CentER

Discussion Paper

No. 2007–96

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By Yukihiko Funaki, Takehiko Yamato

November 2007

ISSN 0924-7815

Tilburg University

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Yukihiko Funaki
School of Political Science and Economics
Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-ku
Tokyo 169-8050, Japan.
E-mail: funaki@waseda.jp

Takehiko Yamato
Department of Value and Decision Science
and Department of Social Engineering
Graduate School of Decision Science and Technology
Tokyo Institute of Technology, 2-12-1 Ookayama
Meguro-ku, Tokyo 152-8552, Japan.
E-mail: yamato@valdes.titech.ac.jp

November 14, 2007

Abstract

In this paper, we examine the question of which coalition structures farsighted players form in coalition formation games with externalities. We introduce a stability concept for a coalition structure called a *sequentially stable* coalition structure. Our concept of domination between two coalition structures is based on a "step-by-step" approach to describe negotiation steps concretely by restricting how coalition structures can change: when one coalition structure is changed to another one, either (i) only one merging of two separate coalitions into a coalition occurs, or (ii) only one breaking up of a coalition into two separate coalitions happens. As applications of our stability notion, we show that the efficient grand coalition structure can be sequentially stable in simple partition function form games and common pool resource games.

JEL classification codes: C70; C71; D62.

*This paper was completed when the first author was visiting the CentER of Tilburg University. He is grateful for its support. This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research on Priority Areas 19046002 and (C2) 16530126. We are grateful for the many helpful comments and suggestions provided by participants at the 2005 SING1 conference, Game Theory seminar at Tilburg University and seminars at University of Amsterdam, Maastricht University, Montpellier University and Tinbergen Institute, especially, Reinoud Joosten, Yuan Ju and Yoshio Kamijo.

Keywords: coalition formation ; common pool resource ; partition function ; stability.

1 Introduction

This paper examines the question of which coalition structures farsighted players form in coalition formation games with externalities. In a novel paper, Ray and Vohra (1997) introduce a stability concept for a coalition structure called an *equilibrium binding agreement* (EBA). They capture explicitly credibility of blocking coalitions, and then induce a recursive definition of a stable coalition structure. Efficiency can no longer be guaranteed even with free negotiation among farsighted players. In particular, the efficient grand coalition structure in which all players form one coalition may not be an EBA in various economic situations such as public goods economies and Cournot oligopolies.

In their definition, however, coalitions can only break up into smaller sizes of coalitions, but cannot merge into larger sizes of coalitions. This especially means that the singleton coalition structure consisting only of one-person coalitions is always an EBA. Diamantoudi and Xue (2007) recently propose an extension of the EBA notion, called an *extended EBA* (EEBA), to allow for any coalitional deviations, so that breaking up as well as merging are possible for coalitions. They strengthen Ray-Vohra's negative result by providing a robust example in which every stable EEBA coalition structure is inefficient even when negotiation processes are open and unrestricted.

These pessimistic conclusions on efficiency are quite striking, but allowing arbitrary changes in coalition structures may not be appropriate when we look for a positive result that an efficient coalition structure is attained. Under unrestricted coalitional deviations, for example, the inefficient singleton coalition structure in which no cooperation among players is formed can be suddenly changed into the efficient grand coalition structure in which all players cooperate. Then the following question naturally arises: through which negotiation steps is the singleton coalition structure changed to the grand coalition structure? More generally, how is some coalition structure transformed into another coalition structure when the two coalition structures are quite different? Answers to these questions are not clear under the EEBA concept. Furthermore, unrestricted changes in coalition structures may not be feasible in many practical situations. For instance, mergers of more than two firms have been little often observed in comparison with bilateral mergers of two firms, because their possibilities of going against anti-trust laws are higher.

Taking account of the above points, we use a ”step-by-step” approach to describe negotiation steps concretely by restricting how coalition structures can change at each step: when one coalition structure is changed to another one, either (i) only one merging of two separate coalitions into a single coalition occurs, or (ii) only one breaking up of a coalition into two separate coalitions happens. More specifically, we consider the following definition of domination between two coalition structures with farsighted players. The coalition structure $\mathcal{P}$ is said to *sequentially dominate* the coalition structure $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ starting from $\mathcal{P}_0 = \mathcal{P}$ to $\mathcal{P}_T = \mathcal{P}'$ such that at each step t, one of the following holds:

(1) Two separate coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$ and no other change occurs. All members in the two merging coalitions prefer the payoffs under the final

coalition structure $\mathcal{P}'$ to those under the coalition structure $\mathcal{P}_t$ before merging; or

(2) One coalition in $\mathcal{P}_t$ breaks up into two separate coalitions in $\mathcal{P}_{t+1}$ and no other change occurs. All members in at least one breaking up coalitions prefer the payoffs under the final coalition structure $\mathcal{P}'$ to those under the coalition structure $\mathcal{P}_t$ before breaking up.

A coalition structure is said to be *sequentially stable* if it sequentially dominates all other coalition structures.

We compare the three stability notions, EBA's, EEBA's, and sequential stability. First, all of them are characterized by the von Neumann-Morgenstern stable sets with respect to different domination relations. Second, sequential stability is a refinement of the EEBA notion in the sense that if a coalition structure $\mathcal{P}$ is sequentially stable, then the singleton set consisting only of $\mathcal{P}$ is an EEBA. However, the converse is not true: the singleton set consisting of one coalition structure that is not sequentially stable may be an EEBA. Moreover, there is no logical relation between sequential stability and the EBA notion. Finally, our condition of sequential stability is much simpler and relatively easier to check than those of an EBA and an EEBA are. This point is important for applications.

We also identify a simple condition for which the efficient grand coalition structure is a unique sequentially stable coalition structure in a partition function form game. Furthermore, as another application of our stability concept, we study a model of an economy with a common pool resource which has been often examined (e.g., Weitzman (1974), Roemer (1989), and Funaki and Yamato (1999)). We show that the efficient grand coalition structure can be sequentially stable in common pool resource games. Our positive results on efficiency contrast with the previous negative ones by Ray and Vohra (1997) and Diamantoudi and Xue (2007).

The rest of the paper is organized as follows. In Section 2, we introduce notation and definitions. In Section 3, we define sequential stability of coalition structures and compare our notion with EBA's and EEBA's. In Section 4, as an application, we examine sequential stability in common pool resource games. Section 5 contains some concluding remarks.

2 Basic Concepts

Let $N = \{1, 2, ..., n\}$ be a set of players. A subset S of N is called a coalition. We use the concept of a coalition structure to express how players form coalitions. Here a *coalition structure* $\mathcal{P}$ is a partition $\{S_1, S_2, ..., S_k\}$ of N, where $S_1, S_2, ..., S_k$ in $\mathcal{P}$ are disjoint and $\cup_{j=1}^{k} S_j = N$. The set of partitions of N is denoted by $\Pi(N)$.

We assume that given any coalition structure $\mathcal{P} \in \Pi(N)$, the feasible payoff vector under $\mathcal{P}$, $u(\mathcal{P}) = (u_1(\mathcal{P}), u_2(\mathcal{P}), ..., u_n(\mathcal{P})) \in \mathbb{R}^n$, is uniquely determined. The triple $(N, \Pi(N), (u_i)_{i \in N})$ is called a game with externalities.

We give two examples of games with externalities.

Example 1. Games in partition function form.

A game in partition function form (N, v) is defined by a pair of a set of players N and a partition function v which assigns to each pair of a partition $\mathcal{P} \in \Pi(N)$ and a coalition $S \in \mathcal{P}$, a real number $v(S|\mathcal{P})$. Given a game in partition function form, the feasible payoff vector under $\mathcal{P}$ is given by $u_i(\mathcal{P}) = \frac{v(S|\mathcal{P})}{|S|}$ $\forall i \in S, \forall S \in \mathcal{P}$. (See Thrall and Lucas(1963)).

Example 2. Hedonic games.

A hedonic game $(N, \{\succ_i\}_{i \in N})$ is defined by a pair of a set of players N and a binary relation $\succ_i$ on $\{S \subset N | S \ni i\}$ for all $i \in N$, which represents i's preference over coalitions that contain i. Consider i's utility function u_i over $\Pi(N)$ defined from $\succ_i$: For $\mathcal{P}$ and $\mathcal{P}' \in \Pi(N)$, we define

$$u_i(\mathcal{P}) > u_i(\mathcal{P}') \iff S \succ_i T,$$

where $i \in S, S \in \mathcal{P}$ and $i \in T, T \in \mathcal{P}'$. Then $(N, \Pi(N), (u_i)_{i \in N})$ becomes a game with externalities. (See, for example, Dreze and Greenberg (1980), Bogomalnia and Jackson(2002), Diamantoudi and Xue (2003).)

We introduce two special types of coalition structures. $\mathcal{P}^N = \{N\}$ is called a *grand* coalition structure, and $\mathcal{P}^I = \{\{1\}, \{2\}, ..., \{n\}\}$ is called a *singleton* coalition structure. We also say that $\mathcal{P}'$ is a *finer* coalition structure of $\mathcal{P}$ ($\mathcal{P}$ is a *coarser* coalition structure of $\mathcal{P}'$) if the coalition structure $\mathcal{P}'$ is given by re-dividing the coalition structure $\mathcal{P}$, that is, $\forall S' \in \mathcal{P}', \exists S \in \mathcal{P}$ such that $S' \subseteq S$ and $|\mathcal{P}'| > |\mathcal{P}|$.

3 Sequentially Stable Coalition Structures

In this section, we give our main stability concept called a "sequentially stable coalition structure". First we give a definition of sequential domination, and after that we give a definition of a sequentially stable coalition structure.

Definition 1. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *sequentially dominates* $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that

(1) $\mathcal{P}_T = \mathcal{P}$ and $\mathcal{P}_0 = \mathcal{P}'$,

(2) for all t $(0 \leq t \leq T-1)$, either $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$, or $\mathcal{P}_{t+1}$ is a coarser coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| - 1$, and

(3) for all t $(0 \leq t \leq T-1)$, for some $S \in \mathcal{P}_{t+1}$ with $S \notin \mathcal{P}_t$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in S.$$

We use the following notation for this sequence of coalition structures:

$$\mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to ... \to \mathcal{P}_T.$$

The condition (3) means that if $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$, for any member i in one of the two divided coalitions S and T such that $S, T \in \mathcal{P}_{t+1}$ and $S \cup T \in \mathcal{P}_t$, his payoff $u_i(\mathcal{P}_t)$ is smaller than his terminal payoff $u_i(\mathcal{P}_T)$; and if $\mathcal{P}_{t+1}$ is a coarser coalition structure of $\mathcal{P}_t$, for any member i in two combined coalitions S

and T such that $S, T \in \mathcal{P}_t$ and $S \cup T \in \mathcal{P}_{t+1}$, his payoff $u_i(\mathcal{P}_t)$ is smaller than his terminal payoff $u_i(\mathcal{P}_T)$.

Definition 2. We say that $\mathcal{P}^* \in \Pi(N)$ is a *sequentially stable coalition structure* if for all other coalition structures $\mathcal{P} \neq \mathcal{P}^*$, $\mathcal{P}^*$ sequentially dominates $\mathcal{P}$.

We will compare our domination relation with that of Ray and Vohra (1997). An important difference is that Ray and Vohra allowed only refinement of coalition structures, but we consider both of refinement and coarseness. We present domination due to Ray and Vohra called *RV-domination*.

Consider a case that only refinement of coalitions is allowed. Then if $\mathcal{P}$ sequentially dominates $\mathcal{P}'$, then there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that $\mathcal{P}_T = \mathcal{P}$, $\mathcal{P}_0 = \mathcal{P}'$ and $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$. Hence for each t, $\mathcal{P}_{t+1} \setminus \mathcal{P}_t$ consists of two coalitions, say S' and S''. Then

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \text{either (1) for all } i \in S' \text{ or (2) for all } i \in S''. \tag{3.1}$$

by Definition 1. We call S' or S'' which satisfies (3.1) the *leading perpetrator* or simply *perpetrator* [1]. Then we can take a sequence of perpetrators $\{S_t\}_{t=0}^{T-1}$ induced from $\{\mathcal{P}_t\}_{t=0}^T$. In this case, we say that $\mathcal{P}$ *sequentially dominates* $\mathcal{P}'$ via a sequence of perpetrators $\{S_t\}_{t=0}^{T-1}$.

Suppose $S_t \in \mathcal{P}'$ for any t. Then if we change the order of the perpetrators, we get another sequence of perpetrators and it also implies the same coalition structure $\mathcal{P}'$ starting from $\mathcal{P}$. If $\mathcal{P}$ *sequentially dominates* $\mathcal{P}'$ via a sequence of perpetrators for all the possible sequences, then we say $\mathcal{P}$ *sequentially dominates* $\mathcal{P}'$ via a collections of perpetrators $\{S_0, S_1, ..., S_{T-1}\}$.

Definition 3.[2] Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *RV-dominates* $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ such that

(1) $\mathcal{P}_T = \mathcal{P}$ and $\mathcal{P}_0 = \mathcal{P}'$,

(2') for all t $(0 \leq t \leq T-1)$, $\mathcal{P}_{t+1}$ is a finer coalition structure of $\mathcal{P}_t$ with $|\mathcal{P}_{t+1}| = |\mathcal{P}_t| + 1$,

(3) for all t $(0 \leq t \leq T-1)$, for some $S \in \mathcal{P}_{t+1}$ with $S \notin \mathcal{P}_t$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in S.$$

(4) Let $\mathcal{S} \equiv \{S_1, S_2, ..., S_{T-1}\}$ be a set of perpetrators starting from $\mathcal{P}_1$, and suppose all the perpetrators $S_0, S_1, S_2, ..., S_{T-1}$ are in $\mathcal{P}'$. Take any subset $\{S_{k_1}, S_{k_2}, ..., S_{k_p}\}$ of $\mathcal{S}$, and let $\mathcal{P}'' = (\mathcal{P} \setminus (S_{k_1} \cup S_{k_2} \cup, ..., \cup S_{k_p}\}) \cup \{S_{k_1}, S_{k_2}, ..., S_{k_p}\}$. Then the coalition structure $\mathcal{P}''$ is sequentially dominated by $\mathcal{P}'$ via a collection of perpetrators $\mathcal{S} \setminus \{S_{k_1}, S_{k_2}, ..., S_{k_p}\}$.

In condition (2'), only refinement of coalition structures is allowed, but both refinement and coarseness are allowed in (2).

[1] If both S' and S'' satisfy (3.1), take any of the two coalitions.

[2] This definition is different from the original one by Ray and Vohra (1997). This is an equivalent condition given by Proposition 1 in Diamantoudi and Xue (2007)

The condition (4) ensures a sequential domination on arbitrary paths or sequences of deviations starting from $\mathcal{P}_1$. This is different from the definition of a sequential domination. In our definition of a sequential domination, we only consider one possible path.

Ray and Vohra (1997) introduce an EBA coalition structure by applying their very general concept of "Equilibrium Bidding Agreement" to a coalition formation problem. Diamantoudi and Xue (2007) characterize the set of EBA coalition structures by RV-domination and the vNM-stable set concept. Let $\mathcal{E}$ be the set of coalition structures such that

(a) *external stability*: for any coalition structure $\mathcal{P}' \notin \mathcal{E}$, there exists $\mathcal{P} \in \mathcal{E}$ such that $\mathcal{P}$ RV-dominates $\mathcal{P}'$, and

(b) *internal stability*: for any coalition structure $\mathcal{P}' \in \mathcal{E}$, there is no $\mathcal{P} \in \mathcal{E}$ such that $\mathcal{P}$ RV-dominates $\mathcal{P}'$.

Diamantoudi and Xue (2007) proved that the set of EBA coalition structures coincide with $\mathcal{E}$. For our notion of sequential domination, the singleton set consisting of the sequentially stable coalition structure is also the vNM-stable set via a sequential domination.

Let us introduce a domination relation of Diamantoudi and Xue (2007) called *DX-domination*.

Definition 4. Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$. We say that $\mathcal{P}$ *DX-dominates* $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^{T}$ such that

(1) $\mathcal{P}_T = \mathcal{P}$, $\mathcal{P}_0 = \mathcal{P}'$, and

(2") for all t $(0 \leq t \leq T-1)$, $\mathcal{P}_{t+1}$ and $\mathcal{P}_t \equiv \{S_1, S_2, ..., S_k\}$ satisfy the following condition; there exists a coalition $Q(t) \subseteq N$ such that

(i) $Q(t) = Q_1 \cup Q_2 \cup ... \cup Q_l$, $Q_j \in \mathcal{P}_{t+1}$ $\forall j = 1, 2, .., l$ and Q_js are disjoint,

(ii) $\forall j = 1, 2, ..., k, S_j \cap Q(t) \neq \emptyset \Rightarrow S_j \setminus Q(t) \in \mathcal{P}_{t+1}$,

(iii) $\forall j = 1, 2, ..., k, S_j \cap Q(t) = \emptyset \Rightarrow S_j \in \mathcal{P}_{t+1}$.

(3') for all t $(0 \leq t \leq T-1)$,

$$u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \quad \forall i \in Q(t).$$

The element of the vNM-stable set of the coalition structures using DX-domination is called the set of *Extended EBA (EEBA)* coalition structures. Condition (2") in Definition 4 implies that all possibilities of refining and merging are allowed for coalitions. On the other hand, the way of changing coalitions should be step by step and no jump are allowed in Definition 1 of a sequential domination. For two coalition structures $\mathcal{P}$ and $\mathcal{P}'$, $\mathcal{P}$ DX-dominates $\mathcal{P}'$ if $\mathcal{P}$ sequentially dominates $\mathcal{P}'$ because (2) and (3) in Definition 1 together imply (2") and (3') in Definition 4. Hence, sequential stability is a refinement of the notion of EEBA's in the sense that if a coalition structure $\mathcal{P}$ is sequentially stable, then the singleton set consisting only of $\mathcal{P}$ is an EEBA. However, the converse is not true: the singleton set consisting of one coalition structure that is not sequentially stable may be an EEBA. Moreover, there is no logical relation between sequential stability and the notion of EBA's. The following example illustrate these facts:

Example 3. Consider a symmetric 5-person game in partition function form (N, v), where $N = \{1, 2, 3, 4, 5\}$ and

$v(N|\mathcal{P}^N) = 50.$ For any $\mathcal{P}_2$ s.t. $|\mathcal{P}_2| = 2$ and for any $S \in \mathcal{P}_2, v(S|\mathcal{P}_2) = 18$.

For any $\mathcal{P}_3$ s.t. $|\mathcal{P}_3| = 3$ and for any $S \in \mathcal{P}_3, v(S|\mathcal{P}_3) = 8$.

For any $\mathcal{P}_4$ s.t. $|\mathcal{P}_4| = 4$ and for any $S \in \mathcal{P}_4, v(S|\mathcal{P}_4) = 5$.

For any $\{i\} \in \mathcal{P}^I, v(\{i\}|\mathcal{P}^I) = 3$.

Figure 1 shows all the possible coalition structures and the feasible payoff vectors under each coalition structure. Here the circle shows the coalition and the number in the circle indicates the cardinality of the coalition. The vector under each coalition shows the feasible payoffs $u_i(\mathcal{P}) = \frac{v(S|\mathcal{P})}{|S|}$.

We will show that the grand coalition structure $\mathcal{P}^N$ is sequentially stable.

The proof consists of 4 steps.

(Step 1)(A) $\{2; 3\} \to \mathcal{P}^N$ because every player in $\{2; 3\}$ gets more payoff at $\mathcal{P}^N$.[3]
(B) $\{1; 1; 3\} \to \{2; 3\} \to \mathcal{P}^N$ because every player in $\{1; 1; 3\}$ gets more payoff at $\mathcal{P}^N$ and (Step 1)(A) holds. Thus $\{1; 1; 3\}$ and $\{2; 3\}$ are sequentially dominated by $\mathcal{P}^N$.
(Step 2) $\{1; 2; 2\} \to \{2; 3\} \to \mathcal{P}^N$ because every player in $\{1; 2; 2\}$ gets more payoff at $\mathcal{P}^N$ and (Step1)(A) holds. Thus $\{1; 2; 2\}$ is sequentially dominated by $\mathcal{P}^N$.
(Step 3) $\{1; 4\} \to \{1; 1; 3\} \to \{2; 3\} \to \mathcal{P}^N$ because the deviation of one person in the 4-person coalition in $\{1; 4\}$ increases his payoff at the final coalition structure $\mathcal{P}^N$ and (Step 1)(B) holds. Thus $\{1; 4\}$ is sequentially dominated by $\mathcal{P}^N$.
(Step 4)(A) $\{1; 1; 1; 2\} \to \{1; 1; 3\} \to \{2; 3\} \to \mathcal{P}^N$ because every player in $\{1; 1; 1; 2\}$ gets more payoff at $\mathcal{P}^N$ and (Step 1)(B) holds. Thus $\{1; 1; 1; 2\}$ is sequentially dominated by $\mathcal{P}^N$.
(B) $\{1; 1; 1; 1; 1\} \to \{1; 1; 1; 2\} \to \{1; 1; 3\} \to \{2; 3\} \to \mathcal{P}^N$ because every player in $\{1; 1; 1; 1; 1\}$ gets more payoff at $\mathcal{P}^N$ and (Step 4)(A) holds. Thus $\{1; 1; 1; 1; 1\}$ is sequentially dominated by $\mathcal{P}^N$.

These observations imply that the grand coalition structure $\mathcal{P}^N$ is sequentially stable.

Moreover, $\mathcal{P}^N$ is only one sequentially stable coalition structure. The reason is as follows: All the coalition structures $\mathcal{P}$ except for $\mathcal{P}^N$ and $\{1; 4\}$ are not sequentially stable because $u_i(\mathcal{P}^N) > u_i(\mathcal{P})$ for any $i \in N$. and then $\mathcal{P}$ cannot sequentially dominate $\mathcal{P}^N$. Next consider $\{1; 4\}$. It is not sequentially stable, because $\{1; 4\}$ cannot sequentially dominate $\{1; 1; 1; 2\}$. The reason is as follows: If $\{1; 4\}$ sequentially dominates $\{1; 1; 1; 2\}$, one of two coalitions in $\{1; 1; 1; 2\}$ should merge in the first step,

[3]Here $\{2; 3\}$ means any coalition structure with one 2-person coalition and one 3-person coalition.

but it is not profitable for all the members in the two coalitions since some of them should be in 4-person coalition in $\{1;4\}$. This is a contradiction.

On the other hand, $\{1;4\}$ DX-dominates $\{1;1;1;2\}$. Hence it is easy to check that $\{1;4\}$ as well as $\mathcal{P}^N$ is an EEBA. Besides, it is not difficult to see that $\{1;4\}$, $\{1;2;2\}$, and $\mathcal{P}^I$ are EBA's.

The properties of EEBA's are examined in Diamantoudi and Xue (2007). In their paper, they give the following proposition:

Definition 5. The coalition structure $\mathcal{P} \in \Pi(N)$ is Pareto efficient if there does not exist $\mathcal{P}' \in \Pi(N)$ such that $u_i(\mathcal{P}') > u_i(\mathcal{P})$ for any $i \in N$.

Proposition 1 (Diamantoudi and Xue (2007)). Let $\mathcal{P}^* \in \Pi(N)$ be Pareto efficient. $\mathcal{P}^*$ is an EEBA if

(a) $u_i(\mathcal{P}^*) > u_i(\mathcal{P}^I)$ $\forall i \in N$, and

(b) for all $\mathcal{P} \in \Pi(N)$ such that $\mathcal{P} \neq \mathcal{P}^*$ and $\mathcal{P} \neq \mathcal{P}^I$, there is a coalition $S \in \mathcal{P}$ such that $|S| > 1$ and $u_i(\mathcal{P}^*) > u_i(\mathcal{P})$ for some $i \in S$.

The similar proposition holds for sequential stability.

Proposition 2. Let $\mathcal{P}^* \in \Pi(N)$ be Pareto efficient. $\mathcal{P}^*$ is sequentially stable if

(a) $\mathcal{P}^*$ sequentially dominates $\mathcal{P}^I$, and

(b) for all $\mathcal{P} \in \Pi$ such that $\mathcal{P} \neq \mathcal{P}^*$ and $\mathcal{P} \neq \mathcal{P}^I$, there is a coalition $S \in \mathcal{P}$ such that $|S| > 1$ and for some member $i \in S$, $u_i(\mathcal{P}^*) > u_i(\mathcal{P})$.

Proof. Take any $\mathcal{P}$ such that $\mathcal{P} \neq \mathcal{P}^*$. We have to find a sequence of coalition structures from any $\mathcal{P}$ to $\mathcal{P}^*$ satisfying (1), (2) and (3) in Definition 5. First we construct a sequence $\{\mathcal{P}_k\}_{k=0}^R$ of coalition structures from $\mathcal{P}$ to $\mathcal{P}^I$, where $\mathcal{P}_0 = \mathcal{P}$ to $\mathcal{P}_R = \mathcal{P}^I$ $(R \leq n)$. In the sequence $\{\mathcal{P}_k\}_{k=0}^R$, for any $\mathcal{P}_k$ such that $\mathcal{P}_k \neq \mathcal{P}^I$, one person deviates from one of the largest coalition in $\mathcal{P}_k$. In this step, the deviated person prefers $\mathcal{P}^*$ to $\mathcal{P}$ because of (b). Second, (a) implies that the existence of a sequence of coalition structures from $\mathcal{P}^I$ to $\mathcal{P}^*$. Combining these sequences, we obtain the desired sequence of coalition structures. This implies $\mathcal{P}^*$ sequentially dominates $\mathcal{P}$. Q.E.D.

We will give a simple condition for which only the grand coalition structure is sequentially stable in a partition function form game.

Proposition 3. Consider an n-person partition function form game which satisfies

$$\frac{v(N|\mathcal{P}^N)}{n} > \frac{v(S|\mathcal{P})}{|S|} \quad \forall S \in \mathcal{P} \ \ \forall \mathcal{P} \in \Pi(N).$$

Then only $\mathcal{P}^N$ is sequentially stable.

Proof. Take any $\mathcal{P} \neq \mathcal{P}^N$. For any S and $\mathcal{P}$ such that $S \in \mathcal{P}$, we have $u_i(\mathcal{P}^N) = \frac{v(N|\mathcal{P}^N)}{n} > \frac{v(S|\mathcal{P})}{|S|} = u_i(\mathcal{P})$ for all $i \in S$. Then first every member of the coalitions in $\mathcal{P}_2$ with 2 coalitions prefers the grand coalition structure, that is, $\mathcal{P}^N$ sequentially dominates $\mathcal{P}_2$. Similarly we can show that $\mathcal{P}^N$ sequentially dominates $\mathcal{P}_3$, $\mathcal{P}_4$,...,$\mathcal{P}^I$, where $|\mathcal{P}_k| = k$. This shows that $\mathcal{P}^N$ is sequentially stable. It is obvious that $\mathcal{P}^N$ cannot be sequentially dominated by any other coalition structure. Q.E.D.

The above result says that in a partition function form game, if the per capita value of the grand coalition is larger than that of any other coalition under any coalition structure, then the set of sequential stable coalition structures consists only of the grand coalition. Moreover, it coincides with the set of EEBA's. However it is different from EBA's because $\mathcal{P}^I$ is also EBA.

4 Applications to Common Pool Resource Games

4.1 The Basic Model

We will apply our concept of sequential stability to the following game of an economy with a common pool resource. For any player $i \in N$, let $x_i \geq 0$ represent the amount of labor input of i. Clearly, the overall amount of labor is given by $\sum_{j\in N} x_j$. The technology that determines the amount of product is considered to be a joint production function of the overall amount of labor $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$f(0) = 0, \lim_{x\to\infty} f'(x) = 0, f'(x) > 0$ and $f''(x) < 0$ for $x > 0$. The distribution of the product is supposed to be proportional to the amount of labor expended by players. In other words, the amount of the product assigned to player i is given by $\frac{x_i}{\sum_{j\in N} x_j} \cdot f(\sum_{j\in N} x_j)$. The price of the product is normalized to be one unit of money and let q be a cost of labor per unit, and we suppose $0 < q < f'(0)$.

Then individual i's income is denoted by

$$m_i(x_1, x_2, ..., x_n) = \frac{x_i}{x_N} f(x_N) - qx_i.$$

The total income of coalition S is denoted by

$$m_S \equiv \sum_{i\in S} m_i = \frac{x_S}{x_N} f(x_N) - qx_S,$$

where $x_S \equiv \sum_{i\in S} x_i$. We consider a game where each coalition is a player. It chooses its total labor input and its payoff is given by the sum of the income over its members. Naturally we can define a Nash equilibrium of that game.

Definition 6. The list $(x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_k})$ is an *equilibrium under* $\mathcal{P}$ if

$$m_{S_j}(x^*_{S_j}, x^*_{S_{-j}}) \geq m_{S_j}(x_{S_j}, x^*_{S_{-j}}), \ \ \forall j, \ \ \forall x_{S_j} \in \mathbb{R}_+.$$

There is a unique equilibrium under every coalition structure:

Proposition 4 (Funaki and Yamato (1999)). *For any* $\mathcal{P} = \{S_1, S_2, ..., S_k\}$, *there exists a unique equilibrium* $(x^*_{S_1}, x^*_{S_2}, ..., x^*_{S_k})$ *under* $\mathcal{P}$ *which satisfies*

$$f'(\sum_{j=1}^{k} x^*_{S_j}) + \frac{(k-1)f(\sum_{j=1}^{k} x^*_{S_j})}{\sum_{j=1}^{k} x^*_{S_j}} = kq, \quad x^*_{S_i} = \frac{\sum_{j=1}^{k} x^*_{S_j}}{k} \ > 0 \ \forall i.$$

Given a coalition structure $\mathcal{P} = \{S_1, ..., S_k\}$, let $(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ be a unique equilibrium under $\mathcal{P}$ and let $x^*_N(\mathcal{P}) = \sum_{i=1}^k x^*_{S_i}(\mathcal{P})$. Moreover, let $m^*_{S_i}(\mathcal{P}) = m_{S_i}(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$ be the equilibrium income of coalition S_i for $i = 1, .., k$ and therefore $m^*_N(\mathcal{P}) = \sum_{i=1}^k m_{S_i}(x^*_{S_1}(\mathcal{P}), ..., x^*_{S_k}(\mathcal{P}))$.

Proposition 5 (Funaki and Yamato (1999)). *For two coalition structures $\mathcal{P}_k = \{S_1, S_2, ..., S_k\}$ and $\mathcal{P}'_{k'} = \{S'_1, S'_2, ..., S'_{k'}\}$ with $k < k'$,*

$$x^*_N(\mathcal{P}_k) < x^*_N(\mathcal{P}'_{k'}), \quad \frac{m^*_N(\mathcal{P}_k)}{n} > \frac{m^*_N(\mathcal{P}'_{k'})}{n},$$

$$S \in \mathcal{P}_k \ \text{ and } \ S \in \mathcal{P}'_{k'} \implies m^*_S(\mathcal{P}_k) > m^*_S(\mathcal{P}'_{k'}).$$

Proposition 5 says that as the number of coalitions decreases, the total amount of labor input decreases, whereas the average income increases. Also, if the number of coalitions in one coalition structure is smaller than that in another coalition structure and coalition S belongs to both coalition structures, then the income of coalition S under the former structure is larger than that under the latter.

We assume that for a common pool resource game, the feasible payoff vector is given by $u_i(\mathcal{P}) = \frac{m^*_{S_j}(\mathcal{P})}{|S_j|} \ \forall i \in S_j, \forall S_j \in \mathcal{P}$. It is natural to consider this because of the symmetry of players.

4.2 Sequentially Stable Coalition Structures in Common Pool Resource Games

We will examine sequential stability of the grand coalition structure in common pool resource games. The following lemma will be useful below, which gives a sufficient condition for which all players prefer the grand coalition structure to another coalition structure.

Lemma 1. In a common pool resource game, let a coalition structure $\mathcal{P}$ be given. Without loss of generality, denote the coalition structure by $\mathcal{P} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$. Let

$$B(k) \equiv \{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})\}/[k^2\{f(x^*_N(\mathcal{P}^N)) - f'(x^*_N(\mathcal{P}^N))x^*_N(\mathcal{P}^N)\}],$$

where $\mathcal{P}^N = \{1, 2, .., n\}$ is the grand coalition structure. Then for each $i \in N$, $u_i(\mathcal{P}) < u^*_i(\mathcal{P}^N)$ if $B(k) < r_1/n$. Moreover, for each $i \in S_2 \cup S_3 \cup ... \cup S_k$, $u_i(\mathcal{P}) < u^*_i(\mathcal{P}^N)$ if $B(k) < r_2/n$.

Proof. By Proposition 4,

$$\begin{aligned} u_i(\mathcal{P}) &= m^*_{S_j}(\mathcal{P})/r_j = [f(x^*_N(\mathcal{P})) - qx^*_N(\mathcal{P})]/(r_j k) \\ &= [f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})]/(r_j k^2), \end{aligned}$$

for $i \in S_j$and $j = 1, ..., k$. Notice that for the grand coalition structure $\mathcal{P}^N$, $k = 1$ and $r_1 = n$, so that $u_i(\mathcal{P}^N) = \left[f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\right]/n$ for $i \in N$. We also remark that a player belonging to the smallest coalition, S_1, obtains the highest payoff among all players, that is, the payoff of each player i, $u_i(\mathcal{P})$, is less than or equal to $u_j(\mathcal{P}) = m_{S_1}^*(\mathcal{P})/r_1$ for $j \in S_1$. Therefore, for each $i \in N$, $u_i(\mathcal{P}) < u_i(\mathcal{P}^N)$ if $B(k) = \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}] < r_1/n$.

In similar, note that a player belonging to the smallest coalition, S_2, among the coalitions, $S_2, S_3, .., S_k$, obtains the highest payoff among players in those $k-1$ coalitions, that is, the payoff of each player $i \in S_2 \cup S_3 \cup ... \cup S_k$, $u_i(\mathcal{P})$, is less than or equal to $u_j(\mathcal{P}) = m_{S_2}^*(\mathcal{P})/r_2$ for $j \in S_2$. Hence, for each $i \in S_2 \cup S_3 \cup ... \cup S_k$, $u_i(\mathcal{P}) < u_i(\mathcal{P}^N)$ if $B(k) = \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}] < r_2/n$. Q.E.D.

We will identify a condition for which the grand coalition structure is sequentially stable. We begin by studying the simple case in which the number of players can be expressed as $n = 2^m (m \geq 2)$.

Theorem 1. Let $n = 2^m (m \geq 2)$. If $B(k) < 1/2^{k-1}$ for all k $(k = 2, ..., m, m+1)$ and $B(k)$ is monotonically decreasing in k, then the grand coalition structure is sequentially stable.

Then we extend Theorem 1 to the general case of the number of players:

Theorem 2. Let $n = 2^m + l$, where $m \geq 2$ and $0 \leq l \leq 2^m - 1$. If the inequalities

$$B(2^{m-h-1} + 2) < \frac{2^{h-1}}{n} \quad (h = 1, 2, ..., m-1) \tag{4.1}$$

and $B(2) < \frac{2^{m-1}}{n}$ hold, and $B(k)$ is monotonically decreasing in k, then the grand coalition structure is sequentially stable.

The basic ideas behind the proofs of Theorems 1 and 2 are the same, although the construction of sequences of coalition structures for the sequential domination by the grand coalition structure becomes more complicated in the proof of Theorem 2. The outline of the proofs of Theorems 1 and 2 is as follows. It consists of 4 steps:
(Step 1) The grand coalition structure $\mathcal{P}^N$ sequentially dominate some key coalition structure $\mathcal{P}^*$.
(Step 2) Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| = |\mathcal{P}^*|$ is sequentially dominated by $\mathcal{P}^N$.
(Step 3) Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| < |\mathcal{P}^*|$ other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$.
(Step 4) Every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| > |\mathcal{P}^*|$ is sequentially dominated by $\mathcal{P}^N$.

By Steps 1-4, every coalition structure other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$. We illustrate these steps by using an example:

Example 4. Let $n = 5$ and $f(x) = \sqrt{x}$.[4] It is easy to check the following:

$m^*_N(\mathcal{P}^N) = 0.5$. For any $\mathcal{P}_2$ s.t. $|\mathcal{P}_2| = 2$ and for any $S \in \mathcal{P}_2, m^*_S(\mathcal{P}_2) = 0.1875$.

For any $\mathcal{P}_3$ s.t. $|\mathcal{P}_3| = 3$ and for any $S \in \mathcal{P}_3$, $m^*_S(\mathcal{P}_3) = 0.0926$.

For any $\mathcal{P}_4$ s.t. $|\mathcal{P}_4| = 4$ and for any $S \in \mathcal{P}_4$, $m^*_S(\mathcal{P}_4) = 0.547$.

For any $\{i\} \in \mathcal{P}^I$, $m^*_{\{i\}}(\mathcal{P}^I) = 0.036$.

Figure 2 depicts all possible coalition structures and the feasible payoff vectors under each coalition structure in this example. Notice that the basic payoff structure of Figure 2 is the same as that of Figure 1 for Example 3, although the payoff values are different. Therefore, by applying the same arguments as those of Steps 1-4 in Example 3, we can see that $\mathcal{P}^N$ is sequentially stable in this example. Here the key coalition structure $\mathcal{P}^*$ is $\{1; 1; 3\}$. In fact, as in Example 3, $\mathcal{P}^N$ is only one sequentially stable coalition structure.[5] On the other hand, $\{1; 4\}$ and $\mathcal{P}^N$ are EEBA's, and $\{1; 4\}$, $\{1; 2; 2\}$ and $\mathcal{P}^I$ are EBA's.

The proof of Theorem 1 is given as follows. The proof of Theorem 2 is in the appendix.

Proof of Theorem 1.

In the following, we denote a coalition structure $\mathcal{P} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$, by $\{r_1; r_2; r_3; ...; r_k\}$, because the payoff is determined by the sizes of all coalitions in a coalition structure.

Consider a coalition structure $\mathcal{P}^*$ consisting of the following $(m + 1)$ coalitions: two 1-person coalitions, one 2-person coalition, one 4-person coalition, one 8-person coalition, ..., and one 2^{m-1}-person coalition. This coalition structure is denoted by $\{1; 1; 2; 4; 8; ....; 2^{m-1}\}$. In what follows we say that $\mathcal{P}$ is *a k-th stage coalition structure* if $|\mathcal{P}| = k$.

The proof consists of four steps.

(Step 1) $\mathcal{P}^*$ is sequentially dominated by $\mathcal{P}^N$:

Consider a sequence of coalition structures $\{\mathcal{P}_t\}^m_{t=0}$ such that $\mathcal{P}_0 = \mathcal{P}^*, \mathcal{P}_m = \mathcal{P}^N$, and the two coalitions of the smallest size in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$ for $t = 0, 1, 2, ..., m - 1$. This sequence is expressed by

$$\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\} \to \mathcal{P}_1 = \{2; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\}$$

$$\to \mathcal{P}_2 = \{4; 4; 8; ...; 2^{m-2}; 2^{m-1}\} \to$$

$$.... \to .... \to \mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\} \to \mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\} \to \mathcal{P}_m = \mathcal{P}^N = \{2^m\}$$

[4] Notice that the conditions in Theorem 2 are satisfied. Here $m = 2$, $l = 1$, and $B(k) = \frac{2k-1}{k^3}$. Clearly, $B(2) = \frac{3}{8} < \frac{2}{5} = \frac{2^{m-1}}{n}$ and $B(k)$ is monotonically decreasing in k.

[5] This is not true in general. Coalition structures other than the grand coalition structure could be sequentially stable. See the example given just before Theorem 3.

First, it follows from Lemma 1 that the 2nd stage coalition structure $\mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-1}/2^m = 1/2 > B(2)$ by the hypothesis.

Next, it follows from Lemma 1 that the 3rd stage coalition structure $\mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-2}/2^m = 1/4 > B(3)$ by the hypothesis.

In general, for $k = 2, ..., m, m+1$, it follows from Lemma 1 that the k-th stage coalition structure $\mathcal{P}_{m-k+1} = \{2^{m-k+1}; 2^{m-k+1}; 2^{m-k+2}; 2^{m-k+3}; ...; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-k+1}/2^m = 1/2^{k-1} > B(k)$ by the hypothesis.

Therefore, the $(m+1)$-th stage coalition structure $\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; ...; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 2) Every $(m+1)$-th stage coalition structure is sequentially dominated by $\mathcal{P}^N$:

Take any $(m+1)$-th stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^T$ such that

1) $\mathcal{P}_0 = \mathcal{P} = \{r_1; r_2; r_3; ...; r_{m-1}; r_m; r_{m+1}\}$

2) $\mathcal{P}_T = \{1; 1; 1; ...; 1; 2^m - m\}$, where $|\mathcal{P}_T| = m+1$.

3) If t is zero or even, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$.

4) If t is odd, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^T$ of coalition structures is given by:

$$\mathcal{P}_0 = \{r_1; r_2; r_3; ..., r_{m-1}; r_m; r_{m+1}\} \ \ ((m+1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_1 = \{r_1; r_2; r_3; ...; r_{m-1}; r_m + r_{m+1}\} \ \ (m\text{-th stage})$$

$$\rightarrow \mathcal{P}_2 = \{1; r_1; r_2; r_3; ...; r_{m-1}; r_m + r_{m+1} - 1\} \ \ ((m+1)\text{-th stage})$$

$$\rightarrow ... \rightarrow ...$$

$$\rightarrow \mathcal{P}_{T-2} = \{1; 1; 1; ...; 1; r_1; \sum_{k=2}^{m+1} r_k - m + 1\} \ \ ((m+1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_{T-1} = \{1; 1; 1; ...; 1; \sum_{k=1}^{m+1} r_k - m + 1\} \ \ (m\text{-th stage})$$

$$\rightarrow \mathcal{P}_T = \{1; 1; 1; 1; ...; 1; \sum_{k=1}^{m+1} r_k - m\} = \{1; 1; 1; ...; 1; 2^m - m\} \ \ ((m+1)\text{-th stage})$$

Next consider $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ such that

1) $\mathcal{P}_T = \{1; 1; 1; ...; 1; 2^m - m\}$,

2) $\mathcal{P}_{T+T'} = \mathcal{P}^* = \{1; 1; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\}$,

3) If $t = T + \lambda$ and λ is zero or even ($\lambda \leq T' - 2$), then the smallest coalition of more than one members and a 1-person coalition in $\mathcal{P}_{T+\lambda}$ merge into one coalition in $\mathcal{P}_{T+\lambda+1}$.

4) If $t = T + \lambda$ and λ is odd ($\lambda \leq T' - 2$), then $2^{m-\frac{\lambda+1}{2}}$ persons in the coalition of $2^{m-\frac{\lambda+1}{2}+1} - (m - \frac{\lambda+1}{2})$ persons in $\mathcal{P}_{T+\lambda}$ deviate and form a coalition in $\mathcal{P}_{T+\lambda+1}$. Note that $2^{m-\frac{\lambda+1}{2}+1} - (m - \frac{\lambda+1}{2}) \geq 1$.

5) If $t = T + T' - 1$, then two one-person coalitions in $\mathcal{P}_{T+T'-1}$ merge into one coalition in $\mathcal{P}_{T+T'}$.

This sequence $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ of coalition structures is given by:

$$\mathcal{P}_T = \{1; 1; 1; 1; ...; 1; 1; 1; 1; 2^m - m\} \ \ ((m+1)\text{-th stage})$$

$$\to \mathcal{P}_{T+1} = \{1; 1; 1; 1; ...; 1; 1; 1; 2^m - m + 1\} \ \ (m\text{-th stage})$$

$$\to \mathcal{P}_{T+2} = \{1; 1; 1; 1; ...; 1; 1; 1; 2^m - m + 1 - 2^{m-1}; 2^{m-1}\}$$

$$= \{1; 1; 1; 1; ...; 1; 1; 1; 2^{m-1} - m + 1; 2^{m-1}\} \ \ ((m+1)\text{-th stage})$$

$$\to \ \ \mathcal{P}_{T+3} = \{1; 1; 1; 1; ...; 1; 1; 2^{m-1} - m + 2; 2^{m-1}\} \ \ (m\text{-th stage})$$

$$\to \ \ \mathcal{P}_{T+4} = \{1; 1; 1; 1; ...; 1; 1; 2^{m-1} - m + 2 - 2^{m-2}; 2^{m-2}; 2^{m-1}\}$$

$$= \{1; 1; 1; 1; ...; 1; 1; 2^{m-2} - m + 2; 2^{m-2}; 2^{m-1}\} \ \ ((m+1)\text{-th stage})$$

$$\to \ \ \mathcal{P}_{T+5} = \{1; 1; 1; 1; ...; 1; 2^{m-2} - m + 3; 2^{m-2}; 2^{m-1}\} \ \ (m\text{-th stage})$$

$$\to ... \to ...$$

$$\to \mathcal{P}_{T+T'-1} = \{1; 1; 1; 1; 4; 8; ...; 2^{m-3}; 2^{m-2}; 2^{m-1}\} \ \ ((m+1)\text{-th stage})$$

$$\to \mathcal{P}_{T+T'} = \{1; 1; 2; 4; 8; ...; 2^{m-3}; 2^{m-2}; 2^{m-1}\} \ \ (m\text{-th stage})$$

This sequence ends at the coalition structure $\mathcal{P}_{T'} = \mathcal{P}^*$.

Hence if we combine two sequences $\{\mathcal{P}_t\}_{t=0}^{T}$ and $\{\mathcal{P}_t\}_{t=T}^{T+T'}$, we can get a sequence $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ from any $(m+1)$-th stage coalition structure $\mathcal{P}$ to $\mathcal{P}^*$. Note that only $(m+1)$-th stage and m-th stage coalition structures appear in this sequence.

Since $B(k) = B(m+1) < 1/2^m = r_1/n$, it follows from Lemma 1 that each member of any coalition in $(m+1)$-th stage coalition structure prefers the payoff under the grand coalition structure $\mathcal{P}^N$ to the payoff under the $(m+1)$-th stage coalition structure.

Also, notice that any deviating coalition in the process from m-th stage coalition structure to $(m+1)$-th stage coalition structure consists of at least two players. Since $B(k) = B(m) < 1/2^{m-1} = 2/2^m \leq r_2/n$, it follows from Lemma 1 that each member of such a deviating coalition prefers the payoff in the grand coalition structure $\mathcal{P}^N$ to the payoff in the m-th stage coalition structure.

Therefore if we combine this sequence $\{\mathcal{P}_t\}_{t=0}^{T'}$ and a sequence from $\mathcal{P}_{T+T'} = \mathcal{P}^*$ to $\mathcal{P}^N$, every coalition structure in the sequence $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ is sequentially dominated by $\mathcal{P}^N$. And so is the $(m+1)$-th stage coalition structure $\mathcal{P}$. This completes the proof of Step 2.

(Step 3) Every coalition structure $\mathcal{P}$ of less than $m+1$ coalitions other than the grand coalition structure $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$.

First, we show that each member of a coalition of the maximal size in any coalition structure $\overline{\mathcal{P}}$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\overline{\mathcal{P}}$. Denote $\overline{\mathcal{P}}$ by $\overline{\mathcal{P}} = \{S_1, S_2, S_3, ..., S_k\}$, where $|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k$. Because $r_k \geq r_i$ for all r_i, $kr_k \geq \sum_{i=1}^{k} r_i = n$, that is, $r_k/n \geq 1/k$. Since $B(k) < 1/2^{k-1}$, it follows that $r_k/n \geq 1/k \geq 1/2^{k-1} > B(k)$. By Lemma 1, we have the desired result.

Take any coalition structure $\mathcal{P}$ of less than $m+1$ coalitions other than $\mathcal{P}^N$. Consider the following sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $(m+1)$-th stage coalition structure $\mathcal{P}'$: one person in a coalition of the maximal size in $\mathcal{P}_t$ deviates and forms a 1-person coalition in $\mathcal{P}_{t+1}$. Notice that such a person in $\mathcal{P}_t$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$ and the sequence from $\mathcal{P}'$ to $\mathcal{P}^N$ in Step 2. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

(Step 4) Every coalition structure $\mathcal{P}$ of more than $m+1$ coalitions is sequentially dominated by $\mathcal{P}^N$.

Take any k-th stage coalition structure $\mathcal{P}$ of more than $m+1$ coalitions. Since B is a decreasing function and $k > m+1$, $B(k) < B(m+1) < 1/2^m = 1/n \leq r_i/n$ holds for any $r_i \geq 1$. This together with Lemma 1 imply that each member of any coalition in $\mathcal{P}$ prefers her payoff under the grand coalition structure $\mathcal{P}^N$ to her payoff under $\mathcal{P}$.

Consider a sequence $\{\mathcal{P}_t\}$ starting from $\mathcal{P}$ to some $(m+1)$-th stage coalition structure $\mathcal{P}'$ such that two coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$. Notice that each member in these two coalitions in $\mathcal{P}_t$ prefers her payoff under $\mathcal{P}^N$ to her payoff under $\mathcal{P}_t$, as shown above. Moreover, it is easy to construct a sequence of coalition structures from $\mathcal{P}$ to $\mathcal{P}^N$ by combining the above sequence from $\mathcal{P}$ to $\mathcal{P}'$ and the sequence from $\mathcal{P}'$ to $\mathcal{P}^N$ in Step 2. These imply that $\mathcal{P}$ is sequentially dominated by $\mathcal{P}^N$.

Q.E.D.

We now apply the above theorem when the production function is given by $f(x) = x^\alpha$ $(0 < \alpha < 1)$. First of all, by Proposition 4, it is easy to check that for any $\mathcal{P}$

$$x_N^*(\mathcal{P}) = (\alpha + k - 1)(x_N^*(\mathcal{P}))^{\alpha-1}/(kq) = \left(\frac{\alpha - 1 + k}{kq}\right)^{1/(1-\alpha)},$$

$$u_i(\mathcal{P}) = m_{S_1}^*(\mathcal{P})/r_1 = \left[f(x_N^*(\mathcal{P})) - qx_N^*(\mathcal{P})\right]/(r_1 k)$$

$$= \left[f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\right]/(r_1 k^2) = (1-\alpha)(x_N^*(\mathcal{P}))^\alpha/(r_1 k^2), \forall i \in S_1.$$

Notice that if $\mathcal{P} = \mathcal{P}^N$, then $k = 1$ and $r_1 = n$, so that

$$x_N^*(\mathcal{P}^N) = \alpha(x_N^*(\mathcal{P}^N))^{\alpha-1}/q = \left(\frac{\alpha}{q}\right)^{1/(1-\alpha)}.$$

$$f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N) = (1-\alpha)(x_N^*(\mathcal{P}^N))^\alpha.$$

This implies

$$B(k) = \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/[k^2\{f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)\}]$$
$$= \frac{1}{k^2}\left(\frac{\alpha - 1 + k}{\alpha k}\right)^{\alpha/(1-\alpha)}.$$

Corollary 1. If $f(x) = x^\alpha$, then for some $\alpha \in (0, 1)$, the grand coalition structure $\mathcal{P}^N$ is sequentially stable for any number of players $n = |N| \geq 4$.

Proof. We will apply Theorem 2. First of all, note that $B(k)$ is an increasing function of α, and $\lim_{\alpha\to 0} B(k) = 1/k^2$ for any k. Hence for sufficiently small $\alpha > 0$, $B(k)$ is very close to $1/k^2$.

Let $m \geq 2$ be given. Consider any integer $n \in [2^m, 2^{m+1})$. First we will show that $\lim_{\alpha\to 0} B(2^{m-h-1} + 2) = 1/(2^{m-h-1} + 2)^2 < 2^{h-1}/n$ for $h = 1, ..., m - 2$. Since $2^{h-1}/n > 2^{h-1}/2^{m+1} = 1/2^{m-h+2}$, it is sufficient to prove that $1/(2^{m-h-1} + 2)^2 \leq 1/2^{m-h+2}$, that is, $(2^{m-h-1} + 2)^2 \geq 2^{m-h+2}$. If $h \leq m - 4$, then $2^{2(m-h-1)} \geq 2^{m-h+2}$, implying the desired result. Also,

for $h = m - 3$, $(2^{m-h-1} + 2)^2 = (2^2 + 2)^2 > 2^5 = 2^{m-h+2}$,
for $h = m - 2$, $(2^{m-h-1} + 2)^2 = (2 + 2)^2 = 2^4 = 2^{m-h+2}$and
for $h = m - 1$, $(2^{m-h-1} + 2)^2 = (1 + 2)^2 = 3^2 > 2^3 = 2^{m-h+2}$.

Moreover, $\lim_{\alpha\to 0} B(2) = 1/4 = 2^{m-1}/2^{m+1} < 2^{m-1}/n$. Finally, it is clear that $B(k)$ is decreasing in k. Therefore, by Theorem 2, $\mathcal{P}^N$ is sequentially stable for some $\alpha \in (0, 1)$. Q.E.D.

This corollary says that if we apply our stability concept to a common pool resource game, the grand coalition structure can be sequentially stable for any number of players.

Coalition structures other than the grand coalition structure could be sequentially stable. For example, in a 6-person game with $f(x) = \sqrt{x}$, the coalition structures consisting of $(n - 1)$-person coalition and one-person coalition, $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}(\{i\} \in N)$ are also sequentially stable. However, such a coalition structure is quite unfair in the sense that the payoff of the player in one-person coalition is equal to the sum of all other players' payoffs. We will examine under which condition these unfair coalition structures are unstable. For $\mathcal{P}$ with $|\mathcal{P}| = k$, let $C(k) \equiv \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/\{f(x_N^*(\mathcal{P}^{N\setminus\{i\}})) - f'(x_N^*(\mathcal{P}^{N\setminus\{i\}}))x_N^*(\mathcal{P}^{N\setminus\{i\}})\}$.

Theorem 3. Let $n \geq 5$. If $C(3) \geq \frac{9}{8}$, then the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$, $(\{i\} \in N)$ are not sequentially stable.

Proof. We will show that any coalition structure containing three coalitions is not sequentially dominated by $\mathcal{P}^{N\setminus\{i\}}$ if $C(3) \geq \frac{9}{8}$. Let $\mathcal{P} = \{S_1, S_2, S_3\}$, $|S_1| \leq |S_2| \leq |S_3|$, be a coalition structure containing 3 coalitions.

In any sequence from $\mathcal{P}$ to $\mathcal{P}^{N\setminus\{i\}}$, two coalitions must merge into one coalition. Thus it is enough to show that the payoff of each player in one of two coalitions is smaller than the payoff in the coalition $N \setminus \{i\}$ of $\mathcal{P}^{N\setminus\{i\}}$. Hence if the largest payoff of a player in the second largest S_2 among all coalition structures with 3 coalitions is smaller than the payoff of a player in $N \setminus \{i\}$, we can attain our purpose.

Then we have to compare the payoff $m_j^*(\mathcal{P})$ of player j in a coalition S_2 of the smallest size with the payoff $m_j^*(\mathcal{P}^{N\setminus\{i\}})$.

Remark that such a coalition structure is given by $|S_1| = 1, |S_2| = |S_3| = \frac{n-1}{2}$ if n is odd, and $|S_1| = 1, |S_2| = \frac{n-2}{2}, |S_3| = \frac{n+2}{2}$ if n is even.

By Proposition 4,

$$m_j^*(\mathcal{P}) = [f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})]/(9r_2),$$

for $j \in S_2$, and

$$m_j^*(\mathcal{P}^{N\setminus\{i\}}) = \left[f(x_N^*(\mathcal{P}^{N\setminus\{i\}})) - f'(x_N^*(\mathcal{P}^{N\setminus\{i\}}))x_N^*(\mathcal{P}^{N\setminus\{i\}})\right]/(4(n-1)),$$

for $j \in N\setminus\{i\}$. Note that for $j \in S_2$, $m_j^*(\mathcal{P}) \geq m_j^*(\mathcal{P}^{N\setminus\{i\}})$ iff $[4(n-1)/(9r_2)]\{f(x_N^*(\mathcal{P}))-f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/\{f(x_N^*(\mathcal{P}^{N\setminus\{i\}}))-f'(x_N^*(\mathcal{P}^{N\setminus\{i\}}))x_N^*(\mathcal{P}^{N\setminus\{i\}})\}] = [4(n-1)/(9r_2)]C(3) \geq 1$. There are two cases to examine. First, if n is even, consider a coalition structure $\mathcal{P}$ with $r_2 = (n-2)/2$. In this case, $4(n-1)/(9r_2) = \frac{8(n-1)}{9(n-2)}$, so that if $C(3) \geq \frac{9}{8}$, then $m_j^*(\mathcal{P}) > m_j^*(\mathcal{P}^{N\setminus\{i\}})$. Second, if n is odd, consider a coalition structure $\mathcal{P}$ with $r_2 = (n-1)/2$. In this case, $4(n-1)/(9r_2) = \frac{8}{9}$, so that if $C(3) \geq \frac{9}{8}$, then $m_j^*(\mathcal{P}) \geq m_j^*(\mathcal{P}^{N\setminus\{i\}})$. Q.E.D.

By applying this theorem to the case in which the production function is give by $f(x) = x^\alpha$ $(0 < \alpha < 1)$, we have the following:

Corollary 2. Let $n \geq 5$. If $f(x) = x^\alpha$ and $\alpha \geq 0.583804$, then the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$, $(\{i\} \in N)$ are not sequentially stable.

Proof. It is easy to see that

$$C(3) = \left(\frac{3(\alpha+1)}{2(\alpha+2)}\right)^{-\alpha/(1-\alpha)}.$$

Therefore, $C(3) > \frac{9}{8}$ iff $1/C(3) = \left(\frac{3(\alpha+1)}{2(\alpha+2)}\right)^{\alpha/(1-\alpha)} < \frac{8}{9}$. Figure 3 illustrates the function $1/C(3) - \frac{8}{9}$. It is not hard to check that if $1/C(3) < \frac{8}{9}$ if $\alpha \geq 0.583804$.

The above result shows that for any number of players, the coalition structures $\mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\}$ cannot be sequentially stable if α is suitably large.

Remark 1. In our definition of domination, either (i) only two coalitions can merge into one coalition, or (ii) one coalition can break up into two coalitions at each step in a sequence. It is possible to define a slightly different notion of domination such that more than two coalitions are allowed to merge into one coalition at each step in a sequence. Our original concept of sequential stability is a refinement of this alternative notion. For this definition of domination, however, we can prove that the unfair coalition structure $\mathcal{P}^{N\setminus\{i\}}$ sequentially dominates any other coalition structure for a sufficiently large n.

Proposition 6. Suppose we allow that singleton coalition structure $\mathcal{P}^I$ can merge into $\mathcal{P}^{N\backslash\{i\}}$ directly at one step. Given $\alpha \in (0,1)$, $\mathcal{P}^{N\backslash\{i\}}$ is sequentially stable for a sufficiently large n.

Proof. $u_k(\mathcal{P}^{N\backslash\{i\}}) > u_k(\mathcal{P}^I)$ for all $k \in N\backslash\{i\} \in \mathcal{P}^{N\backslash\{i\}}$ if $C(n) < \frac{1}{n-1}$, that is,

$$\left(\frac{2(\alpha-1+n)}{(\alpha+1)n}\right)^{\alpha/(1-\alpha)} \frac{1}{n^2} < \frac{1}{n-1}.$$

For any $\alpha \in (0,1)$, this inequality holds if n is sufficiently large for fixed α. Then under the supposition, $\mathcal{P}^{N\backslash\{i\}}$ sequentially dominates $\mathcal{P}^I$. In every coalition structure $\mathcal{P}$ such that $|\mathcal{P}| \leq n-2$, a member in the largest coalition get more payoff in $N\backslash\{i\}$ by Proposition 5. Hence one member deviates from the largest coalition in $\mathcal{P}$. On the other hand, since one-person deviation from the grand coalition is profitable by Lemma 1, $\mathcal{P}^{N\backslash\{i\}}$ sequentially dominates the grand coalition structure $\mathcal{P}^N$.
Q.E.D.

Remark 2. Because our sequential domination implies DX-domination, it follows from Corollary 1 that the grand coalition structure can be an EEBA for any number of players if $|N| \geq 4$. However, a set of EEBA's might contain several other coalition structures. In particular, the unfair coalition structure $\mathcal{P}^{N\backslash\{i\}} = \{\{i\}, N \setminus \{i\}\}$ is an EEBA for a sufficiently large n. In fact, this follows from Proposition 6, because DX-domination is implied by domination under the assumption in Proposition 6 that the singleton coalition structure $\mathcal{P}^I$ can merge into $\mathcal{P}^{N\backslash\{i\}}$ directly at one step. It is difficult to eliminate the possibility that the coalition structures $\mathcal{P}^{N\backslash\{i\}}$ is an EEBA because the singleton player gets the maximal payoff among the payoffs under all coalition structures. (See Diamantoudi and Xue (2007) for a related argument.)

5 Concluding Remarks

We have proposed a sequentially stable coalition structure as a new concept of stability in coalition formation games with externalities. Our concept of domination is based on a step-by-step approach to describe negotiation processes concretely. We have shown that the efficient grand coalition structure can be sequentially stable in simple partition function form games and common pool resource games.

In this paper, each coalition structure corresponds to one payoff vector. For a more general case in which each coalition structure corresponds to many possible payoff vectors, we have to consider a payoff configuration defined by $(z, \mathcal{P})$, which satisfies $z \in \{z | z \in \mathcal{F}(\mathcal{P})\}$. Here $\mathcal{F}(\mathcal{P})$ is a set of feasible payoff vectors under $\mathcal{P}$. In this case, it is not easy to compare the present payoff configuration to the final payoff configuration because of the multiplicity of the final payoff vectors. Then we should take into account sequential domination between two feasible payoff vectors in the same coalition structure. This topic is left for a future research.

We can apply our stability concept to other economic situations like public goods provision games and Cournot oligopoly games. It is generally difficult to check which coalition structures are EBA's in Cournot oligopoly games (Ray and Vohra(1997)).

Examining sequential stability of coalition structures in these economic environments is an open question.

6 Appendix

Proof of Theorem 2.

Consider a coalition structure $\mathcal{P}^{**} = \{1; 1; 2; 2; 2; 2; ....; 2; 2; 2^{m-1} + l\}$ consisting of $2^{m-2} + 2$ coalitions instead of $\mathcal{P}^{*} = \{1; 1; 2; 4; 8; ...; 2^{m-1}\}$ in the proof for Theorem 1.

(Step 1) $\mathcal{P}^{**}$ is sequentially dominated by $\mathcal{P}^{N}$.

We have to find a sequence of coalition structures $\{\mathcal{P}_t\}_{t=1}^{2^{m-2}+2}$ from $\mathcal{P}_1 = \mathcal{P}^{**}$ to $\mathcal{P}_{2^{m-2}+2} = \mathcal{P}^{N}$. We will show the following is a domination sequence of coalition structures.

$$\mathcal{P}^{**} = \mathcal{P}_1 = \{1; 1; 2; 2; 2; 2; 2; ...; 2; 2; 2^{m-1} + l\} \quad ((2^{m-2} + 2)\text{-th stage})$$

$$\rightarrow \mathcal{P}_2 = \{2; 2; 2; 2; 2; 2; ...; 2; 2; 2^{m-1} + l\} \quad ((2^{m-2} + 1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_3 = \{4; 2; 2; 2; 2; ...; 2; 2; 2^{m-1} + l\} \quad (2^{m-2}\text{-th stage})$$

$$\rightarrow \mathcal{P}_4 = \{4; 4; 2; ...; 2; 2; 2^{m-1} + l\} \quad ((2^{m-2} - 1)\text{-th stage})$$

$$\rightarrow .... \rightarrow ....$$

$$\rightarrow \mathcal{P}_{2^{m-3}+1} = \{4; 4; 4; 4; ...; 2; 2; 2^{m-1} + l\} \quad ((2^{m-3} + 2)\text{-th stage})$$

$$\rightarrow \mathcal{P}_{2^{m-3}+2} = \{4; 4; 4; 4; ...; 4; 2^{m-1} + l\} \quad ((2^{m-3} + 1)\text{-th stage})$$

$$\rightarrow \mathcal{P}_{2^{m-3}+3} = \{8; 4; 4; ...; 4; 2^{m-1} + l\} \quad (2^{m-3}\text{-th stage})$$

$$\rightarrow .... \rightarrow ....$$

$$\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+1} = \{8; 8; ...; 8; 4; 4; 2^{m-1} + l\} \quad ((2^{m-4} + 2)\text{-th stage})$$

$$\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+2} = \{8; 8; ...; 8; 8; 2^{m-1} + l\} \quad ((2^{m-4} + 1)\text{-th stage})$$

$$\rightarrow .... \rightarrow ....$$

$$\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+...+2^{m-h-1}+1} = \{2^h; 2^h; 2^h; 2^h; ..; 2^h; 2^{h-1}; 2^{h-1}; 2^{m-1} + l\} \quad ((2^{m-h-1} + 2)\text{-th stage})$$

$$\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+...+2^{m-h-1}+2} = \{2^h; 2^h; 2^h; 2^h; ..; 2^h; 2^h; 2^{m-1} + l\} \quad (2^{m-h-1} + 1\text{-th stage})$$

$$\rightarrow .... \rightarrow ....$$

$$\rightarrow \mathcal{P}_{2^{m-2}-2} = \{2^{m-3}; 2^{m-3}; 2^{m-3}; 2^{m-3}; 2^{m-1} + l\} \quad (5\text{-th stage})$$

$$\rightarrow \mathcal{P}_{2^{m-2}-1} = \{2^{m-2}; 2^{m-3}; 2^{m-3}; 2^{m-1} + l\} \quad (4\text{-th stage})$$

$$\rightarrow \mathcal{P}_{2^{m-2}} = \{2^{m-2}; 2^{m-2}; 2^{m-1} + l\} \quad (3\text{-th stage})$$

$$\to \mathcal{P}_{2^{m-2}+1} = \{2^{m-1}; 2^{m-1}+l\} \text{ (2-th stage)} \quad \to \quad \mathcal{P}_{2^{m-2}+2} = \mathcal{P}^N$$

For $t = 1$, the payoff of every player in the two singletons under $\mathcal{P}_1$ is smaller than that under the final coalition structure $\mathcal{P}^N$ by Lemma 1 because $\frac{1}{n} > B(2^{m-2}+2)$, which is given by (4.1) for $h = 1$. Remark that $|\mathcal{P}_1| = 2^{m-2}+2$.

For $t = 2, 3, ..., 2^{m-3}+1$, the payoff of every player in 2-person coalitions under $\mathcal{P}_t$, $(t = 3, ..., 2^{m-2})$, which is the minimal size among the coalitions, is smaller than that under the final coalition structure $\mathcal{P}^N$ by Lemma 1 because $\frac{2}{n} > B(2^{m-3}+2) > B(2^{m-3}+3) > ... > B(2^{m-2}+1)$. Here this inequality is obtained by $\frac{2}{n} > B(2^{m-3}+2)$ and the monotonicity of $B(k)$.

Let $h \in \{3, 4, ..., m-2\}$. For $t = (2^{m-3}+2^{m-4}+...+2^{m-h})+2, 2^{m-3}+2^{m-4}+...+2^{m-h})+3, ..., (2^{m-3}+2^{m-4}+...+2^{m-h})+1$, the payoff of every member in 2^{h-1}-person coalitions under $\mathcal{P}_t$, $t = (2^{m-3}+2^{m-4}+...+2^{m-h})+2, 2^{m-3}+2^{m-4}+...+2^{m-h})+3, ..., (2^{m-3}+2^{m-4}+...+2^{m-h})+1$ is smaller than that under $\mathcal{P}^N$ by Lemma 1, because $\frac{2^{h-1}}{n} > B(2^{m-h-1}+2)$ and $B(k)$ is decreasing.

For $t = 2^{m-2}-1$ or $2^{m-2}-2$, which correspond to the case of $h = m-2$ above, the payoff of every member in two 2^{m-3}-person coalitions under $\mathcal{P}^{2^{m-2}-1}$ is smaller than that under $\mathcal{P}^N$because $\frac{2^{m-3}}{n} > B(4) = B(2^1+2) > B(5)$.

For $t = 2^{m-2}$, the payoff of every member in two 2^{m-2}-person coalitions under $\mathcal{P}_{2^{m-2}}$ is smaller than that under $\mathcal{P}^N$ because $\frac{2^{m-2}}{n} > B(3) = B(2^0+2)$.

For $t = 2^{m-2}+1$, the payoff of every member in the two coalitions under $\mathcal{P}_{2^{m-2}+1}$ is smaller than that under $\mathcal{P}^N$because $\frac{2^{m-2}}{n} > B(2)$.

Thus we get a sequence from $\mathcal{P}^{**}$ to $\mathcal{P}^N$.

(Step 2) Every $(2^{m-2}+2)$-th stage coalition structure is sequentially dominated.

We denote $M = 2^{m-2}+2$. Take any M-th stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^T$ such that

1) $\mathcal{P}_0 = \mathcal{P} = \{r_1; r_2; r_3; ...; r_{M-2}; r_{M-1}; r_M\}$ $(r_1 \leq r_2 \leq r_3 \leq ... \leq r_{M-1} \leq r_M)$.

2) $\mathcal{P}_T = \{1; 1; 1; ...; 1; n-M+1\}$, where $|\mathcal{P}_T| = M$.

3) If t is zero or even, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.

4) If t is odd, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^T$ of coalition structures is given by:

$$\mathcal{P}_0 = \{r_1; r_2; r_3; ..., r_{M-2}; r_{M-1}; r_M\} \text{ (M-th stage)}$$

$$\to \mathcal{P}_1 = \{1; r_1; r_2; r_3; ...; r_{M-2}; r_{M-1}; r_M - 1\} \text{ (($M+1$)-th stage)}$$

$$\to \mathcal{P}_2 = \{1; r_1; r_2; r_3; ...; r_{M-2}; r_{M-1} + r_M - 1\} \text{ (M-th stage)}$$

$$\to \mathcal{P}_3 = \{1; 1; r_1; r_2; r_3; ...; r_{M-2}; r_{M-1} + r_M - 2\} \text{ (($M+1$)-th stage)}$$

$$\to ... \to ...$$

$$\to \mathcal{P}_{T-2} = \{1; 1; 1; ...; 1; r_1; \sum_{k=2}^{M} r_k - M + 2\} \text{ (M-th stage)}$$

$$\to \mathcal{P}_{T-1} = \{1;1;1;1;...;1;r_1;\sum_{k=2}^{M} r_k - M + 1\} \ \ ((M+1)\text{-th stage})$$

$$\to \mathcal{P}_{T} = \{1;1;1;1;...;1;\sum_{k=1}^{M} r_k - M + 1\} = \{1;1;1;1;...;1;n - M + 1\} \ \ (M\text{-th stage})$$

Next consider $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ such that

1) $\mathcal{P}_T = \{1;1;1;...;1;n-M+1\}$,

2) $\mathcal{P}_{T+T'} = \mathcal{P}^{**} = \{1;1;2;2;2;2;...;2;2;2^{m-1}+l\}$.

3) If $t = T + \lambda$ and λ is 0 or even, then 1-person in the largest coalition in $\mathcal{P}_{T+\lambda}$ deviate and form a singleton in $\mathcal{P}_{T+\lambda+1}$.

4) If $t = T + \lambda$ and λ is odd, then two 1-person coalitions in $\mathcal{P}_{T+\lambda}$ merge into one coalition in $\mathcal{P}_{T+\lambda+1}$.

This sequence $\{\mathcal{P}_t\}_{t=T}^{T+T'}$ of coalition structures is given by:

$$\mathcal{P}_T = \{1;1;1;1;...;1;1;1;n-M+1\} \ \ (M\text{-th stage})$$

$$\to \mathcal{P}_{T+1} = \{1;1;1;1;...;1;1;1;1;n-M\} \ \ ((M+1)\text{-th stage})$$

$$\to \mathcal{P}_{T+2} = \{1;1;1;1;...;1;1;2;n-M\} \ \ (M\text{-th stage})$$

$$\to \ \ \mathcal{P}_{T+3} = \{1;1;1;1;...;1;1;1;2;n-M-1\} \ \ ((M+1)\text{-th stage})$$

$$\to \ \ \mathcal{P}_{T+4} = \{1;1;1;1;...;1;2;2;n-M-1\} \ \ M\text{-th stage})$$

$$\to \ \ \mathcal{P}_{T+5} = \{1;1;1;1;...;1;1;2;2;n-M-2\} \ \ ((M+1)\text{-th stage})$$

$$\to ... \to ...$$

$$\to \mathcal{P}_{T+T'-2} = \{1;1;1;2;...;2;2;n-2M+3\} \ \ M\text{-th stage})$$

$$\to \mathcal{P}_{T+T'-1} = \{1;1;1;1;2;...;2;2;n-2M+4\} \ \ (M+1)\text{-th stage})$$

$$\to \mathcal{P}_{T+T'} = \{1;1;2;2;...;2;2;n-2M+4\} = \{1;1;2;2;...;2;2;2^{m-1}+l\}$$
$$= \mathcal{P}^{**} \ \ (M\text{-th stage})$$

Hence if we combine two sequences $\{\mathcal{P}_t\}_{t=0}^{T}$ and $\{\mathcal{P}_t\}_{t=T}^{T+T'}$, we can get a sequence $\{\mathcal{P}_t\}_{t=0}^{T+T'}$ from any $(m+1)$-th stage coalition structure $\mathcal{P}$ to $\mathcal{P}^{**}$. Note that only deviation of a coalition with 2 or more members appears for all M-th coalition structures in this sequence.

(Step 3) Every coalition structure $\mathcal{P}$ of less than $(2^{m-2}+2)$ coalitions other than the grand coalition structure $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$.

The proof is similar to that in Theorem 1 except for the cardinality of the key coalition structure $\mathcal{P}^{**}$.

(Step 4) Every coalition structure $\mathcal{P}$ of more than $(2^{m-2}+2)$ coalitions is sequentially dominated by $\mathcal{P}^N$.

The proof is the same as that in Theorem 1.

Steps 1-4 show that every coalition structure other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$. Q.E.D.

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