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DELTA AND GAMMA HEDGING OF MORTALITY AND INTEREST RATE RISK

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Delta and Gamma hedging of mortality and interest rate risk

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Abstract

This paper studies the hedging problem of life insurance policies, when the mortality and interest rates are stochastic. We focus primarily on stochastic mortality. We represent death arrival as the first jump time of a doubly stochastic process, i.e. a jump process with stochastic intensity. We propose a Delta-Gamma Hedging technique for mortality risk in this context. The risk factor against which to hedge is the difference between the actual mortality intensity in the future and its "forecast" today, the instantaneous forward intensity. We specialize the hedging technique first to the case in which survival intensities are affine, then to Ornstein-Uhlenbeck and Feller processes, providing actuarial justifications for this restriction. We show that, without imposing no arbitrage, we can get equivalent probability measures under which the HJM condition for no arbitrage is satisfied. Last, we extend our results to the presence of both interest rate and mortality risk, when the forward interest rate follows a constant-parameter Hull and White process. We provide a UK calibrated example of Delta and Gamma Hedging of both mortality and interest rate risk.

1 Introduction

This paper studies the hedging problem of life insurance policies, when the mortality rate is stochastic. In recent years, the literature has focused on the stochastic modeling of mortality rates, in order to deal with unexpected

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changes in the longevity of the sample of policyholders of insurance companies. This kind of risk, due to the stochastic nature of death intensities, is referred to as systematic mortality risk[1]. In the present paper we deal with this, as well as with two other sources of risk life policies are subject to: financial risk and non-systematic mortality risk. The former originates from the stochastic nature of interest rates. The latter is connected to the randomness in the occurrence of death in the sample of insured people and disappears in well diversified portfolios.

The problem of hedging life insurance liabilities in the presence of systematic mortality risk has attracted much attention in recent years. It has been addressed either via risk-minimizing and mean-variance indifference hedging strategies, or through the creation of mortality-linked derivatives and securitization. The first approach has been taken by Dahl and Møller (2006) and Barbarin (2008). The second approach was discussed by Dahl (2004) and Cairns, Blake, Dowd, and MacMinn (2006) and has witnessed a lively debate and a number of recent improvements, see f.i. Blake, De Waegenaere, MacMinn, and Nijman (2010) and references therein.

We study Delta and Gamma hedging. This requires choosing a specific change of measure, but has two main advantages with respect to risk-minimizing and mean-variance indifference strategies. On the one side it represents systematic mortality risk in a very intuitive way, namely as the difference between the actual mortality intensity in the future and its "forecast" today. On the other side, Delta and Gamma hedging is easily implementable and adaptable to self-financing constraints. It indeed ends up in solving a linear system of equations. The comparison with securitization works as follows. The Delta and Gamma hedging complements the securitization approach strongly supported by most academics and industry leaders, in two senses. On the one hand, as is known, the change of measure issue on which hedging relies will not be such an issue any more once the insurance market, thanks to securitization and derivatives, becomes liquid. On the other hand, securitization aims at one-to-one hedging or replication, while we push hedging one step further, through local, but less costly, coverage.

Following a well established stream of actuarial literature, we adapt the setting of risk-neutral interest rate modelling to represent stochastic mortality. We represent death arrival as the first jump time of a doubly stochastic process. To enhance analytical tractability, we assume a pure diffusion of the affine type for the spot mortality intensity. Namely, the process has linear affine drift and instantaneous variance-covariance matrix linear in the intensity itself.

[1] In this paper we do not distinguish between mortality and longevity risk.

In this setting, Cairns, Blake, and Dowd (2006) point out that the HJM no arbitrage condition typical of the financial market can be translated into an equivalent HJM-like condition for forward death intensities. Usually, the respect of the HJM condition on the insurance market is imposed a priori. We show that, for two non-mean reverting processes for the spot intensity, whose appropriateness will be discussed below, there exists an infinity of probability measures – equivalent to the historical one – in which forward death intensities satisfy an HJM condition. No arbitrage holds under any of these measures, even though it is not imposed a priori. These processes belong to the Ornstein Uhlenbeck and the Feller class.
As a consequence, we start by introducing the spot mortality intensities, discuss their soundness as descriptors of the actual – or historical – mortality dynamics, derive the corresponding forward death intensities and tackle the change of measure issue. Among the possible changes, we select the minimal one – which permits to remain in the Ornstein-Uhlenbeck and Feller class – and parameterize it by assuming that the risk premium for mortality risk is constant. By so doing, we can avoid using risk minimizing or mean-variance indifference strategies. We can instead focus on Delta and Gamma hedging. For the sake of simplicity we assume that the market of interest rate bonds is not only arbitrage-free but also complete. First, we consider a pure endowment hedge in the presence of systematic mortality risk only. Then, under independence of mortality and financial risks, we provide an extension of the hedging strategy to both these risks.
To keep the treatment simple, we build Delta and Gamma coverage on pure endowments, using as hedging tools either pure endowments or zero-coupon survival bonds for mortality risk and zero-coupon-bonds for interest rate risk. Since all these assets can be understood as Arrow-Debreu securities – or building blocks – in the insurance and fixed income market, the Delta and Gamma hedge could be extended to more complex and realistic insurance and finance contracts.
In spite of our restriction to pure endowments, the final calibration of the strategies – which uses UK mortality rates for the male generation born in 1945 and the Hull-White interest rates on the UK market – shows that

1. the unhedged effect of a sudden change on mortality rate is remarkable, especially for long time horizons;

2. the corresponding Deltas and Gammas are quite different if one takes into consideration or ignores the stochastic nature of the death intensity:

3. the hedging strategies are easy to implement and customize to self-

financing constraints;

4. Delta and Gamma are bigger for mortality than for financial risk.

The paper is structured as follows: Section 2 recalls the doubly stochastic approach to mortality modelling and introduces the two intensity processese considered in the paper. Section 3 presents the notion of forward death intensity. Section 4 describes the standard financial assumptions on the market for interest rates. Section 5 derives the dynamics of forward intensities and survival probabilities, after the appropriate change of measure. Section 6 shows that the HJM restriction is satisfied without imposing no arbitrage a priori. In Section 7 we discuss the hedging technique for mortality risk. Section 8 addresses mortality and financial risk. Section 9 presents the application to a UK population sample. Section 10 summarizes and concludes.

2 Cox modelling of mortality risk

This Section introduces mortality modelling by specifying the so-called spot mortality intensity (mortality intensity for short). Section 2.1 describes the general framework, while Section 2.2 studies two specific processes which will be considered troughout the paper.

2.1 Instantaneous death intensity

Mortality in the actuarial literature has been recently described by means of Cox or doubly stochastic counting processes, as studied by Brémaud (1981). The modelling technique has been drawn from the financial domain and in particular from the reduced form models of the credit risk literature, where the time to default is described as the first stopping time of a Cox process [2]. In the actuarial literature, mortality modelling via Cox processes has been introduced by Milevsky and Promislow (2001) and Dahl (2004). Intuitively, the time to death - analogously to the time to default in finance - is supposed to be a Poisson process with stochastic intensity. The intensity process may be either a pure diffusion or may present jumps. If in addition it is an affine process, then the survival function can be derived in closed form.
Let us introduce a filtered probability space $(\Omega, \mathbf{F}, \mathbb{P})$, equipped with a filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ which satisfies the usual properties of right-continuity and completeness. On this space, let us consider a non negative, predictable process λ_x, which represents the mortality intensity of an individual or head belonging to generation x at (calendar) time t. We introduce the following

[2]See the seminal paper Lando (1998).

Assumption 1 The mortality intensity λ_x follows a process of the type:

$$d\lambda_x(t) = a(t, \lambda_x(t))dt + \sigma(t, \lambda_x(t))dW_x(t) + dJ_x(t) \tag{1}$$

where J is a pure jump process, W_x is a standard one-dimensional Brownian motion[3] and the regularity properties for ensuring the existence of a strong solution of equation (1) are satisfied for any given initial condition $\lambda_x(0) = \lambda_0 > 0$.

The existence of a stochastic mortality intensity generates systematic mortality risk. Given this assumption on the dynamics of the death intensity, let τ be the time to death of an individual of generation x. We define the survival probability from t to $T \geqslant t$, $S_x(t, T)$, as the survival function of the time to death τ under the probability measure $\mathbb{P}$, conditional on the survival up to time t:

$$S_x(t, T) := \mathbb{P}\left(\tau \geqslant T \mid \tau > t\right)$$

It is known since Brémaud (1981) that - under the previous assumption - the survival probability $S_x(t, T)$ can be represented as

$$S_x(t, T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) \mid \mathcal{F}_t\right] \tag{2}$$

where the expectation is computed under $\mathbb{P}$ and is evidently conditional on $\mathcal{F}_t$. When the evaluation date is zero ($t = 0$), we simply write $S_x(T)$ instead of $S_x(0, T)$.

In this paper, we suppose in addition that

Assumption 2 the drift $a(t, \lambda(t))$, the instantaneous variance-covariance coefficient $\sigma^2(t, \lambda(t))$ and the jump measure η associated with J, which takes values in $\mathbb{R}^+$, have affine dependence on $\lambda(t)$.

Hence, we assume that these coefficients are of the form:

$$a(t, \lambda(t)) = b + c\lambda(t)$$

$$\sigma^2(t, \lambda(t)) = d \cdot \lambda(t)$$

$$\eta(t, \lambda(t)) = l_0 + l_1\lambda(t)$$

where $b, c, d, l_0, l_1 \in \mathbb{R}$. Under this assumption standard results on functionals of affine processes allow us to state that

$$S_x(t, T) = e^{\alpha(T-t)+\beta(T-t)\lambda_x(t)}$$

[3]The extension of the mortality intensity definition to a multidimensional Brownian motion is straightforward.

where α and β solve the following Riccati differential equations (see for instance Duffie and K.Singleton (2000)):

$$\beta'(t) = \beta(t)c + \frac{1}{2}\beta(t)^2 d^2 + l_1 \left[\int_{\mathbb{R}} e^{\beta(t)z} d\nu(z) - 1 \right]$$
$$\alpha'(t) = \beta(t)b + l_0 \left[\int_{\mathbb{R}} e^{\beta(t)z} d\nu(z) - 1 \right]$$

where ν is the distribution function of the jumps of J. The boundary conditions are $\alpha(0) = 0$ and $\beta(0) = 0$.

2.2 Ornstein-Uhlenbeck and Feller processes

In this paper we focus on two intensity processes, which belong to the affine class and are purely diffusive. These processes, together with the solutions α and β of the associated Riccati ODEs, are:

— Ornstein-Uhlenbeck (OU) process without mean reversion:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma dW_x(t) \tag{3}$$
$$\alpha(t) = \frac{\sigma^2}{2a^2}t - \frac{\sigma^2}{a^3}e^{at} + \frac{\sigma^2}{4a^3}e^{2at} + \frac{3\sigma^2}{4a^3} \tag{4}$$
$$\beta(t) = \frac{1}{a}(1 - e^{at}) \tag{5}$$

— Feller Process (FEL) without mean reversion:

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dW_x(t) \tag{6}$$
$$\alpha(t) = 0 \tag{7}$$
$$\beta(t) = \frac{1 - e^{bt}}{c + de^{bt}} \tag{8}$$

with $b = -\sqrt{a^2 + 2\sigma^2}, c = \frac{b+a}{2}, d = \frac{b-a}{2}$. Here, we assume $a > 0, \sigma \geq 0$.

A process, in order to describe human survivorship realistically, has to be "biologically reasonable", i.e. it has to satisfy two technical features: the intensity must never be negative and the survival function has to be decreasing in time T.
In the OU case, λ can indeed turn negative, with positive probability:

$$u = \mathbb{P}(\lambda(t) \leqslant 0) = \phi\left(-\frac{\lambda(0)e^{at}}{\sigma\sqrt{\frac{e^{2at}-1}{2a}}} \right)$$

where ϕ is the distribution function of the standard normal. The survival function is always decreasing when time T is below a certain level T^*:

$$T < T^* = \frac{1}{a} ln \left[1 + \frac{a^2\lambda(0)}{\sigma^2} \left(1 + \sqrt{1 + \frac{2\sigma^2}{a^2\lambda(0)}} \right) \right] \tag{9}$$

In practical applications (section 9) we verify that the probability u is negligible and that the length of the time horizon we consider (the duration of a human life) never exceeds T^*.
For the FEL process, instead, the intensity can never turn negative and the survival function is guaranteed to be decreasing in T if and only if the following condition holds:

$$e^{bt}(\sigma^2 + 2d^2) > \sigma^2 - 2dc. \tag{10}$$

We verify this condition, which is satisfied whenever $\sigma^2 - 2dc < 0$, for our calibrated parameters (see section 9).

In spite of the technical restrictions, Luciano and Vigna (2008) and Luciano, Spreeuw, and Vigna (2008) suggest the appropriateness of these processes for describing the intensity of human mortality. In fact, they show that these models meet all but one of the criteria - motivated by Cairns, Blake, and Dowd (2006) - that a good mortality model should meet:

1. the model should be consistent with historical data: the calibrations of Luciano and Vigna (2008) show that the models meet this criterium;

2. the force of mortality should keep positive: the first model does not meet this criterium; however, the probability of negative values of the intensity is shown to be negligible for practical applications;

3. long-term future dynamics of the model should be biologically reasonable: the models meet this criterium, as the calibrated parameters satisfy conditions (9) and (10) above;

4. long-term deviations in mortality improvements should not be mean-reverting to a pre-determined target, even if the target is time-dependent: the models meet this criterium by construction;

5. the model should be comprehensive enough to deal appropriately with pricing valuation and hedging problem: these models meet this criterium, since it is straightforward to extend them in order to deal with pricing, valuation and hedging problems; this is indeed the scope of the present paper;

6. it should be possible to value mortality linked derivatives using analytical methods or fast numerical methods: these models meet this criterium, as they produce survival probabilities in closed form and with a very small number of parameters.

Cairns, Blake, and Dowd (2006) add that no one of the previous criteria dominates the others. Consistently with their view, we claim the validity of the proposed models, which meet five criteria out of six. The violation of the second criterium above in the OU case is the price paid in order to have a simple and parsimonious model. Notice though that this is only a theoretical limit of the model, as a negative force of mortality has a negligible probability of occurring in practical applications. In addition, the fact that survival functions are given in closed form and depend on a very small number of parameters simplifies the calibration procedure enormously. Last but not least, these two processes are natural stochastic generalizations of the Gompertz model for the force of mortality and, thus, they are easy to interpret in the light of the traditional actuarial practice.
These processes (and especially the first one, the OU) turn out to be significantly suitable for the points 5 and 6 above. In fact, in Sections 6, 7 and 8 we will show that the Delta and Gamma OU-coefficients can be expressed in a very simple closed form. Thus, the Delta-Gamma Hedging technique – widely used in the financial context to hedge purely financial assets – turns out to be remarkably easy to apply. This feature renders quite applicable this hedging technique also in the actuarial-financial context. The Delta and Gamma FEL coefficients are more complicated to find, but the technique is still applicable.

3 Forward death intensities

This Section aims at shifting from mortality intensities to their forward counterparts, both for the general affine case and for the OU and FEL processes. The notion of forward instantaneous intensity for counting processes representing firm defaults has been introduced by Duffie (1998) and Duffie and Singleton (1999), following a discrete-time definition in Litterman and Iben (1991). Stochastic modelling of this quantity has been extensively studied in the financial domain. In the credit risk domain indeed the notion of forward intensity is very helpful, since it allows to determine the change of measure or the intensity dynamics useful for pricing and hedging defaultable bonds (the characterization is obtained under a no arbitrage assumption for the

financial market and is unique when the market is also complete).
Suppose that arbitrages are ruled out, that the recovery rate is null and $\lambda_x(t)$ in (1) represents the default intensity of a firm whose debt is traded in a complete market. Then, we would have the following HJM restriction under the (unique) risk-neutral measure corresponding to $\mathbb{P}$:

$$a(t, \lambda_x(t)) = \sigma(t, \lambda_x(t)) \int_0^t \sigma(u, \lambda_x(u)) du \tag{11}$$

In the actuarial domain, forward death intensities have already been introduced by Dahl (2004) and Cairns, Blake, and Dowd (2006), paralleling the financial definition. In section 5 we prove that, even though the restriction (11) can be violated by death intensities in general, it holds true for the OU and FEL intensity processes, even without imposing no arbitrage, but simply restricting the measure change so that the intensity remains OU or FEL under the new measures.

Let us start from the forward death rate over the period $(t, t+\Delta t)$, evaluated at time zero, as the ratio between the conditional probability of death between t and $t+\Delta t$ and the time span Δt, for a head belonging to generation x, conditional on the event of survival until time t:

$$\frac{1}{\Delta t}\left(\frac{S_x(t) - S_x(t+\Delta t)}{S_x(t)}\right)$$

Let us consider its instantaneous version, which we denote as $f_x(0,t)$. We refer to it as to forward death intensity. It is evident from its definition that - if it exists - the forward death intensity is the logarithmic derivative of the (unconditional) survival probability, as implied by the process λ:

$$f_x(0,t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(1 - \frac{S_x(t+\Delta t)}{S_x(t)}\right) = -\frac{\partial}{\partial t} \ln\left(S_x(t)\right)$$

The similarity of this definition with the force of mortality is quite strong.[4]
Similarly, one can define the forward death intensity for the tenor T, as evaluated at time $t < T$, starting from the survival probability $S_x(t,T)$:

$$f_x(t,T) = -\frac{\partial}{\partial T} \ln\left(S_x(t,T)\right) \tag{12}$$

The forward death intensity $f_x(t,T)$ represents the intensity of mortality which will apply instantaneously at time $T > t$, implied by the knowledge

[4]The two concepts coincide when the diffusion coefficient of the intensity process is null.

of the process λ up to t (or under the filtration $\mathcal{F}_t$). This explains the dependence of f_x on the current date t as well as on the future one, T. It can be interpreted as the "best forecast" of the actual mortality intensity, since it coincides with the latter when $T = t$:

$$f_x(t,t) = \lambda_x(t)$$

Please notice also that the forward death intensity definition, and consequently its expression for the affine case, is analogous to the one of forward instantaneous interest rates, the latter being defined starting from discount factors rather than survival probabilities. As in the case of forward instantaneous interest rates, it can be shown that forward intensities, for given t, can be increasing, decreasing or humped functions of the application date T. It follows from the above definition that the survival probabilities from t to $T > t$ can be written as integrals of (deterministic) forward death probabilities:

$$S_x(t,T) = \exp\left(-\int_t^T f_x(t,s)ds\right) \tag{13}$$

and not only as expectations wrt the intensity process λ , as in (2) above.[5]

Let us turn now to the affine case. As it can be easily shown from (13), when λ is an affine process the initial forward intensity depends on the functions α and β:

$$f_x(0,t) = -\alpha'(t) - \beta'(t)\lambda_x(0) = -\alpha'(t) - \beta'(t)f_x(0,0) \tag{14}$$

and at any time $t \geq T \geq 0$:

$$f_x(t,T) = -\alpha'(T-t) - \beta'(T-t)\lambda_x(t) = -\alpha'(T-t) - \beta'(T-t)f_x(t,t)$$

For the processes defined by equations (3) and (6), the instantaneous forward intensities can be computed as:

$$OU \qquad f_x(t,T) = \lambda_x(t)e^{a(T-t)} - \frac{\sigma^2}{2a^2}(e^{a(T-t)} - 1)^2 \tag{15}$$

[5] Notice that, at any initial time t, forward death intensities can be interpreted as the (inhomogeneous) Poisson arrival rates implied in the current Cox process. Indeed, it is quite natural, especially if one wants a description of survivorship without the mathematical complexity of Cox processes, to try to describe mortality via the equivalent survival probability in a simpler (inhomogeneous) Poisson model. Once a λ process has been fixed, and therefore survival probabilities have been computed, according to (2), one can wonder: what would be the intensity of an inhomogeneous Poisson death arrival process, that would produce the same survival probabilities? Recalling that in the Poisson case survival probabilities are of the type (13), one can interpret – and use – $f(t,T)$ exactly as the survival intensity of an (inhomogeneous) Poisson model equivalent to the given, Cox one.

$$FEL \qquad f_x(t,T) = \frac{4\lambda_x(t)b^2 e^{b(T-t)}}{[(a+b)+(b-a)e^{b(T-t)}]^2}$$

4 Financial risk

In order to introduce a valuation framework for insurance policies, we need to provide provide a description of the financial environment. In addition to mortality risk, we assume the existence of a financial risk, in the sense that the interest rate is described by a stochastic process. While in the mortality domain we started from (spot) intensities – for which we were able to motivate specific modelling choices – and then we went to their forward counterpart, here we follow a well established bulk of literature – starting from Heath, Jarrow, and Morton (1992) – and model directly the instantaneous forward rate $F(t,T)$, i.e. the date-t rate which applies instantaneously at T .

Assumption 3 The process for the forward interest rate $F(t,T)$, defined on the probability space $(\Omega, \mathbf{F}, \mathbb{P})$, is:

$$dF(t,T) = A(t,T)dt + \Sigma(t,T)dW_F(t) \tag{16}$$

where the real functions $A(t,T)$ and $\Sigma(t,T)$ satisfy the usual assumptions for the existence of a strong solution to (16), and W_F is a univariate Brownian motion[6] independent of W_x for all x.

The independence between the Brownian motions means, loosely speaking, independence between mortality and financial risk.[7]
Let us also denote as $\{\mathcal{H}_t : 0 \le t \le T\}$ the filtration generated by the interest rate process. As a particular subcase of the forward rate, obtained when $t = T$, one obtains the short rate process, which we will denote as $r(t)$:

$$F(t,t) := r(t) \tag{17}$$

It is known that, when the market is assumed to admit no arbitrages and be complete, there is a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ - which we will characterize in the next section - under which the zero-coupon-bond price for the maturity T, evaluated at time t, $B(t,T)$, is

$$B(t,T) = \exp\left(-\int_t^T F(t,u)du\right) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)\right] \tag{18}$$

[6]We assume a single Brownian motion for the forward rate dynamics, since we reduced the discussion of mortality risk to a single risk source too: however, the extension to a multidimensional Brownian motion is immediate.

[7]This assumption is common in the literature and seems to be intuitively appropriate. See Miltersen and Persson (2006) for a setting in which mortality and financial risks can be correlated.

We will provide a specific choice for the forward interest rate only at a later stage. We will have no need to motivate it, since it corresponds to a very popular model in Finance, the one-factor Hull and White (Hull and White (1990)).

5 Change of measure and insurance valuations

This section discusses the change of measure that allows us to compute the prices of policies subject to mortality risk in a fashion analogous to (18). First, we define the process of death occurrence inside the sample of insured people of interest. As in Dahl and Møller (2006), we represent it as follows. Let $\tau_1, \tau_2, \ldots \tau_N$ be the lifetimes of the N insured in the cohort x, assumed to be i.i.d. with distribution function $S_x(t,T)$ in (2). Let $M(x,t)$ be the (pure jump) process which counts the number of deaths in such an insurance portfolio:

$$M(x,t) := \sum_{i=1}^{N} 1_{\{\tau_i \leq t\}}$$

where 1 is the indicator function. We define a filtration on $(\Omega, \mathbf{F}, \mathbb{P})$ whose σ -algebras $\{\mathcal{G}_t : 0 \leq t \leq T\}$ are generated by $\mathcal{F}_t$ and $\{M(x,s) : 0 \leq s \leq t\}$. This filtration intuitively collects the information on both the past mortality intensity and on actual death occurrence in the portfolio. Let us consider, on the probability space $(\Omega, \mathbf{F}, \mathbb{P})$, the sigma algebras $\mathcal{I}_t := \mathcal{G}_t \vee \mathcal{H}_t$ generated by unions of the type $\mathcal{G}_t \cup \mathcal{H}_t$, where the σ-algebra $\mathcal{G}_t$ collects information on the mortality intensity and actual death process, while $\mathcal{H}_t$, which is independent of $\mathcal{G}_t$, reflects information on the financial market, namely on the forward rate process The filtration $\{\mathcal{I}_t : 0 \leq t \leq T\}$ therefore represents all the available information on both financial and mortality risk. In order to perform insurance policies evaluations in $(\Omega, \mathbf{F}, \mathbb{P})$, equipped with such a filtration, we need to characterize at least one equivalent measure. This can be done using a version of Girsanov's theorem, as in Jacod and Shiryaev (1987)[8]:

Theorem 5.1 *Let the bi-dimensional process* $\theta(t) := [\theta_x(t) \quad \theta_F(t)]$ *and the*

[8]See also Dahl and Møller (2006) for an application to a similar actuarial setting.

univariate, positive one $\varphi(t)$ be predictable, with

$$\begin{aligned} \int_0^T \theta_x^2(t)dt &< \infty, \\ \int_0^T \theta_F^2(t)dt &< \infty, \\ \int_0^T |\varphi(t)| \lambda_x(t)dt &< \infty \end{aligned}$$

Define the likelihood process $L(t)$ by

$$\begin{cases} L(0) = 1 \\ \frac{dL(t)}{L(t-)} = \theta_x(t)dW_x(t) + \theta_F(t)dW_F(t) + (\varphi(t) - 1)\, dM(x,t) \end{cases}$$

and assume $\mathbb{E}^{\mathbb{P}}[L(t)] = 1, t \leq T$. Then there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, such that the restrictions of $\mathbb{P}$ and $\mathbb{Q}$ to $\mathcal{I}_t$, $\mathbb{P}_t := \mathbb{P} \mid \mathcal{I}_t$, $\mathbb{Q}_t := \mathbb{Q} \mid \mathcal{I}_t$, have Radon-Nykodim derivative $L(t)$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L(t)$$

The mortality indicator process has intensity $\varphi(t)\lambda_x(t)$ under $\mathbb{Q}$ and

$$\begin{aligned} dW_x' &: = dW_x - \theta_x(t)dt \\ dW_F' &: = dW_F - \theta_F(t)dt \end{aligned}$$

define $\mathbb{Q}-$Brownian motions. All the probability measures equivalent to $\mathbb{P}$ can be characterized this way.

Actually, the previous theorem characterizes an infinity of equivalent measures, depending on the choices of the processes $\theta_x(t), \theta_F(t)$ and $\varphi(t)$. These processes represent the prices - or premia - given to the three different sources of risk we model.

The first source of risk, the systematic mortality one, is represented by $\theta_x(t)$. This source of risk is not diversifiable, since it originates from the randomness of death intensity. We have no standard choices to apply in the choice of $\theta_x(t)$, see for instance the extensive discussion in Biffis (2005) and Cairns, Blake, Dowd, and MacMinn (2006). For the sake of analytical tractability, as in Dahl and Møller (2006), we restrict it so that the risk-neutral intensity is still affine. Therefore, we substitute Assumptions 1 and 2 with the following

Assumption 4 The intensity process under $\mathbb{P}$ is purely diffusive and affine. The systematic mortality risk premium is such as to leave it affine under $\mathbb{Q}$:

$$\theta_x(t) := \frac{p(t) + q(t)\lambda_x(t)}{\sigma(t, \lambda_x(t))}$$

with $p(t)$ and $q(t)$ continuous functions of time.

Indeed, with such a risk premium, the intensity process under $\mathbb{Q}$ is

$$d\lambda_x(t) = [a(t, \lambda_x(t)) + p(t) + q(t)\lambda_x(t)]\, dt + \sigma(t, \lambda_x(t))dW'_x. \tag{19}$$

which is still affine. This choice boils down to selecting the so-called minimal martingale measure. It can be questioned – as any other choice – but proves to be very helpful for hedging.[9] For the OU and FEL processes we choose the functions $p = 0$ and q constant, so that we have the same type of process under $\mathbb{P}$ and $\mathbb{Q}$, with the coefficient a in equations (3) and (6) replaced by $a' := a + q$.

The second source of risk, the financial one, originates from the stochastic nature of interest rates. The process $\theta_F(t)$ represents the so called premium for financial risk. Assume that the financial market is complete. The only choice consistent with no arbitrage is

$$\theta_F(t) := -A(t, T)\Sigma^{-1}(t, T) + \int_t^T \Sigma(t, u)du$$

Under this premium indeed the drift coefficient of the forward dynamics $A'(t, T)$ is tied to the diffusion by an HJM relationship:

$$A'(t, T) = \Sigma(t, T) \int_t^T \Sigma(t, u)du \tag{20}$$

It follows that, under the measure $\mathbb{Q}$,

$$dF(t, T) = \left[\Sigma(t, T) \int_t^T \Sigma(t, u)du\right] dt + \Sigma(t, T)dW'_F(t) \tag{21}$$

The time-t values of the forward and short rate are respectively (see f.i. Shreve (2004)):

$$F(t, T) = F(0, T) + \int_0^t \Sigma(s, T) \int_s^T \Sigma(s, m)dmds + \int_0^t \Sigma(u, T)dW'_F(u) \tag{22}$$

[9]Its calibration will be straightforward, as soon as the market for mortality derivatives becomes liquid enough.

$$r(t) = F(0,t) + \int_0^t \Sigma(s,T) \int_s^t \Sigma(s,m) dm ds + \int_0^t \Sigma(u,t) dW'_F(u)$$

where

$$\begin{aligned} dW'_F &= dW_F - \theta_F(t)dt \text{ and} \\ \theta_F(t) &= -A(t,T)\Sigma^{-1}(t,T) + \int_t^T \Sigma(t,u)du. \end{aligned}$$

The third source of risk is the non systematic mortality one, arising from the randomness of death occurrence inside the portfolio of insured people. In the presence of well diversified insurance portfolios, insurance companies are uninterested in hedging this idiosyncratic component of mortality risk, since the law of large numbers is expected to apply. Hence, we assume that the market gives no value to it and we make the following assumption for $\varphi(t)$, which represents the premium for idiosyncratic mortality risk:

Assumption 5 $\varphi(t) = 0$ for every t

The fair premium and the reserves of life insurance policies can be computed as expected values under the measure $\mathbb{Q}$.
Consider the case of a pure endowment contract[10] starting at time 0 and paying one unit of account if the head x is alive at time T. The fair premium or price of such an insurance policy, given the independence between the financial and the actuarial risk, is:

$$P(0,T) = S_x(T)B(0,T) = e^{\alpha(T)+\beta(T)\lambda_x(0)} E_{\mathbb{Q}} \left[-\exp\left(\int_0^T r(u)du \right) \right]$$

The value of the same policy at any future date t is:

$$\begin{aligned} P(t,T) &= S_x(t,T)B(t,T) \\ &= E_{\mathbb{Q}} \left[\exp\left(-\int_t^T \lambda(s)ds \right) \right] E_{\mathbb{Q}} \left[-\exp\left(\int_0^T r(u)du \right) \right] \quad (23) \end{aligned}$$

Hence, we can define a "term structure of pure endowment contracts". The last expression, net of the initial premium, is also the time t reserve for the policy, which the insurance company will be interested in hedging. Notice that we did not impose no arbitrage on the market for these instruments. Once the change of measure has been performed, we can write $P(0,T)$ in

[10]We do this recognizing that more complex policies or annuities can be decomposed into these basic contracts.

terms of the instantaneous forward probability and interest rate (f and F respectively):

$$P(t,T) = \exp\left(- \int_t^T \left[f_x(t,u) + F(t,u) \right] du \right)$$

6 HJM restriction on forward death intensities

In this section we show that, if the risk premium for mortality is constant, then the OU and FEL processes for mortality intensity satisfy an HJM-like restriction on the drift and diffusion. This is important, since proving that the HJM condition holds is equivalent to showing that no arbitrage holds, without having assumed it to start with. We keep the head x fixed, and in the notation we drop the dependence on x.

Forward death intensities, being defined as log derivatives of survival probabilities, follow a stochastic process. This process can be derived starting from the one of the survival probabilities themselves, recalling that the process λ is given by (19). Under Assumption 4, Ito's lemma implies that the functional S follows the process:

$$dS(t,T) = S(t,T)m(t,T)dt + S(t,T)n(t,T)dW'(t)$$

where

$$\begin{aligned} m(t,T) &= \frac{1}{S}\left[\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \lambda}\left[a(t,\lambda) + p(t) + q(t)\lambda(t) \right] + \frac{1}{2}\frac{\partial^2 S}{\partial \lambda^2}\sigma^2(t,\lambda) \right] \\ n(t,T) &= \frac{1}{S}\frac{\partial S}{\partial \lambda}\sigma(t,\lambda) \end{aligned}$$

The forward death intensity $f(t,T)$, defined as the logarithmic derivative of $S(t,T)$, can be shown to follow the dynamics:

$$df(t,T) = v(t,T)dt + w(t,T)dW'(t) \tag{24}$$

where the drift and diffusion coefficients are:

$$v(t,T) = \frac{\partial n(t,T)}{\partial T} n(t,T) - \frac{\partial m(t,T)}{\partial T} \tag{25}$$

$$w(t,T) = -\frac{\partial n(t,T)}{\partial T} \tag{26}$$

Since – according to the Assumption 4 – the intensity process is of the affine class, the drift and diffusion of the survival probabilities are

$$\begin{aligned} m(t,T) &= -\alpha'(T-t) - \beta'(T-t)\lambda(t) + \\ &+ \left[a(t,\lambda) + p(t) + q(t)\lambda(t)\right]\beta(T-t) + \frac{1}{2}\sigma^2(t,\lambda)\beta^2(T-t) \\ n(t,T) &= \sigma(t,\lambda)\beta(T-t) \end{aligned}$$

Given that, one can easily derive the forward intensity process coefficients:

$$\begin{aligned} v(t,T) &= \alpha''(T-t) + \beta''(T-t)\lambda(t) - \left[a(t,\lambda) + p(t) + q(t)\lambda\right]\beta'(T-t) \\ w(t,T) &= -\sigma(t,\lambda)\beta'(T-t) \end{aligned}$$

In general, the forward dynamics then depends on the drift and diffusion coefficients of the mortality intensity and on the properties of the solutions of the Riccati equations. One can wonder whether - starting from a mortality intensity process - an HJM-like condition, which works on the forward survival intensities,

$$v(t,T) = w(t,T)\int_t^T w(t,s)ds \tag{27}$$

is satisfied. We provide the following:

Theorem 6.1 *Let λ be a purely diffusive process which satisfies Assumption 4. Then, the HJM condition (27) is satisfied if and only if:*

$$\frac{\partial m(t,T)}{\partial T} = n(t,t)\frac{\partial n(t,T)}{\partial T}.$$

In particular, this condition is satisfied in the cases of the Ornstein-Uhlenbeck process (3) and of the Feller process (6) with $p = 0$ and q constant.

Proof.

Using (26), we get the r.h.s. of the HJM condition (27):

$$w(t,T)\int_t^T w(t,s)ds = \frac{\partial n(t,T)}{\partial T}(n(t,T) - n(t,t)).$$

Hence, plugging (25) into the HJM condition (27) we get [11]:

$$\frac{\partial m(t,T)}{\partial T} = n(t,t)\frac{\partial n(t,T)}{\partial T}.$$

[11]Notice that a similar condition on the drift and diffusion of spot interest rates is in Shreve (2004).

As for the second part, if the intensity follows an OU process, the forward probability f satisfies the HJM condition (27). This result is a straightforward consequence of the fact that – with $p = 0$ and q constant – the functions α and β of the OU process satisfy the system of ODEs':

$$\begin{cases} \beta'(t) = -1 + a'\beta(t) \\ \alpha'(t) = \frac{1}{2}\sigma^2\beta^2(t) \end{cases} \tag{28}$$

with the boundary conditions $\alpha(0) = 0$ and $\beta(0)$=0. In fact,

$$\begin{aligned} v(t,T) &= \alpha''(T-t) + \beta''(T-t)\lambda(t) - \beta'(T-t)a'\lambda(t) \\ &= \sigma^2\beta(T-t)\beta'(T-t) \\ w(t,T) &= -\sigma\beta'(T-t). \end{aligned}$$

and property (27) is satisfied.

Consider now the Feller process (6) and its well-known solution to the Riccati ODE:

$$\begin{cases} \alpha'(t) = 0 \\ \beta'(t) = -1 + a'\beta(t) + \frac{1}{2}\sigma^2\beta^2(t) \end{cases} \tag{29}$$

Again, we can easily show that condition (27) is satisfied, since

$$v(t,T) = \beta''(T-t)\lambda(t) - a'\lambda(t)\beta'(T-t) = \sigma^2\beta(T-t)\beta'(T-t)\lambda(t)$$

$$w(t,T) = -\sigma(t,\lambda)\beta'(T-t) = -\sigma\sqrt{\lambda(t)}\beta'(T-t).$$

The HJM condition is a characterizing feature of some models for interest rates such as the Vasicek (1977), Hull and White (1990), the CIR (Cox, Ingersoll Jr, and Ross (1985)). It is well known that the HJM condition (27), applied to the coefficients of the interest rate process, as in (20), is equivalent to the absence of arbitrage. In our case, since we showed that - under Assumption 4 - the OU and FEL processes satisfy the HJM condition, arbitrage is ruled out without being imposed. Please notice that the dynamics of the forward intensity under for the OU case $\mathbb{Q}$ is

$$df(t,T) = \frac{\sigma^2}{a'}e^{a'(T-t)}\left(e^{a'(T-t)} - 1\right)dt + \sigma e^{a'(T-t)}dW'(t). \tag{30}$$

It reminds of the Hull and White dynamics for forward interest rates, when the parameters are constant.

7 Mortality risk hedging

In order to study the hedging problem of a portfolio of pure endowment contracts, we assume first that the interest rate is deterministic and, without loss of generality, equal to zero. This allows us to focus in this Section on the hedging of systematic mortality risk only. At a later stage, we will introduce again financial risk (section 8) and study the problem of hedging both mortality and financial risk simultaneously.
Once the risk-neutral measure $\mathbb{Q}$ has been defined, in order to introduce an hedging technique for systematic mortality risk we need to derive the dynamics of the reserve, which represents the value of the policy for the issuer (assuming that the unique premium has already been paid). We do this, for the sake of simplicity, assuming an OU behavior for the intensity.

7.1 Dynamics and sensitivity of the reserve

7.1.1 Affine intensity

Let us integrate (24), to obtain the forward death probability:

$$f(t,T) \;=\; f(0,T) + \int_0^t \left[v(u,T)du + w(u,T)dW'(u)\right] \tag{31}$$

Substituting it into the survival probability (13) and recalling that we write $S(u)$ for $S(0,u)$, we obtain an expression for the future survival probability $S(t,T)$ in terms of the time-zero ones:

$$S(t,T) \;=\; \frac{S(T)}{S(t)}\left[\exp - \int_t^T \int_0^z \left[v(u,T)du + w(u,T)dW'(u)\right] dz\right]$$

Considering the expressions for v and w under Assumption 4, we have:

$$\begin{aligned} S(t,T) \;&=\; \frac{S(T)}{S(t)}[\exp - \int_t^T \int_0^z \{\alpha''(T-u) + \beta''(T-u)\lambda(u) + \\ &-\; \beta'(T-u)\left[a(u,\lambda) + p(u) + q(u)\lambda(u)\right]]du + \\ &-\; \beta'(T-u)\sigma(u,\lambda)dW'(u)]\}dz]. \end{aligned}$$

7.1.2 OU and FEL intensities

We focus now on the OU intensity case. We derive the expression for the forward survival probabilities integrating the dynamics (30):

$$\begin{aligned} f(t,T) &= f(0,T) + \frac{\sigma^2}{a'^2}\{e^{2a'T}\left[e^{-2a't}-1\right] - 2e^{a'T}\left[e^{-a't}-1\right]\} \\ &+ \sigma\int_0^t e^{a'(T-s)}dW(s) \end{aligned}$$

Hence, the reserve can be written simply as

$$P(t,T) = S(t,T) = \frac{S(T)}{S(t)}\exp\left[-X(t,T)I(t) - Y(t,T)\right] \tag{32}$$

where[12]

$$X(t,T) = \frac{\exp(a'(T-t))-1}{a'}$$
$$Y(t,T) = -\sigma^2[1-e^{2a'(T-t)}]X(t,T)^2/(4a')$$
$$I(t) := \lambda(t) - f(0,t)$$

We have therefore provided an expression for the future survival probabilities - and reserves - in terms of deterministic quantities (X,Y) and of a stochastic term $I(t)$, defined as the difference between the actual mortality intensity at time t and its forecast today $f(0,t)$. $I(t)$, therefore, represents the systematic mortality risk factor. Let us notice that, as in the corresponding bond expressions of HJM, the risk factor is unique for all the survival probabilities from t onwards, no matter which horizon $T-t$ they cover. Moreover, as Cairns, Blake, and Dowd (2006) point out, if we extend our framework across generations and model the risk factor as an n dimensional Brownian motion, we obtain that the HJM condition is satisfied for each cohort. Applying Ito's lemma to the survival probabilities, considered as functions of time and the risk factor, we have:

$$dPdS \simeq \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial I}dI + \frac{1}{2}\frac{\partial^2 S}{\partial I^2}(dI)^2$$

It follows that the hedging coefficients for mortality risk are

$$\frac{\partial S}{\partial I} = -S(t,T)X(t,T) \leq 0 \tag{33}$$

$$\frac{\partial^2 S}{\partial I^2} = S(t,T)X^2(t,T) \geq 0 \tag{34}$$

[12]Notice that $-X(t,T) = \beta$ as soon as $a = a'$.

or, for given t,

$$\frac{dP(t,T)}{P(t,T)} \simeq -X(t,T)dI + \frac{1}{2}X(t,T)^2(dI)^2$$

We denote (33) as Delta(Δ^M) and (34) Gamma (Γ^M), where the superscript M indicates that the coefficient refers to mortality risk. These factors allow us to hedge mortality risk up to first and second order effects. They are the analogous of the duration and convexity terms in classical financial hedging of zero-coupon-bonds, and they actually collapse into them when $\sigma(t,\lambda) = \sigma = 0$. In this case, in which mortality has no systematic risk component, we have:

$$Y(t,T) = 0$$

Hence, Delta and Gamma are functions of a' only, as in the deterministic case. We have

$$\Delta^{\sigma=0} = \frac{S(T)}{S(t)}X(t,T)$$

$$\Gamma^{\sigma=0} = \frac{S(T)}{S(t)}X^2(t,T)$$

It is straightforward to compute the sensitivity of any pure endowment policy portfolio with respect to mortality risk (evidently, this must be done for each generation separately). If the portfolio, valued Π, is made up of n_i policies with maturity T_i, $i = 1,..n$, each one with value $S(t, T_i)$, we have

$$d\Pi = \sum n_i dS(t,T_i) =$$

$$\sum_{i=1}^{n} n_i \frac{\partial S}{\partial t}dt + \sum_{i=1}^{n} n_i \frac{\partial S}{\partial I}dI + \frac{1}{2}\sum_{i=1}^{n} n_i^2 \frac{\partial^2 S}{\partial I^2}(dI)^2$$

7.2 Hedging

In order to hedge the reserve we have derived in the previous section we assume that the insurer can use either other pure endowments – with different maturities – or zero-coupon longevity bonds on the same generation [13]. Since we did not price idiosyncratic mortality risk, the price/value of a zero-coupon longevity bond is indeed equal to the pure endowment one. The difference, from the standpoint of an insurance company, is that it can sell endowments – or reduce its exposure through reinsurance – and buy longevity bonds,

[13] If there is no longevity bond for a specific generation, basis risk arises: see for instance Cairns, Blake, Dowd, and MacMinn (2006).

while, at least in principle, it cannot do the converse[14]. We could use a number of other instruments to cover the initial pure endowment, starting from life assurances or death bonds, which pay the benefit in case of death of the insured individual. We restrict the attention to pure endowments and longevity bonds for the sake of simplicity. Let us recall also that – together with the life assurance and death bonds – they represent the Arrow Debreu securities of the insurance market. Once hedging is provided for them, it can be extended to every more complicated instrument.

Suppose for instance that, in order to hedge n endowments with maturity T, it is possible to choose the number of endowments/longevity bonds with maturity T_1 and T_2: call them n_1 and n_2. The value of a portfolio made up of the three assets is

$$\Pi(t) = nS(t,T) + n_1 S(t,T_1) + n_2 S(t,T_2).$$

Its Delta and Gamma are respectively

$$\begin{aligned} \Delta_{\Pi}^{M}(t) = \quad & n\frac{\partial S}{\partial I}(t,T) + n_1\frac{\partial S}{\partial I}(t,T_1) + n_2\frac{\partial S}{\partial I}(t,T_2) \\ \Gamma_{\Pi}^{M}(t) = \quad & n\frac{\partial^2 S}{\partial I^2}(t,T) + n_1\frac{\partial^2 S}{\partial I^2}(t,T_1) + n_2\frac{\partial^2 S}{\partial I^2}(t,T_2) \end{aligned}$$

We can set these Delta and Gamma coefficients to zero (or some other precise value) by adjusting the quantities n_1 and n_2. One can easily solve the system of two equations in two unknowns and obtain hedged portfolios:

$$\begin{cases} \Delta_{\Pi}^{M} = 0 \\ \Gamma_{\Pi}^{M} = 0 \end{cases}$$

Any negative solution for n_i has to be interpreted as an endowment sale, since this leaves the insurer exposed to a liability equal to n times the policy fair price. Any positive solution for n_i has to be interpreted as a longevity bond purchase. The cost of setting up the covered portfolio – which is represented by $\Pi(t)$ – can be paid using the pure endowment premium received by the policyholder. Alternatively, the problem can be extended so as to make the hedged portfolio self-financing. Self-financing can be guaranteed by endogenizing n and solving simultaneously the equations $\Pi = 0, \Delta_{\Pi}^{M} = 0$ and $\Gamma_{\Pi}^{M} = 0$ for n, n_1, n_2. As an alternative, if n is fixed, a third pure endowment/bond with maturity T_3 can be issued or purchased, so that the portfolio

[14] Reinsurance companies have less constraints in this respect. For instance, they can swap pure endowments or issue longevity bonds: see for instance Cowley and Cummins (2005).

made up of $S(t,T), S(t,T_1), S(t,T_2)$ and $S(t,T_3)$ is self-financing and Delta and Gamma hedged. Our application in section 9 will cover both the non self-financing and self-financing possibilities.

8 Mortality and financial risk hedging

Let us consider now the case in which both mortality and financial risk exist. Again we develop the technique assuming a OU intensity. We also select a constant-parameter Hull and White model for the interest rate under the risk-neutral measure:

$$\Sigma(t,T) = \Sigma \exp(-g(T-t)) \tag{35}$$

with $\Sigma, g \in \mathbb{R}^+$. Substituting in (21) indeed we have

$$r(t) = F(0,t) + \frac{1}{2}\frac{\Sigma^2}{g^2}(1-e^{-gt})^2 + \Sigma \int_0^t e^{-g(t-s)} dW`_F(s).$$

which allows us to derive an expression for $B(t,T)$ analogous to (32):

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right]$$

where

$$\bar{X}(t,T) := \frac{1-\exp(-g(T-t))}{g}$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g}\left[1-\exp(-2gt)\right]\bar{X}^2(t,T)$$

where K is the financial risk factor, measured by the difference between the short and forward rate:

$$K(t) := r(t) - F(0,t)$$

The pure endowment reserve at time t, according to (5) above, is

$$P(t,T) = \exp\left(-\int_t^T \left[f(t,u) + F(t,u)\right] du\right) = S(t,T)B(t,T)$$

Given the independence stated in Assumption 3, we can apply Ito's lemma and obtain the dynamics of the reserve $P(t,T)$ as

$$\begin{aligned} dP &= BdS + PdB \simeq B\left[\frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial I}dI + \frac{1}{2}\frac{\partial^2 S}{\partial I^2}(dI)^2\right] + \\ &+ S\left[\frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial K}dK + \frac{1}{2}\frac{\partial^2 B}{\partial K^2}(dK)^2\right] \end{aligned}$$

where

$$\frac{\partial B(t,T)}{\partial K} = -B(t,T)\bar{X}(t,T) \leq 0$$
$$\frac{\partial^2 B(t,T)}{\partial K^2} = B(t,T)\bar{X}^2(t,T) \geq 0$$

It follows that, for given t,

$$\frac{dP(t,T)}{P(t,T)} \simeq -X(t,T)dI + \frac{1}{2}X(t,T)^2(dI)^2 - \bar{X}^2(t,T)dK + \frac{1}{2}\bar{X}^2(t,T)(dK)^2$$

Hedging of the reserve is again possible by a proper selection of pure endowment/longevity bond contracts with different maturities and/or zero-coupon-bonds with different maturities. Here we consider the case in which the hedge against mortality and financial risk is obtained either issuing (purchasing) pure endowments (longevity bonds) or using also bonds.
Consider first using mortality linked contracts only. We can see that Delta and Gamma hedging of both the mortality and financial risk of n endowments with maturity T can be obtained via a mix of n_1, n_2, n_3, n_4 endowments/longevity bonds with maturities ranging from T_1 to T_4, by solving simultaneously the following hedging equations:

$$\begin{cases} \Delta_\Pi^M = 0 \\ \Gamma_\Pi^M = 0 \\ \Delta_\Pi^F = 0 \\ \Gamma_\Pi^F = 0 \end{cases} \tag{36}$$

This indeed means solving the system of equations

$$\begin{cases} \Delta_\Pi^M = nBSX + n_1B_1S_1X_1 + n_2B_2S_2X_2 + n_3B_3S_3X_3 + n_4B_4S_4X_4 = 0 \\ \Gamma_\Pi^M = nBSX^2 + n_1B_1S_1X_1^2 + n_2B_2S_2X_2^2 + n_3B_3S_3X_3^2 + n_4B_4S_4X_4^2 = 0 \\ \Delta_\Pi^F = nBS\bar{X} + n_1B_1S_1\bar{X}_1 + n_2B_2S_2\bar{X}_2 + n_3B_3S_3\bar{X}_3 + n_4B_4S_4\bar{X}_4 = 0 \\ \Gamma_\Pi^F = nBS\bar{X}^2 + n_1B_1S_1\bar{X}_1^{\,2} + n_2B_2S_2\bar{X}_2^{\,2} + n_3B_3S_3\bar{X}_3^{\,2} + n_4B_4S_4\bar{X}_4^{\,2} = 0 \end{cases} \tag{37}$$

where B denotes $B(t,T)$ and $B_i, X_i, \bar{X}_i$ denote $B(t,T_i), X(t,T_i), \bar{X}(t,T_i)$ for $i = 1, ..., 4$.

Consider now using both mortality-linked contracts and zero-coupon-bonds. In this case, the hedging equations (36) become:

$$\begin{cases} \Delta_\Pi^M = nBSX + n_1B_1S_1X_1 + n_2B_2S_2X_2 = 0 \\ \Gamma_\Pi^M = nBSX^2 + n_1B_1S_1X_1^2 + n_2B_2S_2X_2^2 = 0 \\ \Delta_\Pi^F = nBS\bar{X} + n_1B_1S_1\bar{X}_1 + n_2B_2S_2\bar{X}_2 + n_3B_3\bar{X}_3 + n_4B_4\bar{X}_4 = 0 \\ \Gamma_\Pi^F = nBS\bar{X}^2 + n_1B_1S_1\bar{X}_1^{\,2} + n_2B_2S_2\bar{X}_2^{\,2} + n_3B_3\bar{X}_3^{\,2} + n_4B_4\bar{X}_4^{\,2} = 0 \end{cases} \tag{38}$$

These equations can be solved either all together or sequentially (the first 2 with respect to n_1, n_2, the others with respect to n_3 and n_4), covering mortality risk at the first step and financial risk at the second step.
Both problems outlined in (37) and (38) can be extended to self-financing considerations. In both cases the value of the hedged portfolio is given by

$$\Pi(t) = nBS + n_1 B_1 S_1 + n_2 B_2 S_2 + n_3 B_3 S_3 + n_4 B_4 S_4$$

It is self-financing if $\Pi(0) = 0$ or if an additional contract is inserted, so that the enlarged portfolio value is null. In our applications we will explore both possibilities.

9 Application to a UK sample

In this Section, we present an application of our hedging model to a sample of UK insured people. We exploit our minimal change of measure, which preserves the biological and historically reasonable behaviour of the intensity. We also assume that $a' = a$, i.e. that the risk premium on mortality risk is null. This assumption could be easily removed by calibrating the model parameters to actual insurance products, most likely derivatives. We take the view that their market is not liquid enough to permit such calibration (see also Biffis (2005), Cairns, Blake, Dowd, and MacMinn (2006), to mention a few). We therefore calibrate the mortality parameters to historical data (the IML tables, that are projected tables for English annuitants). We assume also - at first - that the interest rate is constant and, without loss of generality, null - as in Section 7. We derive a "term structure of pure endowments" and the values of coefficients Delta and Gamma of the contracts. Afterwards, we introduce also a stochastic interest rate.

9.1 Mortality risk hedging

We keep the head fixed, considering contracts written on the lives of male individuals who were 65 years old on 31/12/2010. Hence, we set $t = 0$ and we calibrate our parameters $a_{65}, \sigma_{65}, \lambda_{65}(0) = -\ln p_{65}$ from our data set, considering the generation of individuals that were born in 1945. For this generation, a is calibrated to 10.94 %, while σ is 0.07 %. $\lambda_{65}(0)$ is instead 0.885%.[15]

First of all, we analyze the effect of a shock of one standard deviation on the Wiener driving the intensity process. Figure 1 shows graphically the

[15] We refer the reader to Luciano and Vigna (2008) for a full description of the data set and the calibration procedure.

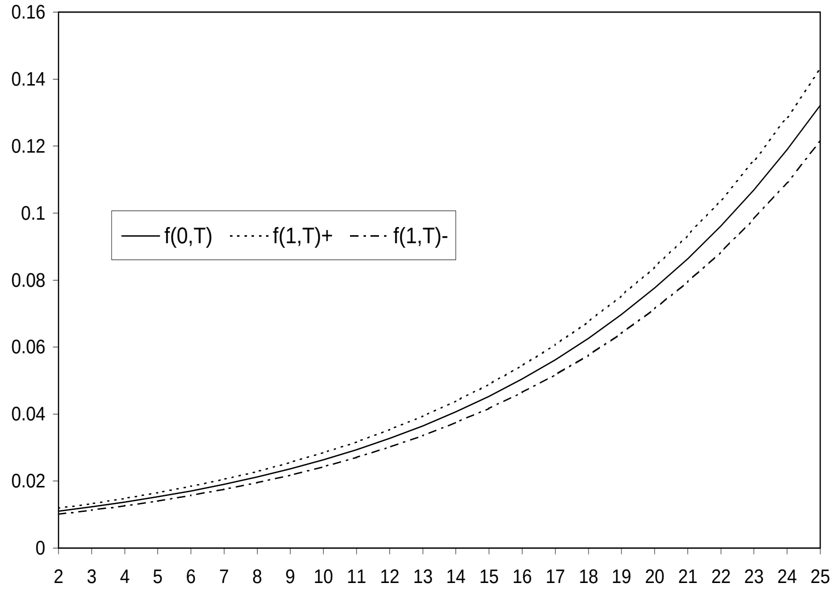

Figure 1: This figure shows the effect on the forward death intensity $f(1, T)$ of a shock equal to one standard deviation as a function of T. The central – solid – line represents the initial forward mortality intensity curve $f(0, T)$

impact of an upward and downward shock of one standard deviation on the forward intensity at $t = 1$ for different time horizons T. The forward mortality structure is derived from (3) using (15). The Figure clearly highlights that the effect becomes more and more evident – the trumpet opens up – as soon as the time horizon of the forward mortality becomes longer. Please notice that the behaviour is – as it should, from the economic point of view – opposite to the one of the corresponding Hull-White interest rates. In the rates case indeed the trumpet is reversed, since short-term forward rates are affected more than longer ones.

The following Table 1 reports the "term structure of pure endowment contracts" and compares the Delta and Gamma coefficients associated with contracts of different maturity in the stochastic case with the deterministic ones.

It appears clearly from the previous Table that the model gives hedging

Table 1: Stochastic vs. deterministic hedging coefficients

	Stochastic hedge			Deterministic hedge	
Maturity	$S(t,T)$	Δ^M	Γ^M	$\Delta^{\sigma=0}$	$\Gamma^{\sigma=0}$
1	0.99069	-1.04691	1.10633	-1.04691	1.10633
2	0.98041	-2.19187	4.90030	-2.19187	4.90030
5	0.94282	-6.27449	41.75698	-6.27439	41.75633
7	0.91116	-9.58396	100.80807	-9.58347	100.80284
10	0.85174	-15.46366	280.74803	-15.46053	280.69129
12	0.80306	-19.94108	495.16678	-19.93255	494.95501
15	0.71505	-27.19228	1034.08392	-27.16108	1032.89754
18	0.60899	-34.31821	1933.91002	-34.22325	1928.55907
20	0.52957	-38.32543	2773.64051	-38.14219	2760.37929
25	0.31713	-41.77104	5501.91988	-41.05700	5407.86868
27	0.23633	-39.27090	6525.53620	-38.18393	6344.91753
30	0.13319	-31.20142	7309.51024	-29.46466	6902.64225
35	0.03144	-12.93603	5322.98669	-10.78469	4437.74408

coefficients for mortality-linked contracts which are quite remarkably different from the deterministic ones for long maturities. For instance, the Δ^M and Γ^M hedging coefficients for a contract with maturity 30 years are respectively 6% smaller and larger than their deterministic counterparts. Contracts with long maturities are clearly very interesting from an insurer's point of view and hence their proper hedging is important.

As an example, imagine that an insurer has issued a pure endowment contract with maturity 15 years. Suppose that he wants to Delta-Gamma hedge this position using as cover instruments mortality-linked contracts with maturity 10 and 20 years. At a cost of 0.37, the insurer can instantaneously Delta-Gamma hedge its portfolio, by purchasing, respectively, 1.11 and 0.26 zero-coupon longevity bonds on these maturities. Having at disposal also the possibility of using contracts with a maturity of 30 years on the same population of individuals, a self-financing Delta-Gamma hedging strategy can be implemented by purchasing 0.48 and 0.60 longevity bonds with maturity respectively 10 and 20 years, and issuing 0.10 pure endowments with maturity 30 years.

Table 2: Hedging coefficients for stochastic financial risk

Maturity	$P(t,T)$	Δ^F	Γ^F
1	0.98395	-0.9798	0.9666
2	0.96214	-1.9103	3.7185
5	0.86696	-4.2988	20.0963
7	0.78430	-5.4865	34.9707
10	0.64372	-6.6170	57.9341
12	0.54597	-6.9606	71.2657
15	0.40404	-6.9596	85.7216
20	0.20649	-6.0149	92.7836
25	0.07972	-4.5599	82.7129
27	0.04902	-3.9667	75.8645
30	0.02037	-3.1366	64.3246
35	0.00278	-1.9995	45.1377

9.2 Mortality and financial risk hedging

The same procedure, as shown in Section 8, can be followed to hedge simultaneously the risks deriving from both stochastic mortality intensities and interest rates. Notice that, if we consider that the interest rate is stochastic (or at least different from zero), prices of pure endowment contracts no longer coincide with survival probabilities. Nonetheless, their Δ^M and Γ^M, the factors associated to mortality risk, remain unchanged when we introduce financial risk (see Section 8). Once one has estimated the coefficients underlying the interest rate process, we can easily derive the values of Δ^F and Γ^F, the factors associated to the financial risk, and the prices $P(t,T)$ of pure endowment/longevity bond contracts.
We calibrate our constant-parameter Hull and White model for forward interest rates to the observed zero-coupon UK government bonds at 31/12/2010.[16] Table 2 shows prices and financial risk hedging factors of pure endowment contracts subject to both financial and mortality risks. Please notice that the absolute values of the factors related to the financial market are smaller than the ones related to the mortality risk.

These factors Δ^F and Γ^F, together with their mortality risk counterparts, Δ^M and Γ^M, allow us to hedge pure endowment contracts from both financial and mortality risk by setting up a portfolio - even self-financing - which instantaneously presents null values of all the Delta and Gamma factors.
As an example, consider again the hedging of a pure endowment with ma-

[16] The parameter g is 3.23%, while the diffusion parameter Σ is calibrated to 1.25 %

turity 15 years. In order to Delta-Gamma hedge against both risks, we need to use four instruments (five if we want to self-finance the strategy). We can either use four pure endowments/longevity bonds written on the lives of the 65 year-old individuals or two mortality-linked contracts and two zero-coupon-bonds. In the first case, imagine to use contracts with maturity 10, 20, 25 and 30 years. The hedging strategy consists then in purchasing 0.35 longevity bonds with maturity 10 years, 1.27 with maturity 20 years and 0.30 with maturity 30 years, while issuing 0.87 pure endowment policies with maturity 25 years. In the second case, imagine the hedging instruments are mortality contracts with maturities 10 and 20 years and two zero-coupon-bonds with maturities 5 and 20 years. The strategy consists in purchasing 1.11 longevity bonds with maturity 10 years and 0.26 with maturity 20 years and in taking a short position on 0.60 zero-coupon-bonds with maturity 5 years and a long one on 0.10 zero-coupon-bonds with maturity 20 years. A self-financing hedge can be easily obtained by adding an instrument to the portfolio. For example, such a self-financing hedge can be obtained by purchasing 0.41 longevity bonds with maturity 10 years, 0.98 with maturity 20 years and 0.22 with maturity 35 years and issuing 0.38 pure endowments with maturity 25 years and 0.13 with maturity 30 years.

10 Summary and conclusions

This paper develops a Delta and Gamma hedging framework for mortality and interest rate risk.
We have shown that, consistently with the interest rate market, when the spot intensity of stochastic mortality follows an OU or FEL process, an HJM condition on its drift holds for every constant risk premium, without assuming no arbitrage. Hence, we have shown that it is possible to hedge systematic mortality risk in a way which is identical to the Delta and Gamma hedging approach in the HJM framework for interest rates. Delta and Gamma are very easy to compute, at least in the OU case. Similarly, the hedging quantities are easily obtained as solutions to linear systems of equations. Hence, this hedging model can be very attractive for practical applications.
Adding financial risk is a straightforward extension in terms of insurance pricing, if the bond market is assumed to be without arbitrages (and complete, so that the financial change of measure is unique). Delta and Gamma hedging is straightforward too if - as in the examples - the risk-neutral dynamics of the forward interest rate is constant-parameter Hull and White. Our application shows that the unhedged effect of a sudden change on the mortality rate is remarkable and the stochastic and deterministic Deltas and

Gammas are quite different, especially for long time horizons. Last but not least, calibrated Deltas and Gammas are bigger for mortality than for financial risk. The Delta and Gamma computation can be performed in the presence of FEL stochastic mortality. The whole hedging technique can also be extended using the same change of measure to the case of a CIR mortality intensity, reverting to a function of time.

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