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Finite State Markov-Chain Approximations to Highly Persistent Processes*

Karen A. Kopecky† Richard M. H. Suen‡

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Abstract

This paper re-examines the Rouwenhorst method of approximating first-order autoregressive processes. This method is appealing because it can match the conditional and unconditional mean, the conditional and unconditional variance and the first-order autocorrelation of any AR(1) process. This paper provides the first formal proof of this and other results. When comparing to five other methods, the Rouwenhorst method has the best performance in approximating the business cycle moments generated by the stochastic growth model. It is shown that, equipped with the Rouwenhorst method, an alternative approach to generating these moments has a higher degree of accuracy than the simulation method.

Keywords: Numerical Methods, Finite State Approximations, Optimal Growth Model

JEL classification: C63.

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1 Introduction

In macroeconomic models, the exogenous stochastic process is typically assumed to follow a stationary first-order autoregressive process. Two well-known examples are the asset pricing model à la Lucas (1978), and the standard real business cycle (RBC) model. In Lucas' model, the stochastic dividend stream is assumed to follow a Markov process. In the RBC model, the logarithm of the productivity shock is assumed to follow a Gaussian AR(1) process. In order to solve these models numerically, the continuous-valued autoregressive process is usually approximated by a discrete state-space Markov chain. To this end, researchers typically employ the approximation method proposed by Tauchen (1986), or the quadrature-based method developed in Tauchen and Hussey (1991). Although these methods differ substantially in details, the underlying idea is the same, that is to construct a discrete state-space Markov chain with transition probabilities that provide a good approximation for the conditional density of the autoregressive process. For AR(1) processes with low persistence, these methods can generate an accurate approximation even when a very coarse state space is used in the approximate Markov chain. However, the performance of these methods deteriorates when the serial correlation is very close to one.[1] This particular problem has been examined closely in a recent study by Flodén (2008). This author shows that the accuracies of the Tauchen (1986) method and the Tauchen-Hussey method are significantly lowered when the serial correlation of the underlying process is greater than 0.95. This problem persists even if one significantly increases the number of states in the discrete state-space Markov chain.

The findings in Flodén (2008) raise concerns because macroeconomic studies often employ highly persistent processes. These findings thus call for a more reliable technique to approximate highly autocorrelated processes. The main objective of this paper is to consider such a technique. More specifically, the current study re-examines a discrete approximation method first proposed in Rouwenhorst (1995) but largely overlooked by the existing literature.[2] Similar to the aforementioned methods, the Rouwenhorst method is about the construction of an approximate discrete state-space Markov chain. But unlike the other methods, the transition

[1] This weakness is acknowledged in the original papers. In Tauchen (1986, p.179), the author notes that "Experimentation showed that the quality of the approximation remains good except when λ [the serial correlation] is very close to unity." In Tauchen and Hussey (1991), the authors note that for processes with high persistence, "adequate approximation requires successively finer state spaces."

[2] An exception is Lkhagvasuren and Galindev (2008) which uses the Rouwenhorst method to approximate first-order vector autoregressive processes.

probabilities of the approximate Markov chain are not intended to mimic the conditional distribution of the underlying AR(1) process. This might seem like a weakness at first, but the Rouwenhorst method has a number of desirable features that are not matched by the other methods. First, only a few parameters are used in constructing the approximate Markov chain under this method. It is thus much more parsimonious and much easier to implement than the quadrature-based method developed in Tauchen and Hussey (1991). Second, the constructed Markov chain can be calibrated to match five important statistics of *any* stationary AR(1) process. These are the conditional and unconditional mean, the conditional and unconditional variance, and the first-order autocorrelation. Thus, even though the transition probabilities of the approximate Markov chain do not mimic the conditional distribution of the underlying AR(1) process, it can still exactly match the first two moments. Third, the Rouwenhorst method is particularly desirable for approximating Gaussian AR(1) processes. This is because the invariant distribution of the constructed Markov chain is a binomial distribution, which converges to the standard normal distribution when the number of states in the state space is sufficiently large.

Some of these features have been mentioned in Rouwenhorst (1995). But a formal proof of these results is still lacking. It is also unclear whether matching the moments of the AR(1) process is important in terms of solving dynamic general equilibrium models. In quantitative studies, obtaining a good approximation for the AR(1) process is seldom an end in itself. Thus a more appropriate metric for evaluating approximation methods in general would be their impact on the computed solutions of the general equilibrium models. Very few attempts have been made to assess the relative performance of the Rouwenhorst method and other approximation methods on this ground. Thus it remains unclear how the choice of approximation method would affect the accuracies of the computed solutions in these models. The current study is intended to fill these gaps.

The main contribution of this paper is two-fold. First, this paper provides formal proofs of all the results mentioned above. These results encompass the claims made in Rouwenhorst (1995). They also extend and generalize those claims in two ways. (i) Rouwenhorst mentions that when the transition matrix of the approximate Markov chain is symmetric, the invariant distribution is given by a binomial distribution. The current study shows that the invariant distribution is binomial even if the symmetric assumption is relaxed. (ii) Rouwenhorst also

claims that in the symmetric case, the approximate Markov chain can be calibrated to match the unconditional mean, the unconditional variance and the first-order autocorrelation of any stationary AR(1) process. This paper shows that the Markov chain can also match the conditional mean and the conditional variance.

The second contribution of this paper is to compare the Rouwenhorst method to five other approximation methods that are commonly used in the literature. These include the Tauchen (1986) method, the original quadrature-based method developed in Tauchen and Hussey (1991), two variations of this method considered in Flodén (2008), and the Adda-Cooper (2003) method. To achieve this, the prototypical stochastic neoclassical growth model without leisure is used as the analytical vehicle.[3] There are two main reasons why we choose this particular model. First, the neoclassical growth model is by far the most common analytical framework in macroeconomics. Variations of the original model have been used to study a wide range of economic issues. Second, it is possible to derive closed-form solutions for the neoclassical growth model under certain specifications. This property of the model provides tremendous convenience for evaluating the accuracy of the approximation methods.

The main criterion for evaluating the six approximation methods is the accuracy in approximating the business cycle moments as predicted by the stochastic growth model. Two approaches to generating these moments are considered. In the baseline approach, an approximation for the stationary distribution of the state variables is first derived. The moments of interest are then computed directly from this distribution. In the second approach, the business cycle moments are generated using the Monte Carlo simulation method. This involves simulating the model repeatedly using the actual AR(1) process and the computed policy function, and thus does not require approximating the stationary distribution. One major difference between these two approaches is the sources of the errors that they introduce. While both methods suffer from errors in the computation of the policy function, under the baseline approach, additional errors arise when approximating the stationary distribution. However, this approach does not suffer from the sampling errors that the simulation method generates.

One important finding of this paper is that, regardless of which approach is taken, the choice of approximation method can have a large impact on the accuracy of the business

[3] The same model is used in Taylor and Uhlig (1990) and the companion papers to illustrate and compare different solution methods. More recently, Aruoba, Fernández-Villaverde and Rubio-Ramírez (2006) use the same model, but with labor-leisure choice, to compare different solution methods.

cycle moments computed. Under the baseline approach, the choice of discretization method has a large impact on the accuracy of the stationary distribution approximation that is used to compute the moments. In general, a method that generates a good approximation for the moments of the AR(1) process also tends to yield an accurate approximation for the stationary distribution. The Rouwenhorst method has the best performance in this regard, followed by a variation of the quadrature-based method considered in Flodén (2008). In the sensitivity analysis, it is shown that the superior performance of the Rouwenhorst method is robust under a wide range of parameter values.

When the Monte Carlo simulation method is used to generate the business cycle moments, no single method dominates all others in all cases. With a logarithmic utility function and full depreciation, the six methods yield almost identical results. When a more realistic value of the depreciation rate is used, the relative performance of the six methods depends on the number of states in the Markov chain. When a rather coarse state space is used, the Rouwenhorst method again has the best overall performance. However, when the fineness of the state space increases, the Adda-Cooper method improves significantly and yields the best overall performance.

Another important finding of this paper is that the baseline approach, equipped with the Rouwenhorst method, has a higher degree of accuracy than the simulation method. This result is one of interest because the simulation method is considered standard practice in estimating unknown statistics of stochastic models. Our results, however, show that this is not the most effective method for generating business cycle moments in the neoclassical growth model.

The rest of this paper is organized as follows. Section 2 presents the Rouwenhorst method. Section 3 presents the analytical results pertaining to this method. Section 4 presents the numerical results. Section 5 concludes.

2 The Rouwenhorst Method

Consider the following AR(1) process

$$z_t = \rho z_{t-1} + \varepsilon_t, \tag{1}$$

where ε_t is a white noise with variance σ_ε^2. If $|\rho| < 1$, then the AR(1) process is stationary and the random variable z_t has a mean of zero, a variance of

$$\sigma_z^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}, \tag{2}$$

and autocorrelations given by

$$\rho_s = \frac{E\left(z_t z_{t-s}\right)}{var\left(z_t\right)} = \rho^s, \quad \text{for } s = 0, 1, 2, ...$$

If, in addition, ε_t is normally distributed in each time period, then z_t is also normally distributed.

Rouwenhorst (1995) proposes a discrete approximation to the AR(1) process in (1). This involves constructing an N-state Markov chain characterized by (i) a symmetric and evenly-spaced state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$, with $\overline{y}_1 = -\psi$ and $\overline{y}_N = \psi$, and (ii) a transition matrix Θ_N. For any $N \geq 2$, the transition matrix is determined by two parameters, $p, q \in (0, 1)$, and is defined recursively as follows:

Step 1: When $N = 2$, define Θ_2 as

$$\Theta_2 = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}.$$

Step 2: For $N \geq 3$, first construct the N-by-N matrix

$$\begin{aligned} &p \begin{bmatrix} \Theta_{N-1} & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} + (1-p) \begin{bmatrix} \mathbf{0} & \Theta_{N-1} \\ 0 & \mathbf{0}' \end{bmatrix} \\ &+ (1-q) \begin{bmatrix} \mathbf{0}' & 0 \\ \Theta_{N-1} & \mathbf{0} \end{bmatrix} + q \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \Theta_{N-1} \end{bmatrix}, \end{aligned} \tag{3}$$

where $\mathbf{0}$ is a $(N-1)$-by-1 column vector of zeros.

Step 3: Divide all but the top and bottom rows by two so that the elements in each row sum to one.

Rouwenhorst mentions two important attributes of this Markov chain. First, for any $N \geq 2$, the first-order autocorrelation is always given by $p + q - 1$. Second, in the symmetric case where $p = q$, the invariant distribution of Θ_N is a binomial distribution with parameters $N-1$ and $1/2$. Formally, let $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N)$ be the invariant distribution of Θ_N. When $p = q$ holds, the elements in $\boldsymbol{\lambda}$ are given by

$$\lambda_i = \frac{\binom{N-1}{i-1}}{2^{N-1}}, \qquad \text{for } i = 1, 2, ..., N.$$

It follows that the unconditional mean of the Markov process is zero and the unconditional variance is

$$\sigma_y^2 = \frac{\psi^2}{N-1}.$$

These properties are particularly useful when it comes to approximating Gaussian AR(1) processes. First, the first-order autocorrelation of the original AR(1) process (ρ) and its variance $\left(\sigma_z^2\right)$ can be perfectly matched by setting

$$\psi = \sqrt{(N-1)}\sigma_z \quad \text{and} \quad p = q = \frac{1+\rho}{2}.$$

Second, since the invariant distribution is a binomial distribution, it is a close approximation for the standard normal distribution when N is sufficiently large. This feature is desirable because for a Gaussian AR(1) process $\{z_t\}$, the invariant distribution of the standardized process $\{z_t/\sigma_z\}$ is the standard normal distribution. Thus, the invariant distribution of the process $\{y_t/\sigma_y\}$ can be made arbitrarily close to the invariant distribution of $\{z_t/\sigma_z\}$ by increasing the number of grid points, N.

3 Analytical Results

The objective of this section is to derive and generalize the results mentioned in the previous section. One problem with the Rouwenhorst method is that the matrix Θ_N generated by the three-step procedure is very difficult to work with analytically. For this reason, this section begins by offering a new and simpler procedure for generating the Rouwenhorst matrix. Using this new procedure, it is shown that a Markov chain with state space Y_N and transition

matrix Θ_N has a unique invariant distribution in the form of a binomial distribution. Unlike Rouwenhorst (1995), which only considers the case when $p = q$, the current study shows that the invariant distribution is binomial for any $p, q \in (0,1)$. The result reported in here thus encompasses the symmetric case as a special case. Once the invariant distribution is determined, it is used to derive a set of conditional and unconditional moments for the Markov chain.

3.1 Reconstructing the Rouwenhorst Matrix

For any $p, q \in (0,1)$, and for any integer $N \geq 2$, define a system of polynomials as follows

$$\Phi(t; N, i) \equiv [p + (1-p)t]^{N-i} (1 - q + qt)^{i-1}, \tag{4}$$

for $i = 1, 2, ..., N$. The polynomials in (4) can be expanded to become

$$\Phi(t; N, i) = \sum_{j=1}^{N} \pi_{i,j}^{(N)} t^{j-1}, \qquad \text{for } i = 1, 2, ..., N. \tag{5}$$

Define an N-by-N matrix $\Pi_N = \left[\pi_{i,j}^{(N)}\right]$ using the coefficients in (5). Using the generating function in (4), one can derive the elements in Π_N recursively using the elements in Π_{N-1}, for $N - 1 \geq 2$. The details of this procedure are described in Appendix A. The main result of this subsection is Proposition 1 which states that the matrix Π_N is identical to the Rouwenhorst matrix Θ_N for any integer $N \geq 2$. Before proceeding to the main result, let's consider a couple of simple examples.

For $N = 2$, the system of polynomials in (4) can be expressed as

$$\begin{bmatrix} \Phi(t; 2, 1) \\ \Phi(t; 2, 2) \end{bmatrix} = \underbrace{\begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}}_{\Pi_2} \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

Obviously Π_2 is identical to the Rouwenhorst matrix Θ_2. For $N = 3$, the system of polynomials

is

$$\begin{bmatrix} \Phi(t;3,1) \\ \Phi(t;3,2) \\ \Phi(t;3,3) \end{bmatrix} = \underbrace{\begin{bmatrix} p^2 & 2p(1-p) & (1-p)^2 \\ p(1-q) & pq+(1-p)(1-q) & q(1-p) \\ (1-q)^2 & 2q(1-q) & q^2 \end{bmatrix}}_{\Pi_3} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}.$$

Again Π_3 is identical to the Rouwenhorst matrix Θ_3 (see Rouwenhorst, 1995, p.327). The general result is established in Proposition 1. All proofs can be found in Appendix B.

Proposition 1 *For any $N \geq 2$, and for any $p, q \in (0,1)$, the matrix Π_N defined above is identical to the Rouwenhorst matrix Θ_N generated by Steps 1-3.*

The next result states that Π_N is a stochastic matrix with non-zero entries. To begin with, set $t = 1$ in both (4) and (5) to obtain

$$\sum_{j=1}^{N} \pi_{i,j}^{(N)} = 1, \quad \text{for } i = 1, 2, ..., N.$$

This means the elements in any row of Π_N sum to one. If, in addition, $\pi_{i,j}^{(N)} \geq 0$ for all i and j, then Π_N is a stochastic matrix. This is proved in the following lemma.

Lemma 2 *For any $N \geq 2$, the matrix Π_N defined above is a stochastic matrix with no zero entries.*

3.2 Discrete State-Space Markov Chain

Consider a Markov chain with a symmetric and evenly-spaced state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$ defined over the interval $[-\psi, \psi]$. In other words, the elements in the state space are given by

$$\overline{y}_i = -\psi + \frac{2\psi}{N-1}(i-1), \qquad \text{for } i = 1, 2, ..., N.$$

The transition matrix of the Markov chain is given by $\Pi_N = \left[\pi_{i,j}^{(N)}\right]$ as defined above. The following result follows immediately from Lemma 2.

Proposition 3 *For any $N \geq 2$, the Markov chain with state space $Y_N = \{\overline{y}_1, ..., \overline{y}_N\}$ and transition matrix Π_N has a unique invariant distribution $\boldsymbol{\lambda}^{(N)} = \left(\lambda_1^{(N)}, ..., \lambda_N^{(N)}\right)$, where*

$\lambda_i^{(N)} \geq 0$ *and* $\sum_{i=1}^{N} \lambda_i^{(N)} = 1$.

Since the invariant distribution is unique, it can be solved by the guess-and-verify method. Let $s \equiv \frac{1-q}{2-(p+q)} \in (0,1)$. The guess for $\boldsymbol{\lambda}^{(N)}$, represented by $\widehat{\boldsymbol{\lambda}}^{(N)}$, is a binomial distribution with parameters $N-1$ and $1-s$. This means

$$\widehat{\lambda}_i^{(N)} = \binom{N-1}{i-1} s^{N-i} (1-s)^{i-1}, \quad \text{for } i = 1, 2, ..., N. \tag{6}$$

It is easy to check that this is the actual solution when $N = 2$. In other words,

$$\lambda_1^{(2)} = \frac{1-q}{2-(p+q)} \quad \text{and} \quad \lambda_2^{(2)} = \frac{1-p}{2-(p+q)}.$$

The result for the general case is established in Proposition 4.

Proposition 4 *For any* $N \geq 2$, *the invariant distribution of the Markov chain defined above is a binomial distribution with parameters* $N-1$ *and* $1-s$.

Some of the conditional and unconditional moments of the Markov chain are listed in Table 1. The mathematical derivations of these results can be found in Appendix C.

Table 1: Selected Moments of the Markov Chain

Conditional Mean	$E(y_{t+1}\|y_t = \overline{y}_i)$	$(q-p)\psi + (p+q-1)\overline{y}_i$
Conditional Variance	$var(y_{t+1}\|y_t = \overline{y}_i)$	$\frac{4\psi^2}{(N-1)^2}[(N-i)(1-p)p + (i-1)q(1-q)]$
Unconditional Mean	$E(y_t)$	$\frac{(q-p)\psi}{2-(p+q)}$
Unconditional Second Moment	$E(y_t^2)$	$\psi^2\left\{1 - 4s(1-s) + \frac{4s(1-s)}{N-1}\right\}$
First-order Autocovariance	$Cov(y_t, y_{t+1})$	$(p+q-1)var(y_t)$
First-order Autocorrelation	$Corr(y_t, y_{t+1})$	$p+q-1$

3.3 Approximating AR(1) Processes

The task at hand is to approximate a given stationary AR(1) process with an N-state Markov chain. Let $\{z_t\}$ be a stationary AR(1) process as defined in (1). The random disturbance term ε_t is assumed to follow an i.i.d. process with finite variance σ_ε^2. As mentioned above, the unconditional mean of z_t is zero, the unconditional variance is given by (2) and the first-order autocorrelation is ρ. Conditional on the realization of z_{t-1}, the mean and variance of z_t are given by

$$E\left(z_t|z_{t-1}\right) = \rho z_{t-1} \qquad \text{and} \qquad var\left(z_t|z_{t-1}\right) = \sigma_\varepsilon^2.$$

Next, define an N-state discrete Markov process $\{y_t\}$ as in section 3.2 with the following restrictions imposed:

$$p = q = \frac{1+\rho}{2} \qquad \text{and} \qquad \psi = \sqrt{N-1}\sigma_\varepsilon. \tag{7}$$

Using the equations listed on Table 1, it is immediate to see that the resulting Markov chain $\{y_t\}$ has the same unconditional mean, unconditional variance and first-order autocorrelation as $\{z_t\}$. Suppose $y_t = \overline{y}_i$ for some $t \geq 0$ and for some $\overline{y}_i$ in the state space Y_N. The conditional mean and conditional variance of y_{t+1} are given by

$$E\left(y_{t+1}|y_t = \overline{y}_i\right) = \rho\overline{y}_i \qquad \text{and} \qquad var\left(y_{t+1}|y_t = \overline{y}_i\right) = \sigma_\varepsilon^2.$$

Thus the Markov chain $\{y_t\}$ has the same conditional mean and conditional variance as the AR(1) process $\{z_t\}$.

Two remarks regarding this procedure are worth mentioning. First, under the Rouwenhorst method, the approximate Markov chain is constructed using ρ and σ_ε^2 alone. In particular, the transition matrix Π_N is not a discretized version of the conditional distribution of z_t. This is the fundamental difference between this method and the ones proposed by Tauchen (1986) and Tauchen and Hussey (1991). Second, the above procedure can be applied to *any* stationary AR(1) process, including those with very high persistence. Thus, unlike the other two methods, the one proposed by Rouwenhorst can always match the unconditional variance and the first-order autocorrelation of z_t.

Suppose now the random disturbances term ε_t in the AR(1) process is also normally

distributed in each time period t. Then the distribution of z_t is a normal distribution. In this case, the invariant distribution of the Markov chain $\{y_t\}$ can provide a good approximation for the distribution of z_t. As shown in Proposition 4, the invariant distribution of y_t is always given by a binomial distribution. Under (7), the mean and variance of the invariant distribution are zero and $\sigma^2 \equiv \sigma_\varepsilon^2/\left(1-\rho^2\right)$, respectively. Thus the standardized process $\{y_t/\sigma\}$ would converge to the standard normal distribution when N is made sufficiently large. According to the Berry-Esséen Theorem, the rate of convergence is on the order of $N^{-1/2}$.

4 Stochastic Neoclassical Growth Model

Consider the planner's problem in the stochastic neoclassical growth model,

$$\max_{\{C_t,K_{t+1}\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty}\beta^t U\left(C_t\right)\right]$$

subject to

$$C_t + K_{t+1} = A_t K_t^{\alpha} + (1-\delta) K_t,$$

$$C_t, K_{t+1} \geq 0,$$

where C_t denotes consumption at time t, K_t denotes capital and A_t is the stochastic technological factor. The function $U(\cdot)$ is the per-period utility function. The parameter $\beta \in (0,1)$ is the subjective discount factor, $\alpha \in (0,1)$ is the share of capital income in total output and $\delta \in (0,1]$ is the depreciation rate of capital. The logarithm of the technological shock, represented by $a_t \equiv \ln A_t$, is assumed to follow an AR(1) process,

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \tag{8}$$

where $\varepsilon_{t+1} \sim$ i.i.d. $N\left(0,\sigma_\varepsilon^2\right)$ and $\rho \in (0,1)$. Conditional on $a_t = a$, the random variable a_{t+1} is normally distributed with mean ρa and variance σ_ε^2. Let $F(\cdot|a)$ be the conditional distribution function. For any given value of a, define $\overline{K}(a)$ by

$$\overline{K}(a) = \left[\frac{\exp(a)}{\delta}\right]^{\frac{1}{1-\alpha}}.$$

Then, conditional on $a_t = a$, the state space of capital can be restricted to $\mathcal{K}(a) = \left[0, \overline{K}(a)\right]$. The state space of the stochastic growth model is given by

$$\mathcal{S} = \{(K, a) : K \in \mathcal{K}(a), a \in \mathbb{R}\}.$$

The Bellman equation for the planner's problem can be written as

$$V(K, a) = \max_{K' \in \mathcal{K}(a)} \left\{ U\left[\exp(a) K^{\alpha} + (1-\delta) K - K'\right] + \beta \int V\left(K', a'\right) dF\left(a'|a\right) \right\}. \quad (9)$$

The solution of this problem includes a value function $V : \mathcal{S} \to \mathbb{R}$ and a policy function $g : \mathcal{S} \to \mathbb{R}$. The latter specifies the law of motion for capital.

4.1 Discretizing the AR(1) Process

The first step in solving the Bellman equation is to devise an approximation for the integral in the objective function. This typically involves replacing the AR(1) process in (8) with a discrete state-space Markov chain. Formally, define an N-state Markov chain with state space $\mathcal{A} = \{\overline{a}_1, ..., \overline{a}_N\}$ and transition matrix $\Pi = [\pi_{i,j}]$. The Bellman equation can then be written as

$$\widetilde{V}(K, \overline{a}_i) = \max_{K' \in \mathcal{K}(\overline{a}_i)} \left\{ U\left[\exp(\overline{a}_i) K^{\alpha} + (1-\delta) K - K'\right] + \beta \sum_{j=1}^{N} \widetilde{V}\left(K', \overline{a}_j\right) \pi_{i,j} \right\}, \quad (10)$$

for every $\overline{a}_i$ in $\mathcal{A}$. The solution of this problem, $\widetilde{V}$, is an approximation of the *actual* value function V.

In the following section, six different methods for constructing the Markov chain will be considered. These include the Rouwenhorst method, the Tauchen (1986) method, the quadrature method developed in Tauchen and Hussey (1991), two variations of the original Tauchen-Hussey method considered in Flodén (2008), and the method described in Adda and Cooper (2003, p.56-58). The Rouwenhorst method for approximating AR(1) processes has been described in section 3.3. Details of the other methods are provided below.

Tauchen (1986) method

Under this method, an evenly-spaced state space $\mathcal{A} = \{\overline{a}_1, ..., \overline{a}_N\}$ is used, with

$$\overline{a}_N = -\overline{a}_1 = \frac{M\sigma_\varepsilon^2}{1-\rho^2},$$

where M is a positive real number. The step between any two grid points is given by $h = (\overline{a}_N - \overline{a}_1)/(N-1)$. Let Φ be the probability distribution function for the standard normal distribution. For any $i = 1, ..., N$, the transition probabilities of the Markov chain are given by

$$\pi_{i,1} = \Phi\left(\frac{\overline{a}_1 - \rho\overline{a}_i + h/2}{\sigma_\varepsilon}\right),$$

$$\pi_{i,N} = 1 - \Phi\left(\frac{\overline{a}_N - \rho\overline{a}_i - h/2}{\sigma_\varepsilon}\right),$$

and

$$\pi_{i,j} = \Phi\left(\frac{\overline{a}_j - \rho\overline{a}_i + h/2}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\overline{a}_j - \rho\overline{a}_i - h/2}{\sigma_\varepsilon}\right),$$

for $j = 2, ..., N-1$. Tauchen states that if the state space $\mathcal{A}$ is sufficiently fine, then the conditional distribution of the discrete process will converge to the conditional distribution function $F(a'|\overline{a}_i)$.

The Quadrature-Based Methods

This class of methods is built upon the Gauss-Hermite quadrature method for approximating the value of integrals. Let z be a normally distributed random variable with mean zero and variance σ^2. Under the Gauss-Hermite quadrature method, the expectation of a function of z is approximated by

$$E[g(z)] \approx \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N} \phi_i g\left(\sqrt{2}\sigma x_i\right),$$

where $\{\phi_i\}$ are the Gauss-Hermite weights and $\{x_i\}$ are the Gauss-Hermite nodes over $[-\infty, \infty]$.[4]

The general procedure of the quadrature-based methods can be summarized as follows.

[4] For a formal discussion on the Gauss-Hermite quadrature method, see Davis and Rabinowitz (1984) Chapter 3.

First, the elements of the state space $\mathcal{A}$ are determined by

$$\overline{a}_i = \sqrt{2}\sigma x_i, \qquad \text{for } i = 1, 2, ..., N.$$

Second, the elements in the transition matrix Π are given by

$$\pi_{i,j} = \frac{f\left(\overline{a}_j|\overline{a}_i\right)}{f\left(\overline{a}_j|0\right)}\frac{\overline{w}_j}{s_i},$$

where $\overline{w}_j = \phi_j/\sqrt{\pi}$, the function $f\left(\overline{a}_j|\overline{a}_i\right)$ is the density function for a normal distribution with mean $\rho\overline{a}_i$ and variance σ^2, and

$$s_i = \sum_{n=1}^{N} \frac{f\left(\overline{a}_n|\overline{a}_i\right)}{f\left(\overline{a}_n|0\right)}\overline{w}_n.$$

The only difference between the original method considered in Tauchen and Hussey (1991) and the variations considered in Flodén (2008) is the choice of σ. In the original version, the standard deviation σ is taken to be σ_ε. In other words, the transition probabilities of the Markov chain are constructed using the conditional density function of a. In the first variation, the standard deviation of a_t is used instead, i.e., $\sigma = \sigma_a = \sigma_\varepsilon/\sqrt{1-\rho^2}$. In the second variation, σ is a weighted average between σ_a and σ_ε. In particular,

$$\sigma = \omega\sigma_\varepsilon + (1-\omega)\,\sigma_a,$$

with $\omega = 0.5 + 0.25\rho$.

The Adda-Cooper Method

The first step of this method is to partition the real line into N intervals. These intervals are constructed so that the random variable a_t has an equal probability of falling into them. Formally, let $I_n = [x_n, x_{n+1}]$ be the nth interval with $x_1 = -\infty$ and $x_{N+1} = +\infty$. The cut-off points $\{x_n\}_{n=2}^{N}$ are obtained by solving the following system of equations:

$$\Phi\left(\frac{x_{n+1}}{\sigma_a}\right) - \Phi\left(\frac{x_n}{\sigma_a}\right) = \frac{1}{N}, \qquad \text{for } n = 1, 2, ..., N,$$

where Φ is the probability distribution function for the standard normal distribution. The nth element in the state space $\mathcal{A} = \{\overline{a}_1, ..., \overline{a}_N\}$ is then given by the mean value of the nth interval, i.e.,

$$\overline{a}_n = E\left[a | a \in I_n\right].$$

For any $i, j \in \{1, 2, ..., N\}$, the transition probability $\pi_{i,j}$ is defined as the probability of moving from interval I_i to interval I_j in one period. Formally, this is given by

$$\pi_{i,j} = \Pr\left[a' \in I_j | a \in I_i\right].$$

4.2 Experiments and Evaluation

The objective of this section is to evaluate the performance of different discretization methods. To achieve this, we focus on the business cycle moments generated by the stochastic growth model. The main criteria for evaluating the six discretization methods is the accuracy in approximating these moments.

Solution Method

The first step in computing the business cycle moments is to choose a specific form for the utility function and a set of values for the parameters $\{\alpha, \beta, \delta, \sigma_\varepsilon, \rho\}$. In the baseline model, the utility function is logarithmic and there is full depreciation. The full depreciation assumption is later relaxed in section 4.4. Under the baseline specifications, it is possible to derive analytically (i) the policy functions for investment and consumption, (ii) the stationary distribution of the state variables, and (iii) the variances and first-order autocorrelations of the endogenous variables. These closed-form solutions play a key role in evaluating the discretization methods. This will become clear in subsequent discussions. The other parameter values are chosen to be the same as in King and Rebelo (1999): $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$ and $\rho = 0.979$.

The next step is to discretize the state space $\mathcal{S}$ of the stochastic growth model. First, the AR(1) process in (8) is approximated using the methods mentioned above. The resulting N-state Markov chain is characterized by a state space $\mathcal{A} = \{\overline{a}_1, ..., \overline{a}_N\}$ and a transition matrix $\Pi = [\pi_{i,j}]$. Second, the continuous state space for capital is replaced by an evenly-spaced grid.

Define the variable $k \equiv \ln K$. The set of grid points for k is represented by $\mathcal{K} = \left\{\overline{k}_1, ..., \overline{k}_M\right\}$. The discretized state space for the stochastic growth model can be expressed by

$$\widehat{\mathcal{S}} = \left\{\left(\overline{k}_m, \overline{a}_n\right) : \overline{k}_m \in \mathcal{K}, \overline{a}_n \in \mathcal{A}\right\}. \tag{11}$$

In the baseline case, the number of states in the Markov chain is set to five and the number of grid points for capital is 1000. As reported in Flodén (2008), the performance of the quadrature-based methods in approximating highly persistent processes is very sensitive to the number of points in $\mathcal{A}$. As a robustness check, we also consider the cases when $N = 2$ and $N = 10$ in section 4.3. After the discrete state space $\widehat{\mathcal{S}}$ is formed, the value function and the associated policy function are solved using the value-function iteration method described in Tauchen (1990) and Burnside (1999). The outcome of this procedure includes a set of $N \times M$ values of the policy function evaluated on $\widehat{\mathcal{S}}$. This set of values is represented by $\left\{\widehat{g}\left(\overline{k}_m, \overline{a}_n\right)\right\}$.

The final task is to compute the stationary distribution of the state variables (k, a). The first step to achieve this is to construct the transition matrix for these variables. Under the discrete state-space method, the probability of moving from state $\left(\overline{k}_m, \overline{a}_n\right)$ in $\widehat{\mathcal{S}}$ to another state $\left(\overline{k}_l, \overline{a}_j\right)$ in $\widehat{\mathcal{S}}$ in one period is specified by

$$\Pr\left[\left(k', a'\right) = \left(\overline{k}_l, \overline{a}_j\right) | \left(k, a\right) = \left(\overline{k}_m, \overline{a}_n\right)\right] = \begin{cases} \pi_{n,j} & \text{if } \overline{k}_l = \widehat{g}\left(\overline{k}_m, \overline{a}_n\right) \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

The resulting NM-by-NM transition matrix is denoted P. Let $\widehat{\boldsymbol{\pi}} = (\widehat{\pi}_1, ..., \widehat{\pi}_{NM})$ be the stationary distribution associated with P. Formally, this is defined by

$$\widehat{\boldsymbol{\pi}} P = \widehat{\boldsymbol{\pi}}.$$

In principle, $\widehat{\boldsymbol{\pi}}$ can be obtained as the eigenvector of P corresponding to eigenvalue 1, with the normalization $\sum_{i=1}^{NM} \widehat{\pi}_i = 1$. This method, however, is not practical when the number of grid points for capital (M) or the number of grid points in the discrete Markov chain (N) is large. In the following experiments, an approximation for the stationary distribution is obtained by iterating the equation

$$\widetilde{\boldsymbol{\pi}}^l P = \widetilde{\boldsymbol{\pi}}^{l+1}. \tag{13}$$

A good approximation for $\widehat{\boldsymbol{\pi}}$ can be obtained when l is sufficiently large. Given the approximate stationary distribution $\widetilde{\boldsymbol{\pi}}^l$ and the policy function $\widehat{g}$, the business cycle moments of interest can be computed. This process of computing the business cycle moments is referred to below as the baseline approach.

An alternative route to compute the business cycle moments is to use Monte Carlo simulations. The standard procedure involves the following steps. Simulate the random variable a_t using the *actual* AR(1) process given in (8) over a long period of time, say T. The resulting sequence is denoted $\{\widetilde{a}_t\}_{t=0}^{T}$. Construct a sequence of capital $\left\{\widetilde{k}_t\right\}_{t=0}^{T}$ according to

$$\widetilde{k}_{t+1} = \widehat{g}\left(\widetilde{k}_t, \widetilde{a}_t\right), \quad \text{with } \widetilde{k}_0 \text{ given.}$$

In general, the generated values of $\widetilde{k}_t$ and $\widetilde{a}_t$ will not coincide with the grid points in $\widehat{\mathcal{S}}$. In this case, linear interpolation is used to compute the value of $\widehat{g}\left(\widetilde{k}_t, \widetilde{a}_t\right)$. Next, compute the sample variances and covariance as follows,

$$s_{xx} = \frac{1}{T}\sum_{t=\tau}^{T}\widetilde{x}_t^2 - \left(\frac{1}{T}\sum_{t=\tau}^{T}\widetilde{x}_t\right)^2, \quad \text{for } x = k, a,$$

$$s_{ak} = \frac{1}{T}\sum_{t=\tau}^{T}\widetilde{a}_t\widetilde{k}_t - \left(\frac{1}{T}\sum_{t=\tau}^{T}\widetilde{a}_t\right)\left(\frac{1}{T}\sum_{t=\tau}^{T}\widetilde{k}_t\right).$$

To ensure that the generated values of $\widetilde{k}_t$ and $\widetilde{a}_t$ are drawn from the stationary distribution, the first τ observations in either sequence are excluded. Repeat the above procedure L times to obtain

$$\overline{s}_{kk} = \frac{1}{L}\sum_{i=1}^{L}s_{kk}^i, \quad \overline{s}_{aa} = \frac{1}{L}\sum_{i=1}^{L}s_{aa}^i, \quad \text{and} \quad \overline{s}_{ak} = \frac{1}{L}\sum_{i=1}^{L}s_{ak}^i,$$

where s_{kk}^i is the sample variance for k in the ith simulation. The simulated moments, $(\overline{s}_{kk}, \overline{s}_{aa}, \overline{s}_{ak})$, then serve as an estimate for the variance-covariance matrix of (k, a). The moments for the other variables are obtained in the same fashion. To put this in practice, 2000 sequences of a_t are drawn from the actual AR(1) process. Each sequence contains 3000 observations. The first 500 observations are discarded when computing the sample variances and covariances.

The business cycle moments computed under these two approaches are then compared to

their true values obtained using the closed-form solutions. It turns out that the two approaches would yield very different results. These differences are reported in the error analysis section.

Baseline Results

Table 2 presents the baseline results obtained under the above procedure. The six discretization methods are compared on three grounds: (i) the accuracy in approximating the AR(1) process, (ii) the precision in approximating the stationary distribution of the state variables, and (iii) the accuracy in approximating the business cycle moments. The true values obtained under the closed-form solutions are used as the yardstick for comparison in each step.

Panel (A) of Table 2 shows the performance of these methods in approximating the AR(1) process.[5] As explained in section 3.3, the transition matrix in the Rouwenhorst method (R) can be calibrated to match exactly the persistence parameter, the standard deviation of ε and the standard deviation of a. Among the other five methods, the Adda-Cooper method (A-C) has the highest accuracy in terms of matching the persistence parameter. This is followed by the second variation of the Tauchen-Hussey method (F-2), the Tauchen (1986) method and the first variation of the Tauchen-Hussey method (F-1). The original Tauchen-Hussey method has the lowest accuracy in terms of approximating the persistence parameter.

When it comes to matching the standard deviation of a, all five methods (excluding the Rouwenhorst method) have difficulties in replicating the true value. With a relative error of 2.15 percent, the F-1 method has the best performance within this group. The other four methods have relative errors ranging from five percent to 60 percent. In particular, the original Tauchen-Hussey method can only replicate 40 percent of the actual value of σ_a. This problem of the Tauchen-Hussey method is also reported in Flodén (2008). For the Tauchen (1986) method, the F-2 method and the Adda-Cooper method, the low precision in approximating σ_a is associated with a low precision in approximating σ_ε.

Next, we consider the accuracies of these methods in approximating the stationary distribution of the state variables. With a logarithmic utility function, full depreciation, and ε_t following a Gaussian white noise process, the *actual* stationary distribution of (k, a) is a

[5] The relative errors reported in panel (A) are directly comparable to those reported in Flodén (2008) Table 2 for $n = 5$ and $\rho = 0.98$. The only difference is Flodén did not consider the Rouwenhorst method.

bivariate normal distribution with mean vector

$$\boldsymbol{\mu}' = \begin{bmatrix} \frac{\ln(\alpha\beta)}{1-\alpha} & 0 \end{bmatrix},$$

and variance-covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_k^2 & \sigma_{ka} \\ \sigma_{ka} & \sigma_a^2 \end{bmatrix},$$

where

$$\sigma_k^2 = \frac{(1+\alpha\beta)\,\sigma_a^2}{(1-\alpha^2)\,(1-\alpha\rho)},$$

$$\sigma_{ka} = \frac{\rho\sigma_a^2}{1-\alpha\rho}, \qquad \text{and} \qquad \sigma_a^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}.$$

Panel (B) of Table 2 shows the performance of these methods in approximating the standard deviation of k and the covariance between a and k. In general, a discretization method that generates an accurate approximation for σ_a also has high precision in approximating these two moments. Among these six methods, the Rouwenhorst method has the highest accuracy in approximating these two moments. The relative errors for the two are about 0.14 percent. This outperforms the other methods by a significant margin. The F-1 method, which is the second best, has a relative error of about three percent in approximating σ_k and an error of eight percent in approximating σ_{ka}.

Finally, we compare the performance of these methods in approximating the business cycle moments. In particular, we focus on the standard deviation of output, consumption and investment (in logarithmic terms) and the first-order autocorrelation of output (in logarithmic terms).[6] The results are shown in panel (C) of Table 2. Again the Rouwenhorst method has the best performance in terms of approximating all these moments. The F-1 method is the second best method in terms of "overall" performance. In terms of approximating the first-order autocorrelation of output, the Tauchen (1986) method and the F-2 method are actually more accurate than the F-1 method. However, the F-1 method performs better in approximating the standard deviation of the endogenous variables.

[6] The first-order autocorrelation of consumption and investment (in logarithmic terms), and the cross-correlation between output and these variables are not shown in the paper. These results are available from the authors upon request.

Two things can be observed when comparing across all three panels. First, the relative errors in approximating σ_a are very similar to those in approximating the standard deviation of capital, output, consumption and investment. Second, the relative errors in approximating ρ are close to those in approximating the first-order autocorrelation for output. These results suggest that a good approximation for the moments of the AR(1) process is important in obtaining an accurate approximation for the business cycle moments.

Error Analysis

The relative errors reported in Table 2 have a number of sources. For the purpose of this discussion, we classify these into two groups. The first group of errors arises when solving the Bellman equation in (9). This includes the errors that arise when we restrict the choice of next-period capital to a discrete set of values, and the truncation errors that emerge when we approximate the fixed point of the Bellman equation using a finite number of iterations. The second group of errors occurs during the computation of the stationary distribution of the state variables. First, the transition matrix P, constructed using the discrete Markov chain and the computed policy function, is an approximation of the actual transition function. Second, truncation errors arise when we approximate the stationary distribution using a finite number of iterations. The second group of errors would not occur if Monte Carlo simulations are used to generate the business cycle moments. In this case, however, a new source of error arises when we estimate the actual moments by a finite sample.

When the actual policy function is known, it is possible to disentangle the two groups of errors. With logarithmic utility function and full depreciation, the policy function for next-period capital (in logarithmic terms) is given by

$$k_{t+1} = g\left(k_t, a_t\right) \equiv \ln \alpha\beta + a_t + \alpha k_t. \tag{14}$$

Now consider the following experiment. Construct a discrete state space $\widehat{\mathcal{S}}$ as in (11) using one of the six discretization methods. Construct the transition matrix P as in (12) *but replace the computed policy function* $\widehat{g}\left(k, a\right)$ *with the actual one in (14).* Iterate equation (13) successively to obtain an approximation for the stationary distribution of the state variables. Finally, use the approximate stationary distribution and the *actual* policy function $g\left(k, a\right)$ to compute the

business cycle moments. By replacing $\widehat{g}(k,a)$ with the actual policy function, this procedure effectively removes all the errors involved in solving the Bellman equation. The remaining errors are thus due to the approximation of the stationary distribution of the state variables. The results of this procedure are reported in panel (B) of Table 3. To facilitate comparison, the baseline results are shown in panel (A) of the same table.

It is immediate to see that the figures in the two panels are almost identical. Replacing the computed policy function with the actual one does not affect the approximation of the technology shock process. As a result, the approximated values for ρ, σ_ε and σ_a are identical in the two sets of results. As for the standard deviations of the endogenous variables, only minor discrepancies are observed in the two panels. In other words, even though we have removed all the errors in computing the policy function, the baseline results remain largely unchanged. This has two implications. First, this implies that almost all the relative errors in the baseline case are due to the approximation of the stationary distribution $\widehat{\boldsymbol{\pi}}$. Second, this means the choice of discretization method has only a relatively minor impact on the solution of the Bellman equation. In sum, this experiment illustrates that the choice of discretization method matters because it would significantly affect the approximation of the stationary distribution.

The same conclusion can be drawn from another experiment. Suppose now the business cycle moments are computed using Monte Carlo simulations. More specifically, after solving the dynamic programming problem in (9), the model is simulated using the actual AR(1) process and the computed policy function $\widehat{g}(k,a)$. Under this procedure, the choice of discretization method only affects the simulated moments through the computed policy function. Table 4 presents the relative errors obtained under this procedure alongside with the baseline results. The two methods of generating business cycle moments have produced very different results. When the model is simulated using the actual AR(1) process, all six discretization methods generate almost identical results. This again implies that the differences in the baseline results across the six discretization methods are due to the approximation of the stationary distribution $\widehat{\boldsymbol{\pi}}$.

The results in Table 4 also show that the accuracy of the Monte Carlo simulation method cannot be taken for granted. This method is able to yield highly accurate estimates for ρ, σ_ε and ρ_y. But it also yields a relative error of 2.6 percent when approximating the standard

deviations and an error of four percent when approximating σ_{ka}. When comparing between these and the baseline results, it is obvious that the baseline approach, equipped with the Rouwenhorst method, outperforms the Monte Carlo simulation method.

4.3 Robustness Check

In this section, it is shown that the relative performance of the six discretization methods are robust to changes in (i) the number of points in the discrete state space N, (ii) the persistence parameter ρ, and (iii) the standard deviation of the white noise process σ_ε.

Changing the Number of States

Table 5 compares the performance of the six methods under different choices of N. Intuitively, increasing the number of states in the Markov chain should improve the performance of the discretization methods. This is true for the Rouwenhorst method, the original Tauchen-Hussey method, the F-2 method, and the Adda-Cooper method. However, this is not true for the Tauchen (1986) method and the F-1 method.

The results in Table 5 show that the superior performance of the Rouwenhorst method is robust even when there are only two states in the discrete Markov chain. As explained in section 3.3, this method can always match the values of ρ, σ_ε and σ_a regardless of the choice of N. The relative errors in approximating the standard deviations of output, capital, consumption and investment are similar in all three cases. In particular, increasing the number of states from five to ten increases the precision only marginally. The original Tauchen-Hussey method has the lowest precision among the six in all three cases. Even when the number of states is increased to ten, the Tauchen-Hussey method can only replicate 57 percent of the actual value of σ_y. The performance of this method is much better when approximating ρ_y but the precision is still the lowest among the six.

The performance of the F-2 method and the Adda-Cooper method improves significantly when the number of states increases. Similar to the baseline results, the F-2 method performs better in terms of approximating the standard deviations of the endogenous variables, whereas the Adda-Cooper method performs better in approximating ρ_y.

Next, we consider the performance of the Tauchen (1986) method. As mentioned above, the precision of this method does not necessarily improve when the number of states increases.

When there are only two states, the relative errors in approximating the standard deviations are about sixteen percent. These drop to twelve percent when there are five states but rise back to eighteen percent when there are ten states.[7] In either case, the Tauchen (1986) method has a lower precision than the Rouwenhorst method, the F-1 method and the Adda-Cooper method. As in the baseline case, the Tauchen (1986) method performs better when approximating ρ_y. With a ten-state Markov chain, the relative error is about 0.16 percent, which is among the lowest in the group. Finally, Table 5 shows that, in terms of approximating the standard deviations, the F-1 method actually works best with a two-state Markov chain. The relative error in approximating σ_y is a mere 0.65 percent when there are only two states. Whereas, the relative error in approximating ρ_y remains the same in all three cases.

Changing the Persistence Parameter

Table 6 compares the performance of the six methods under different values of ρ. The superior performance of the Rouwenhorst method is robust to changes in this parameter. In particular, increasing the persistence of the AR(1) process from 0.5 to 0.979 has very little impact on its precision. This shows that the Rouwenhorst method is a reliable technique for approximating stationary AR(1) process in general.

The performance of the three quadrature-based methods and the Tauchen (1986) method is very sensitive to the value of ρ. Similar to Flodén (2008), our results show that the quadrature-based method and the Tauchen (1986) method work best in approximating AR(1) processes with low persistence. But unlike Flodén (2008) which only focuses on the parameters of the AR(1) process, the current study also considers the impact of these methods on the moments of the endogenous variables. When ρ equals to 0.5 or 0.6, the original Tauchen-Hussey method and its two variations can generate highly accurate approximations that are comparable to those generated by the Rouwenhorst method. The relative errors for the business cycle moments are all less than one percent. Within this range of ρ, the three quadrature-based methods are more accurate than the Tauchen (1986) method. When ρ equals to 0.5, the Tauchen (1986) method has a relative error of five percent in approximating σ_{ka} and an error of two percent in approximating ρ_y. However, the accuracies of the Tauchen-Hussey method

[7] A similar pattern is also observed in Flodén (2008) Table 2. The table shows that when $\rho = 0.98$, the relative error in approximating σ_a under the Tauchen (1986) method is 11.7 percent when $N = 5$ and 18.9 percent when $N = 10$.

and the F-2 method deteriorate quickly when the persistence parameter approaches one. For instance, the Tauchen-Hussey method has a relative error of 25 percent in approximating σ_y when ρ equals to 0.9 and an error of 61 percent when ρ is 0.979. A similar but less dramatic pattern is observed for the F-2 method. Among the three quadrature-based methods, the F-1 method is least sensitive to changes in the persistence parameter. Increasing this parameter from 0.7 to 0.979 raises the relative errors in approximating σ_y from 0.39 percent to three percent. The relative error in approximating ρ_y increases from 0.88 percent to 1.1 percent under the same change.

Unlike the quadrature-based methods, the Adda-Cooper method is more accurate when the underlying AR(1) process is more persistent. When ρ equals to 0.5, the relative errors in approximating σ_{ka} and ρ_y are 20 percent and four percent, respectively. These reduce to sixteen percent and two percent, respectively, when ρ is 0.979. The precision in approximating the standard deviations does not seem to be affected by the changes in ρ.

Finally, it is worth mentioning that the results of the two experiments conducted in the error analysis section are also robust to different values of the persistence parameter. These results are summarized as follow.[8] First, the figures reported in Table 6 are largely unaffected when we replace the computed policy function with the actual one. Second, when the business cycle moments are computed using Monte Carlo simulations, all six discretization methods generate very similar results.

Changing the Standard Deviation of the White Noise Process

The performance of the six methods under different values of σ_ε are shown in Table 7. In terms of approximating the AR(1) process, increasing the value of σ_ε from 0.001 to 0.1 does not seem to affect the performance of these methods. In terms of approximating the standard deviations of the endogenous variables and the covariance between a and k, the accuracies of the Tauchen (1986) method, the original Tauchen-Hussey method, the F-2 method and the Adda-Cooper method improve when the AR(1) process is less volatile. The opposite is true for the Rouwenhorst method. The variations in the relative errors, however, are not significant. More specifically, increasing σ_ε from 0.001 to 0.1 changes the relative errors by less than two percentage points in most cases. Unlike the other methods, the performance of

[8] The numerical results are not shown in the paper but are available from the authors upon request.

the F-1 method is more sensitive to the value of σ_ε. For instance, when σ_ε equals to 0.001 the relative errors in approximating σ_k and σ_{ka} are 0.5 percent and five percent, respectively. These become 1.6 percent and 2.3 percent, respectively, when σ_ε is 0.1. Finally, the precision of all six methods in approximating ρ_y is not sensitive to changes in the value of σ_ε.

4.4 Relaxing the Assumption of Full Depreciation

This section evaluates the performance of the six discretization methods in solving the stochastic growth model when the full depreciation assumption is relaxed. The rate of depreciation is now taken to be 2.5 percent, which is the same as in King and Rebelo (1999). All other parameters remain the same as in the baseline case. The same evaluation process is performed as in section 4.2. For each of the six discretization methods, we compute the business cycle moments using the baseline approach and the Monte Carlo simulation method. Without full depreciation, however, a closed-form solution for the policy function is not available and the actual values of the business cycle moments are unknown. Thus we derive a highly accurate approximation for the actual moments which is then used as our yardstick for comparison. To achieve this, we first construct an extremely fine discrete state space with 2000 grid points for capital and 400 states in the Markov chain constructed by the Rouwenhorst method. We then compute the business cycle moments using the baseline approach described earlier. The rationale for this procedure is as follows. As explained in the error analysis section, the baseline approach involves two groups of errors: (i) errors that arise when solving the Bellman equation, and (ii) errors that arise when computing the stationary distribution. When the number of grid points in the discrete state space is sufficiently large, the value function iteration method is able to yield highly accurate solutions for the Bellman equation. Thus, by adopting an extremely fine state space, the above procedure should render the first group of errors very small. As for the second group of errors, our baseline results for the full depreciation case show that combining the Rouwenhorst method and the baseline approach can yield a highly accurate approximation for the stationary distribution. As a robustness check on this procedure, we double the size of the state space and find that it has no effect on the computed statistics. The business cycle moments obtained under this procedure are referred to below as the true solutions.

The main findings of this exercise are as follows. First, the superior performance of the

baseline approach combined with the Rouwenhorst discretization method is robust to relaxing the full depreciation assumption. Second, the overall performance of the other methods deteriorates significantly when δ is less than one. Panel (A) of Table 8 shows the results obtained under the baseline approach for three different values of N. Panel (B) of the same table reports the simulation results. First note that the Rouwenhorst method has the best overall performance for each grid size N when comparing both across columns in Panel (A) and between Panels (A) and (B). Thus the Rouwenhorst method under the baseline approach is not only superior to the other methods but also to computing the statistics using Monte Carlo simulations. Second, note that the overall performance of the other methods, as measured by the size of the relative errors in their estimates, is substantially worse with δ set at 2.5 percent than in the full depreciation case (Tables 2 and 5). This is particularly true for the estimates of σ_{ka} and σ_i. For example, consider the F-1 method which has the second highest precision in the full depreciation case. With only five states in the Markov chain and full depreciation, this method generates a relative error of eight percent in approximating σ_{ka} and an error of about three percent in approximating σ_i (see Table 2). These become 26 percent and 21 percent, respectively, when δ equals 0.025. In contrast, relaxing the full depreciation assumption has only a negligible effect on the estimates of ρ_y.

A closer look at Panel (A) of Table 8 reveals that, similar to the results in Table 5, increasing the number of states in the Markov chain usually improves the accuracy of the approximations. However, the performance of the methods varies significantly when it comes to approximating the standard deviations and the covariance between k and a, even when N is large. For the Rouwenhorst method, a five-fold increase in the number of states only marginally affects the precision of the results. However, unlike the full depreciation case, increasing the number of states does not always improve the precision. In particular, the relatively large error in approximating σ_i remains even when there are 25 states. For the original Tauchen-Hussey method, its performance improves significantly when the fineness of the state space increases. However, even when there are 25 states, this method can only replicate 67 percent of the true value of σ_{ka} and 83 percent of the true value of σ_y. The overall performance of the Tauchen (1986) method and the F-1 method is also rather disappointing in this case. A five-fold increase in the number of states does not seem to have a significant impact on their precision. On the other hand, when N is large the F-2 method is able to yield

highly accurate approximations that are comparable to those generated by the Rouwenhorst method. It thus has the best performance among the three quadrature-based methods. As for the Adda-Cooper method, relatively large errors remain even when there are 25 states. For instance, the relative errors in approximating σ_{ka} and σ_i are about five percent.

Unlike the full depreciation case, the six discretization methods under the Monte Carlo simulation approach do not generate near identical results. This can be seen by comparing the columns in Panel (B) of Table 8. Thus the choice of discretization method matters even when the business cycle moments are computed using Monte Carlo simulations. This is due to the following reason. In the absence of full depreciation, the policy function for next-period capital (in logarithms), represented by

$$k_{t+1} = g(k_t, a_t),$$

is no longer a linear function. Consequently, additional approximation errors arise when we compute $g(k_t, a_t)$ for values of k_t and a_t that are outside the discrete state space. The size of these errors depends on the location of the grid points and hence the choice of the discretization method. As the number of states in the Markov chain increases, the state space becomes finer and the errors associated with the interpolation procedure falls. For this reason, a five-fold increase in N significantly reduces the relative errors of the discretization methods. Finally, under the Monte Carlo simulation approach, no single method dominates all others in all three choices of N. When there are five states in the Markov chain, the Rouwenhorst method has the best overall performance within the group. But when there are 25 states, the Adda-Cooper method has the best overall performance. In this case, the Tauchen (1986) method, the original Tauchen-Hussey method and the F-2 method all perform equally well as the Rouwenhorst method.

5 Conclusions

This paper re-examines the Rouwenhorst method of constructing a discrete-valued Markov chain to approximate a given first-order autoregressive process. Under this method, the constructed Markov chain can be calibrated to match the conditional and unconditional mean, the conditional and unconditional variance and the first-order autocorrelation of any station-

ary AR(1) process. Because of this distinctive feature, the Rouwenhorst method is more reliable than the Tauchen (1986) method and the Tauchen-Hussey method to approximate highly persistent processes. In this paper, a new and simpler procedure for generating the transition matrix in the Rouwenhorst method is developed and the first formal proof for all the important properties of the constructed Markov chain is provided.

In the quantitative analysis, the Rouwenhorst method is compared to five other discretization methods. These methods are evaluated based on their performance in approximating the business cycle moments generated by the standard neoclassical growth model without leisure. Two approaches to generate these moments are considered. In the baseline approach, an approximation for the stationary distribution of the state variables is first computed. In the second approach, the moments of interest are generated using Monte Carlo simulations. Our quantitative analysis yields two important messages. *First, under both approaches, the choice of approximation method can have a large impact on the accuracy of the solutions.* Under the baseline approach, an accurate approximation of the moments of the AR(1) process is important in accurately approximating the business cycle moments. The Rouwenhorst method has the best performance in this regard and outperforms the other methods by a significant margin. Its superior performance is robust under a wide range of parameter values. Under the second approach, no single method dominates all others in all cases. When a realistic value of the depreciation rate is used, the Rouwenhorst method again has the best overall performance when there are only five states in the Markov chain. However, when the fineness of the state space increases, the Adda-Cooper method improves significantly and yields the best overall performance. *Second, the simulation method is not the best approach to generate the business cycle statistics in the neoclassical growth model.* Our results show that the combination of the baseline approach and the Rouwenhorst method has a higher degree of accuracy than the simulation method.

In this paper, we use a standard representative-agent model as our test model. We believe that similar results can be obtained in heterogeneous-agent economies. However, we leave a detailed exploration of these models for future research.

Table 2 Baseline Results

(A) Approximating the AR(1) process

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
ρ	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000
σ_ε	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000
σ_a	1.1159	0.4006	1.0215	0.7742	0.9471	1.0000

(B) Approximating the Variance-Covariance Matrix for State Variables

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
σ_k	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
σ_{ka}	1.2741	0.1401	1.0818	0.6071	0.8464	0.9986

(C) Approximating Business Cycle Moments

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
σ_y	1.1223	0.3880	1.0310	0.7763	0.9338	0.9995
σ_c	1.1219	0.3879	1.0295	0.7776	0.9343	1.0000
σ_i	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
ρ_y	1.0074	0.9538	1.0107	1.0063	0.9807	1.0000

T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $\rho = 0.979$, $N = 5$.

Table 3 Error Analysis

(A) Using Computed Policy Function (Baseline case)

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
ρ	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000
σ_ε	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000
σ_a	1.1159	0.4006	1.0215	0.7742	0.9471	1.0000
σ_k	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
σ_{ka}	1.2741	0.1401	1.0818	0.6071	0.8464	0.9986
σ_y	1.1223	0.3880	1.0310	0.7763	0.9338	0.9995
σ_c	1.1219	0.3879	1.0295	0.7776	0.9343	1.0000
σ_i	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
ρ_y	1.0074	0.9538	1.0107	1.0063	0.9807	1.0000

(B) Using Actual Policy Function

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
ρ	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000
σ_ε	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000
σ_a	1.1159	0.4006	1.0212	0.7742	0.9471	1.0000
σ_k	1.1219	0.3880	1.0292	0.7777	0.9343	1.0000
σ_{ka}	1.2726	0.1400	1.0762	0.6104	0.8475	1.0000
σ_y	1.1219	0.3879	1.0292	0.7777	0.9343	1.0000
σ_c	1.1219	0.3879	1.0292	0.7777	0.9343	1.0000
σ_i	1.1219	0.3880	1.0292	0.7777	0.9343	1.0000
ρ_y	1.0074	0.9537	1.0107	1.0063	0.9807	1.0000

T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, σ_ε= 0.0072, $\rho = 0.979$, $N = 5$.

Table 4 Baseline Approach vs. Monte Carlo Simulations

(A) Baseline case

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
ρ	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000
σ_ε	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000
σ_a	1.1159	0.4006	1.0215	0.7742	0.9471	1.0000
σ_k	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
σ_{ka}	1.2741	0.1401	1.0818	0.6071	0.8464	0.9986
σ_y	1.1223	0.3880	1.0310	0.7763	0.9338	0.9995
σ_c	1.1219	0.3879	1.0295	0.7776	0.9343	1.0000
σ_i	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
ρ_y	1.0074	0.9538	1.0107	1.0063	0.9807	1.0000

(B) Monte Carlo Simulations

	Generated Values Relative to True Values					
	Tauchen	T-H	F-1	F-2	A-C	R
ρ	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983
σ_ε	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001
σ_a	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748
σ_k	0.9744	0.9745	0.9745	0.9745	0.9746	0.9745
σ_{ka}	0.9575	0.9575	0.9575	0.9575	0.9576	0.9575
σ_y	0.9744	0.9744	0.9744	0.9744	0.9744	0.9744
σ_c	0.9744	0.9744	0.9744	0.9744	0.9744	0.9744
σ_i	0.9744	0.9745	0.9745	0.9745	0.9746	0.9745
ρ_y	0.9991	0.9991	0.9991	0.9991	0.9991	0.9991

T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, σ_ε= 0.0072, $\rho = 0.979$, $N = 5$.

Table 5 Changing the Number of States in the Markov Chain

	$N = 2$						$N = 5$ (Baseline)						$N = 10$					
	Generated Values Relative to True Values						Generated Values Relative to True Values						Generated Values Relative to True Values					
	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R
ρ	1.0214	0.7688	1.0215	1.0206	0.8879	1.0000	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000	1.0045	0.9867	1.0214	1.0006	1.0038	1.0000
σ_ε	0.0465	0.6584	0.0000	0.0805	1.9346	1.0000	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000	1.0963	0.9493	0.0225	0.8886	1.2781	1.0000
σ_a	0.8318	0.2039	1.0000	0.4071	0.7979	1.0000	1.1159	0.4006	1.0215	0.7742	0.9471	1.0000	1.1874	0.5860	1.0076	0.9558	0.9793	1.0000
σ_k	0.8433	0.1844	1.0040	0.4095	0.7718	0.9966	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986	1.1892	0.5794	1.0122	0.9583	0.9753	0.9988
σ_{ka}	0.7183	0.0283	1.0281	0.1705	0.5400	0.9966	1.2741	0.1401	1.0818	0.6071	0.8464	0.9986	1.4171	0.3260	1.0444	0.9190	0.9386	0.9988
σ_y	0.8399	0.1870	1.0065	0.4099	0.7682	0.9989	1.1223	0.3880	1.0310	0.7763	0.9338	0.9995	1.1889	0.5788	1.0144	0.9573	0.9743	0.9996
σ_c	0.8383	0.1882	1.0078	0.4101	0.7664	1.0000	1.1219	0.3879	1.0295	0.7776	0.9343	1.0000	1.1888	0.5784	1.0154	0.9569	0.9739	1.0000
σ_i	0.8433	0.1844	1.0040	0.4095	0.7718	0.9966	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986	1.1892	0.5794	1.0122	0.9583	0.9753	0.9988
ρ_y	1.0107	0.8752	1.0107	1.0103	0.9422	1.0000	1.0074	0.9538	1.0107	1.0063	0.9807	1.0000	1.0016	0.9817	1.0107	1.0015	0.9922	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $\rho = 0.979$.

Table 6 Changing the Persistence Parameter

	$\rho = 0.5$						$\rho = 0.6$						$\rho = 0.7$					
	Generated Values Relative to True Values						Generated Values Relative to True Values						Generated Values Relative to True Values					
	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R
ρ	0.9686	0.9997	1.0007	1.0000	0.9310	1.0000	0.9739	0.9986	1.0039	0.9999	0.9471	1.0000	0.9798	0.9953	1.0174	0.9997	0.9665	1.0000
σ_ε	1.0139	0.9994	0.9998	0.9999	0.9737	1.0000	1.0234	0.9972	0.9977	0.9993	0.9888	1.0000	1.0392	0.9905	0.9819	0.9969	1.0112	1.0000
σ_a	1.0012	0.9990	0.9999	0.9999	0.9471	1.0000	1.0035	0.9950	0.9996	0.9993	0.9471	1.0000	1.0085	0.9793	0.9982	0.9963	0.9471	1.0000
σ_k	0.9973	1.0025	0.9973	1.0048	0.9364	1.0010	0.9983	0.9973	0.9988	1.0017	0.9345	0.9991	0.9996	0.9788	1.0085	0.9969	0.9243	0.9983
σ_{ka}	0.9581	0.9995	0.9971	1.0035	0.8050	1.0011	0.9629	0.9873	1.0021	0.9998	0.8072	0.9981	0.9734	0.9456	1.0238	0.9925	0.8050	0.9985
σ_y	0.9952	0.9996	0.9993	1.0009	0.9336	1.0002	0.9964	0.9948	1.0000	0.9997	0.9316	0.9996	1.0003	0.9769	1.0039	0.9963	0.9282	0.9996
σ_c	0.9944	0.9983	1.0004	0.9992	0.9324	1.0001	0.9957	0.9938	1.0006	0.9989	0.9304	1.0000	1.0007	0.9761	1.0018	0.9962	0.9302	1.0003
σ_i	0.9973	1.0025	0.9973	1.0048	0.9364	1.0010	0.9983	0.9973	0.9988	1.0017	0.9345	0.9991	0.9996	0.9788	1.0085	0.9969	0.9243	0.9983
ρ_y	0.9824	1.0002	0.9997	1.0009	0.9588	1.0001	0.9828	0.9985	1.0014	1.0003	0.9599	0.9999	0.9836	0.9943	1.0088	0.9997	0.9613	0.9998

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, $\sigma_\varepsilon = 0.0072$, $N = 5$.

Table 6 (continued) Changing the Persistence Parameter

	$\rho = 0.9$						$\rho = 0.95$						$\rho = 0.979$ (baseline)					
	Generated Values Relative to True Values						Generated Values Relative to True Values						Generated Values Relative to True Values					
	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R
ρ	0.9944	0.9689	1.0911	0.9986	1.0060	1.0000	1.0081	0.9550	1.0524	1.0025	1.0067	1.0000	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000
σ_ε	1.1260	0.9379	0.4217	0.9379	1.1403	1.0000	1.0643	0.9101	0.0586	0.8142	1.2822	1.0000	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000
σ_a	1.0543	0.7701	1.0008	0.9347	0.9471	1.0000	1.0883	0.5904	1.0135	0.8639	0.9471	1.0000	1.1159	0.4006	1.0215	0.7742	0.9471	1.0000
σ_k	1.0511	0.7564	1.0259	0.9337	0.9258	1.0024	1.0908	0.5753	1.0326	0.8660	0.9346	0.9948	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
σ_{ka}	1.0878	0.5453	1.1319	0.8716	0.8248	1.0024	1.1901	0.3108	1.1082	0.7530	0.8398	0.9948	1.2741	0.1401	1.0818	0.6071	0.8464	0.9986
σ_y	1.0493	0.7558	1.0297	0.9341	0.9291	1.0008	1.0897	0.5750	1.0324	0.8658	0.9328	0.9983	1.1223	0.3880	1.0310	0.7763	0.9338	0.9995
σ_c	1.0485	0.7556	1.0316	0.9344	0.9307	1.0000	1.0892	0.5748	1.0323	0.8657	0.9319	1.0000	1.1219	0.3879	1.0295	0.7776	0.9343	1.0000
σ_i	1.0511	0.7564	1.0259	0.9337	0.9258	1.0024	1.0908	0.5753	1.0326	0.8660	0.9346	0.9948	1.1232	0.3882	1.0342	0.7734	0.9330	0.9986
ρ_y	0.9917	0.9702	1.0454	0.9996	0.9718	1.0000	1.0012	0.9602	1.0260	1.0030	0.9765	0.9999	1.0074	0.9538	1.0107	1.0063	0.9807	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, σ_ε= 0.0072, $N = 5$.

Table 7 Changing the Standard Deviation of the White Noise Process

	σ_ε= 0.001						σ_ε= 0.01						σ_ε= 0.1					
	Generated Values Relative to True Values						Generated Values Relative to True Values						Generated Values Relative to True Values					
	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R
ρ	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000
σ_ε	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000
σ_a	1.1159	0.4006	1.0213	0.7742	0.9471	1.0000	1.1159	0.4006	1.0216	0.7742	0.9471	1.0000	1.1159	0.4006	1.0212	0.7742	0.9471	1.0000
σ_k	1.1126	0.4166	1.0052	0.7911	0.9546	0.9910	1.1246	0.3864	1.0318	0.7798	0.9328	1.0022	1.1218	0.3879	0.9838	0.7775	0.9342	1.0000
σ_{ka}	1.2614	0.1497	1.0503	0.6203	0.8651	0.9905	1.2756	0.1395	1.0794	0.6121	0.8462	1.0022	1.2725	0.1400	1.0236	0.6103	0.8475	1.0000
σ_y	1.1187	0.3965	1.0211	0.7819	0.9405	0.9969	1.1228	0.3875	1.0303	0.7784	0.9338	1.0007	1.1219	0.3879	1.0130	0.7776	0.9342	1.0000
σ_c	1.1218	0.3875	1.0290	0.7778	0.9340	1.0000	1.1219	0.3880	1.0296	0.7777	0.9343	1.0000	1.1219	0.3879	1.0297	0.7777	0.9343	1.0000
σ_i	1.1126	0.4166	1.0052	0.7911	0.9546	0.9910	1.1246	0.3864	1.0318	0.7798	0.9328	1.0022	1.1218	0.3879	0.9838	0.7775	0.9342	1.0000
ρ_y	1.0074	0.9554	1.0107	1.0063	0.9810	0.9999	1.0074	0.9536	1.0107	1.0063	0.9807	1.0000	1.0074	0.9537	1.0107	1.0063	0.9807	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 1$, $\alpha = 0.33$, $\beta = 0.984$, ρ= 0.979, $N = 5$.

Table 8 Results obtained when $\delta = 0.025$.

(A) Baseline Approach

	$N = 5$						$N = 10$						$N = 25$					
	Generated Values Relative to True Values						Generated Values Relative to True Values						Generated Values Relative to True Values					
	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R
ρ	1.0157	0.9453	1.0215	1.0096	0.9993	1.0000	1.0045	0.9867	1.0214	1.0006	1.0038	1.0000	1.0000	0.9980	1.0176	1.0000	1.0012	1.0000
σ_ε	0.6260	0.8905	0.0002	0.5019	1.5599	1.0000	1.0963	0.9493	0.0225	0.8886	1.2781	1.0000	1.0982	0.9877	0.4192	0.9994	1.0958	1.0000
σ_a	1.1159	0.4006	1.0212	0.7742	0.9471	1.0000	1.1874	0.5860	1.0076	0.9558	0.9793	1.0000	1.0969	0.8481	0.9998	0.9996	0.9937	1.0000
σ_k	1.0969	0.3306	1.0123	0.7430	0.8816	0.9907	1.1924	0.5457	1.0090	0.9572	0.9528	1.0027	1.0973	0.8381	1.0508	0.9996	0.9864	0.9983
σ_{ka}	1.4000	0.0803	1.2629	0.6479	0.6580	0.9906	1.4553	0.2538	1.2444	0.9357	0.8461	1.0033	1.2025	0.6721	1.2313	0.9983	0.9518	0.9957
σ_y	1.1406	0.3506	1.0616	0.7828	0.8882	0.9970	1.1955	0.5502	1.0514	0.9605	0.9532	1.0010	1.0968	0.8359	1.0502	0.9994	0.9858	0.9990
σ_c	1.2251	0.2903	1.2024	0.8405	0.7934	1.0035	1.2148	0.5000	1.1752	0.9772	0.9135	1.0051	1.0998	0.8183	1.1401	1.0033	0.9759	1.0032
σ_i	0.8625	0.6467	0.7902	0.6459	1.2673	1.0109	1.1562	0.7899	0.7217	0.9341	1.1336	1.0245	1.1131	0.9395	0.7716	1.0161	1.0564	1.0134
ρ_y	1.0072	0.9411	1.0101	1.0061	0.9779	0.9999	1.0016	0.9790	1.0102	1.0015	0.9915	1.0000	1.0000	0.9958	1.0084	1.0000	0.9975	1.0000

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 0.025$, $\alpha = 0.33$, $\beta = 0.984$, ρ= 0.979, σ_ε= 0.0072.

Table 8 Results obtained when $\delta = 0.025$.

(B) Monte Carlo Simulations

	$N = 5$						$N = 10$						$N = 25$					
	Generated Values Relative to True Values						Generated Values Relative to True Values						Generated Values Relative to True Values					
	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R	Tau	T-H	F-1	F-2	A-C	R
ρ	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983
σ_ε	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001
σ_a	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748	0.9748
σ_k	0.8500	0.5265	0.7932	0.7879	1.1018	0.9571	0.9403	0.7791	0.7962	0.9374	1.0197	0.9662	09631	0.9549	0.8396	0.9623	0.9816	0.9636
σ_{ka}	0.8307	0.4885	0.7725	0.7677	1.0754	0.9331	0.9205	0.7439	0.7743	0.9153	0.9968	0.9452	0.9417	0.9305	0.8289	0.9405	0.9601	0.9422
σ_y	0.9335	0.8296	0.9152	0.9146	1.0121	0.9664	0.9618	0.9090	0.9159	0.9607	0.9864	0.9699	0.9688	0.9660	0.9320	0.9684	0.9745	0.9689
σ_c	0.9569	0.9653	0.9512	0.9632	0.9857	0.9670	0.9651	0.9725	0.9516	0.9646	0.9734	0.9669	0.9653	0.9709	0.9570	0.9650	0.9685	0.9660
σ_i	0.8903	0.7433	0.8304	0.8866	1.1423	0.9939	0.9782	0.9202	0.8337	0.9731	1.0546	1.0001	0.9908	1.0095	0.8885	0.9891	1.0145	0.9943
ρ_y	0.9980	0.9949	0.9975	0.9976	0.9998	0.9988	0.9987	0.9975	0.9975	0.9987	0.9993	0.9989	0.9989	0.9989	0.9980	0.9989	0.9990	0.9989

Tau stands for the Tauchen (1986) method; T-H stands for the original Tauchen-Hussey method; F-1 stands for the first variation of T-H; F-2 stands for the second variation; A-C stands for the Adda-Cooper method; R stands for the Rouwenhorst method.

Parameter values: $\delta = 0.025$, $\alpha = 0.33$, $\beta = 0.984$, ρ= 0.979, σ_ε= 0.0072.

Appendix A

Fix $N \geq 3$. The objective of this section is to derive a set of equations that can be used to describe the elements in Π_N. The proof of Proposition 1 is built upon these equations.

To begin with, the elements in the first and the last rows of Π_N can be obtained by expanding the polynomials $[p + (1-p)\,t]^{N-1}$ and $(1-q+qt)^{N-1}$, respectively. Using the binomial formula, we can obtain

$$\pi_{1,j}^{(N)} = \binom{N-1}{j-1} p^{N-j} (1-p)^{j-1}, \tag{15}$$

and

$$\pi_{N,j}^{(N)} = \binom{N-1}{j-1} (1-q)^{N-j} q^{j-1}, \tag{16}$$

for $j = 1, 2, ..., N$.

For all other rows, i.e., $i = 2, ..., N-1$, the elements in Π_N can be defined recursively using the elements in Π_{N-1}. Begin with the system for $N - 1 \geq 2$. The system of polynomials is given by

$$\Phi(t; N-1, i) = [p + (1-p)\,t]^{N-1-i} (1-q+qt)^{i-1} = \sum_{j=1}^{N-1} \pi_{i,j}^{(N-1)} t^{j-1},$$

for $i = 1, ..., N-1$. There are two ways to relate this system to the one for N:

$$\Phi(t; N, i) = [p + (1-p)\,t]\,\Phi(t; N-1, i), \tag{17}$$

for $i = 1, ..., N-1$, and

$$\Phi(t; N, i) = (1-q+qt)\,\Phi(t; N-1, i-1), \tag{18}$$

for $i = 2, ..., N$. Substituting (5) into (17) gives

$$\begin{aligned} \sum_{j=1}^{N} \pi_{i,j}^{(N)} t^{j-1} &= [p + (1-p)\,t] \sum_{j=1}^{N-1} \pi_{i,j}^{(N-1)} t^{j-1} \\ &= \sum_{j=1}^{N-1} p\pi_{i,j}^{(N-1)} t^{j-1} + \sum_{j=1}^{N-1} (1-p)\,\pi_{i,j}^{(N-1)} t^{j}, \end{aligned}$$

for $i = 1, ..., N-1$. Similarly, substituting (5) into (18) would give

$$\begin{aligned}\sum_{j=1}^{N} \pi_{i,j}^{(N)} t^{j-1} &= (1-q+qt) \sum_{j=1}^{N-1} \pi_{(i-1),j}^{(N-1)} t^{j-1} \\ &= \sum_{j=1}^{N-1} (1-q)\, \pi_{(i-1),j}^{(N-1)} t^{j-1} + \sum_{j=1}^{N-1} q\pi_{(i-1),j}^{(N-1)} t^{j},\end{aligned}$$

for $i = 2, ..., N$. The following can be obtained by comparing the coefficients for $i = 1, 2, ..., N-1$,

$$\pi_{i,1}^{(N)} = p\pi_{i,1}^{(N-1)} = (1-q)\, \pi_{(i-1),1}^{(N-1)} \tag{19}$$

$$\begin{aligned}\pi_{i,j}^{(N)} &= p\pi_{i,j}^{(N-1)} + (1-p)\, \pi_{i,(j-1)}^{(N-1)} \\ &= (1-q)\, \pi_{(i-1),j}^{(N)} + q\pi_{(i-1),(j-1)}^{(N)}, \quad \text{for } j = 2, ..., N-1,\end{aligned} \tag{20}$$

and

$$\pi_{i,N}^{(N)} = (1-p)\, \pi_{i,(N-1)}^{(N-1)} = q\pi_{(i-1),N}^{(N-1)}. \tag{21}$$

Appendix B

Proof of Proposition 1

The case for $N = 2$ and $N = 3$ have already been proved in the text. So fix $N \geq 4$. The elements in the Rouwenhorst matrix $\Theta_N = \left[\theta_{i,j}^{(N)}\right]$ are governed by the following sets of equations:

For the elements in the first row,

$$\theta_{1,j}^{(N)} = \begin{cases} p\theta_{1,j}^{(N-1)} & \text{if } j = 1 \\ p\theta_{1,j}^{(N-1)} + (1-p)\, \theta_{1,(j-1)}^{(N-1)} & \text{if } j = 2, ..., N-1 \\ (1-p)\, \theta_{1,(j-1)}^{(N-1)} & \text{if } j = N. \end{cases} \tag{22}$$

For the elements in the final row,

$$\theta_{N,j}^{(N)} = \begin{cases} (1-q)\,\theta_{(N-1),j}^{(N-1)} & \text{if } j = 1 \\ (1-q)\,\theta_{(N-1),j}^{(N-1)} + q\theta_{(N-1),(j-1)}^{(N-1)} & \text{if } j = 2, ..., N-1 \\ q\theta_{(N-1),(j-1)}^{(N-1)} & \text{if } j = N. \end{cases} \tag{23}$$

For the elements in row $i = 2, ..., N-1$,

$$\theta_{i,j}^{(N)} = \begin{cases} \frac{1}{2}\left[p\theta_{i,j}^{(N-1)} + (1-q)\,\theta_{(i-1),j}^{(N-1)}\right] & \text{if } j = 1 \\ \frac{1}{2}\left[(1-p)\,\theta_{i,(j-1)}^{(N-1)} + q\theta_{(i-1),(j-1)}^{(N-1)}\right] & \text{if } j = N, \end{cases} \tag{24}$$

and for $j = 2, ..., N-1$,

$$\begin{aligned} \theta_{i,j}^{(N)} &= \frac{1}{2}\left[p\theta_{i,j}^{(N-1)} + (1-p)\,\theta_{i,(j-1)}^{(N-1)} + (1-q)\,\theta_{(i-1),j}^{(N-1)}\right. \\ &\quad \left. + q\theta_{(i-1),(j-1)}^{(N-1)}\right], \end{aligned} \tag{25}$$

For any given Θ_{N-1}, the system of equations (22)-(25) defines a unique Θ_N. Similarly, for any given Π_{N-1}, the system of equations (15)-(21) defines a unique Π_N. Since $\Theta_2 = \Pi_2$, it suffice to show that the elements in Π_N generated by (15)-(21) satisfies the system (22)-(25).

Consider the first row (i.e., $i = 1$) in Π_N. According to (15),

$$\pi_{11}^{(N)} = p^{N-1} = p\pi_{11}^{(N-1)},$$

and

$$\pi_{1,N}^{(N)} = (1-p)^{N-1} = (1-p)\,\pi_{1,(N-1)}^{(N-1)}.$$

For $j = 2, ..., N-1$, since

$$\pi_{1,j}^{(N-1)} = \binom{N-2}{j-1} p^{N-1-j}\,(1-p)^{j-1},$$

$$\pi_{1,(j-1)}^{(N-1)} = \binom{N-2}{j-2} p^{N-j}\,(1-p)^{j-2},$$

and

$$\binom{N-1}{j-1} = \binom{N-2}{j-1} + \binom{N-2}{j-2},$$

we have

$$\pi_{1,j}^{(N)} = p\pi_{1,j}^{(N-1)} + (1-p)\,\pi_{1,(j-1)}^{(N-1)}.$$

This shows that the elements in the first row of Π_N satisfies (22). Using (16) and the same procedure, one can show that the elements in the last row of Π_N satisfies (23).

The rest of the proof follows immediately from (19)-(21). For any row $i = 2, ..., N-1$ in Π_N, (19) implies

$$\pi_{i,1}^{(N)} = \frac{1}{2}\left[p\pi_{i,1}^{(N-1)} + (1-q)\,\pi_{(i-1),1}^{(N-1)}\right].$$

Similarly, (20) and (21) imply

$$\pi_{i,N}^{(N)} = \frac{1}{2}\left[(1-p)\,\pi_{i,(N-1)}^{(N-1)} + q\pi_{(i-1),N}^{(N-1)}\right],$$

and

$$\begin{aligned}\pi_{ij}^{(N)} &= \frac{1}{2}\left[p\pi_{ij}^{(N-1)} + (1-p)\,\pi_{i,(j-1)}^{(N-1)} + (1-q)\,\pi_{(i-1),j}^{(N-1)}\right. \\ &\quad \left. + q\pi_{(i-1),(j-1)}^{(N-1)}\right], \end{aligned} \tag{26}$$

for $j = 2, ..., N-1$, respectively. Thus all the elements in row $i = 2, ..., N-1$ in Π_N satisfies (24) and (25). This completes the proof.

Proof of Lemma 2

It suffice to check that all the elements of Π_N are strictly positive. From (15) and (16), it is obvious that the elements in the first and the last rows are strictly positive. For the other rows, a simple induction argument is used. First, Π_2 is a stochastic matrix with non-zero entries. Suppose the result is true for $N-1 \geq 2$. It follows from (19)-(21) that $\pi_{ij}^{(N)} > 0$ for $i = 2, ..., N-1$ and for $j = 1, 2, ..., N$. This completes the proof.

Proof of Proposition 4

As mentioned in the proof of Proposition 1, the first column of Π_N is given by

$$\pi_{i,1}^{(N)} = p^{N-i}\left(1-q\right)^{i-1},$$

for $i = 1, 2, ..., N$. Define $\widehat{\lambda}_i^{(N)}$ as in (6). Then

$$\begin{aligned}
\sum_{i=1}^{N} \widehat{\lambda}_i^{(N)} \pi_{i,1}^{(N)} &= \sum_{i=1}^{N} \binom{N-1}{i-1} s^{N-i}\left(1-s\right)^{i-1} p^{N-i}\left(1-q\right)^{i-1} \\
&= \sum_{i=1}^{N} \binom{N-1}{i-1} \left(sp\right)^{N-i}\left(1-s\right)^{i-1}\left(1-q\right)^{i-1} \\
&= \left[sp + \left(1-s\right)\left(1-q\right)\right]^{N} \\
&= s^N = \widehat{\lambda}_1^{(N)}.
\end{aligned}$$

For all other columns except the first one, an induction argument is used to prove the result. As mentioned in the text, the guess is correct when $N = 2$. Suppose the guess is correct for some $N \geq 2$, i.e.,

$$\widehat{\lambda}_j^{(N)} = \sum_{i=1}^{N} \widehat{\lambda}_i^{(N)} \pi_{i,j}^{(N)}, \qquad \text{for } j = 1, 2, ..., N. \tag{27}$$

We have already proved that this is true when $j = 1$, so proceeds to $j = 2, ..., N+1$.

Using (6), the following can be derived

$$\widehat{\lambda}_i^{(N+1)} = \begin{cases} s\widehat{\lambda}_i^{(N)} & \text{for } i = 1 \\ s\widehat{\lambda}_i^{(N)} + \left(1-s\right)\widehat{\lambda}_{i-1}^{(N)} & \text{for } i = 2, ..., N, \\ \left(1-s\right)\widehat{\lambda}_{i-1}^{(N)} & \text{for } i = N+1. \end{cases} \tag{28}$$

Using these one can obtain

$$\begin{aligned}
&\sum_{i=1}^{N+1}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)} \\
&= \widehat{\lambda}_1^{(N+1)}\pi_{1,j}^{(N+1)}+\sum_{i=2}^{N}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)}+\widehat{\lambda}_{N+1}^{(N+1)}\pi_{(N+1),j}^{(N+1)} \\
&= s\widehat{\lambda}_1^{(N)}\pi_{1,j}^{(N+1)}+\sum_{i=2}^{N}\left[s\widehat{\lambda}_i^{(N)}+(1-s)\,\widehat{\lambda}_{i-1}^{(N)}\right]\pi_{i,j}^{(N+1)}+(1-s)\,\widehat{\lambda}_N^{(N+1)}\pi_{(N+1),j}^{(N+1)} \\
&= \sum_{i=1}^{N}s\widehat{\lambda}_i^{(N)}\pi_{i,j}^{(N+1)}+\sum_{i=1}^{N-1}(1-s)\,\widehat{\lambda}_i^{(N)}\pi_{(i+1),j}^{(N+1)}+(1-s)\,\widehat{\lambda}_N^{(N+1)}\pi_{(N+1),j}^{(N+1)} \\
&= \sum_{i=1}^{N}s\widehat{\lambda}_i^{(N)}\pi_{i,j}^{(N+1)}+\sum_{i=1}^{N}(1-s)\,\widehat{\lambda}_i^{(N)}\pi_{(i+1),j}^{(N+1)}. \qquad (29)
\end{aligned}$$

Based on (20), the following can be obtained

$$\pi_{i,j}^{(N+1)}=p\pi_{i,j}^{(N)}+(1-p)\,\pi_{i,j-1}^{(N)},$$

and

$$\pi_{i+1,j}^{(N+1)}=(1-q)\,\pi_{i,j}^{(N)}+q\pi_{i,(j-1)}^{(N)},$$

for $j=2,3,...,N$. Substituting these into (29) gives

$$\begin{aligned}
&\sum_{i=1}^{N+1}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)} \\
&= s\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\left[p\pi_{i,j}^{(N)}+(1-p)\,\pi_{i,(j-1)}^{(N)}\right]+(1-s)\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\left[(1-q)\,\pi_{i,j}^{(N)}+q\pi_{i,(j-1)}^{(N)}\right] \\
&= [sp+(1-s)\,(1-q)]\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\pi_{i,j}^{(N)}+[s\,(1-p)+(1-s)\,q]\sum_{i=1}^{N}\widehat{\lambda}_i^{(N)}\pi_{i,(j-1)}^{(N)}.
\end{aligned}$$

Using the induction hypothesis (27), the following can be obtained

$$\begin{aligned}
\sum_{i=1}^{N+1}\widehat{\lambda}_i^{(N+1)}\pi_{i,j}^{(N+1)} &= [sp+(1-s)\,(1-q)]\,\widehat{\lambda}_j^{(N)}+[s\,(1-p)+(1-s)\,q]\,\widehat{\lambda}_{j-1}^{(N)} \\
&= s\widehat{\lambda}_j^{(N)}+(1-s)\,\widehat{\lambda}_{j-1}^{(N)} \\
&= \widehat{\lambda}_j^{(N+1)},
\end{aligned}$$

for $j = 2, 3, ..., N$. The last line is obtained by using (28). Since $\sum_{i=1}^{N+1} \widehat{\lambda}_i^{(N+1)} = 1$ and $\sum_{j=1}^{N+1} \pi_{i,j}^{(N+1)} = 1$, the remaining equation

$$\sum_{i=1}^{N+1} \widehat{\lambda}_i^{(N+1)} \pi_{i,j}^{(N+1)} = \widehat{\lambda}_j^{(N+1)}, \qquad \text{for } j = N+1,$$

must be satisfied. This completes the proof.

Appendix C

The objective of this section is to derive the moments listed on Table 1. Since it is understood that these are moments for an N-state Markov chain, the notations $\pi_{i,j}^{(N)}$ and $\lambda_j^{(N)}$ are simplified to become $\pi_{i,j}$ and λ_j, respectively.

Preliminaries

The following result is used in deriving the conditional mean for the Markov chain.

Lemma 5 *For any $N \geq 2$, and for $i = 1, ..., N$,*

$$\sum_{j=1}^{N} \pi_{i,j} (j-1) = (1-p)(N-i) + (i-1) q, \tag{30}$$

$$\sum_{j=1}^{N} \pi_{1,j} (j-1)^2 = \left[\sum_{j=1}^{N} \pi_{i,j} (j-1) \right]^2 + (N-i)(1-p) p + (i-1) q (1-q). \tag{31}$$

Proof. Recall the following expression

$$[p + (1-p) t]^{N-i} (1 - q + qt)^{i-1} = \sum_{j=1}^{N} \pi_{i,j} t^{j-1}, \tag{32}$$

for $i = 1, ..., N$. Equation (30) can be obtained in two steps: (i) Differentiate both sides of (32) with respect to t. (ii) Set $t = 1$.

Equation (31) can be obtained as follows: Fix $i = 1, ..., N$. Differentiate both sides of (32) with respect to t twice and set $t = 1$. This gives

$$\begin{aligned}\sum_{j=1}^{N} \pi_{i,j} (j-1)(j-2) &= \sum_{j=1}^{N} \pi_{i,j} (j-1)^2 - \sum_{j=1}^{N} \pi_{i,j} (j-1) \\ &= [(N-i)(1-p) + (i-1)q]^2 - (N-i)(1-p)^2 - (i-1)q^2 \\ &= \left[\sum_{j=1}^{N} \pi_{i,j} (j-1)\right]^2 - (N-i)(1-p)^2 - (i-1)q^2.\end{aligned}$$

Equation (31) can be obtained by combining this and equation (30). This completes the proof of Lemma 5. ■

The following equations are useful in deriving the other moments. For a binomial distribution with parameters $N-1$ and $1-s$, the first two moments are given by

$$\sum_{i=1}^{N} \binom{N-1}{i-1} s^{N-i} (1-s)^{i-1} (i-1) = (N-1)(1-s), \tag{33}$$

$$\begin{aligned}&\sum_{i=1}^{N} \binom{N-1}{i-1} s^{N-i} (1-s)^{i-1} (i-1)^2 \\ = \; &(N-1)(1-s)s + (N-1)^2 (1-s)^2. \end{aligned} \tag{34}$$

Conditional Mean

We are now ready to compute the conditional means. Conditional on $y_t = \overline{y}_i$, the mean value of y_{t+1} is given by

$$\begin{aligned}E(y_{t+1} | y_t = \overline{y}_i) &= \sum_{j=1}^{N} \pi_{i,j} \overline{y}_j = \sum_{j=1}^{N} \pi_{i,j} \left[-\psi + \frac{2\psi}{N-1}(j-1)\right] \\ &= -\psi + \frac{2\psi}{N-1} \sum_{j=1}^{N} \pi_{i,j} (j-1).\end{aligned}$$

It follows from (30) that

$$\begin{aligned}\sum_{j=1}^{N} \pi_{i,j} (j-1) &= (1-p)(N-i) + (i-1)q \\ &= (1-p)(N-1) + (q+p-1)(i-1).\end{aligned}$$

Hence

$$\begin{aligned} E\left(y_{t+1}|y_t=\overline{y}_i\right) &= -\psi+\frac{2\psi}{N-1}\left[\left(1-p\right)\left(N-1\right)+\left(q+p-1\right)\left(i-1\right)\right] \\ &= -\psi+2\psi\left(1-p\right)+\left(q+p-1\right)\frac{2\psi}{N-1}\left(i-1\right) \\ &= \left(q-p\right)\psi+\left(q+p-1\right)\overline{y}_i. \end{aligned} \tag{35}$$

Conditional Variance

Conditional on $y_t=\overline{y}_i$, the variance of y_{t+1} is given by

$$var\left(y_{t+1}|\overline{y}_i\right)=\sum_{j=1}^{N}\pi_{i,j}\overline{y}_j^2-\left(\sum_{j=1}^{N}\pi_{i,j}\overline{y}_j\right)^2,$$

where

$$\sum_{j=1}^{N}\pi_{i,j}\overline{y}_j^2=\psi^2-\frac{4\psi^2}{N-1}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)+\frac{4\psi^2}{\left(N-1\right)^2}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)^2,$$

and

$$\left(\sum_{j=1}^{N}\pi_{ij}\overline{y}_j\right)^2=\psi^2-\frac{4\psi^2}{N-1}\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)+\frac{4\psi^2}{\left(N-1\right)^2}\left[\sum_{j=1}^{N}\pi_{i,j}\left(j-1\right)\right]^2.$$

It follows from (31) that

$$var\left(y_{t+1}|\overline{y}_i\right)=\frac{4\psi^2}{\left(N-1\right)^2}\left[\left(N-i\right)\left(1-p\right)p+\left(i-1\right)q\left(1-q\right)\right].$$

Unconditional Mean

The unconditional mean of the Markov chain is given by

$$\begin{aligned} \sum_{i=1}^{N}\lambda_i\overline{y}_i &= \sum_{i=1}^{N}\lambda_i E\left(y_{t+1}|y_t=\overline{y}_i\right) \\ &= \sum_{i=1}^{N}\lambda_i\left[\left(q-p\right)\psi+\left(q+p-1\right)\overline{y}_i\right] \\ &= \left(q-p\right)\psi+\left(q+p-1\right)\sum_{i=1}^{N}\lambda_i\overline{y}_i. \end{aligned}$$

Hence

$$\sum_{i=1}^{N} \lambda_i \overline{y}_i = \frac{(q-p)\,\psi}{2-(p+q)} \equiv \mu. \tag{36}$$

Unconditional Second Moment

$$\begin{aligned}
\sum_{i=1}^{N} \lambda_i \overline{y}_i^2 &= \sum_{i=1}^{N} \lambda_i \left[-\psi + \frac{2\psi}{N-1}(i-1)\right]^2 \\
&= \sum_{i=1}^{N} \lambda_i \left[\psi^2 - \frac{4\psi^2}{N-1}(i-1) + \frac{4\psi^2}{(N-1)^2}(i-1)^2\right] \\
&= \psi^2 - \frac{4\psi^2}{N-1}\sum_{i=1}^{N} \lambda_i (i-1) + \frac{4\psi^2}{(N-1)^2}\sum_{i=1}^{N} \lambda_i (i-1)^2 .
\end{aligned}$$

Using (33) and (34), we have

$$\begin{aligned}
\sum_{i=1}^{N} \lambda_i \overline{y}_i^2 &= \psi^2 - 4\psi^2(1-s) + \frac{4\psi^2(1-s)\,s}{N-1} + 4\psi^2(1-s)^2 \\
&= \psi^2\left[1 - 4(1-s)\,s + \frac{4(1-s)\,s}{N-1}\right].
\end{aligned}$$

First-order Autocovariance

First consider the following expression,

$$\begin{aligned}
E(y_t y_{t+1}) &= \sum_{i=1}^{N} \lambda_i E(y_{t+1} y_t | y_t = \overline{y}_i) \\
&= \sum_{i=1}^{N} \lambda_i \overline{y}_i E(y_{t+1} | y_t = \overline{y}_i).
\end{aligned}$$

Using (35), we have

$$\begin{aligned}
E(y_t y_{t+1}) &= \sum_{i=1}^{N} \lambda_i \overline{y}_i \left[(q-p)\,\psi + (q+p-1)\,\overline{y}_i\right] \\
&= (q-p)\,\psi \sum_{i=1}^{N} \lambda_i \overline{y}_i + (q+p-1)\sum_{i=1}^{N} \lambda_i \overline{y}_i^2 .
\end{aligned} \tag{37}$$

Let σ_y^2 be the unconditional variance of the Markov chain so that

$$\sigma_y^2 = \sum_{i=1}^{N} \lambda_i \overline{y}_i^2 - \mu^2,$$

where μ is the unconditional mean defined in (36). Substituting this into (37) gives

$$\begin{aligned} & E\left(y_t y_{t+1}\right) \\ = \; & (q-p)\,\psi\mu + (q+p-1)\left[\sigma_y^2 + \mu^2\right] \\ = \; & \left[(q-p)\,\psi + (q+p-1)\,\mu\right]\mu + (q+p-1)\,\sigma_y^2, \end{aligned}$$

where

$$(q-p)\,\psi + (q+p-1)\,\mu = \frac{(q-p)\,\psi}{2-(p+q)} = \mu.$$

Hence

$$E\left(y_t y_{t+1}\right) = \mu^2 + (q+p-1)\,\sigma_y^2.$$

Thus the first-order autocovariance is given by

$$E\left[\left(y_t - \mu\right)\left(y_{t+1} - \mu\right)\right] = E\left(y_t y_{t+1}\right) - \mu^2 = (q+p-1)\,\sigma_y^2.$$

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