correspondence B^i is continuous on the set A^i. Define for $i \in I$ the set $D^i = \{(y', q') \in Y \times Q : (\hat{w}^i(y'), q') \in A^i\}$ and the correspondence $\tilde{\zeta}^i : D^i \to X^i$ by

$$\tilde{\zeta}^i(y, q) = \operatorname{argmax}\left\{V^i(x^i, y, q) : x^i \in B^i(\hat{w}^i(y), q)\right\}$$

and $\tilde{\zeta} = (\tilde{\zeta}^i)_{i\in I}$. Therefore, we can apply the Berge Maximum Theorem for the set X^i and D^i to conclude that $\tilde{\zeta}^i$ is upper hemicontinuous.[53] Take any subsequence[54] $(\hat{x}(y_n), \hat{q}(y_n))_{n\in N} \in X \times Q$ with $N \subset \mathbb{N}$. This sequence has a subsequence[55] $(\hat{x}(y_n), \hat{q}(y_n))_{n\in N'}$ converging to $(x, q) \in X \times Q$ because $X \times Q$ is compact. Since $q \in Q$ then there exists one coordinate q_l such that $q_l > 0$. Moreover, by hypothesis, there exists an agent $j \in I$ such that $\hat{w}^j_l(y) > 0$ and hence $(\hat{w}^j(y), q) \in A^j$, that is, $(y, q) \in D^j$. Since $\tilde{\zeta}^j$ is upper hemicontinuous on (y, q), then $\hat{x}^j(y_n) \in \tilde{\zeta}^j(y_n, \hat{q}(y_n))$ for $n \in N'$ implies that $x^j \in \tilde{\zeta}^j(y, q)$. This implies that q is positive by local non satiation because x^j is optimal and belongs to the interior of X^j since the allocation x satisfies the market clearing conditions.[56] The positivity of the price q assures that $(y, q) \in D^i$ and hence $x^i \in \zeta^i(y, q)$ is optimal for each $i \in I$. Therefore x and q constitutes an equilibrium for the configuration y. By the uniqueness, $(x, q) = (\hat{x}(y), \hat{q}(y))$ and since it is independent of the initial subsequence $(\hat{x}(y_n), \hat{q}(y_n))_{n\in N}$ we have that $\hat{x}(y_n) \to \hat{x}(y)$ and $\hat{q}(y_n) \to \hat{q}(y)$ as $n \to \infty$.

□

Theorem 7.7. *Define $Y = \{(\bar{\theta}, s) \in R^{LI}_+ \times S : \sum_{i\in I} \bar{\theta}^i = 1\}$ and $\mathscr{Y}$ its borelians. Write $X^i = C^i \times \Theta^i$ with $\Theta^i = \mathbb{R}^H_+$ and $C^i = \mathbb{R}_+$. Suppose that $V^i : X^i \times Y \times Q \to \mathbb{R}$ is given by*

$$V^i(c^i, \theta^i, y, q) = u^i(c^i) + \int_Y v^i(\theta^i, y', \tilde{q}^i(y', q))\lambda(y, q, dy')$$

where $\lambda : Y \times Q \to \mathrm{Prob}(Y)$ is continuous and the bounded continuous functions $u^i : C^i \to \mathbb{R}_+$ and $v^i : \Theta^i \times Y \times Q^\circ \to \mathbb{R}_+$ are strictly increasing and concave on C^i and Θ^i respectively. If $\hat{w}^i(y) = (\bar{\theta}^i \hat{d}(s), \bar{\theta}^i)$ for $y \in Y$ and $\tilde{q}^i : Y \times Q \to Q^\circ$ is continuous then Theorem 7.5 holds.

Proof: It follows directly[57] from Lemma 7.1 that for each $y \in Y$ the function $V^i(\,\cdot\,, \cdot\,, y, \cdot\,)$ is continuous. Moreover, since $\sum_{i\in I} \bar{\theta}^i = 1$ then for each $y = (\bar{\theta}, s) \in Y$ and each coordinate $l \leq L$ there exists some agent j with $\bar{\theta}^j_l > 0$ and hence positive endowment

[53] Notice that the composition of the function $(y, q) \mapsto (\hat{w}^i(y), q)$ defined on D^i and the correspondence $B^i : A^i \to X^i$ is well defined and hence is continuous as the composition of continuous correspondences.

[54] Observe that $\hat{x}(y_n) \in \tilde{\zeta}(y_n, \hat{q}(y_n))$ for all $n \in \mathbb{N}$.

[55] With $N' \subset N$.

[56] The allocation x satisfies the market clearing conditions because $\hat{q}$ is a recursive equilibrium and hence $\hat{x}(y_n) \in \tilde{\zeta}(y_n, \hat{q}(y_n))$ satisfies the market clearing conditions for each $n \in N'$. Recall that $\hat{w}$ is continuous.

[57] In Lemma 7.1, choose $Z = X^i \times Y \times Q$ and $f : Y \times Z \to \mathbb{R}_+$ defined by $f(y', x^i, y, q) = v^i(\theta^i, y', \tilde{q}^i(y', q))$.

of the respective good ($l = 1$ and $\bar{\theta}^j \hat{d}(s) > 0$) or asset ($2 \leq l \leq H+1$ and $\bar{\theta}^j_l > 0$). Therefore, $\{V^i, \hat{w}^i\}_{i \in I}$ satisfies the hypothesis of Theorem 7.5 since the concavity of V^i on x^i is trivial and u^i is strictly increasing and $v^i(\,\cdot\,, y, q)$ is strictly increasing for each $(y, q) \in Y \times Q^\circ$.

$\square$

Theorem 7.8. *If $(\hat{c}, \hat{\theta}, \hat{q})$ is a recursive equilibrium as in Definition 3.3 then its implemented process $\{\boldsymbol{c}_t, \boldsymbol{\theta}_t, \boldsymbol{q}_t\}_{t \in \mathbb{N}}$ starting from $\boldsymbol{\theta}_0 \in \overline{\Theta}$ is a sequential equilibrium of the economy with initial asset holdings $\boldsymbol{\theta}_0 \in \overline{\Theta}$.*

Proof: It is sufficient to prove that agent i choices $\{\boldsymbol{c}^i_t, \boldsymbol{\theta}^i_t\}_{t \in \mathbb{N}}$ are optimal given the prices $\{\boldsymbol{q}_t\}_{t \in \mathbb{N}}$. Fix an arbitrary $(\bar{\theta}^i, y, q) \in \Theta^i \times Y \times Q$ with $y = (\bar{\theta}, s)$. Choose $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\bar{\theta}^i, s, q)$ and write

$$\boldsymbol{U}^i_n(\boldsymbol{c}^i, y) = u^i(\boldsymbol{c}^i_0) + \sum_{r=1}^{n} \int_{Y^r} \beta^r u^i(\boldsymbol{c}^i_r(y^r)) \mu^i_r(y, q, dy^r).$$

Therefore

$$\begin{aligned} v^i(\bar{\theta}^i, y, q) &\stackrel{def}{=} \sup \left\{ u^i(c^i) + \beta \int_Y v^i(\theta^i, y_1, \tilde{q}^i(y_1, q)) \lambda^i(y, q, dy_1) \right\} \\ &\geq u^i(\boldsymbol{c}^i_0) + \beta \int_Y v^i(\boldsymbol{\theta}^i_0, y_1, \tilde{q}^i(y_1, q)) \lambda^i(y, q, dy_1). \end{aligned} \tag{22}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in B^i(\bar{\theta}^i, s, q)$.

Write $\tilde{v}^i(\boldsymbol{\theta}^i_0, y_1, q) = v^i(\boldsymbol{\theta}^i_0, y_1, \tilde{q}^i(y_1, q))$. Using the Bellman Equation again, we have that

$$\begin{aligned} \tilde{v}^i(\boldsymbol{\theta}^i_0, y_1, q) &= \sup \left\{ u^i(c^i) + \beta \int_Y v^i[\theta^i, y_2, \boldsymbol{q}^i_2(y^2, q)] \lambda^i(y_1, \tilde{q}^i(y_1, q), dy_2) \right\} \\ &\geq u^i(\boldsymbol{c}^i_1(y_1)) + \beta \int_Y v^i[\boldsymbol{\theta}^i_1(y_1), y_2, \boldsymbol{q}^i_2(y^2, q)] \lambda^i(y_1, \tilde{q}^i(y_1, q), dy_2) \end{aligned}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in B^i(\boldsymbol{\theta}^i_0, s_1, \tilde{q}^i(y_1, q))$ and $\boldsymbol{q}^i_2(y^2, q) = \tilde{q}^i(y_2, \tilde{q}^i(y_1, q))$ according to Definition 2.2. The second inequality comes from the fact that $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ is feasible and hence we have $(\boldsymbol{c}^i_1(y_1), \boldsymbol{\theta}^i_1(y_1)) \in B^i(\boldsymbol{\theta}^i_0, s_1, \tilde{q}^i(y_1, q))$. Replacing the previous inequality of $v^i(\boldsymbol{\theta}^i_0, y_1, \tilde{q}^i(y_1, q))$ in (22) then[58]

$$\begin{aligned} v^i(\bar{\theta}^i, y, q) &\geq u^i(\boldsymbol{c}^i_0) + \beta \int_Y u^i(\boldsymbol{c}^i_1(y_1))\lambda^i(y, q, dy_1) \\ &+ \beta^2 \int_{Y^2} v^i[\boldsymbol{\theta}^i_1(y_1), y_2, \boldsymbol{q}^i_2(y^2, q)]\lambda^i(y_1, \tilde{q}^i(y_1, q), dy_2)\lambda^i(y, q, dy_1) \\ &= u^i(\boldsymbol{c}^i_0) + \beta \int_Y u^i(\boldsymbol{c}^i_1(y_1))\mu^i_1(y, q, dy_1) \\ &+ \beta^2 \int_{Y^2} v^i[\boldsymbol{\theta}^i_1(y_1), y_2, \boldsymbol{q}^i_2(y^2, q)]\mu^i_2(y, q, dy^2) \\ &= \boldsymbol{U}^i_1(\boldsymbol{c}^i, y, q) + \beta^2 \int_{Y^2} v^i[\boldsymbol{\theta}^i_1(y_1), y_2, \boldsymbol{q}^i_2(y^2, q)]\mu^i_2(y, q, dy^2). \end{aligned}$$

It follows from induction on n that

$$v^i(\bar{\theta}^i, y, q) \geq \boldsymbol{U}^i_{n-1}(\boldsymbol{c}^i, y, q) + \beta^n \int_{Y^n} v^i[\boldsymbol{\theta}^i_{n-1}(y^{n-1}), y_n, \boldsymbol{q}^i_n(y^n, q))]\mu^i_n(y, q, dy^n).$$

Taking the limit and using that v is bounded we have $v^i(\bar{\theta}^i, y, q) \geq \boldsymbol{U}^i(\boldsymbol{c}^i, y)$ for all $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i(\theta^i, s, q)$ since $(\boldsymbol{c}^i, \boldsymbol{\theta}^i)$ was chosen arbitrary.

Let $\tilde{\theta}^i : \Theta^i \times Y \times Q \to \Theta^i$ and $\tilde{c}^i : \Theta^i \times Y \times Q \to C^i$ be the argmax of the agent i's Bellman Equation (2) according to Definition 3.2. Choose $\{\hat{\boldsymbol{c}}_t, \hat{\boldsymbol{\theta}}_t, \hat{\boldsymbol{q}}_t\}_{t\in\mathbb{N}}$ according to equations (4) and (5). For each fixed realized period $t \in \mathbb{N}$, we follow the arguments above taking $y = (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t)$ and[59] $q = \hat{\boldsymbol{q}}_t(s^t) = \hat{q}(\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t)$. Consider the plan $(\tilde{\boldsymbol{c}}^i_0, \tilde{\boldsymbol{\theta}}^i_0) = (\hat{\boldsymbol{c}}^i_t(s^t), \hat{\boldsymbol{\theta}}^i_t(s^t))$ and recursively the measurable selectors[60]

$$\begin{aligned} \tilde{\boldsymbol{\theta}}^i_r(y^r) &\in \tilde{\theta}^i[\tilde{\boldsymbol{\theta}}^i_{r-1}(y^{r-1}), y_r, \boldsymbol{q}^i_r(y^r, \hat{\boldsymbol{q}}_t(s^t))] \\ \tilde{\boldsymbol{c}}^i_r(y^r) &\in \tilde{c}^i[\tilde{\boldsymbol{\theta}}^i_{r-1}(y^{r-1}), y_r, \boldsymbol{q}^i_r(y^r, \hat{\boldsymbol{q}}_t(s^t))] \end{aligned} \tag{23}$$

for all $r \in \mathbb{N}$ where $\boldsymbol{q}^i_r$ is given by Definition 2.2. Using Definitions 3.3 and 3.4 we have that

$$\begin{aligned} \hat{\boldsymbol{\theta}}^i_t(s^t) &= \hat{\theta}^i(\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t) \in \tilde{\theta}^i(\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t), \hat{\boldsymbol{q}}_t(s^t)) \\ \hat{\boldsymbol{c}}^i_t(s^t) &= \hat{c}^i(\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t) \in \tilde{c}^i(\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t), \hat{\boldsymbol{q}}_t(s^t)). \end{aligned} \tag{24}$$

Using that the policy correspondences $\tilde{c}^i$ and $\tilde{\theta}^i$ satisfy the relation (24) then $(\tilde{\boldsymbol{c}}^i_0, \tilde{\boldsymbol{\theta}}^i_0) \in$

[58] Recall that

$$\mu^i_r(y, q)(A_1, ..., A_r) = \int_{A_1} \cdots \int_{A_r} \lambda^i(y_{r-1}, \boldsymbol{q}^i_{r-1}(y^{r-1}, q), dy_r) \cdots \lambda^i(y, q, dy_1)$$

for each rectangle $A_1 \times ... \times A_r$. See Stokey and Lucas Chapter 9 for more details about the composition of the stochastic kernels λ^i.

[59] See Definition 3.4.

[60] The Measurable Maximum Theorem assures that the policy correspondences have measurable selectors because v^i is continuous and B^i is lower hemicontinuous on A^i and hence weakly measurable. Notice that we are using that $\mathscr{Y}_t$ are the Borelians of Y_t for all $t \in \mathbb{N}$. See Aliprantis & Border (1999) for further details about the Measurable Maximum Theorem.

$B^i(\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), s_t, \hat{\boldsymbol{q}}_t(s^t))$. Moreover, by equation (23) we have that $(\tilde{\boldsymbol{c}}^i_r(y^r), \tilde{\boldsymbol{\theta}}^i_r(y^r)) \in B^i(\tilde{\boldsymbol{\theta}}^i_{r-1}(y^{r-1}), s_r, \boldsymbol{q}^i_r(y^r, q))$ for all $y^r \in Y^r$, that is, $(\tilde{\boldsymbol{c}}^i, \tilde{\boldsymbol{\theta}}^i) \in \boldsymbol{F}^i[\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), s_t, \hat{\boldsymbol{q}}_t(s^t)]$. Using the arguments above we get

$$v^i[\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t), \hat{\boldsymbol{q}}_t(s^t)] \geq \boldsymbol{U}^i[\boldsymbol{c}^i, (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t, \hat{\boldsymbol{q}}_t(s^t))]$$

for all $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \boldsymbol{F}^i[\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), s_t, \hat{\boldsymbol{q}}_t(s^t)]$. The construction of the plan $(\tilde{\boldsymbol{c}}^i, \tilde{\boldsymbol{\theta}}^i)$ implies that all inequalities of the above arguments must bind. Thus

$$v^i[\boldsymbol{\theta}^i_{t-1}(s^{t-1}), (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t), \hat{\boldsymbol{q}}_t(s^t)] = \boldsymbol{U}^i[\tilde{\boldsymbol{c}}^i, (\boldsymbol{\theta}_{t-1}(s^{t-1}), s_t)]$$

and hence $(\tilde{\boldsymbol{c}}^i, \tilde{\boldsymbol{\theta}}^i) \in \widehat{\boldsymbol{F}}^i[\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t), \hat{\boldsymbol{q}}_t(s^t)]$, that is,

$$(\hat{\boldsymbol{c}}^i_t(s^t), \hat{\boldsymbol{\theta}}^i_t(s^t)) \in \delta^i(\hat{\boldsymbol{\theta}}^i_{t-1}(s^{t-1}), (\hat{\boldsymbol{\theta}}_{t-1}(s^{t-1}), s_t), \hat{\boldsymbol{q}}_t(s^t)).$$

Therefore $\{\hat{\boldsymbol{c}}_t, \hat{\boldsymbol{\theta}}_t, \hat{\boldsymbol{q}}_t\}_{t\in\mathbb{N}}$ is an equilibrium for the economy $\mathcal{E}$ because the recursive equilibrium satisfies all market clearing conditions.

□

7.2 Results related to Section 4

Lemma 7.9. *Let $\tilde{\theta}^i : \Theta^i \times \overline{\Theta} \times Q^\circ \to \Theta^i$ be the asset policy function as in Definition 3.2. Then $\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) > 0$ for all $(\theta^i_-, \bar{\theta}, q) \in \Theta^i \times \overline{\Theta} \times Q^\circ$ with $\theta^i_- > 0$.*

Proof: Fix $\theta^i_- > 0$, $q \in Q^\circ$ and write $p = q^a/q^c > 0$. Since $u^i \geq 0$ and $u^i(0) = 0$ the subset $\{v^i \in \mathbb{V} : v^i(0, \cdot, \cdot) = 0 \text{ and } v^i \geq 0\}$ of $\mathbb{V}$ is closed and nonempty and hence the value function v^i of the Bellman equation satisfies $v^i(0, \cdot, \cdot) = 0$ and $v^i \geq 0$. Write $M = \max\{\partial u^i(c^i) : c^i \in [\hat{d}\theta^i_-, (p+\hat{d})\theta^i_-]\}$ and take $\tilde{c}^i > 0$ such that[61] $\partial u^i(c^i) > (\beta\hat{d})^{-1}pM$ for $c^i \leq \tilde{c}^i$. Choose $\tilde{\theta}^i_-$ with $\tilde{\theta}^i_- < \theta^i_-$ and $\hat{d}\tilde{\theta}^i_- \leq \tilde{c}^i$. Since $(\hat{d}\tilde{\theta}^i_-, 0) \in B^i(\tilde{\theta}^i_-, \dot{q})$ for each $\dot{q} \in Q^\circ$ then evaluating the sup over all $(c^i, \theta^i) \in B^i(\tilde{\theta}^i_-, \dot{q})$:

$$\begin{aligned}
v^i(\tilde{\theta}^i_-, \bar{\theta}, \dot{q}) &= \sup\left\{u^i(c^i) + \beta\int_Y v^i(\theta^i, \bar{\theta}', \tilde{q}^i(\bar{\theta}', \dot{q}))\lambda^i(\bar{\theta}, \dot{q}, \hat{d}\bar{\theta}')\right\} \\
&\geq u^i(\hat{d}\tilde{\theta}^i_-) + \beta\int_Y v^i(0, \bar{\theta}', \tilde{q}^i(\bar{\theta}', \dot{q}))\lambda^i(\bar{\theta}, \dot{q}, \hat{d}\bar{\theta}') \\
&= u^i(\hat{d}\tilde{\theta}^i_-) \\
&= \partial u^i(c^i)\hat{d}\tilde{\theta}^i_- \quad \text{for some } c^i > 0 \text{ with } c^i < \hat{d}\tilde{\theta}^i_- \\
&> \beta^{-1}Mp\tilde{\theta}^i_- \quad \text{for all } (\bar{\theta}, \dot{q}) \in \overline{\Theta} \times Q^\circ \text{ since } c^i < \hat{d}\tilde{\theta}^i_- \leq \tilde{c}^i.
\end{aligned}$$

[61] This is possible since u^i is continuously differentiable and satisfies $\lim_{c^i\to 0}\partial u^i(c^i) = \infty$.

If $\tilde{\theta}^i(\theta^i_-, \bar{\theta}, q) = 0$ for some $\bar{\theta} \in \overline{\Theta}$, then $\tilde{c}^i(\theta^i_-, \bar{\theta}, q) = (p + \hat{d})\theta^i_-$ and $v^i(\theta^i_-, \bar{\theta}, q) = u^i((p + \hat{d})\theta^i_-)$. Thus $\tilde{\theta}^i_- < \theta^i_-$ implies that[62] $(-p\tilde{\theta}^i_- + (p + \hat{d})\theta^i_-, \tilde{\theta}^i_-) \in B^i(\theta^i_-, q)$ and hence

$$u^i((p + \hat{d})\theta^i_-) \geq u^i(-p\tilde{\theta}^i + (p + \hat{d})\theta^i_-) + \beta \int_Y v^i(\tilde{\theta}^i_-, \bar{\theta}', \tilde{q}^i(\bar{\theta}', q))\lambda^i(\bar{\theta}, q, d\bar{\theta}').$$

Using that u^i is continuously differentiable in an open interval containing $[\hat{d}\theta^i_-, (p+\hat{d})\theta^i_-]$ then the mean value theorem assures that $|u^i(c^i) - u^i(\bar{c}^i)| \leq M|c^i - \bar{c}^i|$ for all $c^i, \bar{c}^i \in [\hat{d}\theta^i_-, (p + \hat{d})\theta^i_-]$. Therefore

$$\begin{aligned} Mp\tilde{\theta}^i_- &\geq u^i((p + \hat{d})\theta^i_-) - u^i(-p\tilde{\theta}^i + (p + \hat{d})\theta^i_-) \\ &\geq \beta \int_Y v^i(\tilde{\theta}^i_-, \bar{\theta}', \tilde{q}^i(\bar{\theta}', q))\lambda^i(\bar{\theta}, q, d\bar{\theta}') \\ &> Mp\tilde{\theta}^i_- \end{aligned}$$

which is a contradiction.

□

Lemma 7.10. *If* $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \widehat{\boldsymbol{F}}^i(\theta^i_-, y, q)$ *then* $(\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0) \in \tilde{x}^i(\theta^i_-, y, q)$.

Proof: Suppose that $(\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0) \notin \tilde{x}^i(\theta^i_-, y, q)$. Using the same arguments of Theorem 7.8 and that $(\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0) \in B^i(\theta^i_-, s, q)$ we get:

$$\begin{aligned} v^i(\theta^i_-, y, q) &= \sup \left\{ u^i(c^i) + \beta \int_Y v^i(\theta^i, y_1, \tilde{q}^i(y_1, q))\lambda^i(y, q, dy_1) \right\} \\ &> u^i(\boldsymbol{c}^i_0) + \beta \int_Y v^i(\boldsymbol{\theta}^i_0, y_1, \tilde{q}^i(y_1, q))\lambda^i(y, q, dy_1) \\ &\geq \boldsymbol{U}^i_{n-1}(\boldsymbol{c}^i, y, q) \\ &+ \beta^n \int_{Y^n} v^i[\boldsymbol{\theta}^i_{n-1}(y^{n-1}), y_n, \boldsymbol{q}^i_n(y^n, q))]\mu^i_n(y, q, dy^n) \end{aligned}$$

where the sup in the first equation is over all $(c^i, \theta^i) \in B^i(\theta^i_-, s, q)$. Taking the limit as $n \to \infty$ then

$$\begin{aligned} v^i(\theta^i_-, y, q) &> u^i(\boldsymbol{c}^i_0) + \beta \int_Y v^i(\boldsymbol{\theta}^i_0, y_1, \tilde{q}^i(y_1, q))\lambda^i(y, q, dy_1) \\ &\geq \boldsymbol{U}^i(\boldsymbol{c}^i, y, q). \end{aligned}$$

Choose $(\tilde{\boldsymbol{c}}^i_0, \tilde{\boldsymbol{\theta}}^i_0) = (\boldsymbol{c}^i_0, \boldsymbol{\theta}^i_0)$ and for all $r \in \mathbb{N}$ an optimal measurable selection $(\tilde{\boldsymbol{c}}^i_r(\,\cdot\,), \tilde{\boldsymbol{\theta}}^i_r(\,\cdot\,))$ as in Theorem 7.8 equation (23) inductively. Then $\boldsymbol{U}^i(\tilde{\boldsymbol{c}}^i, y, q) = v^i(\theta^i_-, y, q) > \boldsymbol{U}^i(\boldsymbol{c}^i, y, q)$ which is a contradiction since $(\tilde{\boldsymbol{c}}^i, \tilde{\boldsymbol{\theta}}^i)$ is feasible and $(\boldsymbol{c}^i, \boldsymbol{\theta}^i) \in \widehat{\boldsymbol{F}}^i(\theta^i_-, y, q)$.

[62] Notice that $-p\tilde{\theta}^i_- + (p + \hat{d})\theta^i_- = p(\theta^i_- - \tilde{\theta}^i_-) + \hat{d}\theta^i_- > 0$.

□

7.3 Results related to Section 5

Lemma 7.11. *Let $(Y, \mathscr{Y})$ be a measurable space with*[63] *$\mathscr{Y} = \mathscr{P}(Y)$ and W a topological space. Consider a bounded*[64] *$f : Y \times W \to \mathbb{R}_+$ and $\nu \in \mathrm{Prob}(Y)$ such that $f(y, \cdot)$ is continuous for each y belonging to a countable set $A \subset Y$ with $\nu(A) = 1$. Then the function $w \mapsto \int_Y f(y, w)\nu(dy)$ is continuous.*

Proof: Let $A = \{y_n\}_{n\in\mathbb{N}} \subset A$ be a countable set with $\nu(A) = 1$ and M such that $f \leq M$. Consider a net $\{w_\alpha\}_{\alpha\in D} \subset W$ converging to $w \in W$. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ and $A_N = \{y_n\}_{n\leq N} \subset A$ such that $\nu(A_N^c) < \epsilon/(4M)$. The continuity of $f(y_n, \cdot)$ allows us to find for each $n \in \mathbb{N}$ an $\alpha_n \in D$ such that

$$|f(y_n, w_\alpha) - f(y_n, w)| < \epsilon/2 \text{ for all } \alpha \geq \alpha_n.$$

Choose $\alpha' = \max\{\alpha_1, ..., \alpha_N\}$. Then $|f(y, w_\alpha) - f(y, w)| < \epsilon/2$ for all $\alpha \geq \alpha'$ and $y \in A_N$. Therefore, $\alpha \geq \alpha'$ implies

$$\begin{aligned}\left|\int_Y f(y, w_\alpha)\nu(dy) - \int_Y f(y, w)\nu(dy)\right| &\leq \int_Y |f(y, w_\alpha) - f(y, w)|\nu(dy) \\ &= \int_{A_N} |f(y, w_\alpha) - f(y, w)|\nu(dy) \\ &+ \int_{A_N^c} |f(y, w_\alpha) - f(y, w)|\nu(dy) \\ &\leq \epsilon/2 + 2M\epsilon/(4M) = \epsilon.\end{aligned}$$

□

Theorem 7.12. *Under Assumption 5.1, there exists a PPF recursive equilibrium.*

Proof: Take any $(\hat{q}', q') \in \widehat{Q}' \times Q^\circ$ where Q° are given as in notation 7.1, $\widehat{Q}'$ is given as in notation 5.1 and write X^i the compact set containing the set given in (16) in its interior as in Theorem 7.4 with $X^i = C^i \times \Theta^i$. Consider $V^j : X^j \times Y \times Q \to \mathbb{R}$ defined by

$$V^j(c^j, \theta^j, y, q) = u^j(c^j) + \beta \int_Y v^j(\theta^j, y', \tilde{q}^j(y', q))\lambda(y, q, dy')$$

and $V^k : X^k \times Y \times \widehat{Q}' \to \mathbb{R}$ defined by

$$V^k(c^k, \theta^k, y, \hat{q}) = u^k(c^k) + \beta \int_Y v^k_{pf}(\theta^k, y', \hat{q})\lambda^k(y, dy')$$

[63] The set $\mathscr{P}(Y)$ is the set of all subsets of Y.

[64] Every function defined on Y is measurable on $\mathscr{Y}$.

where v^j is given by (2) and v^k_{pf} is given by (14). Define $Y = \{(\bar\theta, s) \in R^{LI} \times S : \sum_{i\in I} \bar\theta^i = 1\}$ and $\hat w^i : Y \to \mathbb{R}^{H+1}$ by $\hat w^i(y) = (\bar\theta^i \hat d(s), \bar\theta^i)$.

Let $\hat q^* : \widehat Q \times [0,1] \to \widehat Q$ be the function defined for each $(\hat q, \epsilon) \in \widehat Q \times [0,1]$ by $\hat q^*(\hat q, \epsilon)(y) = (1-\epsilon)\hat q(y) + \epsilon \hat q'(y)$ and $\tilde w^i : Y \times [0,1] \to W^i \cap \mathbb{R}_{++}$ by $\tilde w^i(y,\epsilon) = (1-\epsilon)\hat w^i(y) + \epsilon \bar w$ for some $\bar w \in W^i \cap \mathbb{R}^L_{++}$. Write the sets $D^j = \{(y,q,\epsilon) \in Y \times Q \times [0,1] : (\tilde w^j(y,\epsilon), q) \in A^j\}$ and

$$D^k = \{(y, \hat q, \epsilon) \in Y \times \widehat Q \times [0,1] : \hat q^*(\hat q, \epsilon) \in \widehat Q' \text{ and } (\tilde w^k(y,\epsilon), \hat q(y)) \in A^k\}.$$

We can apply the arguments of Theorems 7.4 and 7.5 for $\{\tilde w^i\}_{i\in I}$, replacing the correspondence given in (17) by the correspondences $\tilde\zeta^j : D^j \to X^j$ and $\tilde\zeta^k : D^k \to X^k$ defined respectively as[65]

$$\tilde\zeta^j(y,q,\epsilon) = \operatorname{argmax}\left\{V^j(x^j, y, q) : x^j \in B^j(\tilde w^j(y,\epsilon), q)\right\}$$

and

$$\tilde\zeta^k(y,\hat q,\epsilon) = \operatorname{argmax}\left\{V^k(x^k, y, \hat q^*(\hat q,\epsilon)) : x^k \in B^k(\tilde w^k(y,\epsilon), \hat q(y))\right\}.$$

Choosing $\epsilon_n = 1/n$ we find the functions $(\hat x_n, \hat q_n)$ in the compact set $\widehat X \times \widehat Q$ satisfying the market clearing conditions for $\{\tilde w^i\}_{i\in I}$ with $x^k_n(y) \in \tilde\zeta^k(y, \hat q_n, 1/n)$ and $x^j_n(y) \in \tilde\zeta^j(y, \hat q_n(y), 1/n)$ for all $y \in Y$ and $n \in \mathbb{N}$. Therefore we can choose a subnet $(\hat x_\alpha, \hat q_\alpha, \epsilon_\alpha)_{\alpha\in\Lambda}$ of the (net) sequence $(\hat x_n, \hat q_n, 1/n)_{n\in\mathbb{N}}$ converging[66] to $(\hat x, \hat q, 0) \in \widehat X \times \widehat Q \times [0,1]$. Clearly, $\hat x$ satisfies the market clearing conditions for $\{\hat w^i\}_{i\in I}$ because $\sum_{i\in I} \hat x^i_\alpha(y) = \sum_{i\in I} \tilde w^i(y,\epsilon_\alpha)$ for all $\alpha \in \Lambda$ and all $y \in Z$. Therefore $\hat x(y)$ belongs to the interior of X for each $y \in Y$.

Write $Z' = \{y \in Y : (y, \hat q(y), 0) \in D^j$ for some $j \in J\}$ and notice that[67] $Z \subset Z'$. Take $y \in Z'$ and $j \in J$ such that $(y, \hat q(y), 0) \in D^j$. From the construction of the equilibrium sequence, $\hat x^j_\alpha(y) \in \tilde\zeta^j(y, \hat q_\alpha(y), \epsilon_\alpha)$ for all $\alpha \in \Lambda$. Since $\tilde\zeta^j(y, \cdot, \cdot)$ is upper hemicontinuous then $\hat x^j(y) \in \tilde\zeta^j(y, \hat q(y), 0)$ and hence $\hat x^j(y)$ is optimal. Since $\hat x^j(y)$ is interior and the utility is strictly increasing, then $\hat q(y) \in Q^\circ$ and hence $\hat q \in \widehat Q'$ because $Z \subset Z'$ and y was chosen arbitrarily. Suppose that $y \notin Z'$. Since there exists $l \le L$ such that $\hat q_l(y) > 0$ and using that $\sum_{i\in I} \bar\theta^i_l = 1$ then $\sum_{k\in K} \bar\theta^k_l = 1$ (otherwise $y \in Z'$) and hence $(w^k(y), \hat q(y)) \in A^k$ for some $k \in K$. Thus $(y, \hat q, 0) \in D^k$ and $\hat x^k_\alpha(y) \in \tilde\zeta^k(y, \hat q_\alpha, \epsilon_\alpha)$ for all $\alpha \in \Lambda$ implies that $\hat x^k(y) \in \tilde\zeta^k(y, \hat q, 0)$, that is, $\hat x^k(y)$ is optimal and hence $\hat q(y) \in Q^\circ$ by local non satiation. Therefore $\hat q \in \widehat Q^\circ$ since y was chosen arbitrarily and consequently $(y, \hat q(y), 0) \in D^j$ for all $j \in J$ and each fixed $y \in Y$. Since $\hat x^j_\alpha(y) \in \tilde\zeta^j(y, \hat q_\alpha(y), \epsilon_\alpha)$ for all $\alpha \in \Lambda$ then $\hat x^j(y) \in \tilde\zeta^j(y, \hat q(y), 0)$ and hence $\hat x^j(y)$ is optimal for $j \in J$ and all $y \in Y$.

[65] Observe that $\tilde\zeta^j(y,\cdot)$ and $\tilde\zeta^k(y,\cdot,\cdot)$ are upper hemicontinuous for each $y \in Y$ such that they are defined by the Berge Maximum Theorem. Moreover $Y \times \widehat Q \times (0,1] \subset D^k$ and $Y \times Q \times (0,1] \subset D^j$.

[66] Since $(1/n)_{n\in\mathbb{N}}$ converges to zero then $\{\epsilon_\alpha\}_{\alpha\in\Lambda}$ converges to zero too.

[67] To see that $Z \subset Z'$, notice that $y = (\bar\theta, s) \in Z$ and $\hat q(y) \in Q$ imply that there exists $l \le L = H+1$ such that $\hat q_l(y) > 0$ and $\sum_{k\in K} \bar\theta^k_l \ne 1$. Since $\sum_{i\in I} \bar\theta^i_l = 1$ then there exists $j \in J$ such that $\bar\theta^j_l > 0$ and hence $(y, \hat q(y), 0) \in D^j$.

Moreover, $(y, \hat{q}(y), 0) \in D^k$ implies $\hat{x}^k(y) \in \tilde{\zeta}^k(y, \hat{q}, 0)$ for all $k \in K$. This is the same to state that $\hat{x}(y)$ and $\hat{q}(y)$ constitutes a recursive equilibrium for the configuration y and $\epsilon = 0$, that is, in the economy with the initial wealth functions $\{\hat{w}^i\}_{i \in I}$. We use arguments analogous to Theorem 7.4 to prove the optimality of the recursive equilibrium if $X^i = \mathbb{R}_+^{H+1}$.

$\square$

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