Existence of pseudo-equilibria in a financial economy

Philippe Bich and Bernard Cornet

Dedicated to Steve Smale on the occasion of his eightieth birthday

Abstract. This paper proves the existence of a pseudo-equilibrium in a financial economy with incomplete markets in which the agents may have nonordered preferences. We will use a fixed-point-like theorem of [4] that generalizes the results by Hirsch, Magill, Mas-Colell [18] and Husseini, Lasry, Magill [19] to encompass the framework considered by Gale and Mas-Colell ([14],[15]).

1. Introduction

Since Arrow-Debreu [3], fixed-point theory has been a standard tool in general equilibrium theory to prove the existence of a Walrasian equilibrium. The first existence result has been derived from fixed-point theorems à la Brouwer-Kakutani in the case where agents have preferences represented by a complete preorder. The book *Theory of value* of Gérard Debreu [11] summarizes the work in the 50s, by Kenneth Arrow, Gérard Debreu, John Nash, John Von Neumann et Oskar Morgenstern, that essentially used arguments from general topology, convex analysis, together with the fixed-point theorems of Brouwer [6] and Kakutani [20]. The work by Gale and Mas-Colell ([14], [15]) then extended Kakutani's result to treat the existence problem when the agents have nonordered preferences. Another approach to the existence problem, which relies on degree theory or Poincaré-Hopf's theorem was initiated in the 70s by Steve Smale in a sequence of papers on "Global

analysis and economics" (see [25],[26],[27],[28],[29]). This approach had a tremendous influence in the literature including the present paper and is presented in the books by Balasko [5], Dierker [10] and Mas-Colell [22]).

General Equilibrium Theory with *Incomplete markets* aims to study the interactions of the financial part of the economy and its commodity part in a world where time and uncertainty play a fundamental role (see, for example, Radner [24], Magill-Quinzii [21], or Geanakoplos [16] for a survey on GEI model). The mathematical tools used to prove the existence still rely on fixed-point-like theorems but of a different kind since some variables may belong to a Grassmann manifold $G^k(V)$, that is, the set of vector spaces of fixed dimension k in a given Euclidean space V. The main difficulty to deal with this extended framework comes from the fact that, in general, the set $G^k(V)$ does not satisfy properties of convexity or acyclicity that would allow to use standard fixed-point theorems of Brouwer, Kakutani or Eilenberg-Montgomery [13] to deduce the existence of equilibria as in the standard case.

The aim of this paper is to prove the existence of a pseudo-equilibrium in a financial economy with incomplete markets in which the agents may have nonordered preferences and thus extend the existence result by Duffie-Shafer [12]. We will use a fixed-point-like theorem of [4] that generalizes the results by Hirsch, Magill, Mas-Colell [18] and Husseiny, Lasry, Magill [19] to encompass the framework considered by Gale and Mas-Colell ([14],[15]). The article is organized as follows. In Section 2, we present the general equilibrium model with incomplete markets and recall the definitions of an equilibrium and of a pseudo-equilibrium. We also state the main existence result (Theorem 1) for the existence of a pseudo-equilibrium. The proof of the main existence result is given in Section 3.

2. The model and the main result

2.1. The model

We first give the formal definition of the model. There is a finite set I of consumers (by abuse of notation, I will also denote the cardinal of the set I) and we consider the simplest case of intertemporal model with two dates, $t = 0$ (today) and $t = 1$ (tomorrow) (see [21], [1], [8] for a generalization to several periods). There is no uncertainty at date 0, and the uncertainty at date $t = 1$ is represented by finitely many states of nature S ($s = 1, \ldots, S$); only one state happens at $t = 1$ and it is only known at date $t = 1$. For convenience, the unique state of nature at $t = 0$ will be denoted $s = 0$. There is a positive number ℓ of divisible goods available at each date and each state $s = 0, \ldots, S$; hence, the number of commodities available either at $t = 0$ (with certainty) or at $t = 1$ (contingent on each possible state of nature) is $L = \ell(1 + S)$. An economy $\mathcal{E}$ can be summarized by a list $\mathcal{E} = \big((X_i, P_i, e_i)_{i \in I}, V\big)$ where for every $i \in I$:

- $X_i \subset \mathbb{R}^L$ is the consumption set of the i-th consumer, that is the set of possible consumptions of this consumer.
- For all $x \in X := \Pi_{i \in I} X_i$, $P_i(x) \subset X_i$ is the set of consumptions which are strictly preferred to x_i by the i-th consumer, given the consumptions $(x_j)_{j \neq i}$ of the other consumers.[1] Thus, P_i is a correspondence from X to X_i.
- $e_i \in \mathbb{R}^L$ is the initial endowment vector of the i-th consumer.
- At time $t = 0$, there exist financial markets for a positive number J of assets, $J \leq S$. An asset j ($j = 1, \ldots, J$) can be bought at time $t = 0$ and delivers at time $t = 1$ a financial payoff $V_s^j(p)$ (in unit of account) if state s prevails, given the commodity price vector $p \in \mathbb{R}^L$. We denote by $V(p) = \left(V_s^j(p)\right)$ the $S \times J$ payoff matrix across states $s = 1, \ldots, S$.

Given the asset price vector $q = (q_1, \ldots, q_J) \in \mathbb{R}^J$ specifying the price q_j of the j-th asset for $j = 1, \ldots, J$, we denote by $W(p, q)$ the $(1 + S) \times J$ total payoff matrix across states $s = 0, \ldots, S$, that is

$$W(p, q) = \begin{pmatrix} -q \\ V(p) \end{pmatrix}.$$

The vector $z = (z_1, \ldots, z_J) \in \mathbb{R}^J$ denotes the portfolio of a consumer and specifies the quantity $|z_j|$ of each asset j in the portfolio, with the usual convention that if $z_j > 0$ then $|z_j|$ represents the quantity of asset j bought at date 0 and if $z_j < 0$ then $|z_j|$ represents the quantity of asset j sold at date 0. Then $W(p, q)z$ is a $(1 + S)$-column vector (or by extension a vector in $\mathbb{R}^{1+S}$) which describes the financial stream of the portfolio z across the $1 + S$ states of nature.

2.2. Equilibria and pseudo-equilibria

We now formally define the notions of equilibrium and of pseudo-equilibrium.[2]

Definition 2.1. *An equilibrium of the economy $\mathcal{E}$ is a list $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ in $\mathbb{R}^{LI} \times \mathbb{R}^{JI} \times \mathbb{R}^L \times \mathbb{R}^J$ such that, if we let $\bar{x} = (\bar{x}_i)_{i \in I}$ and $\bar{z} = (\bar{z}_i)_{i \in I}$, one has:*

(i) for every $i \in I$, $(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q})$ and $(P_i(\bar{x}) \times \mathbb{R}^J) \cap B_i(\bar{p}, \bar{q}) = \emptyset$,
where $B_i(\bar{p}, \bar{q}) = \{(x_i, z_i) \in X_i \times \mathbb{R}^J : \bar{p} \square (x_i - e_i) \leq W(\bar{p}, \bar{q})z_i\}$;

(ii) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$;

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[1]This general way to describe the tastes of the consumers encompasses the case where the consumer i has a preference relation $\preceq_i$ which is a complete preorder on X_i. In this case, for all $x \in \Pi_{i \in I} X_i$, $P_i(x) = \{x_i' \in X_i : x_i \prec_i x_i'\}$ where the strict preference relation $\prec_i$ is defined by $x_i \prec_i x_i'$ if $[\, x_i \preceq_i x_i'$ and not $x_i' \preceq_i x_i]$.

[2]In this paper, we will often denote a vector $x \in \mathbb{R}^L$ by $x = (x(0), x(1), \ldots, x(S))$ where $x(s) \in \mathbb{R}^\ell$ for $s = 0, 1, \ldots, S$. If $p = (p(0), \ldots, p(S)) \in \mathbb{R}^L$ [resp. $\lambda = (\lambda(0), \ldots, \lambda(S)) \in \mathbb{R}^{1+S}$] and $x = (x(0), \ldots, x(S)) \in \mathbb{R}^L$, we denote $p \square x$ [resp. $\lambda \square p$] the vector in $\mathbb{R}^{1+S}$ [$\mathbb{R}^L$] defined by

$$p \square x = (p(0) \cdot x(0), p(1) \cdot x(1), \ldots, p(S) \cdot x(S)) [\text{resp. } \lambda \square p = (\lambda(0)p(0), \lambda(1)p(1), \ldots, \lambda(S)p(S))],$$

where $x \cdot y := \sum_{i=1}^{\ell} x_i y_i$ denotes the dot product of two vectors $x = (x_i), y = (y_i)$ in $\mathbb{R}^\ell$. Besides, for every integer n and every $\lambda \in \mathbb{R}^n$, $\lambda^\perp := \{(x_1, ..., x_n) \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i x_i = 0\}$. Last, if $x = (x(0), \ldots, x(S))$ and $y = (y(0), \ldots, y(S))$ are two vectors of $\mathbb{R}^{1+S}$, or by extension two $(1 + S)$-column vectors, then $x \leq y$ (resp. $x << y$) means that for every $s = 0, 1, \ldots, S$, one has $x(s) \leq y(s)$ (resp. $x(s) < y(s)$).

(iii) $\sum_{i\in I} \bar{z}_i = 0$.

Condition *(i)* states that $(\bar{x}_i, \bar{z}_i)$ is optimal in consumer i's budget set $B_i(\bar{p}, \bar{q})$. The so-called Market Clearing Conditions *(ii)* and *(iii)* guarantee the feasibility of the equilibrium allocation, and of the portfolio allocation.

We recall that even under strong assumptions on the economy, an equilibrium may fail to exist, because of possible discontinuities of the column span of $W(p, q)$, denoted $\operatorname{Im} W(p, q)$ when prices vary (see, for example, Hart [17]). A standard answer (see Duffie, Shaffer [12]) is to relax the previous equilibrium notion by the notion of pseudo-equilibrium and we refer to Aouani, Cornet [2] for another approach that assumes that the dimension of $\operatorname{Im} W(p, q)$ remains constant when prices vary. In the following, if V is a Euclidean space and k is an integer such that $0 \leq k \leq \dim(V)$, then we denote by $G^k(V)$ the set consisting of all the linear subspaces of V of dimension k.

Definition 2.2. *A pseudo-equilibrium of the economy $\mathcal{E}$ is a list $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ in $\mathbb{R}^{LI} \times \mathbb{R}^L \times \mathbb{R}^J \times G^J(\mathbb{R}^{1+S})$ such that, if we let $\bar{x} = (\bar{x}_i)_{i\in I}$, one has:*
(i) for every $i \in I$, $\bar{x}_i \in B_i(\bar{p}, \bar{E})$ and $P_i(\bar{x}) \cap B_i(\bar{p}, \bar{E}) = \emptyset$,
where $B_i(\bar{p}, \bar{E}) := \left\{ x_i \in X_i : \exists t_i \in \bar{E}, \ \bar{p} \,\square\, (x_i - e_i) \leq t_i \right\}$;
(ii) $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} e_i$;
(iii) $\operatorname{Im} W(\bar{p}, \bar{q}) \subset \bar{E}$.

The following proposition shows the relationship between equilibria and pseudo-equilibria.

Proposition 2.1. *(a) Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be an equilibrium of the economy $\mathcal{E}$ such that $\operatorname{rank} V(\bar{p}) = J$, then $(\bar{x}, \bar{p}, \bar{q}, \operatorname{Im} W(\bar{p}, \bar{q}))$ is a pseudo-equilibrium of $\mathcal{E}$.*

(b) Let $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ be a pseudo-equilibrium of $\mathcal{E}$ such that $\bar{E} = \operatorname{Im} W(\bar{p}, \bar{q})$, hen there exist $\bar{z} = (\bar{z}_i)_{i\in I} \in \mathbb{R}^{JI}$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of $\mathcal{E}$ if we additionally assume that

Non-Satiation: *for every $i \in I$, for every $s_0 \in \{0, 1, \dots, S\}$, there exists $x_i \in P_i(\bar{x})$ such that $x_i(s) = \bar{x}_i(s)$ for every $s \neq s_0$.*

We will use the following lemma to prove Proposition 2.1.

Lemma 2.1. *For every pseudo-equilibrium $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ of $\mathcal{E}$ satisfying the above Non-Satiation Assumption, one has $\bar{E} \cap \mathbb{R}_+^{1+S} = \{0\}$, or equivalently there exists $\lambda \in \mathbb{R}_{++}^{1+S}$ such that $\bar{E} \subset \lambda^{\perp}$.*

Proof. First notice that the two conclusions of Lemma 2.1 are equivalent from a standard separation theorem. Let $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ be a pseudo-equilibrium and suppose that $\bar{E} \cap \mathbb{R}_+^{1+S} \neq \{0\}$. Then there exists $t = (t(s))_{s=0,\dots,S}$ in $\bar{E}$ such that $t(s) \geq 0$ for every $s = 0, \dots, S$ and $t(s_0) > 0$ for some s_0, $0 \leq s_0 \leq S$. Under the Non-Satiation Assumption for consumer 1, there exists $x_1 \in P_1(\bar{x})$ such that $x_1(s) = \bar{x}_1(s)$ for every $s \neq s_0$. But, from the Pseudo-equilibrium Condition *(i)*, there exists $\bar{t}_1 \in \bar{E}$ such that $\bar{p} \,\square\, (\bar{x}_1 - e_1) \leq \bar{t}_1$. Hence, for every integer n large enough,

$\bar{p} \square (x_1 - e_1) \leq \bar{t}_1 + nt$ and $\bar{t}_1 + nt \in \bar{E}$. Thus $x_1 \in P_1(\bar{x}) \cap B_1(\bar{p}, \bar{E})$ which is empty from the Pseudo-equilibrium Condition of Agent 1. $\qquad \square$

Proof of Proposition 2.1. *Part* (a). Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be an equilibrium of the economy $\mathcal{E}$ such that rank $V(\bar{p}) = J$. Clearly, one has rank$V(\bar{p})$ =rank$W(\bar{p}, \bar{q}) = J$, which guarantees that Im$W(\bar{p}, \bar{q}) \in G^J(\mathbb{R}^{1+S})$ and that $(\bar{x}, \bar{p}, \bar{q}, \text{Im}W(\bar{p}, \bar{q}))$ is a pseudo-equilibrium of $\mathcal{E}$. $\qquad \square$

Part (b). Let $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ be a pseudo-equilibrium of $\mathcal{E}$ such that $\bar{E}$ =Im$W(\bar{p}, \bar{q})$. For every $i \in I$, there exists $\bar{t}_i \in \bar{E}$ such that $\bar{p} \square (\bar{x}_i - e_i) \leq \bar{t}_i$. Summing up over i these inequalities, we obtain

$$\bar{p} \square \sum_{i \in I}(\bar{x}_i - e_i) \leq \sum_{i \in I} \bar{t}_i.$$

But $\sum_{i \in I}(\bar{x}_i - e_i) = 0$, from the Market Clearing Condition (ii), thus $\sum_{i \in I} \bar{t}_i \geq 0$, and we also have $\sum_{i \in I} \bar{t}_i \in \bar{E}$. From Lemma 2.1, one has $\bar{E} \cap \mathbb{R}_+^{1+S} = \{0\}$ and one deduces that $\sum_{i \in I} \bar{t}_i = 0$. Finally, $\sum_{i \in I}(\bar{p} \square (\bar{x}_i - e_i) - \bar{t}_i) = 0$, and each term of this sum belongs to $-\mathbb{R}_+^{1+S}$. Hence, each of these terms is null, that is:

$$\bar{p} \square (\bar{x}_i - e_i) = \bar{t}_i, \text{ for every } i \in I.$$

Since, for every $i \in I$, $\bar{t}_i \in \bar{E}$ =Im$W(\bar{p}, \bar{q})$, there exists $z_i \in \mathbb{R}^J$ such that $\bar{t}_i = W(\bar{p}, \bar{q})z_i$. So we obtain

$$0 = \bar{p} \square \sum_{i \in I}(\bar{x}_i - e_i) = \sum_{i \in I} \bar{t}_i = W(\bar{p}, \bar{q})(\sum_{i \in I} z_i).$$

We now let $\bar{z}_i = z_i - (1/I)(\sum_{i \in I} z_i)$ for every $i \in I$. Then, $\sum_{i \in I} \bar{z}_i = 0$, that is, the Market Clearing Condition (iii) holds. Furthermore, we have

$$\bar{p} \square (\bar{x}_i - e_i) = \bar{t}_i = W(\bar{p}, \bar{q})z_i = W(\bar{p}, \bar{q})\bar{z}_i.$$

This shows that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of $\mathcal{E}$. $\qquad \square$

2.3. The main existence theorem

We now set the basic assumptions on the economy $\mathcal{E}$ we will consider hereafter. We first define the set of attainable allocations of the economy, that is,

$$\mathcal{A}(\mathcal{E}) := \{(x_1, \ldots, x_I) \in X_1 \times \ldots \times X_I : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}.$$

Consumption Assumption C: *For every* $i \in I$*:*

(i) [Consumption sets] X_i is closed, convex and bounded below, in the sense that there exists $\underline{x}_i \in \mathbb{R}^L$ such that $X_i \subset \underline{x}_i + \mathbb{R}_+^L$;

(ii) [Continuity] the correspondence P_i is lower semicontinuous [3]*;*

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[3]A correspondence F, from X to Y, is said to be lower semicontinuous (l.s.c.) [resp. upper semicontinuous (u.s.c.)] if the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ [resp. $\{x \in X : F(x) \subset U\}$] is open in X for every open set $U \subset Y$. We define the graph of F by $G(F) := \{(x, y) \in X \times Y : y \in F(x)\}$. The correspondence F is said to be open-graph (resp. closed-graph) if $G(F)$ is open (resp. closed) in $X \times Y$ (for its relative topology).

(iii) [Openess-type Assumption][4]*: for all $\bar{x} \in X$, for all $x_i \in P_i(\bar{x})$, for all $x_i' \in X_i, x_i' \neq x_i$ then $[x_i', x_i) \cap P_i(\bar{x}) \neq \emptyset$;*

(iv) [Convexity] for every $x \in X$, $P_i(x)$ is convex;

(v) [Irreflexivity] for every $x \in X$, $x_i \notin P_i(x)$;

(vi) [Non-Satiation] for every $\bar{x} \in \mathcal{A}(\mathcal{E})$, for every $s_0 \in \{0, 1, \ldots, S\}$, there exists $x_i \in X_i$ such that $x_i(s) = \bar{x}_i(s)$ for every $s \neq s_0$ and $x_i \in P_i(\bar{x})$;

(vii) [Strong Survival Assumption] $e_i \in \mathrm{int} X_i$.

Financial Assumption F: *The mapping $p \to V(p)$ is continuous on $\mathbb{R}^L$.*

We now state the main existence theorem which extends the existence result of Duffie-Shafer [12] to the case of agents with nonordered preferences.

Theorem 2.1. *Assume the economy $\mathcal{E}$ satisfy Assumptions **C** and **F**. For every $\lambda \in \mathbb{R}^{1+S}_{++}$, there exists a pseudo-equilibrium $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ of $\mathcal{E}$ such that $\bar{E} \subset \lambda^\perp$.*

The proof of Theorem 2.1 is given in Section 3.

When the payoff matrix $V(p)$ does not depend upon the price p, the financial structure is said to be *nominal*. We can now deduce the existence result of Cass [7], Duffie [9], Werner [30] in the case of a nominal financial structure.

Corollary 2.1. *Let $\mathcal{E} = \left((X_i, P_i, e_i)_{i=1\ldots,I}, V\right)$ satisfy Assumptions **C** and **F**, where V is a nominal asset structure. For every $\lambda \in \mathbb{R}^{1+S}_{++}$ such that $\lambda_0 = 1$, there exists an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ of $\mathcal{E}$ such that $q_j = \sum_{s=1}^{S} \lambda_s V_s^j$ for every $j = 1, \ldots, J$.*

Proof. For every $\lambda \in \mathbb{R}^{1+S}_{++}$ such that $\lambda_0 = 1$, define $q_j = \sum_{s=1}^{S} \lambda_s V_s^j$ for every j. Let $J' = \mathrm{rank}\, V$ and consider an economy $\mathcal{E}' = \left((X_i, P_i, e_i)_{i=1\ldots,I}, V'\right)$, where V' is a $S \times J'$ full-rank submatrix of V. From Theorem 2.1 and Proposition 2.1 we deduce the existence of an equilibrium $(\bar{x}, \bar{z}', \bar{p}, \bar{q}')$ of $\mathcal{E}'$. Then, one can easily modify the agents portfolios $\bar{z}'$ and the asset prices $\bar{q}'$ to get an equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ of $\mathcal{E}$. $\square$

3. Proof of Theorem 2.1

3.1. Proof of Theorem 2.1 under additional assumptions

In this section, we first prove Theorem 2.1 under the following additional assumptions. The next Section 3.2 will then provide a proof of Theorem 2.1 in the general case.

Assumptions A: *(i) [Local Non-Satiation] For every $\bar{x} = (\bar{x}_i)_{i \in I} \in \mathcal{A}(\mathcal{E})$, for every $i \in I$, for every $x_i \in P_i(\bar{x})$, then $(\bar{x}_i, x_i] \subset P_i(\bar{x})$.*

(ii) [Compactness] For every $i \in I$, X_i is a compact subset of $\mathbb{R}^L$.

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[4]This assumption, which is satisfied, in particular, when $P_i(\bar{x})$ is open in X_i (for its relative topology) is discussed in Remark 3.1 in Section 3.2.1.

3.1.1. Definition of the reaction correspondences. Let B be the closed unit ball of $\mathbb{R}^L$, let $\mathbb{1}$ be the vector in $\mathbb{R}^{1+S}$ whose coordinates are all equal to 1, and let $\lambda \in \mathbb{R}_{++}^{1+S}$ such that $\lambda(0) = 1$. We now introduce some definitions in which (x, p, E) is given in $\Pi_{i \in I} X_i \times B \times G^J(\lambda^\perp)$:

- for $i = 2, \ldots, I$, we consider the "augmented" budget sets:

$$\beta_i(p, E) = \{x_i \in X_i : \exists t_i \in E, \ p \square (x_i - e_i) \le t_i + (1 - \|p\|)\mathbb{1}\},$$

$$\breve{\beta}_i(p, E) = \{x_i \in X_i : \exists t_i \in E, \ p \square (x_i - e_i) \ll t_i + (1 - \|p\|)\mathbb{1}\};$$

- for $i = 1$, following the so-called "Cass trick," we consider the "augmented" Walrasian budget set (which does not depend on E):

$$\beta_1(p, E) = \{x_1 \in X_1 : (\lambda \square p) \cdot (x_1 - e_1) \le 1 - \|p\|\},$$

$$\breve{\beta}_1(p, E) = \{x_1 \in X_1 : (\lambda \square p) \cdot (x_1 - e_1) \ll 1 - \|p\|\};$$

- for $i = 1, \ldots, I$, we let:

$$\Phi_i(x, p, E) = \begin{cases} \{e_i\} & \text{if } x_i \notin \beta_i(p, E) \text{ and } \breve{\beta}_i(p, E) = \emptyset, \\ \beta_i(p, E) & \text{if } x_i \notin \beta_i(p, E) \text{ and } \breve{\beta}_i(p, E) \ne \emptyset, \\ \breve{\beta}_i(p, E) \cap P_i(x) & \text{if } x_i \in \beta_i(p, E); \end{cases}$$

- for $i = 0$, the revision of prices is done according to the standard rule:

$$\Phi_0(x, p, E) = \{p' \in B : p' \cdot \sum_{i=1}^I (x_i - e_i) > p \cdot \sum_{i=1}^I (x_i - e_i)\};$$

- for $j = 1, \ldots, J$, we let:

$$\psi_j(x, p, E) = (-\sum_{s=1}^S \lambda(s) V_s^j(p), V_1^j(p), \ldots, V_S^j(p)).$$

The properties of the above correspondences and mappings are summarized in the following lemma. Throughout this paper, for every Euclidean space V and every integer $k \le \dim V$, we will consider on the Grassmann manifold $G^k(V)$ the standard topology defined as follows. First, define the Stiefel manifold $V^k(V)$ as the collection of all k-tuples of linearly independent vectors of V. Clearly, $V^k(V)$ is an open subset of V^k and it is endowed with the topology induced by the one of V^k. Then, define the mapping $f : V^k(V) \to G^k(V)$ by $f(e_1, \ldots, e_k) = \text{span}\{e_1, \ldots, e_k\}$, for every $(e_1, \ldots, e_k) \in V^k(V)$, and define on $G^k(V)$ the topology as follows; a subset $U \subset G^k(V)$ is open if and only if its inverse image $f^{-1}(U)$ is open in $V^k(V)$ (see, for example, [23]).

Lemma 3.1. *For every $i = 0, \ldots, I$, the correspondence Φ_i is lower semicontinuous and has convex (possibly empty) values. For every $j = 1, \ldots, J$, ψ_j is a continuous mapping.*

Proof. The assertions on Φ_0 and the ψ_j, $j = 1, \ldots, J$, are straightforward. We notice that, for every $i = 1, \ldots, I$, Φ_i has convex values and we now prove that, for every $i \in \{2, \ldots, I\}$, Φ_i is lower semicontinuous, the proof being similar for Φ_1.

We will use the following claim, the proof of which relies on the definition of the topology of $G^J(\mathbb{R}^{1+S})$ and is given in [4].

Claim 3.1. *For every Euclidean space V and every integer $k \leq \dim V$, the set $\{(t, E) \in V \times G^k(V) : t \in E\}$ is closed.*

Step (i). First remark that $\breve{\beta}_i$ is an open-graph correspondence, so the set $A_i = \{(x, p, E) \in \Pi_{k \in I} X_k \times B \times G^J(\lambda^\perp) : \breve{\beta}_i(p, E) \neq \emptyset\}$ is open.

Step (ii). We now prove that β_i is a closed-graph correspondence for $i = 2, \ldots, I$. Indeed, consider a sequence (x_i^n, p^n, E^n) in $X_i \times B \times G^J(\lambda^\perp)$ such that $x_i^n \in \beta_i(p^n, E^n)$ for every integer n, and suppose that (x_i^n, p^n, E^n) converges to $(x_i, p, E) \in X_i \times B \times G^J(\lambda^\perp)$. Then there exists a sequence (t^n) in $\mathbb{R}^{1+S}$, such that $t^n \in E^n$ for every integer n, and

$$p^n \,\square\, (x_i^n - e_i) \leq t^n + (1 - \|p^n\|)\mathbb{1}.$$

Now, remark that the sequence (t^n) is bounded. Indeed, if it is not the case, without any loss of generality one can suppose that $\lim_{n \to \infty} \|t^n\| = \infty$ and that $t^n / \|t^n\|$ converges to some $t \in \mathbb{R}^{1+S}$ such that $\|t\| = 1$. For n large enough, we have

$$\frac{p^n \,\square\, (x_i^n - e_i)}{\|t^n\|} \leq \frac{t^n}{\|t^n\|} + \frac{(1 - \|p^n\|)\mathbb{1}}{\|t^n\|}.$$

Hence, passing to limit in the above inequality, we would obtain $0 \leq t$ (since the sequences (p^n) and (x_i^n) belong to the bounded sets B and X_i), $t \neq 0$ (since $\|t\| = 1$) and $t \cdot \lambda = 0$ (since $t^n \in E^n \subset \lambda^\perp$ for every n). This contradicts $\lambda \gg 0$ and ends the proof that (t^n) is bounded.

Now, since the sequence (t^n) is bounded, without any loss of generality one can suppose that (t^n) converges to some $t \in \lambda^\perp$. From the above inequality, we obtain

$$p \,\square\, (x_i - e_i) \leq t + (1 - \|p\|)\mathbb{1}.$$

Besides, recalling that $t^n \in E^n$ for every n, from Claim 3.1, one gets $t \in E$. This proves that β_i is a closed-graph correspondence for $i = 2, \ldots, I$. Consequently, the set $A_i' = \{(x, p, E) \in \Pi_{k \in I} X_k \times B \times G^J(\lambda^\perp) : x_i \notin \beta_i(p, E)\}$ is open.

Step (iii). Now, notice that on the set A_i, β_i is the closure of the open-graph correspondence $\breve{\beta}_i$; so β_i is lower semicontinuous on A_i.

Step (iv). Thus, using Steps (i), (ii), (iii), the facts that $e_i \in \beta_i(p, E)$ and $\breve{\beta}_i(p, E) \cap P_i(x) \subset \beta_i(p, E)$, one easily proves that Φ_i is lower semicontinuous. $\square$

3.1.2. The fixed-point argument. The existence proof relies on the following fixed-point-like theorem by Bich and Cornet (see [4]).

Theorem 3.1. *Let V be a finite dimensional Euclidean space, for $i = 1, \ldots, n$, let X_i be a nonempty, compact, convex subset of some Euclidean space, let J be a positive integer such that $J \leq \dim V$, and let $X = \Pi_{i=1}^n X_i \times G^J(V)$. For $i = 1, \ldots, n$, let Φ_i be a correspondence from X to X_i, which is lower semicontinuous and convex-valued (possibly empty-valued), and for $j = 1, \ldots, J$, let $\psi_j : X \to V$ be a continuous mapping.*

Then, there exists $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{E}) \in X_1 \times .. \times X_n \times G^J(V)$ such that

(a) for every $i = 1, \ldots, n$, $[$either $\bar{x}_i \in \Phi_i(\bar{x})$ or $\Phi_i(\bar{x}) = \emptyset]$;

(b) for every $j = 1, \ldots, J$, $\psi_j(\bar{x}) \in \bar{E}$.

The proof of Theorem 3.1 can be found in [4]. Remark that this theorem is a generalization first of Gale and Mas-Colell Theorem (see [14] and [15]), and also of Hirsch, Magill and Mas-Colell [18], Husseini, Lasry and Magill [19].

We will now use Theorem 3.1 to prove the existence of a pseudo-equilibrium. Let $V = \lambda^\perp$ and $n = I+1$. For every $i = 1, \ldots, I$, X_i is taken to be the consumption set of the i-th consumer which is convex, compact and nonempty [from Assumption **C**], and for $i = 0$, $X_0 = B$, the closed unit ball of $\mathbb{R}^L$. From Lemma 3.1, the correspondences Φ_i $(i = 0, 1, \ldots, I)$ and the mappings ψ_j $(j = 1, \ldots, J)$, defined in the previous section, satisfy the assumptions of Theorem 3.1. Consequently, from Theorem 3.1, there exists $(\bar{x}_1, \ldots, \bar{x}_I, \bar{p}, \bar{E}) \in X_1 \times \ldots \times X_I \times B \times G^J(V)$ satisfying the above conditions (a) and (b) and we let $\bar{x} := (\bar{x}_1, \ldots, \bar{x}_I)$. We first notice that, by construction and from the Irreflexivity Assumption in **C**, $\bar{x}_i \notin \Phi_i(\bar{x}, \bar{p}, \bar{E})$ for every $i = 1, \ldots, I$, and $\bar{p} \notin \Phi_0(\bar{x}, \bar{p}, \bar{E})$. Consequently

(a') for every $i = 0, \ldots, I$, $\Phi_i(\bar{x}, \bar{p}, \bar{E}) = \emptyset$;

(b') for every $j = 1, \ldots, J$, $\psi_j(\bar{x}, \bar{p}, \bar{E}) \in \bar{E}$.

We easily deduce the following conditions (using the fact that $e_i \in \beta_i(\bar{p}, \bar{E})$):

$$\bar{x}_i \in \beta_i(\bar{p}, \bar{E}) \text{ and } P_i(\bar{x}) \cap \breve{\beta}_i(\bar{p}, \bar{E}) = \emptyset \text{ for every } i \in I; \qquad \text{(FP1)}$$

$$p \cdot \sum_{i=1}^I (\bar{x}_i - e_i) \leq \bar{p} \cdot \sum_{i=1}^I (\bar{x}_i - e_i) \quad \text{for every } p \in B; \qquad \text{(FP2)}$$

$$\psi_j(\bar{x}, \bar{p}, \bar{E}) \in \bar{E} \text{ for every } j = 1, \ldots, J. \qquad \text{(FP3)}$$

We now let $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_J)$ by $\bar{q}_j = \sum_{s=1}^S \lambda(s) V_s^j(\bar{p})$ for every $j = 1, \ldots, J$.

3.1.3. The vector $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ is a pseudo-equilibrium of $\mathcal{E}$. This is a consequence of the claims proved hereafter.

Claim 3.2. Im $W(\bar{p}, \bar{q}) \subset \bar{E}$.

Proof. From our choice of $\bar{q}$ we notice that for every $j = 1, \ldots, J$,

$$\psi_j(\bar{x}, \bar{p}, \bar{E}) = (-\bar{q}_j, V_1^j(\bar{p}), \ldots, V_S^j(\bar{p})) = W^j(\bar{p}, \bar{q}),$$

where $W^j(\bar{p}, \bar{q})$ denotes the j-th column of the matrix $W(\bar{p}, \bar{q})$. Hence, Claim 3.2 follows from the Fixed-Point Condition FP3. $\square$

The Fixed-Point Condition FP1 implies that $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$ for every $i \in I$, hence, from the definition of the sets $\beta_i(\bar{p}, \bar{E})$, there exists $\bar{t}_i \in \bar{E}$ such that:

$$\bar{p} \,\square\, (\bar{x}_i - e_i) \leq \bar{t}_i + (1 - \|\bar{p}\|)\mathbb{1}, \text{ for every i} = 2, \ldots, \mathrm{I}; \tag{3.1}$$

$$(\lambda \,\square\, \bar{p}) \cdot (\bar{x}_1 - e_1) \leq 1 - \|\bar{p}\|. \tag{3.2}$$

Since $\bar{t}_i \in \bar{E} \subset \lambda^\perp$ for every $i \in I$, taking the scalar product of each side of (3.1) with the vector $\lambda \in \mathbb{R}_{++}^{S+1}$, we obtain:

$$(\lambda \,\square\, \bar{p}) \cdot (\bar{x}_i - e_i) \leq (1 - \|\bar{p}\|) \sum_{s=0}^{S} \lambda(s) \text{ for every } i = 2, \ldots I. \tag{3.3}$$

Claim 3.3. $\sum_{i=1}^{I} \bar{x}_i = \sum_{i=1}^{I} \bar{e}_i$.

Proof. If $\sum_{i=1}^{I}(\bar{x}_i - e_i) \neq 0$, it follows from Condition FP2 that

$$\bar{p} = \frac{\sum_{i \in I}(\bar{x}_i - e_i)}{\|\sum_{i \in I}(\bar{x}_i - e_i)\|}.$$

Hence, $\|\bar{p}\|=1$ and

$$(\lambda \,\square\, \bar{p}) \cdot \sum_{i=1}^{I}(\bar{x}_i - e_i) = \sum_{s=0}^{S} \lambda(s)p(s) \cdot (\sum_i(\bar{x}_i(s) - e_i(s))) = \sum_{s=0}^{S} \lambda(s) \frac{\|\sum_i(\bar{x}_i(s) - e_i(s))\|^2}{\|\sum_i(\bar{x}_i - e_i)\|}$$

and consequently

$$(\lambda \,\square\, \bar{p}) \cdot \sum_{i=1}^{I}(\bar{x}_i - e_i) \geq (\min_s \lambda(s))\|\sum_{i \in I}(\bar{x}_i - e_i)\| > 0.$$

But summing up over i the above Inequalities (3.2) and (3.3), and recalling that $\|\bar{p}\|=1$, we get

$$(\lambda \,\square\, \bar{p}) \cdot \sum_{i=1}^{I}(\bar{x}_i - e_i) \leq (1 - \|\bar{p}\|)(1 + \sum_{s=0}^{S} \lambda(s)) = 0,$$

which contradicts the above inequality. $\square$

Claim 3.4. *For every* $s = 0, 1, \ldots, S$, $\bar{p}(s) \neq 0$.

Proof. We first notice that $\bar{p} \neq 0$. Indeed, if $\bar{p} = 0$, then $\breve{\beta}_i(\bar{p}, \bar{E}) = \beta_i(\bar{p}, \bar{E}) = X_i$, for every $i \in I$. But, from Claim 3.3, $\bar{x} \in \mathcal{A}(\mathcal{E})$ and from the Non-Satiation Assumption in **C**, there exists $x_i \in P_i(\bar{x}) = P_i(\bar{x}) \cap \breve{\beta}_i(\bar{p}, \bar{E})$. This contradicts the Fixed-Point Condition FP1.

We now prove that, for every $s = 0, \ldots, S$, we have $\bar{p}(s) \neq 0$. Indeed, suppose that, for some s, one has $\bar{p}(s) = 0$. From Claim 3.3, $\bar{x} \in \mathcal{A}(\mathcal{E})$, and from the Non-Satiation Assumption in **C**, there exists $x_1 \in P_1(\bar{x})$ such that $x_1(s') = \bar{x}_1(s')$ for $s' \neq s$; from the Fixed-Point Condition FP1, $\bar{x}_1 \in \beta_1(\bar{p}, \bar{E})$ and, recalling that $\bar{p}(s) = 0$, one deduces that $x_1 \in \beta_1(\bar{p}, \bar{E})$. But $\breve{\beta}_1(\bar{p}, \bar{E}) \neq \emptyset$, since $e_i \in \mathrm{int} X_i$ (by the Survival Assumption in **C**), and since $\bar{p} \neq 0$ (from above). We now let $y_1 \in \breve{\beta}_1(\bar{p}, \bar{E})$ and we notice that $[y_1, x_1) \subset \breve{\beta}_1(\bar{p}, \bar{E})$. Now, from the Openess-type Assumption in **C**, recalling that $x_1 \in P_1(\bar{x})$ and $y_1 \in X_1$, we have $[y_1, x_1) \cap P_1(\bar{x}) \neq \emptyset$. Consequently, $P_1(\bar{x}) \cap \breve{\beta}_1(\bar{p}, \bar{E}) \neq \emptyset$, which contradicts the Fixed-Point Condition FP1. $\qquad\square$

Claim 3.5. *For every $i \in I$, $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$ and $P_i(\bar{x}) \cap \beta_i(\bar{p}, \bar{E}) = \emptyset$.*

Proof. From the Fixed-Point Condition FP1, one has $\bar{x}_i \in \beta_i(\bar{p}, \bar{E})$. Now, suppose that there exists i such that $P_i(\bar{x}) \cap \beta_i(\bar{p}, \bar{E}) \neq \emptyset$. Let $x_i \in P_i(\bar{x}) \cap \beta_i(\bar{p}, \bar{E})$. Since $e_i \in \mathrm{int} X_i$ (Survival Assumption in **C**) and $\bar{p}(s) \neq 0$ for every $s = 0, \ldots, S$, (Claim 3.4) one deduces that $\breve{\beta}_i(\bar{p}, \bar{E}) \neq \emptyset$ and we let $y_i \in \breve{\beta}_i(\bar{p}, \bar{E})$. We notice that $[y_i, x_i) \subset \breve{\beta}_i(\bar{p})$. But, $[y_i, x_i) \cap P_i(\bar{x}) \neq \emptyset$, from the Openess-type Assumption in **C**, recalling that $x_i \in P_i(\bar{x})$ and $y_i \in X_i$. Consequently, $P_i(\bar{x}) \cap \breve{\beta}_i(\bar{p}, \bar{E}) \neq \emptyset$, which contradicts the Fixed-Point Condition FP1. $\qquad\square$

Claim 3.6. $\|\bar{p}\| = 1$.

Proof. We first prove that each consumer budget constraint is binding, i.e.:

$$\bar{p} \,\square\, (\bar{x}_i - e_i) = \bar{t}_i + (1 - \|\bar{p}\|)\mathbb{1} \text{ for every } i = 2, \ldots, I. \tag{3.4}$$

$$(\lambda \,\square\, \bar{p}) \cdot (\bar{x}_1 - e_1) = 1 - \|\bar{p}\|. \tag{3.5}$$

Indeed, suppose that (3.4) does not hold, then there exists $i = 2, \ldots, I$ such that $\bar{p} \,\square\, (\bar{x}_i - e_i) \leq \bar{t}_i + (1 - \|\bar{p}\|)\mathbb{1}$, with a strict inequality for some component, say the s-th component. But $\bar{x} \in \mathcal{A}(\mathcal{E})$ (Claim 3.3) and from the Non-Satiation Assumption in **C** and the additional Local Non-Satiation Assumption in **A**, there exists $x_i \in X_i$ such that $x_i \in P_i(\bar{x})$, $x_i(s') = \bar{x}_i(s')$ for every $s' \neq s$, and $[x_i, \bar{x}_i) \subset P_i(\bar{x})$. Besides, we can choose $x_i' \in [x_i, \bar{x}_i)$ close enough to $\bar{x}_i$ so that $x_i' \in \beta_i(\bar{p}, \bar{E})$. Consequently, $x_i' \in P_i(\bar{x}) \cap \beta_i(\bar{p}, \bar{E})$, which contradicts the fact that $P_i(\bar{x}) \cap \beta_i(\bar{p}, \bar{E}) = \emptyset$ (by Claim 3.5). This ends the proof of (3.4) and the proof of (3.5) is similar.

Now, taking the scalar product of both sides of the equalities (3.4) with $\lambda \in \mathbb{R}_{++}^{S+1}$, recalling that $\bar{t}_i \in \bar{E} \subset \lambda^\perp$, we obtain

$$(\lambda \,\square\, \bar{p}) \cdot (\bar{x}_i - e_i) = (1 - \|p\|)\sum_{s=0}^{S} \lambda(s) \text{ for every } i = 2, \ldots, I. \tag{3.6}$$

Summing up the equalities (3.6) (for $i = 2, \ldots, I$) and the equality (3.5) (for $i = 1$), using the fact that $\sum_{i=1}^{I} \bar{x}_i = \sum_{i=1}^{I} \bar{e}_i$ (Claim 3.3), we get

$$0 = (\lambda \,\square\, \bar{p}) \cdot \sum_{i=1}^{I} (\bar{x}_i - e_i) = (1 - \|p\|)(1 + \sum_{s=0}^{S} \lambda(s)).$$

Consequently, $\|\bar{p}\| = 1$. □

Claim 3.7. *For every $i \in I$, $\bar{x}_i \in B_i(\bar{p}, \bar{E})$ and $P_i(\bar{x}) \cap B_i(\bar{p}, \bar{E}) = \emptyset$.*

Proof. Since $\|\bar{p}\| = 1$ (from Claim 3.6), we have $B_i(\bar{p}, \bar{E}) = \beta_i(\bar{p}, \bar{E})$ for every $i = 2, \ldots, I$. Hence, Claim 3.5 implies the result for every consumer $i = 2, \ldots, I$.

Let us now consider the first consumer. Since $\bar{E} \subset \lambda^{\perp}$, one easily obtains that $B_1(\bar{p}, \bar{E}) \subset \beta_1(\bar{p}, \bar{E})$. From Claim 3.5, we get $\bar{x}_1 \in \beta_1(\bar{p}, \bar{E})$ and $P_1(\bar{x}) \cap \beta_1(\bar{p}, \bar{E}) = \emptyset$, hence $P_1(\bar{x}) \cap B_1(\bar{p}, \bar{E}) = \emptyset$. The proof of Claim 3.7 will thus be complete if we show that $\bar{x}_1 \in B_1(\bar{p}, \bar{E})$, or the stronger assertion that

$$\bar{p} \,\square\, (\bar{x}_1 - e_1) \in \bar{E}.$$

Since $\sum_{i=1}^{I} (\bar{x}_i - e_i) = 0$ (Claim 3.3) and since the budget constraint of every consumer is binding (cf. the proof of Claim 3.6), for every $i \neq 1$, there exists $\bar{t}_i \in \bar{E}$ such that:

$$\bar{p} \,\square\, (\bar{x}_1 - e_1) = -\sum_{i \neq 1} \bar{p} \,\square\, (\bar{x}_i - e_i) = -\sum_{i \neq 1} \bar{t}_i \in \bar{E},$$

which ends the proof of the claim. □

3.2. Proof of Theorem 2.1 in the general case

We now give the proof of Theorem 2.1 without considering the additional assumptions **A**. First, given an economy $\mathcal{E}$ which satisfies Assumptions **C** and **F**, we will consider a new economy $\hat{\mathcal{E}}$ with enlarged strictly preferred sets as in Gale-Mas Colell ([14], [15]). Second, we will truncate the economy $\hat{\mathcal{E}}$, to define a new economy $\hat{\mathcal{E}}^r$, which satisfies Assumptions **C**, **F** and **A**. Thus, Theorem 2.1 (with the additional Assumption **A**) will provide the existence of a pseudo-equilibrium of $\hat{\mathcal{E}}^r$, and we will check that it is also a pseudo-equilibrium of $\mathcal{E}$.

3.2.1. Enlarging the preferences. Consider an economy $\mathcal{E} = \big((X_i, P_i, e_i)_{i \in I}, V\big)$. Following Gale and Mas-Colell ([14], [15]), for every $x \in \Pi_{i \in I} X_i$, we define the "augmented preferences" $\hat{P}_i$ by:

$$\hat{P}_i(x) := \cup_{x_i' \in P_i(x)} (x_i, x_i'] = \{x_i + \lambda(x_i' - x_i) : 0 < \lambda \leq 1, \ x_i' \in P_i(x)\} \subset X_i.$$

We will need the following lemma, the proof of which is similar to the one given by Gale and Mas-Colell ([14],[15])) (see also Angeloni-Cornet [1]).

Lemma 3.2. *If $\mathcal{E} = \big((X_i, P_i, e_i)_{i \in I}, V\big)$ satisfies Assumptions **C** and **F**, then the economy $\hat{\mathcal{E}} = \big((X_i, \hat{P}_i, e_i)_{i \in I}, V\big)$ satisfies Assumptions **C**, **F** and the additional Local Non-Satiation Assumption in **A**. Moreover, every pseudo-equilibrium of $\hat{\mathcal{E}}$ is a pseudo-equilibrium of $\mathcal{E}$.*

Remark 3.1. *The following counterexample shows that $\hat{P}_i$ may not have open values (relatively to X_i) when P_i has open values (relatively to X_i). This is the main reason why we use the Openess-type Assumption **C**-(iii) (instead of assuming that P_i has open values) in both Lemma 3.2 and Theorem 2.1.*

Consider the set $X = \text{co}\{X_0 \cup \{S, N\}\}$, which is the convex hull of the set X_0 and the points N and S defined as follows:

$$N = (0, 0, 1), S = (0, 0, -1), X_0 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, (x-1)^2 + y^2 \leq 1\},$$

and a correspondence P from X to X such that $P(N) := \{(x, y, z) \in X : z < 0\}$. Then $P(N)$ is open for the relative topology of X and $\hat{P}(N)$ is not open in X.

3.2.2. Truncating the economy. Consider an economy $\mathcal{E} = \big((X_i, P_i, e_i)_{i \in I}, V\big)$ which satisfies Assumption **C**. Since the consumption sets X_i are bounded below, we deduce that the attainable set $\mathcal{A}(\mathcal{E})$ is compact. Denoting $\hat{X}_i$ the projection of $\mathcal{A}(\mathcal{E})$ on X_i, we deduce that the set $\hat{X}_i$ is bounded for every $i \in I$. Consequently, one can choose $r > 0$ large enough such that

$$\hat{X}_i \subset \text{int}\bar{B}(0, r) \text{ for every } i \in I.$$

For every $i \in I$, we let

$$X_i^r = X_i \cap \bar{B}(0, r), \quad P_i^r(x) = P_i(x) \cap X_i^r,$$

and we define a new economy $\mathcal{E}^r$ which only differs from $\mathcal{E}$ by the fact the consumption sets X_i have been replaced by the above sets X_i^r and the correspondences P_i by P_i^r. To summarize, we let the truncated economy of $\mathcal{E}$ be

$$\mathcal{E}^r := \big((X_i^r, P_i^r, e_i)_{i \in I}, V\big).$$

Then, one can easily prove the following lemma:

Lemma 3.3. *Let $\mathcal{E}$ satisfy Assumptions **C**, **F**, and the Local Non-Satiation Assumption in **A** and let $r > 0$ be chosen as above. Then $\mathcal{E}^r$ satisfies Assumptions **C**, **F** and **A** and every pseudo-equilibrium of $\mathcal{E}^r$ is a pseudo-equilibrium of $\mathcal{E}$.*

3.2.3. Proof of Theorem 2.1 in the general case. Let $\mathcal{E} = \big((X_i, P_i, e_i)_{i \in I}, V\big)$ satisfy Assumptions **C** and **F**. Applying Lemma 3.2 and Lemma 3.3, one can define the enlarged economy $\hat{\mathcal{E}}$ which satisfies Assumptions **C**, **F** and the local Non Satiation Assumption in **A** and one can choose $r > 0$ large enough so that the truncated enlarged economy $\hat{\mathcal{E}}^r$ satisfies Assumptions **C**, **F** and **A**. From Section 3.1 (providing the proof of Theorem 2.1 with the additional Assumptions **A**), there exists a pseudo-equilibrium $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ of $\hat{\mathcal{E}}^r$ such that $\bar{E} \subset \lambda^\perp$. From Lemma 3.2 and Lemma 3.3, $(\bar{x}, \bar{p}, \bar{q}, \bar{E})$ is also a pseudo-equilibrium of the economy $\mathcal{E}$. This ends the proof of Theorem 2.1.

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Philippe Bich
Paris School of Economics, Centre d'Economie de la Sorbonne
Université Paris 1 Panthéon Sorbonne
75647 Paris Cedex 13, France
e-mail: `bich@univ-paris1.fr`

Bernard Cornet
Paris School of Economics, Centre d'Economie de la Sorbonne
Université Paris 1 Panthéon Sorbonne
75647 Paris Cedex 13, France
and
Department of Economics
University of Kansas
66045 Kansas, USA
e-mail: `cornet@univ-paris1.fr, cornet@ku.edu`