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A Strategic Selection Procedure

Toru Suzuki*

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Abstract

A decision maker (DM) wishes to select a competent candidate to fill a position. However, since the wage and task of the position is predetermined, the DM cannot use contract as a screening device. This paper formulates the problem as a class of selection problem and derives the optimal selection procedure. The key element of our selection procedure is voluntary testing. That is, unlike statistical selection procedures, the signal generating process is endogenous. Then, the optimal selection rule takes into account not only the test performances but also signaling element of the test. We analyze the selection procedure as a signaling game and derive the optimal selection rule. Moreover, the optimal size of candidate pool and the selection efficiency are also analyzed. It is shown that, by making the test voluntary, the selection efficiency can be dramatically improved.

Keywords. Signaling, Screening, Selection problem, Selection procedure, Testing
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1 Introduction

Consider a situation in which a decision maker (DM) wishes to select a qualified candidate to fill a position. For example, a firm wishes to select a worker who has a sufficient ability to perform certain task. A university wishes to fund students whose academic ability is higher than certain level. The standard approach for such problem is screening, that is, the DM designs a contract which satisfies incentive compatible condition in order to sort out qualified candidates. However, it is often the case that there is little room to design contract. For example, in many public positions, the task and wage are predetermined. In fellowship/scholarship positions, there is often no task and the amount of the scholarship is already fixed. The purpose of this paper is to propose a simple selection procedure for such environments.

We formulate the problem as a class of selection problem in mathematical statistics literature, e.g., Lehmann (1961). Selection problem is to select a population which posses certain unobservable characteristics from a grand population. In other words, the purpose of selection is to find "qualified" candidates. Then, selection procedure is defined to be a selection rule which maximizes the objective of the DM given available information. This paper extends the standard selection procedure to a strategic environment. In the standard selection procedure, all available information about candidates is exogenously given. On the other hand, in our selection procedure, since each candidate chooses whether to take a test, the signal generating process is endogenous. Then, since the action of candidates can reveal private information, the DM can utilize information not only from test performances but also from their actions.

We analyze the selection procedure as a signaling game between candidates and the DM. Given private information, each candidate decides whether to take the test. Then, the DM selects one candidate after observing the test results. The test technology is assumed to be monotonic, that is, higher type tends to perform better. Then, the optimal selection rule is the DM's equilibrium strategy. We focus on "testing equilibrium," which is symmetric perfect Bayesian equilibrium where some types take the test with strictly positive probability.

In Section 3, it is shown that there exists unique testing equilibrium if the cost of selecting unqualified types, i.e., "false positive," is sufficiently high for the DM. More intuitively, whenever the DM prefers "selecting no candidate" to "a random choice" given the prior probability, there exists a unique testing equilibrium. In the testing equilibrium, each candidate takes the test only if his type is higher than a cutoff type. Hence, whenever one candidate takes the test in equilibrium, his type is higher than the cutoff type. Then, based on the equilibrium posterior belief, the optimal selection rule is derived. We show that the optimal selection rule is based on both relative and absolute evaluation of the test performances.

In our selection procedure, the size of candidate pool affects the level of information revelation in the testing equilibrium and determines the selection efficiency. Hence, the DM may wish to choose not only the selection rule but also the size of candidate pool. For instance, the DM can increase the size of candidate pool by notifying the opportunity to larger population. If the size of candidate pool is too large, the DM can reduce the size by a fair lottery. Section 4 defines optimal candidate pool to be the size of candidate pool which maximizes the DM's interest. First, we analyze the case where the DM cannot control the cost of test for candidates. Then, we provide the upper and lower bounds of the optimal size. Second, we analyze the case in which the DM can control the cost of test for candidates. In this case, we can easily pin down the optimal size. It is shown that the optimal size only depends on the DM's cost of testing and the distribution of types.

In Section 5, we define selection power to be the probability of "true positives," i.e., the selected candidate is one of qualified types. Then, given a selection power, we analyze the minimum size of candidate pool which attains the selection power. It is shown that, whenever the DM's cost of "false positives" is sufficiently high, our selection procedure can attain any level of selection power with a finite number of candidates. Moreover, we compare our selection procedure with a purely statistical selection procedure, i.e., the signal generating process is exogenous. It is shown that the minimum size of candidate pool to attain the certain selection power is always smaller in our selection procedure. Moreover, we show that, whenever the DM's cost of "false positive" is sufficiently high, the selection procedure with exogenous signals has no selection power. On the other hand, with the same cost of "false positive," our procedure can attain any level of selection power as long as the size of candidate pool is sufficiently large.

Related literature. The formulation of selection problem is based on Lehmann (1961). His paper provides the optimum properties of selection procedure for non-strategic environment, e.g., selection of products, plants etc. Our paper is an extension of the selection procedure to a strategic environment.

Testing is already incorporated to some earlier screening models, e.g., Guasch and Weiss (1980) and Nalebuff and Scharfstein (1987). However, in their models, qualified candidates can be sorted out by the wage schedule which depends on the test result. Hence, the spirit of their models is different from that of ours. Unlike their models and other screening models, our paper focuses on the environment in which there is little room to design contract, i.e., predetermined wage and task. Hence, the contribution of our paper is to provide a simple selection procedure for such non-standard screening environment. Laffont and Martimort (2002) discuss the importance of incentive design for economic environments where the set of feasible contracts is limited.

We analyze the selection procedure as a signaling game in which the DM observes not only action but also the test result. Since the payoff function is the same across all types in our signaling game, the standard single crossing condition is violated. However, since the test performance statistically reflects private information and the candidate has to outperform other candidates to be selected, the test becomes effective signaling device as competition among candidates becomes stronger. That is, when the number of candidates is sufficiently large, each candidate takes the test only if his type is higher than a certain level. On the other hand, when the number of candidates is small, the test is not effective signaling device and all equilibria can be pooling.

Finally, in our model, since candidates compete to get an opportunity, there is some similarity to contest models, e.g., Siegel (2009). There are some contest models which study the selection efficiency of contest, e.g., Hvide and Kristiansen (2003). However, the spirit of our model is different from contest models. In contest models, the main interest is in the effort level of contestants given a fixed "selection rule," i.e., contest. On the other hand, in our model, the main interest is in the optimal selection rule to sort out qualified types. That is, the selection rule is chosen by the DM to select "qualified" candidates. Moreover, since the selection rule is not designed to induce higher effort, our paper is also different from optimal contest literatures.

2 Model

First, we define selection problem based on Lehmann (1961). Let $\mathcal{I} = \{1, 2, .., I\}$ be a set of candidates and $\Theta = [\theta_{\min}, \theta_{\max}]$ be a set of types. Given $\theta^* \in int(\Theta)$, the set of **qualified types** is defined to be $\{\theta \in \Theta | \theta \geq \theta^*\}$. For each candidate, the type is private information and independently drawn from an absolutely continuous probability distribution $G(\theta)$ with $\text{supp}(g) = \Theta$. Then, a **selection problem** is $\langle \mathcal{I}, \Theta, \theta^*, K, g \rangle$ in which a decision maker (DM) selects at most K candidates so that the selected candidates are qualified types. For simplicity, we focus on the case $K = 1$.[1]

The payoff function of the DM is as follows. If the DM selects candidate i and $\theta_i \geq \theta^*$, then the DM's payoff is 1. On the other hand, if the DM selects candidate i and $\theta_i < \theta^*$, then the DM's payoff is $-\beta$ where $\beta > 0$ is the cost of "false positives." For example, if unqualified types can produce nothing, β may be the wage the firm pays. If unqualified types can damage the firm, β may be the damage and the wage. Finally, the DM's payoff is 0 if he rejects all candidates. All candidates wish to be selected and the position has common value $w > 0$ which can be interpreted as wage for job candidates and scholarship for students.

The DM applies a **selection procedure** $\langle X, Z, f, r \rangle$ which consists of two stages. The first stage is a signal generating process. Concretely, each candidate decides whether to take a test. If candidate i takes the test, the performance $x_i \in X = [\underline{x}, \bar{x}]$ is drawn from probability distribution $F(.|\theta_i)$. Then, let $Z = X \cup \{\varnothing\}$ be the set of signals that each candidate can generate and $z_i \in Z$ denote the signal of candidate i. Concretely, if candidate i takes the test, $z_i = x_i$ and, if candidate i does not take the test, $z_i = \varnothing$. The second stage is a selection process. Given available information $(z_1, z_2, .., z_I) \in Z^I$, the DM selects one candidate or rejects all based on a **selection rule** $r(z_1, z_2, .., z_I)$ where $r : Z^I \to \mathcal{I} \cup \{\varnothing\}$.

We assume that the test technology f satisfies the following two assumptions.

Assumption 1. $\text{supp}(f(.|\theta)) = X$ *for any* θ.
Assumption 2. $\frac{f(.|\theta')}{f(.|\theta)}$ *is strictly increasing in* x *if* $\theta' > \theta$.

Assumption 1 says that, there is no test performance which perfectly reveals the type.

[1] For $K > 1$, most of results are qualitatively preserved .

Assumption 2 states that the test technology f has the monotone likelihood ratio (MLR) property.

The test costs $c \in (0, w)$ for any candidate. When a candidate does not take the test, his payoff is w if he is selected and 0 if he is not selected. When a candidate takes the test, his payoff is $w - c$ if he is selected and $-c$ if he is not selected. Note that the payoff function is the same across types.

It is assumed that each candidate takes his action to maximize his expected payoff and the DM selects a candidate to maximize his expected payoff. Then, we analyze the selection procedure as a signaling game in which candidates are the senders and the DM is the receiver. Let $A_i = \{0, 1\}$ be candidate i's set of actions where 0 denotes "not taking the test" and 1 denotes "taking the test." Then, candidate i's strategy is a mapping $s_i : \Theta \to A_i$. On the other hand, the DM's strategy is selection rule $r(z_1, z_2, .., z_I)$. We define **testing equilibrium** to be perfect Bayesian equilibrium in which (i) the strategy profile of candidates is symmetric, i.e., $s_i(\theta_i) = s_j(\theta_j)$ if $\theta_i = \theta_j$ (ii) the probability that some candidates take the test is strictly positive, i.e., $\int_{\{\theta | s(\theta)=1\}} g(\theta)d\theta > 0$. Then, this paper focuses on testing equilibrium.

The rest of this paper investigates the followings: (i) the selection rule which maximizes the DM's interest (ii) the size of candidate pool which maximizes the DM's interest (iii) the selection efficiency of the procedure.

3 Optimal selection rule

Since we analyze the selection procedure as a signaling game, the optimal selection rule is the DM's equilibrium strategy. This section analyzes testing equilibrium and characterizes the optimal selection rule given size of candidate pool I.

The following lemma provides the properties of the DM's optimal reaction given candidates' strategy profile s.

Lemma 1. *Given candidates' strategy profile* s*, there exists* $x(s) \in X$ *such that*
(i) *if* $r(z_1, z_2, .., z_I) = i$ *and* $a_i = 1$*, then* $x_i \geq x_j$ *for any* j *and* $x_i \geq x(s)$*.*
(ii) *if* $r(z_1, z_2, .., z_I) = \varnothing$ *and* $a_j = 1$*, then* $x_j < x(s)$*.*

Proof. Let $\Theta(s) = \{\theta | s(\theta) = 1\}$ and $\mu_s(\theta_i|z_i)$ be the posterior probability density of θ_i given candidate i's test result z_i and strategy profile s. That is,

$$\mu_s(\theta_i|z_i) = \begin{cases} \frac{f(x|\theta_i)g(\theta_i)}{\int_{\theta_i' \in \Theta(s)} f(x|\theta_i')g(\theta_i')d\theta_i'} \text{ if } z_i = x_i \\ \frac{g(\theta_i)}{\int_{\theta_i' \in \Theta / \Theta(s)} g(\theta_i')d\theta_i'} \text{ if } z_i = \varnothing \end{cases}$$

Then, when the DM chooses i, the expected payoff of the DM given z_i is

$$\int_{\theta_i \geq \theta^*} \mu_s(\theta_i|z_i)d\theta_i - \beta \int_{\theta_i < \theta^*} \mu_s(\theta_i|z_i)d\theta_i$$

Thus, whenever candidate i is selected, two conditions have to be satisfied. First, the DM's expected payoff from candidate i has to be higher than that from other candidates. That is,

$$\begin{aligned} &\int_{\theta_i \geq \theta^*} \mu_s(\theta_i|z_i)d\theta_i - \beta \int_{\theta_i < \theta^*} \mu_s(\theta_i|z_i)d\theta_i \\ \geq &\int_{\theta_j \geq \theta^*} \mu_s(\theta_j|z_j)d\theta_j - \beta \int_{\theta_j < \theta^*} \mu_s(\theta_j|z_j)d\theta_j. \end{aligned}$$

for all j. Then, by the MLR property, $x_i \geq x_j$. This proves the first part of (i).

Second, the expected payoff of the DM from selecting i has to be positive. That is,

$$\int_{\theta_i \geq \theta^*} \mu_s(\theta_i|z_i)d\theta_i \geq \beta \int_{\theta_i < \theta^*} \mu_s(\theta_i|z_i)d\theta_i$$

Observe that, by the MLR property, $\int_{\theta_i \geq \theta^*} \mu_s(\theta_i|x_i)d\theta_i$ is increasing in x_i and $\int_{\theta_i < \theta^*} \mu_s(\theta_i|x_i)d\theta_i$ is decreasing in x_i. Then, let

$$x(s) = \begin{cases} \widetilde{x} \text{ if there exists } \int_{\theta_i \geq \theta^*} \mu_s(\theta_i|\widetilde{x})d\theta_i = \beta \int_{\theta_i < \theta^*} \mu_s(\theta_i|\widetilde{x})d\theta_i \\ \underline{x} \text{ if } \int_{\theta_i \geq \theta^*} \mu_s(\theta_i|x_i)d\theta_i > \beta \int_{\theta_i < \theta^*} \mu_s(\theta_i|x_i)d\theta_i \text{ for all } x_i \\ \bar{x} \text{ if } \int_{\theta_i \geq \theta^*} \mu_s(\theta_i|x_i)d\theta_i < \beta \int_{\theta_i < \theta^*} \mu_s(\theta_i|x_i)d\theta_i \text{ for all } x_i \end{cases}.$$

Hence, the expected payoff from selecting candidate i is positive if and only if $x_i \geq x(s)$. Q.E.D.

Remark. If $K > 1$, whenever a candidate is selected, his performance is at least as good as K-th highest performance. On the other hand, $x(s)$ is the same as that of $K = 1$.

Now, we introduce a candidate's strategy which plays a key role in this paper. A candidate's strategy is $\hat{\theta}$-*cutoff strategy* if

$$s_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > \hat{\theta} \\ 0 & \text{if } \theta_i < \hat{\theta} \end{cases}$$

The next lemma claims that, in any testing equilibria, all candidates play a cutoff strategy.

Lemma 2. *Any testing equilibrium consists of a cutoff strategy profile.*

Proof. Suppose not. Then, there exists $\theta' > \theta''$ such that $s_i(\theta') = 0$ and $s_i(\theta'') = 1$. Then, let $\Theta(s) = \{\theta | s(\theta) = 1\}$ and

$$H(x_i|\theta_j, s) = \begin{cases} F(x_i|\theta_j) \text{ if } \theta_j \in \Theta(s) \\ 1 \text{ if } \theta_j \notin \Theta(s) \end{cases}.$$

Then, since the distribution of types is independent, the probability that candidate i's performance x_i is the highest among all candidates is

$$\Pr(x_i \geq x_j, \forall j | s, I) = \left(\int_{\theta_j} H(x_i|\theta_j, s) dG(\theta_j) \right)^I.$$

Hence, by Lemma 1, the expected payoff of type θ from the test given s and I is

$$\int_{x_i \geq x(s)} w \Pr(x_i \geq x_j, \forall j | s, I) dF(x_i|\theta_i) - c.$$

Then, by the MLR property,

$$\int_{x_i \geq x(s)} w \Pr(x_i \geq x_j, \forall j | s, I) dF(x_i|\theta_i') > \int_{x_i \geq x(s)} w \Pr(x_i \geq x_j, \forall j | s, I) dF(x_i|\theta_i'').$$

and type θ' has incentive to deviate, a contradiction. Q.E.D.

The next proposition states that whenever the cost of "false positive" is sufficiently high for the DM, there exists a unique testing equilibrium for any I. On the other hand, even if the cost of "false positive" is low, whenever the number of candidates is sufficiently large, there exists a testing equilibrium.

Proposition 1.
(i) *If* $(1+\beta)G(\theta^*) > 1$, *there exists a unique testing equilibrium for any* I.
(ii) *If* $(1+\beta)G(\theta^*) \leq 1$, *testing equilibrium exists for sufficiently large* I

Proof. By abusing notation, let

$$H(x_i|\theta_j,\hat{\theta}) = \begin{cases} F(x_i|\theta_j) \text{ if } \theta_j > \hat{\theta} \\ 1 \text{ if } \theta_j < \hat{\theta} \end{cases}$$

Then, given $\hat{\theta}$-cutoff strategy, the probability that candidate i's performance x_i is the highest among all candidates is

$$\Pr(x_i \geq x_j, \forall j|\hat{\theta}, I) = \left(\int_{\theta_j} H(x_i|\theta_j,\hat{\theta})dG(\theta_j)\right)^I.$$

Moreover, let $U(\hat{\theta}|\hat{\theta}, I, c)$ be the expected payoff from the test for candidate i with $\theta_i = \hat{\theta}$ given $\hat{\theta}$-cutoff strategy profile, c and I. That is,

$$U(\hat{\theta}|\hat{\theta}, I, c) = \int_{x_i \geq x(\hat{\theta})} w \Pr(x_i \geq x_j, \forall j|\hat{\theta}, I)dF(x_i|\hat{\theta}) - c$$

Then the proof consists of three claims.

Claim 1. $U(\hat{\theta}|\hat{\theta}, I, c)$ *is continuous and strictly increasing in* $\hat{\theta}$.

First, it is easy to see that $\Pr(x_i \geq x_j, \forall j|\hat{\theta}, I)$ is continuous and strictly increasing in $\hat{\theta}$. Second, by abusing notation, let

$$x(\hat{\theta}) = \begin{cases} \widetilde{x} \text{ if there exists } \int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|\widetilde{x})d\theta_i = \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|\widetilde{x})d\theta_i \\ \underline{x} \text{ if } \int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|x_i)d\theta_i > \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|x_i)d\theta_i \text{ for all } x_i \\ \bar{x} \text{ if } \int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|x_i)d\theta_i < \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|x_i)d\theta_i \text{ for all } x_i \end{cases}$$

where

$$\mu_{\hat{\theta}}(\theta_i|z_i) = \begin{cases} \frac{f(x|\theta_i)g(\theta_i)}{\int_{\theta_i' \geq \hat{\theta}} f(x|\theta_i')g(\theta_i')d\theta_i'} \text{ if } z_i = x_i \\ \frac{g(\theta_i)}{\int_{\theta_i' < \hat{\theta}} g(\theta_i')d\theta_i'} \text{ if } z_i = \varnothing \end{cases}$$

Then, obviously, $\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|x)d\theta_i$ and $\int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|x)d\theta_i$ are both continuous in $\hat{\theta}$. Moreover, by the MLR property, $\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|x)d\theta_i$ is strictly increasing in $\hat{\theta}$ and $\int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|x)d\theta_i$

is strictly decreasing in $\hat{\theta}$. Hence, $x(\hat{\theta})$ is decreasing in $\hat{\theta}$. Therefore, $U(\hat{\theta}|\hat{\theta}, I, c)$ is continuous and strictly increasing in $\hat{\theta}$.

Claim 2. There exits a unique cutoff equilibrium if $(1+\beta)G(\theta^*) > 1$.

Note that, as $\hat{\theta} \to \theta_{\max}$, the probability that some candidate takes the test goes to 0. Hence, by choosing $\hat{\theta}$ close to $\theta_{\max}$, we can make $\Pr(x_i \geq x_j, \forall j|\hat{\theta}, I)$ arbitrarily close to 1. Then, $\lim_{\hat{\theta}\to\theta_{\max}} U(\hat{\theta}|\hat{\theta}, I, c) = w - c > 0$.

To construct the cutoff equilibrium, first, suppose $\lim_{\hat{\theta}\to\theta_{\min}} U(\hat{\theta}|\hat{\theta}, I, c) \geq 0$. Then, by the MLR property, $\lim_{\hat{\theta}\to\theta_{\min}} U(\theta|\hat{\theta}, I, c) \geq 0$ for any θ and, by Claim 1, $U(\hat{\theta}|\hat{\theta}, I, c) \geq 0$ for any $\hat{\theta}$. Thus, the equilibrium cutoff is $\theta_{\min}$, that is, the equilibrium strategy is such that $s_i(\theta) = 1$ for all θ. Second, consider the case where $\lim_{\hat{\theta}\to\theta_{\min}} U(\hat{\theta}|\hat{\theta}, I, c) < 0$, then, by Claim 1, there exists a unique $\hat{\theta} \in int(\Theta)$ such that $U(\hat{\theta}|\hat{\theta}, I, c) = 0$ and this is the equilibrium cutoff type.

Claim 3. Suppose $(1+\beta)G(\theta^*) < 1$. *There exits a cutoff equilibrium if* I *is sufficiently large.*

Note that, whenever no one takes the test, all candidates who do not take the test are indifferent for the DM. Moreover, if $(1+\beta)G(\theta^*) < 1$, the DM's expected payoff from choosing any candidate is strictly larger than 0. Then, suppose the DM selects each candidate with the same probability, $\frac{1}{I}$, when no one takes the test. Observe that, given a cutoff strategy with $\hat{\theta} \geq \theta^*$, the expected payoff of each candidate from "not taking test" is $\frac{w}{I}G(\hat{\theta})^{I-1}$ if $\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i > \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i$. On the other hand, the expected payoff is 0 if $\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i < \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i$. Note that $\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i > \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i$ for large $\hat{\theta}$ since $(1+\beta)G(\theta^*) < 1$. Moreover, it is easy to see that the expected payoff from "not taking test" is increasing in $\hat{\theta}$. Then, since $\frac{w}{I}G(\hat{\theta})^{I-1}$ is increasing in $\hat{\theta}$ and $\lim_{\hat{\theta}\to\theta_{\max}} \frac{w}{I}G(\hat{\theta})^{I-1} = \frac{w}{I}$, we have $\lim_{\hat{\theta}\to\theta_{\max}}\{U(\hat{\theta}|\hat{\theta}, I, c) - \frac{w}{I}G(\hat{\theta})^{I-1}\} > 0$ for $I > \frac{w}{w-c}$.

Now, choose a large I so that $\lim_{\hat{\theta}\to\theta^*} U(\hat{\theta}|\hat{\theta}, I, c) < 0$. Then, since $U(\hat{\theta}|\hat{\theta}, I, c) - \frac{w}{I}G(\hat{\theta})^{I-1} < 0$ for small $\hat{\theta} > \theta^*$, there exists $\hat{\theta} \in (\theta^*, \theta_{\max})$ such that $U(\hat{\theta}|\hat{\theta}, I, c) = \frac{w}{I}G(\hat{\theta})^{I-1}$. If $\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i > \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta_i|\varnothing)d\theta_i$ for such $\hat{\theta}$, then let such $\hat{\theta}$ be the equilibrium cutoff type. Otherwise, let the solution of $U(\hat{\theta}|\hat{\theta}, I, c) = 0$ be the equilibrium cutoff type. Q.E.D.

To get an intuition of condition $(1+\beta)G(\theta^*) > 1$, suppose that the DM has to make his decision only based on the prior probability. Then, the DM strictly prefers "rejecting all" to "random choice" if and only if $(1+\beta)G(\theta^*) > 1$. Hence, there exists a unique testing equilibrium whenever the DM prefers not to select any candidate without information. If $(1+\beta)G(\theta^*) < 1$, the existence of testing equilibrium is not guaranteed for small I. For example, there is no testing equilibrium if $(1+\beta)G(\theta^*) < 1$ and $I = 1$.

As we mentioned before, the optimal selection rule is the DM's equilibrium strategy. To be more explicit, whenever a testing equilibrium exists, the optimal selection rule is

$$r^*(z_1, z_2, .., z_I) = \begin{cases} i \text{ if } x_i \geq x_j \text{ for any } j \text{ and } x_i \geq x(\hat{\theta}(c, I)) \\ \varnothing \text{ if, for any } j \in \mathcal{I},\ x_j < x(\hat{\theta}(c, I)) \end{cases} .$$

where $\hat{\theta}(c, I)$ is the equilibrium cutoff type in the testing equilibrium given c and I.

Note that the selection rule is based on both absolute and relative performances. Moreover, the minimum performance criterion $x(\hat{\theta}(c, I))$ depends on the cost of test c and the size of candidate pool I. Then, the next section analyzes the optimal size I for the optimal selection rule.

4 Optimal candidate pool

In the last section, we derived the optimal selection rule given I and c. This section focuses on the case in which there exists a unique testing equilibrium, i.e. $(1+\beta)G(\theta^*) > 1$, and analyzes the optimal size of candidate pool. We assume that testing is costly for the DM. Concretely, suppose that the testing costs $\delta > 0$ per candidate for the DM. Then, let $W(I, \delta)$ denote the ex ante expected payoff of the DM in which candidates play the testing equilibrium. Then, **optimal candidate pool** I^* is defined to be the size of candidate pool such that $W(I^*, \delta) \geq W(I, \delta)$ for any I.

The optimal size of candidate pool is analyzed in two cases. First, we analyze the case in which cost c is fixed, e.g., c is the cost of a standard test. Second, we analyze the case in which the DM can control both I and c.

4.1 Optimal candidate pool given c

Suppose that the DM chooses I to maximize his expected payoff given c. The next lemma clarifies the relationship between the equilibrium cutoff type and the size of candidate pool. Let $\hat{\theta}(c, I)$ be the equilibrium cutoff given c and I.

Lemma 3. *Given* c, $\hat{\theta}(c, I)$ is increasing in I. Moreover, there exists I' such that $\hat{\theta}(c, I)$ is strictly increasing in $I > I'$.

Proof. Given a cutoff strategy profile, $\Pr(x_i \geq x_j, \forall j|\hat{\theta}, I) > \Pr(x_i \geq x_j, \forall j|\hat{\theta}, I + 1)$. Hence, $U(\hat{\theta}|\hat{\theta}, I, c) > U(\hat{\theta}|\hat{\theta}, I + 1, c)$. Note that, from Claim 1 in the proof of Proposition 1, $U(\hat{\theta}|\hat{\theta}, I, c)$ is strictly increasing in $\hat{\theta}$. Then, if there exists a solution of $U(\hat{\theta}|\hat{\theta}, I, c) = 0$, the solution of $U(\hat{\theta}|\hat{\theta}, I + 1, c) = 0$ is strictly higher. Hence, if $\hat{\theta}(c, I) \in int(\Theta)$, then, $\hat{\theta}(c, I) < \hat{\theta}(c, I + 1)$. On the other hand, if $\hat{\theta}(c, I) = \theta_{\min}$, then, $\hat{\theta}(c, I) \leq \hat{\theta}(c, I + 1)$.

For the second part, observe that $\lim_{\hat{\theta}\to\theta_{\min}} U(\hat{\theta}|\hat{\theta}, I, c) < 0$ if I is sufficiently large, then $\hat{\theta}(c, I) \in int(\Theta)$ for such I. Thus, if I' is defined to be the smallest I such that $\lim_{\hat{\theta}\to\theta_{\min}} U(\hat{\theta}|\hat{\theta}, I, c) < 0$, $\hat{\theta}(c, I)$ is strictly increasing in $I > I'$. Q.E.D.

Remark. Any $I < \infty$, $\hat{\theta}(c, I) < \theta_{\max}$.

Lemma 3 says that if there exists a pooling equilibrium in which all types take the test for small I, then, there exists I from which the testing equilibrium is semi-pooling for larger I.

The next lemma shows that larger candidate pool always increases the probability of "successful selection" as long as the size of candidate pool is smaller than a certain level. Let

$$\hat{I}(c) = \max\{I|\theta(c, I) < \theta^*\}.$$

Moreover, let $\pi_+(I, \hat{\theta})$ be the probability that the DM selects a candidate with $\theta \geq \theta^*$given I and $\hat{\theta}$.

Lemma 4. $\pi_+(I, \hat{\theta}(c, I)) < \pi_+(I + 1, \hat{\theta}(c, I + 1))$ for $I \leq \hat{I}(c)$ and $\pi_+(I, \hat{\theta}(c, I)) = 1 - G(\hat{\theta}(c, I))^I$ for $I > \hat{I}(c)$.

Proof. To prove the first part, note the probability that the DM selects candidate i and $\theta_i \geq \theta$ is

$$q_i(I, \hat{\theta}) = \int_{x_i \geq x(\hat{\theta})} \int_{\theta_i > \max\{\hat{\theta}, \theta^*\}} \Pr(x_i \geq x_j, \forall j|\hat{\theta}, I) f(x_i|\theta_i) dG(\theta_i) dx_i.$$

Then, by symmetry, $\pi_+(I, \hat{\theta}) = \sum_i q_i(I, \hat{\theta}) = I q_i(I, \hat{\theta})$.

Claim 1. $\pi_+(I,\hat{\theta})$ is increasing in I given $\hat{\theta}$.

To prove the claim, let $\pi_+(I,\hat{\theta},x)$ be

$$\pi_+(I,\hat{\theta},x)=\begin{cases} I\int_{\theta_i\geq\max\{\hat{\theta},\theta^*\}}\Pr(x_i=x\geq x_j,\forall j|\hat{\theta},I)f(x_i|\theta_i)dG(\theta_i) & x\geq x(\hat{\theta}) \\ 0 & \text{if } x<x(\hat{\theta})\end{cases}.$$

Then, for $x,x'\geq x(\hat{\theta})$,

$$\frac{\pi_+(I,\hat{\theta},x)}{\pi_+(I,\hat{\theta},x')}=\frac{I\int_{\theta_i\geq\max\{\hat{\theta},\theta^*\}}\left(\int_{\theta_j}H(x|\theta_j,\hat{\theta})dG(\theta_j)\right)^I f(x|\theta_i)dG(\theta_i)}{I\int_{\theta_i\geq\max\{\hat{\theta},\theta^*\}}\left(\int_{\theta_j}H(x'|\theta_j,\hat{\theta})dG(\theta_j)\right)^I f(x'|\theta_i)dG(\theta_i)}$$

$$\frac{\pi_+(I+1,\hat{\theta},x)}{\pi_+(I+1,\hat{\theta},x')}=\frac{(I+1)\int_{\theta_i\geq\max\{\hat{\theta},\theta^*\}}\left(\int_{\theta_j}H(x|\theta_j,\hat{\theta})dG(\theta_j)\right)^{I+1} f(x|\theta_i)dG(\theta_i)}{(I+1)\int_{\theta_i\geq\max\{\hat{\theta},\theta^*\}}\left(\int_{\theta_j}H(x'|\theta_j,\hat{\theta})dG(\theta_j)\right)^{I+1} f(x'|\theta_i)dG(\theta_i)}$$

By the MLR property, if $x'>x$, then

$$\int_{\theta_j}H(x|\theta_j,\hat{\theta}(c,I))dG(\theta_j)<\int_{\theta_j}H(x'|\theta_j,\hat{\theta}(c,I))dG(\theta_j).$$

and thus

$$\frac{\pi_+(I,\hat{\theta},x)}{\pi_+(I,\hat{\theta},x')}\geq\frac{\pi_+(I+1,\hat{\theta},x)}{\pi_+(I+1,\hat{\theta},x')}.$$

and the inequality is strict for $x'>x(\hat{\theta})$

Hence,

$$\int_{x_i\geq x(\hat{\theta})}\pi_+(I+1,\hat{\theta},x_i)dx_i>\int_{x_i\geq x(\hat{\theta})}\pi_+(I,\hat{\theta},x_i)dx_i.$$

Claim 2. $\pi_+(I,\hat{\theta})$ is increasing in $\hat{\theta}<\theta^*$ given I.

For $\hat{\theta}<\theta^*$, $\hat{\theta}$ affects $\pi_+(I,\hat{\theta})$ through two channels, i.e., $\Pr(x_i\geq x_j,\forall j|\hat{\theta},I)$ and $x(\hat{\theta})$. First, observe that since the number of competitors becomes smaller, $\Pr(x_i\geq x_j,\forall j|\hat{\theta},I)$ is increasing in $\hat{\theta}$. Second, as we see in Claim 1 in the proof of Proposition 1, $x(\hat{\theta})$ is decreasing in $\hat{\theta}$. Hence, $\pi_+(I,\hat{\theta})$ is increasing in $\hat{\theta}<\theta^*$.

Claim 3. $\pi_+(I,\hat{\theta}(c,I))<\pi_+(I+1,\hat{\theta}(c,I+1))$ for $I\leq\hat{I}(c)$.

First, by Claim 1, $\pi_+(I, \hat{\theta}(c, I)) < \pi_+(I+1, \hat{\theta}(c, I))$. Note that $\hat{\theta}(c, I) < \theta^*$ for all $I \leq \hat{I}(c)$. Then, by Claim 2, $\pi_+(I+1, \hat{\theta}(c, I)) < \pi_+(I+1, \hat{\theta}(c, I+1))$ for $I \leq \hat{I}(c)$. Hence, $\pi_+(I, \hat{\theta}(c, I)) < \pi_+(I+1, \hat{\theta}(c, I+1))$ for $I \leq \hat{I}(c)$.

For the second part, note that if $I > \hat{I}(c)$, then $\theta^* \leq \hat{\theta}(c, I)$ and thus $x(\hat{\theta}) = \underline{x}$. Hence, whenever the DM fails to select a qualified candidate, this is only because no one takes the test. Then, the probability of such event is $G(\hat{\theta}(c, I))^I$. Q.E.D.

To get an intuition of Lemma 4, note that, if $I \leq \hat{I}$, not only qualified types but also unqualified types take the test. Then, when the number of candidates becomes larger, the chance that a qualified candidate outperforms other candidates becomes higher. Moreover, as we showed in Lemma 3, when the number of candidates is larger, the set of unqualified types who take the test becomes smaller in equilibrium. Hence, more competition increases the probability of "successful selection." When $I > \hat{I}(c)$, candidates who take the test are all qualified types in equilibrium. Hence, as long as some candidates take the test, the DM can select a qualified type. On the other hand, if $I > \hat{I}(c)$, qualified types in $[\theta^*, \hat{\theta}(c, I))$ do not take the test in equilibrium. Hence, if all qualified types in the candidate pool are in $[\theta^*, \hat{\theta}(c, I))$, no one takes the test and the DM cannot select any qualified candidate.

The next lemma claims that, whenever the size of candidate pool is sufficiently large, the probability of selecting a unqualified candidate is zero. Let $\pi_-(I, \hat{\theta})$ be the probability that the DM selects a candidate whose type is $\theta < \theta^*$.

Lemma 5. $\pi_-(I, \hat{\theta}(c, I)) > 0$ for $I \leq \hat{I}(c)$ and $\pi_-(I, \hat{\theta}(c, I)) = 0$ if $I > \hat{I}(c)$.

Proof. Observe that, by the similar reasoning as $\pi_+(I, \hat{\theta})$,

$$\pi_-(I, \hat{\theta}) = I \int_{x_i \geq x(\hat{\theta})} \int_{\theta_i \in [\min\{\hat{\theta}, \theta^*\}, \theta^*]} \Pr(x_i \geq x_j, \forall j | \hat{\theta}, I) f(x_i|\theta_i) dG(\theta_i) dx_i$$

First, if $I \leq \hat{I}(c)$, then $\hat{\theta}(c, I) < \theta^*$. Thus, $\pi_-(I, \hat{\theta}(c, I)) > 0$. Second, if $I > \hat{I}(c)$, then $\min\{\hat{\theta}, \theta^*\} = \theta^*$ and thus $\pi_-(I, \hat{\theta}(c, I)) = 0$ for any I. Q.E.D.

The idea of Lemma 5 is as follows. When $I > \hat{I}(c)$, only qualified types take the test in the testing equilibrium. Note that, since $(1+\beta)G(\theta^*) > 1$, the DM selects a candidate only if he takes the test. Then, no candidate whose type is $\theta < \theta^*$ can be selected.

Remark. It is not obvious that $\pi_-(I, \hat{\theta}(c, I))$ is decreasing in $I \leq \hat{I}(c)$. To see the reason, observe that the minimum level of performance for the selection $x(\hat{\theta}(c, I))$ is decreasing in I. Thus, the probability that unqualified types can satisfy the minimum performance level becomes higher. Then, even though the probability that unqualified types take the test is smaller for larger $I \leq \hat{I}(c)$, the net effect on $\pi_-(I, \hat{\theta}(c, I))$ is not obvious.

Before the main result of this section, we need to establish the following lemma. Let $\Gamma(I, \delta, c) = 1 - G(\hat{\theta}(c, I))^I - \delta I$.

Lemma 6. *If there exists $I > \hat{I}(c)$ such that $G(\hat{\theta}(c, I))^I - G(\hat{\theta}(c, I+1))^{I+1} > \delta$, then there exists $\widetilde{I}(\delta) < \infty$ such that $\widetilde{I}(\delta) > \hat{I}(c)$ and $\Gamma(\widetilde{I}(\delta), \delta, c) \geq \Gamma(I', \delta, c)$ for any $I' > \hat{I}(c)$.*

Proof. Note that $\Gamma(I+1, \delta, c) - \Gamma(I, \delta, c) = G(\hat{\theta}(c, I))^I - G(\hat{\theta}(c, I+1))^{I+1} - \delta$ if $I > \hat{I}(c)$. Then, by Lemma 3, $G(\hat{\theta}(c, I))^I - G(\hat{\theta}(c, I+1))^{I+1} < G(\hat{\theta}(c, I))^I(1 - G(\hat{\theta}(c, I))$. Note that, since $\lim_{I\to\infty} G(\hat{\theta}(c, I)) = 1$, $G(\hat{\theta}(c, I))^I - G(\hat{\theta}(c, I+1))^{I+1} < \delta$ for sufficiently large I. Then, given δ, there exists $\max_{I>\hat{I}(c)} \Gamma(I, \delta, c)$ and, then, let $\widetilde{I}(\delta, c) = \arg\max_{I>\hat{I}(c)} \Gamma(I, \delta, c)$. Q.E.D.

The next proposition provides the properties of the optimal candidate pool.

Proposition 2. Let I^* be the optimal size of candidate pool.
(i) $I^* \leq \widetilde{I}(\delta, c)$.
(ii) *For sufficiently small δ, $I^* = \hat{I}(c)$* or $\widetilde{I}(\delta, c)$.
(iii) *For sufficiently small δ and sufficiently large β, $I^* = \widetilde{I}(\delta, c)$.*

Proof. Note that

$$W(I, \delta) = \pi_+(I, \hat{\theta}(c, I)) - \beta\pi_-(I, \hat{\theta}(c, I)) - \delta I.$$

(i) To see the upper bound of I^*, recall that, by Lemma 4, $\pi_+(I, \hat{\theta}(c, I)) = 1 - G(\hat{\theta}(c, I))^I$ for any $I > \hat{I}(c)$. Moreover, by Lemma 5, $\pi_-(I, \hat{\theta}(c, I)) = 0$ whenever $I > \hat{I}(c)$. Hence, $W(I, \delta) = 1 - G(\hat{\theta})^I - \delta I$ for $I > \hat{I}(c)$. Then, by Lemma 6, $I^* = \widetilde{I}(\delta, c)$ whenever $I^* > \hat{I}(c)$.

(ii) Note that, by (i), if we establish that $W(I,\delta)$ is increasing in $I < \hat{I}(c)$, the result immediately follows. Let $\hat{W}(\hat{\theta}, I)$ be the expected payoff of the DM in which candidates' strategy profile is $\hat{\theta}$-cutoff and the number of candidates is I.

Claim 1. Given $I < \hat{I}(c)$, $\hat{W}(\hat{\theta}, I)$ is increasing in $\hat{\theta} < \theta^*$.

Let y be the highest performance among candidates. First, consider $\hat{\theta}', \hat{\theta}$ such that $\theta^* > \hat{\theta}' > \hat{\theta}$. Then,

$$\int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}'}(\theta|y)d\theta - \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}'}(\theta|y)d\theta \geq \int_{\theta_i \geq \theta^*} \mu_{\hat{\theta}}(\theta|y)d\theta - \beta \int_{\theta_i < \theta^*} \mu_{\hat{\theta}}(\theta|y)d\theta.$$

for any $y \geq x(\hat{\theta}')$. Moreover, the inequality is strict if $x(\hat{\theta}') \leq y < x(\hat{\theta})$. Hence, $\hat{W}(\hat{\theta}', I) \geq \hat{W}(\hat{\theta}, I)$. Finally, if $y < x(\hat{\theta}')$, then the DM's payoff is 0 for both $\hat{\theta}'$and $\hat{\theta}$. Hence, $W(\hat{\theta}, I)$ is increasing in $\hat{\theta} < \theta^*$ given I.

Claim 2. Given $\hat{\theta} < \theta^*, W(\hat{\theta}, I)$ is increasing in I.

Again, let y be the highest performance among candidates. Then, by the MLR property, the expected payoff of the DM is increasing in y.

Now, consider the distribution of y given I. To see how additional candidate affects the distribution of y, suppose we add a candidate j' to a candidate pool with size I. Then, given the probability of having y conditional on I, $p(y|I)$, we can write the probability of having y conditional on $I+1$, $p(y|I+1)$, as follows.

$$\begin{aligned} p(y|I+1) &= \Pr(y \geq x_j)p(y|I) \\ &= \int_{\theta_{j'}} H(y|\theta_{j'}, \hat{\theta})dG(\theta_{j'})p(y|I). \end{aligned}$$

where

$$H(y|\theta_{j'}, \hat{\theta}) = \begin{cases} F(y|\theta_j) \text{ if } \theta_{j'} > \hat{\theta} \\ 1 \text{ if } \theta_{j'} < \hat{\theta} \end{cases}$$

Then,

$$\frac{p(y'|I+1)}{p(y|I+1)} = \frac{\int_{\theta_{j'}} H(y'|\theta_{j'}, \hat{\theta})dG(\theta_{j'})p(y'|I)}{\int_{\theta_{j'}} H(y|\theta_{j'}, \hat{\theta})dG(\theta_{j'})p(y|I)}$$

Note that, by the MLR property, if $y' > y$,

$$\frac{\int_{\theta_{j'}} H(y'|\theta_{j'},\hat{\theta})dG(\theta_{j'})}{\int_{\theta_{j'}} H(y|\theta_{j'},\hat{\theta})dG(\theta_{j'})} > 1.$$

Hence,

$$\frac{p(y'|I+1)}{p(y|I+1)} \geq \frac{p(y'|I)}{p(y|I)}$$

Thus, $p(y|I+1)$ first-order-stochastically dominates $p(y|I)$. Then, since the expected payoff of the DM is increasing in y, the expected payoff is higher in $I+1$.

Therefore, from Claim 1 and Claim 2, $W(I,\delta)$ is increasing in $I < \hat{I}(c)$.

(iii) By Lemma 5, $\pi_{-}(I,\hat{\theta}(c,I)) = 0$ if $I > \hat{I}(c)$ and $\pi_{-}(I,\hat{\theta}(c,I)) > 0$ if $I \leq \hat{I}(c)$. Hence, if β is sufficiently large and δ is small, any $I \leq \hat{I}(c)$ cannot be optimal. Then, by (i), $I^* = \widetilde{I}(\delta)$. Q.E.D.

4.2 Optimal candidate pool with controllable cost

Now, suppose the DM can choose both I and c to maximize his interest.

Lemma 7. *$\hat{\theta}(c,I)$ is increasing and continuous in c. Moreover, there exists c' such that $\hat{\theta}(c,I)$ is strictly increasing in $c > c'$.*

Proof. From Claim 1 in the proof of Proposition 1, $U(\hat{\theta}|\hat{\theta},I,c)$ is strictly increasing in $\hat{\theta}$. Moreover, obviously, $U(\hat{\theta}|\hat{\theta},I,c)$ is strictly decreasing in c. Thus, if there exists $\hat{\theta}$ which solves $U(\hat{\theta}|\hat{\theta},I,c) = 0$, $\hat{\theta}$ which solves $U(\hat{\theta}|\hat{\theta},I,c+\varepsilon) = 0$ for $\varepsilon > 0$ is strictly higher. Hence, if $\hat{\theta}(c,I) \in int(\Theta)$, then, $\hat{\theta}(c,I) < \hat{\theta}(c+\varepsilon,I)$. On the other hand, if $\hat{\theta}(c,I) = \theta_{\min}$, then, $\hat{\theta}(c,I) \leq \hat{\theta}(c+\varepsilon,I)$. Moreover, since $U(\hat{\theta}|\hat{\theta},I,c)$ is strictly increasing and continuous in $\hat{\theta}$, $\hat{\theta}(c,I)$ is continuous in c.

For the second part, observe that $\lim_{\hat{\theta}\to\theta_{\min}} U(\hat{\theta}|\hat{\theta},I,c) < 0$ if c is sufficiently large, then $\hat{\theta}(c,I) \in int(\Theta)$ for such I. Thus, if c' is defined to be the smallest c such that $\lim_{\hat{\theta}\to\theta_{\min}} U(\hat{\theta}|\hat{\theta},I,c) \leq 0$, $\hat{\theta}(c,I)$ is strictly increasing in $c > c'$. Q.E.D.

Now, let

$$c^*(I) = \int_{x_i \geq x(\hat{\theta})} w \Pr(x_i \geq x_j, \forall j|\theta^*, I)dF(x_i|\theta^*)$$

The next lemma provides the "optimal cost" given I.

Lemma 8. *Given I and the optimal selection rule r^*, $c^*(I)$ maximizes the DM's expected payoff.*

Proof. Observe that $\hat{\theta}(c^*(I), I) = \theta^*$. First, if $c < c^*(I)$, then, from Lemma 7, the probability that the DM selects a candidate with $\theta < \theta^*$ is strictly positive. Second, if $c > c^*(I)$, then, $\theta^* < \hat{\theta}(c, I)$ from Lemma 7. Thus, when all qualified candidates are in $[\theta^*, \hat{\theta}(c, I)]$, the DM fails to select any qualified candidates. If $c = c^*(I)$, the probability that the DM selects a candidate with $\theta < \theta^*$ is 0. Moreover, whenever there are some qualified candidates in the candidate pool, the probability that the DM selects one of them is 1. Q.E.D.

Remark. Since the expected payoff of the cutoff type is decreasing in I, $c^*(I)$ is also decreasing in I.

Now, we are ready to derive the optimal size of candidate pool. The next proposition shows that, when c is controllable, we can easily pin down the optimal size. Let

$$\bar{I}(\delta) = \arg\max_I \{1 - G(\theta^*)^I - \delta I\}.$$

Proposition 3. $I^* = \bar{I}(\delta)$.

Proof. By Lemma 8, the DM chooses $c^*(I)$ given I. Then, for any I, the DM fails to select any qualified candidate only if there is no candidate in the candidate pool. Since the probability of such event is $G(\theta^*)^I$, the expected payoff of the DM from $c^*(I)$ is $1 - G(\theta^*)^I - \delta I$. Thus, the optimal size of candidate pool is $\bar{I}(\delta)$. Q.E.D.

Proposition 3 says that whenever the DM can control both c and I, the optimal size of candidate pool only depends on $G(\theta^*)$ and δ. To see why the optimal size does not depend on other factors such as test technology f and payoff parameter w, recall that, when the DM maximizes his expected payoff, he chooses $c = c^*(I^*)$. Then, $c^*(I^*)$ reflects f and w.

5 Selection power

This section analyzes the relationship between the size of candidate pool and "selection power" of the procedure given cost c and test technology f. We define **selection power** to be the probability that the selected candidate is $\theta \geq \theta^*$ given I, that is, $\Pr(\theta_r \geq \theta^* | r \neq \varnothing, I)$. Then, let $I(\varepsilon)$ be the smallest I in which the selection power is at least $1 - \varepsilon$. Formally,

$$I(\varepsilon) = \begin{cases} \min\{I | 1 - \varepsilon < \Pr(\theta_r \geq \theta^* | r \neq \varnothing, I)\} & \text{if } \{I | 1 - \varepsilon < \Pr(\theta_r \geq \theta^* | r \neq \varnothing, I)\} \neq \emptyset \\ \varnothing & \text{if } \{I | 1 - \varepsilon < \Pr(\theta_r \geq \theta^* | r \neq \varnothing, I)\} = \emptyset \end{cases}.$$

The following proposition provides the properties of $I(\varepsilon)$.

Proposition 4. *Suppose* $(1 + \beta)G(\theta^*) > 1$. *For any* $\varepsilon > 0$, $I\ (\varepsilon) \leq \hat{I}(c) + 1$. *Moreover,* $I(\varepsilon) = \hat{I}(c) + 1$ *for sufficiently small* $\varepsilon > 0$.

Proof. Observe that, if $I > \hat{I}(c)$, all candidates who take the test are $\theta \geq \theta^*$. Hence, whenever the DM selects a candidate, he is $\theta \geq \theta^*$. That is, $\Pr(\theta_r \geq \theta^* | r \neq \varnothing, I)\} = 1$ for any $I > \hat{I}(c)$. Therefore, for any $\varepsilon > 0$, $I(\varepsilon) \leq \hat{I}(c) + 1$. On the other hand, note that, for any $I \leq \hat{I}(c)$, the probability that the DM selects $\theta \in [\hat{\theta}(c, I), \theta^*)$ is positive. Hence, $\Pr(\theta_r \geq \theta^* | r \neq \varnothing, I) < 1$ for any $I \leq \hat{I}(c)$. Thus, for sufficiently small $\varepsilon > 0$, $\Pr(\theta_r \geq \theta^* | r \neq \varnothing, I)\} < 1 - \varepsilon$ for all $I \leq \hat{I}(c)$. Therefore, $I(\varepsilon) = \hat{I}(c) + 1$ for sufficiently small $\varepsilon > 0$. Q.E.D.

In the rest of this section, we compare our selection procedure with a purely statistical selection procedure in terms of $I(\varepsilon)$. A selection procedure is **passive selection procedure** if it is mandatory for candidates to take the test. More preciously, in the passive selection procedure, $z_i = x_i$ for all i. Thus, the signal generating process is exogenous and there is no signaling element in the selection procedure. Let $I_P(\varepsilon)$ be the minimum size of candidate pool for the passive selection procedure to attain $1 - \varepsilon$. To distinguish our procedure to the passive procedure, let us call our procedure "strategic selection procedure."

Proposition 5. *Suppose* $(1 + \beta)G(\theta^*) > 1$.
(i) $I_P(\varepsilon) \geq I(\varepsilon)$.
(ii) There exists $\hat{\varepsilon} < 1$ *such that* $I_P(\hat{\varepsilon}) = \varnothing$ for any $\varepsilon < \hat{\varepsilon}$.

Proof. Let $\Pr(x_i \geq x_j, \forall j|\theta_{\min}, I)$ be $\Pr(x_i \geq x_j, \forall j|\hat{\theta}, I)$ with $\hat{\theta} = \theta_{\min}$ and $x(\theta_{\min})$ be $x(\hat{\theta})$ with $\hat{\theta} = \theta_{\min}$. Then, in the passive procedure, the probability that $r = i$ and $\theta_i \geq \theta^*$ is

$$q_i(I, \theta_{\min}) = \int_{x_i \geq x(\theta_{\min})} \int_{\theta_i \geq \theta^*} \Pr(x_i \geq x_j, \forall j|\theta_{\min}, I) f(x_i|\theta_i) dG(\theta_i) dx_i.$$

Then, $\Pr(\theta_r \geq \theta^* : \theta_{\min}, I) = \sum_i q_i(I, \theta_{\min}) = I q_i(I, \theta_{\min})$.

Turning to the strategic selection procedure, suppose candidate i takes the test. Then, the probability that $r = i$ and $\theta_i \geq \theta^*$ is

$$q_i(I, \hat{\theta}) = \int_{x_i \geq x(\hat{\theta})} \int_{\theta_i \geq \theta^*} \Pr(x_i \geq x_j, \forall j|\hat{\theta}, I) f(x_i|\theta_i) dG(\theta_i|\theta_i \geq \hat{\theta}) dx_i.$$

Then, $\Pr(\theta_r \geq \theta^* : \hat{\theta}, I) = \sum_i q_i(I, \hat{\theta}) = I q_i(I, \hat{\theta})$.

First, it is easy to see that $\Pr(x_i \geq x_j, \forall j|\theta_{\min}, I) \leq \Pr(x_i \geq x_j, \forall j|\hat{\theta}, I)$. Second, $x(\theta_{\min}) \geq x(\hat{\theta})$. Moreover, $G(\theta_i|\theta_i \geq \hat{\theta})$ first-order-stochastically dominates $G(\theta_i)$. Hence, $\Pr(\theta_r \geq \theta^* : \theta_{\min}, I) \leq \Pr(\theta_r \geq \theta^* : \hat{\theta}, I)$. That is, whenever $I' = I_P(\varepsilon)$, $\Pr(\theta_r \geq \theta^* : \hat{\theta}, I') < 1 - \varepsilon$.

(ii) First, by the analogous argument to the proof in Lemma 4, we can show that $\Pr(\theta_r \geq \theta^* : \theta_{\min}, I)$ is increasing in I. Then, since the highest performance converges to $\bar{x}$ as $I \to \infty$, whenever the DM selects a candidate,

$$\lim_{I \to \infty} \Pr(\theta_r > \theta^* : \theta_{\min}, I) = \int_{\theta > \theta^*} \frac{f(\bar{x}|\theta) g(\theta)}{\int_{\theta'} f(\bar{x}|\theta') g(\theta') d\theta'} d\theta < 1.$$

Let $\hat{\varepsilon} = 1 - \lim_{I \to \infty} \Pr(\theta_r > \theta^* : \theta_{\min}, I)$. Then, for any $\varepsilon < \hat{\varepsilon}$, $\{I | 1 - \varepsilon < \Pr(\theta_r > \theta^* : \theta_{\min}, I)\} = \emptyset$. Q.E.D.

An intuition of the result is the following. Since the passive selection procedure has no signaling element, the selection power is restricted by the test technology f. On the other hand, in the strategic selection procedure, since larger number of candidates can make the signaling more informative, the DM can sort out qualified types with higher probability given the same test technology f.

Remark. Note that, if the test is very noisy and the distribution of types has a large mass over $\theta < \theta^*$, we have

$$\int_{\theta \geq \theta^*} \frac{f(\bar{x}|\theta) g(\theta)}{\int_{\theta'} f(\bar{x}|\theta') g(\theta') d\theta'} d\theta < \beta \int_{\theta < \theta^*} \frac{f(\bar{x}|\theta) g(\theta)}{\int_{\theta'} f(\bar{x}|\theta') g(\theta') d\theta'} d\theta.$$

In this case, even if there is a candidate with $\bar{x}$, the DM rejects all candidates in the passive selection procedure, i.e., $I_P(\varepsilon) = \varnothing$ for any $\varepsilon < 1$. On the other hand, the strategic selection procedure can select a qualified candidate with a high probability in the same environment as long as the size of candidate pool is sufficiently large.

Remark. Whenever testing equilibrium does not exist, i.e., $(1+\beta)G(\theta^*) < 1$ and I is small, the DM has to make a decision based on the prior probability. In this case, the passive selection procedure, which utilizes test performances, can outperform the strategic selection procedure. Hence, when δ is small, the DM may prefer the passive procedure to the strategic procedure.

6 Summary

This section summarizes the main results.

1. Whenever a testing equilibrium exists, the optimal selection rule given c and I is

$$r(z_1, z_2, .., z_I) = \begin{cases} i \text{ if } x_i \geq x_j \text{ for any } j \text{ and } x_i \geq x(\hat{\theta}(c, I)) \\ \varnothing \text{ if, for any } j \in \mathcal{I},\ x_j < x(\hat{\theta}(c, I)) \end{cases}$$

   where $\hat{\theta}(c, I)$ is the equilibrium cutoff type.

2. Suppose $(1+\beta)G(\theta^*) > 1$.

   (a) If c is not controllable and δ is sufficiently small, $I^*(\delta) = \hat{I}(c)$ or $\widetilde{I}(\delta, c)$. Moreover, if β is large, then $I^*(\delta) = \widetilde{I}(\delta, c)$.

   (b) If c is controllable, $I^* = \bar{I}(\delta)$ and $c^* = c^*(\bar{I}(\delta))$

3. Suppose $(1+\beta)G(\theta^*) \leq 1$. If I and δ are sufficiently small, the DM can be better off by making the test "mandatory."

7 Concluding remarks

This paper extends selection problem to a strategic environment and analyzes the property of the optimal selection procedure. The motivation of our paper is mainly rooted in the practice. Unlike usual screening models which aim "perfect revelation" of each type, the selection procedure is designed to sort out "competent" types. On the other hand, unlike screening which requires a flexible environment to design contract, the selection procedure can be applied to environments where the set of feasible contracts is quite restricted, e.g., predetermined wage and task. In other words, the approach of our paper is to formulate a "milder" problem and develop a simple procedure which can be applicable to wider range of environments.

There is another advantage of selection procedures. In screening models, it is assumed that the agent knows own "type." However, when the type is "ability," such assumption can be too strong since people are often overconfident about own ability. When the DM's interest is not in selecting a "confident" candidate but selecting a "qualified" candidate, it is important to employ testing, which reflects true ability, in the selection process. In fact, many recruiting and admission processes in the real world are based on testing or past performances.

One possible future direction of "strategic selection procedure" is to explore various kinds of signal generating process. For example, the testing can be sequential rather than simultaneous. That is, each candidate sequentially decides whether to take the test given the history of other candidates' test performances. In this way, the DM may save some cost for testing. On the other hand, it is not obvious that the DM can extract more private information from such sequential procedure.

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