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Optimal Collusion with Internal Contracting

Gea Myoung Lee

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Gea M. Lee*

School of Economics and Social Sciences
Singapore Management University
90 Stamford Road, Singapore 178903
gmlee@smu.edu.sg
Tel: 65-6828-0857
Fax: 65-6828-0833

Abstract

In this paper, we develop a model of collusion in which two firms play an infinitely-repeated Bertrand game when each firm has a privately-informed agent. The colluding firms, fixing prices, allocate market shares based on the agent's information as to cost types. We emphasize that the presence of privately-informed agents may provide firms with a strategic opportunity to exploit an interaction between internal contracting and market-sharing arrangement: the contracts with agents may be used to induce firms' truthful communication in their collusion, and collusive market-share allocation may act to reduce the agents' information rents.

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1. Introduction

Collusions in practice are often characterized by price-fixing and market-share allocation.[1] Along with such actual features, recent theoretical work has shed light on diverse aspects of collusion in prices and quantities. An important feature in recent theoretical work is that the state of firms' production costs is regarded as private information. Aoyagi (2003), Athey and Bagwell (2001, 2006), Athey et al. (2004) and Skrzypacz and Hopenhayn (2004) develop models of this kind, where firms play a repeated Bertrand pricing game or repeated procurement auctions.[2] Despite their rich analyses of collusion, those theoretical models have given no attention to the possibility that the *location* private information is the hired agents who produce firms' outputs. By contrast, private information, held by agents, as to the state of production has received extensive attention in the literature, and in practice, productions are often accomplished by agents on a contractual basis. In this paper, building on recent work on theory of collusion, we investigate how the presence of privately-informed agents affects a commonly observed collusive behavior, price-fixing with market-share allocation.

Our paper develops a model of collusion in which two firms play an infinitely-repeated Bertrand game, and uses work by Athey and Bagwell (2001) and Athey et al. (2004) as a benchmark. Following their models, we consider the environment in which firm actions are publicly observed. A novelty of our model is that each firm has a privately-informed agent: private information is held by the agent who produces output for the firm. Employing Perfect Public Equilibrium (PPE), we establish two different classes of equilibria, *asymmetric* and *symmetric* PPE, to describe the features that would not be observed without the presence of privately-informed agents.

There is a two-tier relationship in our model: each firm writes to its agent a law-enforced contract and makes a self-enforced agreement with its rival firm. In each period, each agent privately observes its cost type. The cost type is high or low and i.i.d. across agents and time.[3] Each agent makes a report of cost type to its firm. The firm then makes a cost announcement to its rival firm, sets prices and allocates market shares. In an ideal collusive scheme, setting high prices, firms would allocate market shares by the criterion of productive efficiency, whereby all production is assigned to the agent(s) with the lowest production cost. Given the two types of relationship, a major difficulty with finding an optimal collusion is to establish a two-tier revelation mechanism that induces agents to make truthful reports and firms to make truthful announcements.

We firstly analyze an asymmetric PPE (APPE), and show how the presence of privately-

[1] Whinston (2006) surveys theoretical and empirical literature on price-fixing collusions, and Harrington (2006) provides patterns of price and market allocation in real cartels.

[2] Some recent literature explores self-enforcing trade agreements among privately-informed countries. See, e.g., Bagwell and Staiger (2005), Lee (2007) and Martin and Vergote (2007).

[3] Private information in our model is transitory. If private information is persistent, then the analysis will be more complicated; a fim's action may signal its information and affect its rival's belief. For related recent work, see Athey and Bagwell (2006).

informed agents affects the APPE-value set when firm strategies are unrestricted (possibly asymmetric). Colluding firms would intend to communicate truthfully to allocate market shares by the criterion of productive efficiency. To achieve productive efficiency, the firm that announces high cost must give up its market share when the rival announces low cost. There are contrasting incentive problems in the model. At the inter-firm level, a high-cost firm has an incentive to *understate* the reported high cost in the hope of increasing its market share, given that each firm privately observes its agent's report. To elicit the high-cost firm's truthfulness today, a high continuation value (as future reward) is afforded to the firm that reports high cost.[4] At the intra-firm level, a low-cost agent has an incentive to *overstate* the observed low cost in the hope of receiving the greater transfer payment for a given level of production. To elicit the low-cost agent's truthful reports, internal contract grants information rents to the agent who reports low cost.[5]

Interestingly, these contrasting incentives can work to the colluding firms' advantage.[6] Consider first the effect of collusion on internal contract. If colluding firms coordinate to allocate market shares by the criterion of productive efficiency, then a low-cost agent who reports high cost will be paid nothing (because of no production) when the other firm announces low cost. Market-allocation collusion may thus soften the low-cost agent's incentive to overstate the observed cost type and make it less costly to induce the agent's truthfulness in terms of information rents. Consider next the effect of internal contract on collusion. If an internal contract specifies that a high-cost agent receives a large payment when the agent produces more than a predetermined level of output, then the contract acts to soften the high-cost firm's current-period incentive to understate the reported cost type and thus reduce the corresponding continuation-value (future) reward.[7]

Our analysis of APPE, building on the "no-agent" model by Athey and Bagwell (2001), has the following distinct features. We show that the existence of privately-informed agents may significantly affect the APPE-value set. As in their paper, we establish a Pareto-frontier line

[4] In practice, the cartels prosecuted by the U.S. Antitrust Division are found to use rather sophisticated schemes. For instance, many cartels have used "future markets" as a channel of exchanging direct side-payments. They used a compensation scheme, whereby any firm that had sold more than its allotted share was required in the following budget period to purchase the excess from an underbudget firm that had not reached its allocation target in the preceding period (Business Week, July 27, 1998).

[5] The internal incentive problem is not new; a similar and more generalized incentive problem is widely found in the mechanism design literature (e.g., Fudenberg and Tirole, 1991) and in the extensive "transfer pricing" literature. For a classic evidence on agent's overstating behavior, see Schiff and Lewin (1970), and for agent's cost-padding behavior, see Laffont and Tirole (1993).

[6] The contrasting incentives between informed and uninformed parties differ from the "countervailing incentives" faced only by an informed party, as seen in Lewis and Sappington (1989), Spiegel and Spulber (1997) and others.

[7] Continuation values in our paper play the role of side-payments in a legalized cartel. The models with legalized cartel (e.g., Roberts, 1985; Cramton and Palfrey, 1990; Kihlstrom and Vives, 1992) show that communication helps firms to identify the most efficient firm, and side-payments provide firms with truth-telling incentives. Our analysis, in its relation to literature on repeated procurement auctions, may describe the case in which (i) two collusive bidders play a knockout auction, prior to actual bidding, to find who will be a lowest-cost supplier (e.g., McAfee and McMillan, 1992), and (ii) each bidder suffers some costs of distorting information, were it to lie.

segment of APPE values. Construction of the line segment is possible only when the segment is sufficiently long, so that a high-cost firm is persuaded to be truthful today by a sufficiently high continuation value drawn within the segment even if the firm may end up with zero market share.[8] In the model, agents hold private information and firms deliver it following the agents' report. Having an incentive to distort the agents' private information, firms use the internal contract and endogenize the level of costs a high-cost firm suffers when it falsifies its agent's report. As argued above, such a contractual arrangement reduces the continuation-value reward that is necessary to induce the high-cost firms' current-period truthfulness, which relaxes the constraint that the segment must be sufficiently long. In this way, the internal contract acts as a commitment device that elicits the firm's truthfulness. As a result, an interaction between collusion and internal contracting can be exploited: a contractual arrangement is used to achieve productive efficiency in market-allocation collusion, and productive efficiency in market allocation, in turn, enhances the contractual efficiency by reducing information rents for agents. This interaction also applies to the analysis of a symmetric PPE.

We secondly analyze a symmetric PPE (SPPE), and investigate how the presence of privately-informed agents affects the SPPE-value set when there is a symmetry restriction on firm strategies. Symmetry here means that current-period prices and market-share allocations must be symmetric across firms for all histories. The corresponding value set is then restricted to the 45-degree line, and the Pareto frontier is reduced to a point, not a line segment. In the APPE we construct, a continuation-value loss for one firm implies a continuation-value gain for another along the Pareto-frontier segment; continuation-value transfers do not cause inefficiency along the segment. In any SPPE, by contrast, continuation-value variations entail some waste of values for all firms together; continuation-value transfers are wasteful on the 45-degree line.

We construct an SPPE in which to prevent the high-cost firm's understatement today, a low continuation value (as future penalty) is given to the firms that announce low cost together. In our model, a simple commitment device of contract is used so that a high-cost firm gains nothing today when it lies and increases its market share above the predetermined level. It then becomes unnecessary to penalize the low-cost firms with the low-continuation value. Thus, with a simple contractual arrangement, firms are induced to be truthful without depending on wasteful continuation-value transfers. This finding shows that the symmetry restriction affects characteristics of SPPE differently between our model and the no-agent model: SPPE suffers a waste of equilibrium values in the no-agent model, but it can approximate the optimal monopoly profit despite the symmetry restriction in our model.

Our analysis of APPE and SPPE is based on the assumption that information as to the state of production cost is asymmetrically held by agents. If firms can observe their agents' cost types at no cost, then our model becomes the no-agent model. If firms can observe their agents' cost types

[8] We establish a self-generating set of APPE values, following the recursive structure explored by Abreu et al. (1986, 1990), whereby after any history, the set of continuation values are always equal to the equilibrium value set.

only after incurring informational costs, then our findings imply that colluding firms may find it beneficial to deliberately restrict their own observability of agents' cost types until signing the internal contract. At a broad level, our paper predicts that in the presence of privately-informed agents, firms may find it relatively easy to achieve an optimal APPE and save a potential waste of optimal SPPE values.[9] In the literature, firms own private information. Firms there can observe and distort private information at no extra costs. In this paper, agents hold private information and firms deliver it. The contractual arrangement in this paper captures the circumstance in which firms deliberately reduce the degree to which they control of private information; their incentive to distort the agents' private information is bounded by the contract with agents. We find that a simple contractual arrangement that reduces the firms' incentive to distort the agents' private information makes it possible to establish a sufficiently long Pareto-frontier segment in APPE and avoid wasteful continuation-value transfers in SPPE.

Our model also contrasts with the no-agent model by Athey et al. (2004) in which cost types are continuously distributed. They predict that when the distribution of cost types is log-concave, optimal SPPE is characterized by a pooling equilibrium in which market shares are constant regardless of cost types; firms sacrifice productive efficiency and instead save informational costs that would be necessary to deter higher-cost firms from mimicking lower-cost firms. In our two-type model, firms achieve productive efficiency in market allocation, which, in turn, reduces informational costs.

Our findings provide a new perspective on collusive behavior: the presence of privately-informed agents may provide firms with a strategic opportunity to exploit the interaction between internal contracting and market-sharing arrangement. A variety of strategic contracting devices have been highlighted by the literature.[10] There is a broad analogy between our analysis and the work done by Fershtman and Judd (1987) or by Fershtman et al. (1991). They show that a firm may compete more effectively in a Cournot oligopoly game, or collude more effectively with the other firm, if its manager enters this game and is bounded by a wage contract. Likewise, we show that firms may collude more effectively when they are bounded by a strategic use of internal contract.

The rest of this paper is organized as follows. Section 2 introduces the basic model, and describes the approach we use in the paper. Section 3 describes the constraints that equilibrium strategies must satisfy. Section 4 characterizes an APPE, where firm strategies are unrestricted. Section 5 describes an SPPE, where firm strategies are restricted to be symmetric. Section 6 discusses possible extensions of the model. Section 7 provides conclusions.

[9] Related theme is found in the literature. Using an principal-agent setting, Dewatripont and Maskin (1995) show that the contracting parties may find it desirable to deliberately restrict the observability of principal. Lee (2003) argues that the scope of intertemporal price discrimination may diminish if a monopolist has more information as to consumers' past purchasing history.

[10] Vickers (1985), Fershtman and Judd (1987), Katz (1991), Reitman (1993), Sklivas (1987), Spagnolo (2000), and Kockesen and Ok (2004) show that incentive contracts with delegated agents can serve as a strategic commitment device.

2. The Model

There are two ex ante identical firms. A novel feature in our model is that each firm has a privately-informed agent: private information is held by the agent who produces output for the firm. Prices and quantities are publicly observed, but unit costs are privately observed by the agent. In each period, costs are independently drawn from the identical common-knowledge distribution with discrete support $\{\theta_L, \theta_H\}$. A cost type θ_L (θ_H) is drawn with probability μ (probability $1-\mu$). For notational simplicity, define the cost differential as $\triangle \equiv \theta_H - \theta_L > 0$ and denote the discrete support as $\{L, H\}$. The main purpose of our analysis is to highlight how the presence of privately-informed agents affects a commonly observed collusive behavior, price-fixing with market-share allocation. To this end, we assume that there is a unit mass of homogeneous consumers whose valuation of the good is ρ. This assumption greatly simplifies our analysis, since the problem of finding an optimal collusion can be reduced to that of finding market-share allocations, given that patient firms will not undercut the optimal fixed price ρ, which is assumed to be higher than θ_H.

2.1. Optimal Values

In this subsection, as a motivating benchmark, we consider a contracting game in which a monopolist offers a contract to two privately-informed agents. Our later analysis will show that colluding firms may be able to replicate the monopolist's optimal behavior. The timing of the game is as follows: (i) each agent i privately observes its cost type $\theta^i \in \{L, H\}$, (ii) the monopolist offers a single-period contract to each agent, (iii) each agent i makes a report $r^i \in \{L, H\}$ to the firm, (iv) the monopolist determines a production level for each agent, q^i, and each agent produces the quantity and (v) the monetary transfers requested by the contract are enforced.

The firm determines the level of output subsequent to the agents' report. The production level assigned for agent i, q^i, is conditional on the agents' report. Thus, the production level is determined by a production-allocation scheme, $q^i : \{L, H\} \times \{L, H\} \to Q$, where $Q \equiv [0, 1]$. The contract for agent i is a pair, $\left\{t^i, q^i\right\}$, where t^i is the payment for agent i. A type-θ_j agent has utility $t^i - \theta_j q^i$ if the agent produces q^i and receives monetary transfer t^i, and any type of agent gets zero utility if the agent refuses the contract. In a pair of cost types (θ_j, θ_k), θ_j is agent 1's cost type and θ_k is agent 2's cost type. The pair (θ_j, θ_k) is hereafter indexed by $(j, k) \in \{(L, L), (L, H), (H, L), (H, H)\}$. Let p_{jk} represent the price selected for state (j, k) and let t^i_{jk} and q^i_{jk} represent the transfer for agent i in (j, k) and the quantity produced by agent i in (j, k), respectively. Following these notations, we can denote the contracts for agents by $\{(t^1_{jk}, q^1_{jk}), (t^2_{jk}, q^2_{jk})\}$. For later use, letting $\mu_L \equiv \mu$ and $\mu_H \equiv 1 - \mu$, we define the expected quantities:

$$\overline{q}^1_j \equiv \sum_{k \in \{L,H\}} \mu_k q^1_{jk} \text{ and } \overline{q}^2_k \equiv \sum_{j \in \{L,H\}} \mu_j q^2_{jk}.$$

We now find an optimal contract of the monopolist.

Monopolist's Problem: The optimal prices are fixed, $p_{jk} = \rho \ \forall (j,k)$. The monopolist finds the contract $\{(t^1_{jk}, q^1_{jk}), (t^2_{jk}, q^2_{jk})\}$ that maximizes the expected profit

$$\sum_{j\in\{L,H\}} \sum_{k\in\{L,H\}} \mu_j \mu_k \left[\rho \cdot (q^1_{jk} + q^2_{jk}) - t^1_{jk} - t^2_{jk}\right] \tag{1}$$

subject to:

(i) Incentive Compatibility for agent 1: $\forall k \in \{L, H\}$,

$$t^1_{Lk} - \theta_L q^1_{Lk} \geq t^1_{Hk} - \theta_L q^1_{Hk} \tag{IC^1_{Lk}}$$

$$t^1_{Hk} - \theta_H q^1_{Hk} \geq t^1_{Lk} - \theta_H q^1_{Lk} \tag{IC^1_{Hk}}$$

(ii) Individual Rationality for agent 1:

$$\sum_{k\in\{L,H\}} \mu_k \left(t^1_{Lk} - \theta_L q^1_{Lk}\right) \geq 0 \tag{IR^1_L}$$

$$\sum_{k\in\{L,H\}} \mu_k \left(t^1_{Hk} - \theta_H q^1_{Lk}\right) \geq 0 \tag{IR^1_H}$$

(iii) IC^2_{jL}, IC^2_{jH}, IR^2_L and IR^2_H for agent 2.

Note that we use dominant-strategy incentive constraints to find the optimal contract. In this context, however, the contract can be equivalently implemented in Bayesian or in dominant strategy if the expected output decreases in cost type ($\overline{q}^i_L > \overline{q}^i_H$), this being satisfied in the solution.[11] Hence, there is no loss of generality in looking for the optimal contract within the set of dominant-strategy implementation. It is implied by optimality that incentive compatibility for "low-cost" agents, IC^1_{Lk}, and individual rationality for "high-cost" agents, IR^1_H, are binding. Observing that there is some freedom in the choice of t^1_{HL} and t^1_{HH}, we can find many transfer schemes that satisfy these binding constraints. One candidate is that for $k \in \{L, H\}$, high-cost agents participate in all states of nature, $t^1_{Hk} = \theta_H q^1_{Hk}$, and low-cost agents receive $t^1_{Lk} = \theta_L q^1_{Lk} + \triangle \cdot q^1_{Hk}$. The corresponding candidate for agent 2 is that for $j \in \{L, H\}$, $t^2_{jH} = \theta_H q^2_{jH}$ and $t^2_{jL} = \theta_L q^2_{jL} + \triangle \cdot q^2_{jH}$. Thus, a low-cost agent i is induced to be truthful by the expected information rent $\triangle \cdot \overline{q}^i_H$.

Given that all the candidates derive the same expected profit, the Monopolist's Problem now looks for the production-allocation scheme, $\{(q^1_{jk}, q^2_{jk})\}$, that maximizes

$$\sum_{j\in\{L,H\}} \sum_{k\in\{L,H\}} \mu_j \mu_k \left[\rho \cdot (q^1_{jk} + q^2_{jk}) - C_j q^1_{jk} - C_k q^2_{jk}\right], \tag{2}$$

where C_L and C_H represent the virtual (unit) costs associated with productions of low- and high-cost agents:

$$C_L \equiv \theta_L \text{ and } C_H \equiv \theta_H + \frac{\mu}{1-\mu}\triangle. \tag{3}$$

[11] The use of dominant-strategy implementation is due to Mookherjee and Reichelstein (1992). They show that the equivalence between Bayesian and dominant-strategy implementations holds if the agents' cost functions satisfy a generalized single crossing property. This property trivially holds in our model.

The virtual costs include informational costs of eliciting the agents' truthfulness, which are reflected in the additional terms $\frac{\mu}{1-\mu}\triangle$ in (3). It follows that the optimal solution satisfies (i) $q^1_{LH} = q^2_{HL} = 1$, (ii) $q^1_{LL} + q^2_{LL} = 1$, and (iii) $q^1_{HH} + q^2_{HH} = 1$ if $\rho \geq C_H$, and $q^1_{HH} + q^2_{HH} = 0$ otherwise. In this paper, we say that *productive efficiency* is achieved if these three conditions hold. The conditions mean that all production is assigned to the agent(s) with the lowest production cost: in (L, H) or (H, L), the lowest-cost agent alone produces the output, and in (L, L) or (H, H), agents may together produce the total output. In the parameter range $\rho < C_H$, informational costs of eliciting the agents' truthfulness make the production in (H, H) unprofitable.

There is a striking benefit of achieving productive efficiency: production is carried out only by the lowest-cost agent(s), and further, productive efficiency enhances contractual efficiency by reducing information rents. Under productive efficiency, the virtual costs in (L, L), (L, H) or (H, L) are only production costs θ_L; the virtual cost gap between the high- and low-cost agent is then $C_H - C_L = \triangle + \frac{\mu}{1-\mu}\triangle$, which is more than the gap between cost types, $\theta_H - \theta_L = \triangle$. The virtual costs are higher than production costs only in (H, H), and thus the expected information rents are reduced to $(1-\mu)^2(C_H - \theta_H) = \mu(1-\mu)\triangle \cdot (q^1_{HH} + q^2_{HH})$. The overall expected costs are $[1-(1-\mu)^2]\theta_L + (1-\mu)^2 C_H = E(\theta)$ if $\rho \geq C_H$, and $[1-(1-\mu)^2]\theta_L$ otherwise.

Lemma 1. *The optimal monopoly profit is*

$$\pi^m \equiv \begin{cases} \rho - E(\theta) & \text{if } \rho \geq C_H \\ [1-(1-\mu)^2](\rho - \theta_L) & \text{otherwise.} \end{cases} \quad (4)$$

The first-best profit is $\pi^f \equiv \rho - [1-(1-\mu)^2]\theta_L - (1-\mu)^2\theta_H$, which is what the firm could earn if it were able to observe the agents' cost types. In the presence of privately-informed agents, $\pi^f > \pi^m$. If $\rho \geq C_H$, the profit differential is the information rents: $\pi^f - \pi^m = \mu(1-\mu)\triangle$. If $\rho < C_H$, the firm incurs no information rents but sacrifices the profit in (H, H): $\pi^f - \pi^m = (1-\mu)^2(\rho - \theta_H)$.[12] In this paper, we say that firms achieve an optimal collusion if they earn π^m as their per-period joint profit. For later use, we define a value set $V^m \equiv \{(u^1, u^2) : u^1 + u^2 = \frac{\pi^m}{1-\delta}\}$, where δ is common discount factor.

We lastly clarify the assumptions that we have made to derive Lemma 1. The first assumption is that there is no side-contracting collusion between agents; agents across firms cannot form a cartel to make collusive reports using side-payments.[13] The second assumption is that the firm can make an ex ante commitment to production schedules. Under $\rho < C_H$, before the agent's report, the virtual cost of producing $q^1_{HH} + q^2_{HH} = 1$ is C_H, which is too high. After the agents'

[12] The assumption, $\rho > \theta_H$, ensures that $\pi^f > \pi^m$. If $\theta_L < \rho \leq \theta_H$, then $\pi^m = \pi^f$. We ignore the parameter range $\theta_L < \rho \leq \theta_H$, where our analysis becomes trivially simple, given that a high-cost firm has no incentive to produce any output and mimic a low-cost type.

[13] Our paper does not allow any form of side-payment across firms or across agents. Laffont and Martimort (1997, 2000) characterize optimal collusion-proof mechanisms when privately-informed agents are collusive in their side-contracting games. In related work, Che and Kim (2007) and Dequiedt (2007) study collusion-proof mechanisms in auction.

report of (H, H), however, the firm may be tempted to produce $q_{HH}^1 + q_{HH}^2 = 1$ if the contract is renegotiable.[14] In later analysis, production schedule in any state is achieved by a self-enforced agreement.

2.2. Nash-Equilibrium Values

In this subsection, we look for the Nash-equilibrium values. There are two firms. Each firm now has a privately-informed agent. Departing from the monopoly model, we introduce a new notation for the payment scheme for each agent. The payment function for agent i is a mapping, $t^i : \{L, H\} \times Q \to \mathbb{R}$. A typical payment for agent i is denoted by $t^i(q^i, r^i)$ when agent i makes a report of $r^i \in \{L, H\}$ and produces $q^i \in [0, 1]$.

To find Nash-equilibrium values, we proceed with two steps. First, suppose that each firm knows its agent's cost at no information rents. A high-cost firm charges price at θ_H, and a low-cost firm mixes, earning the expected profit $(1-\mu)\triangle$ by slightly undercutting the high-cost firm's price. The ex ante expected profit for each firm is then $\mu(1-\mu)\triangle$. Second, we specify the contract that elicits agent's truthfulness. If agent i reports low cost and produces q^i, then the agent receives $t^i(q^i, L) = \theta_L q^i + \frac{(1-\mu)\triangle}{2}$, and if agent i reports high cost and produces q^i, then the agent receives $t^i(q^i, H) = \theta_H q^i$. We now confirm that agents' incentive compatibility holds. Consider incentive compatibility of a low-cost agent. Given that market shares in states (L, H), (H, L) and (H, H) are determined by prices, $q_{LH}^1 = q_{HL}^2 = 1$ and $q_{HH}^1 = q_{HH}^2 = \frac{1}{2}$, a low-cost agent is induced to be truthful by the expected information rent $\frac{(1-\mu)\triangle}{2} = \triangle \cdot \overline{q}_H^i$. Consider next incentive compatibility of a high-cost agent. If a high-cost agent i mimics a low-cost type, then the agent will get the expected information rents $\frac{(1-\mu)\triangle}{2} = \triangle \cdot \overline{q}_H^i$ but suffer an increase of the expected production cost $\triangle \cdot \overline{q}_L^i$. It follows that $\triangle \cdot \overline{q}_L^i \geq \triangle \cdot \overline{q}_H^i$, since the monotonicity, $\overline{q}_L^i \geq \overline{q}_H^i$, holds for *any* realization of q_{LL}^i under the mixed prices of two low-cost firms. We now find the ex ante expected profit of each firm. The ex ante expected information rents are $\frac{\mu(1-\mu)\triangle}{2}$, and thus the ex ante expected profit, net of such information rents, is $\pi^n \equiv \frac{\mu(1-\mu)\triangle}{2}$. For the punishment phase in the repeated game below, we define the set of Nash-equilibrium values as $V^n \equiv \{(u^1, u^2) : u^1 = u^2 = \underline{v} \equiv \frac{\pi^n}{1-\delta}\}$.

2.3. The Repeated Game

In this subsection, we describe the stage game and the repeated game. Our analysis hereafter is based on the following assumptions: (i) firms do not exchange side-payments in the form of monetary transfers across firms, (ii) agents across firms do not form a cartel to make collusive reports using their side-payments and (iii) no firm secretly renegotiates the contract (collude) with its agent. The model thus addresses a stringent environment for collusive side-contracting

[14] The ex post production in (H, H) is possible only if the contract is renegotiable; under $\rho < C_H$, the ex ante optimal contract specifies payments only for low-cost agents, $t_{jk}^i = \theta_L q_{jk}^i$ for relevant (j, k). The optimal renegotiation-proof contract would be the contract offered under $\rho \geq C_H$, and the associated suboptimal profit would be $\rho - E(\theta)$.

behaviors. Note also that the contract with agent lasts for only one period.[15]

Consider the stage game. The timing of the stage game is as follows: (i) each agent i privately observes its cost type $\theta^i \in \{L, H\}$, (ii) each firm i offers a single-period contract x^i to each agent, (iii) each agent i makes a report $r^i \in \{L, H\}$ to the firm, (iv) each firm i makes an announcement $a^i \in \{L, H\}$ to its rival firm $j \neq i$, (v) each firm i chooses a price p^i and makes a market-share proposal q^i and (vi) each agent i produces the quantity requested by the firm. Market shares are realized and the contracts are enforced.

The stage game is designed to reflect an environment in which equally priced firms, subsequent to the agents' report, communicate with each other to allocate market shares in a state-dependent way, and each agent, following firms' selection of market allocations, produces the corresponding quantity (market share). Each agent i observes $\theta^i \in \{L, H\}$ and reports $r^i \in \{L, H\}$ under a contract x^i. Each firm then announces $a^i \in \{L, H\}$ and sets price p^i and makes market-share proposal q^i. Given that each agent accepts the contract, the vectors, $\mathbf{p} \equiv (p^1, p^2)$ and $\mathbf{q} \equiv (q^1, q^2)$, jointly determine market share for firm i, m^i. If $p^i > \rho$, then $m^i = 0$, and if $p^i < p^j \leq \rho$, then $m^i = 1$. If $p^1 = p^2 \leq \rho$, then $m^i = \frac{1}{2}$ if $q^1 + q^2 \neq 1$, and $m^i = q^i$ otherwise. We can find that market-share proposals matter only for equally priced firms; if prices are different, then the lowest-priced firm captures the entire market for a relevant price range.

To simplify the exposition, we now follow two steps: we first describe the inter-firm game for a given internal contract x^i, and then describe the contract. Then, in the interim stage that follows the agent's report $r^i \in \{L, H\}$, firm i has a finite strategy set:[16]

$$S^i = \left\{\widetilde{a}^i \mid \widetilde{a}^i : \{L, H\} \to \{L, H\}\right\} \times \left\{\widetilde{p}^i \mid \widetilde{p}^i : \{L, H\} \times \{L, H\} \to \mathbb{R}\right\}$$
$$\times \left\{\widetilde{q}^i \mid \widetilde{q}^i : \{L, H\} \times \{L, H\} \to Q\right\}.$$

Announcement function, $\widetilde{a}^i$, is conditional on the agent i's report, and pricing and market-share functions, $\widetilde{p}^i$ and $\widetilde{q}^i$, are conditional on the agent i's report and its rival's announcement. A typical stage-game strategy for firm $i \neq j$ is

$$s^i\left(r^i, a^j\right) \equiv \left\{\widetilde{a}^i\left(r^i\right), \widetilde{p}^i\left(r^i, a^j\right), \widetilde{q}^i\left(r^i, a^j\right)\right\}.$$

The associated vector is denoted by $\mathbf{s}(\mathbf{r}) \equiv (s^1(r^1, a^2), s^2(r^2, a^1))$, where $\mathbf{r}$ is the vector of the agent reports, $\mathbf{r} \equiv (r^1, r^2)$. A strategy vector $\mathbf{s}$ provides an interim stage-game payoff, $\Pi^i(\mathbf{s}) = E_{r^j}\left[\pi^i(\mathbf{s}, \mathbf{r})\right]$, where $\pi^i(\mathbf{s}, \mathbf{r})$ represents the realized profit given the strategies. An ex ante expected stage-game payoff is $\overline{\Pi}^i(\mathbf{s}) = E_{r^i}\left[\Pi^i(\mathbf{s})\right]$.

[15] An example of such environment might be the one in which agents play a one-shot game when each firm, having replaceable potential agents, can easily detect side-contracting behaviors. It is beyond the scope of the present paper to analyze a contracting scheme when each agent has future prospects in various multi-period contractual relationships. In the extension section, however, we will present the case in which agents have future prospects.

[16] Note that it is not ensured for now that agents make truthful report ($r^i = \theta^i$) and firms make truthful announcements, ($a^i = r^i$).

Consider the repeated game. For solution concept, we employ *Perfect Public Equilibrium* (PPE), where strategies are conditional on the publicly observed history of realized choices (Fudenberg et al., 1994). Upon entering a period, each firm publicly observes the realized choices. Each firm also privately observes its current cost type, the history of the cost types it had and the choice functions it used in previous periods. Thus, a firm does not observe its rival firm's current or past cost types and does not observe its rival firm's current or past choice functions. Let $\mathbf{a} \equiv (a^1, a^2)$ represent the vectors of firm announcements. Upon entering a period τ, each firm observes the public history of realized choices, $h_\tau = \{\mathbf{a}_t, \mathbf{p}_t, \mathbf{q}_t\}_{t=1}^{\tau-1}$ and $h_1 = \varnothing$. A strategy of firm i in period τ, denoted by σ_τ^i, is a mapping from the set of potential public histories H_τ to the set of stage-game strategies S^i. A strategy profile in each period τ is then defined by $\sigma_\tau \equiv (\sigma_\tau^1, \sigma_\tau^2)$. Each history h_τ provides the per-period expected payoff $\overline{\Pi}^i(\sigma_\tau(h_\tau))$. Each strategy involves a probability distribution, and thus entails the expected payoff $E[\sum_{\tau=1}^{\infty} \delta^{\tau-1} \overline{\Pi}^i(\sigma_\tau(h_\tau))]$.

We finally describe the contract x^i. As in most of the existing literature that studies a strategic device of contract, we assume that contracts are observed by colluding firms.[17] A one-period contract is chosen from a set of payment schemes, $x^i \in \{\widetilde{t^i} \mid \widetilde{t^i} : \{L, H\} \times Q \to \mathbb{R}\}$. A typical payment for agent i is denoted by $t^i(q^i, r^i)$ when agent i makes a report of $r^i \in \{L, H\}$ and produces q^i. This quantity q^i is conditional on the two firms' announcement, $q^i : \{L, H\} \times \{L, H\} \to Q$. In this paper, we restrict attention to the contracts in which compensations to agents are at least as high as production costs, $t^i(q^i, L) \geq \theta_L q^i$ and $t^i(q^i, H) \geq \theta_H q^i$, which ensures that each agent produces the requested quantity if the agent accepts any one-period contract.

In an ideal collusive scheme, firms would communicate with each other to allocate market shares by the criterion of productive efficiency, whereby all production is assigned to the agent(s) with the lowest production cost. Because of the two-tier communication channels, such market-allocation collusions may pose a challenging problem in regard to finding enforceable contracts. For example, we invoke the payment scheme that we used above under $\rho \geq C_H$:

$$t^i\left(q^i, L\right) = \theta_L q^i + \triangle \cdot \overline{q}_H^i \text{ and } t^i(q^i, H) = \theta_H q^i.$$

In this payment scheme, although the term q^i on the RHS is conditional on the two firms' announcements, it is the "actual" quantity that agent i produces, and thus is verifiable. The information-rent term $\triangle \cdot \overline{q}_H^i$ is, however, determined by the expected quantity that agent i would produce if the agent reported high cost. This quantity is not conditional on the agent's current cost type (low cost) and associated production. Hence, the payment $t^i(q^i, L)$ $\forall i$ is verifiable only if low-cost agents are able to verify what would be the market-allocation schemes in (H, L), (L, H) and (H, H). In this sense, the notation $t^i(q^i, r^i)$ is overly simplified. Another problem with finding enforceable contracts is that agents may find it difficult to verify whether firms truthfully announce what they have reported. If firms have an incentive to falsify the agents' reported information, it may not

[17] In a recent study of strategic delegation, Kockesen and Ok (2004) show that even unobservable contracts may serve as a commitment device.

be ensured that agents make truthful reports of their cost types. Complication thus arises; the payment to an agent (t^i) is determined by the agent's report (r^i) *and* the firms' market-allocation schemes that are conditional on firms' external announcements (a^i and a^j). For now, we simply avoid these difficulties by making the following assumption:

Assumption 1. *Each agent is able to verify both firms' announcements and their market-allocation schedules.*

This assumption is very restrictive. However, after we find a payment scheme that is enforceable only under the assumption, we will establish an alternative scheme that is enforceable *in the absence of Assumption 1.* In the alternative scheme, the payment to agent i will be dependent solely on the real quantity q^i that the agent produces. With this in mind, we use the notation $t^i(q^i, r^i)$ for now, despite its oversimplification.

2.4. Two-Tier Mechanism Design

In this subsection, we describe how the repeated game and internal contract are pulled together. A major difficulty with finding an optimal collusion is to establish a two-tier revelation mechanism: $r^i = \theta^i$ (agent i's truthful report to firm i) and $a^i = r^i$ (firm i's truthful announcement to firm $j \neq i$). Our approach involves two steps. Broadly speaking, in Step 1, we assume that each firm knows its agent's type *for a given contract,* and find collusive market-share schedules, and in Step 2, we find an internal contract such that market-share allocation achieves productive efficiency. This subsection is organized as follows. We first describe Step 1 and introduce a dynamic programming tool developed by Abreu et al. (1986, 1990). Following this tool, each firm's PPE payoff is factored into two components, current-period profit and (discounted) expected continuation values that are conditional on current-period actions, and after any history, the set of continuation values is equal to the equilibrium-value set. We next follow Athey and Bagwell (2001) and Athey et al. (2004), who show that existing tools from (static) mechanism design theory can be used to find the solution of the factored program. We finally establish a two-tier mechanism design program.

Step 1 (Factored Program): Assume that for a given contract, each firm knows its agent's cost type (or equivalently each agent makes truthful report ($r^i = \theta^i$)). The program chooses current-period strategies $\mathbf{s} \in S$ and continuation-value function $\boldsymbol{v} : \{L, H\} \times \{L, H\} \times \mathbb{R}^4 \to co(V)$ to maximize

$$u^i = E_{\theta^i}\left[\Pi^i(\mathbf{s}) + \delta v^i\left(\mathbf{s}(\boldsymbol{\theta})\right)\right]$$

subject to: for firms i and j, and any deviation $\widehat{s}^i = (\widehat{a}^i, \widehat{p}^i, \widehat{q}^i) \in S^i$,

$$E_{\theta^i}\left[\Pi^i(\mathbf{s}) + \delta v^i\left(\mathbf{s}(\boldsymbol{\theta})\right)\right] \geq E_{\theta^i}\left[\Pi^i\left(\widehat{s}^i, s^j\right) + \delta v^i\left(\widehat{s}^i(\theta^i, \widetilde{a}^j(\theta^j)), s^j(\theta^j, \widehat{a}^i(\theta^i))\right)\right].$$

Note that continuation-value function $\boldsymbol{v}$ is conditional on two firms' announcement and current-period strategies $(\mathbf{p}, \mathbf{q})$.

We next adopt the work by Athey and Bagwell (2001) and Athey et al. (2004), who show that the self-generating set (in the spirit of Abreu et al.) can be found by using existing tools from the (static) mechanism design literature. Let the vector $\mathbf{z} \equiv (\mathbf{p}, \mathbf{q}, \mathbf{v})$ represent the equilibrium-path strategy vector for prices, quantities and continuation values. To find the equilibrium-path payoffs, suppose that firm 1 announces cost type $\hat{j}$ when it knows that its agent draws cost type j, and that if $(\hat{j}, k)$ is realized as a result of the stage game, then the firm receives the current-period payoff $\pi^1(p^1_{\hat{j}k}, q^1_{\hat{j}k}, j)$ and continuation value $v^1_{\hat{j}k}$. It then follows that if firm 1, knowing that its agent's cost type is j, announces cost type $\hat{j}$, then it receives the interim current-period payoff and continuation value:

$$\Pi^1(\hat{j}, j) \equiv \sum_{k \in \{L,H\}} \mu_k \pi^1(p^1_{\hat{j}k}, q^1_{\hat{j}k}, j) \text{ and } \overline{v}^1_{\hat{j}} \equiv \sum_{k \in \{L,H\}} \mu_k v^1_{\hat{j}k}.$$

We express the equilibrium-path interim payoff for firm i in a "direct" form: $\Pi^i(\hat{j}, j) + \delta \overline{v}^i_{\hat{j}}$.

Two-Tier Program:

Step 1 (Mechanism Design Program): Assume that for a given contract, each firm knows its agent's cost type (each agent makes truthful report $(r^i = \theta^i)$). The program chooses current-period strategies $(\mathbf{p}, \mathbf{q}) : \{L, H\} \times \{L, H\} \to \mathbb{R}^4$ and continuation-value function $\mathbf{v} : \{L, H\} \times \{L, H\} \times \mathbb{R}^4 \to co(V)$ to maximize the ex ante expected payoff

$$u^i(\mathbf{z}) = \sum_{j \in \{L,H\}} \mu_j \left[\Pi^i(j, j) + \delta \overline{v}^i_j\right]$$

subject to:

(i) On-Schedule Constraints: $\forall \hat{j} \neq j$, $v^i_{\hat{j}k} \in co(V)$,

$$\Pi^i(j, j) + \delta \overline{v}^i_j \geq \Pi^i(\hat{j}, j) + \delta \overline{v}^i_{\hat{j}}. \qquad (\text{On-IC}^i_j)$$

(ii) Off-Schedule Constraints: $\forall (\widehat{p_{jk}}, \widehat{q_{jk}}) \notin \{(p^i_{jk}, q^i_{jk})\}$, $\widehat{v} \in co(V)$,

$$\pi^i\left(p^i_{jk}, q^i_{jk}, j\right) + \delta v^i_{jk} \geq \pi^i\left(\widehat{p_{jk}}, \widehat{q_{jk}}, j\right) + \delta \widehat{v}. \qquad (\text{Off-IC}^i_{jk})$$

(iii) Off-Schedule Constraints: $\forall (\widehat{p_{\hat{j}k}}, \widehat{q_{\hat{j}k}}) \notin \{(p^1_{jk}, q^1_{jk})\}$, $\widehat{v} \in co(V)$,

$$\Pi^1(j, j) + \delta \overline{v}^1_j \geq \sum_{k \epsilon \{L,H\}} \mu_k \left[\pi^1\left(\widehat{p_{\hat{j}k}}, \widehat{q_{\hat{j}k}}, j\right) + \delta \widehat{v}\right]. \qquad (\text{Off-m-IC}^1_j)$$

The constraint Off-m-IC^2 is analogous.

Step 2 (Choice of Contract): Letting $u^i(\mathbf{z}(x^i))$ represent the ex ante expected payoff under a contract x^i, we find a contract x^i that satisfies

(i) Agent's Truthful Reports: $r^i = \theta^i$.

(ii) Optimality Condition: for any alternative contract $\widehat{x}$, $u^i(\mathbf{z}(x^i)) \geq u^i(\mathbf{z}(\widehat{x}))$.

The mechanism design program chooses current-period strategies $(\mathbf{p}, \mathbf{q})$ conditional on two firms' announcement, and chooses continuation-value function $\mathbf{v}$ conditional on two firms' announcement and current-period $(\mathbf{p}, \mathbf{q})$. The vector $\mathbf{z} \equiv (\mathbf{p}, \mathbf{q}, \mathbf{v})$ is chosen to satisfy feasibility and incentive-compatibility constraints. Feasibility constraint means that continuation values are drawn from the equilibrium-value set. Incentive compatibility consists of two parts: (i) the "on-schedule" (truth-telling) incentive compatibility that each firm truthfully announces its cost and (ii) the "off-schedule" (non-deviating) incentive compatibility that each firm cannot gain by choosing a price or market share that is not specified for any cost type. An on-schedule deviation is not detected as a deviation to the rival firm, since it follows the equilibrium vector, whereas an off-schedule deviation is observed. The repeated play of the (noncooperative) Nash equilibrium is always an equilibrium of the repeated game; thus, when firms are sufficiently patient, the Nash reversion can be used as the punishment that follows any off-schedule deviation.

There are two types of off-schedule deviations: (i) a deviation from the vector $(\mathbf{p}, \mathbf{q})$ *after* the announcement (Off-IC$^i_{jk}$), and (ii) a "misrepresentation" *at* the announcement and a subsequent deviation from the vector (Off-m-IC^i_j).[18] The first type of deviation is realized by a firm that slightly undercuts the price (say, $\widehat{p_{jk}} = p^i_{jk} - \varepsilon$) and captures the entire market ($\widehat{q_{jk}} = 1$) after the announcement (j, k). The second type of deviation is realized by a firm that misrepresents its type, aiming to undercut the price subsequently. For example, if the price at the announcement $(\hat{\jmath}, k)$, $p_{\hat{\jmath}k}$, is higher than in other announcements, then firm 1, knowing that its agent's type is j, may be tempted to announce $\hat{\jmath} \neq j$, aiming to undercut the high price ($\widehat{p_{\hat{\jmath}k}} = p_{\hat{\jmath}k} - \varepsilon$). In the section that follows, we will argue that the second type of deviation can be ignored at a price-fixing collusion.

In Step 1, we adopt the work by Athey and Bagwell (2001), whereby the PPE-value set in the Factored Program (say, it is V^*) can be equally established by vectors $\mathbf{z} = (\mathbf{p}, \mathbf{q}, \mathbf{v})$ that satisfy feasibility and incentive-compatibility constraints in the Mechanism Design Program.[19] To be precise, define the set of incentive compatible vectors, where continuation values are drawn from $co(V)$:

$$Z^{IC}(V) \equiv \left\{ \mathbf{z} : \text{On-IC}^i_j,\ \text{Off-IC}^i_{jk} \text{ and Off-m-IC}^i_j \text{ hold, and } (v^1_{jk}, v^2_{jk}) \in co(V)\ \forall i, j, (j,k) \right\}.$$

Athey and Bagwell show that a value set, generated by these vectors, together with the punishment-value set (V^n) is equal to the PPE-value set:

$$\left\{ \left(u^1, u^2\right) : \exists \mathbf{z} \in Z^{IC}(V) \right\} \cup V^n = V^*.$$

This means that for any PPE values $(u^1, u^2) \in V$, there exists $\mathbf{z} \in Z^{IC}(V)$ such that $u^i = u^i(\mathbf{z})$.

In Step 2, we build on this result and select a contract x^i that induces the agent's truthfulness and maximizes the expected profit. In the following section, we will argue that the role of contract is not only a mechanism for agents' truthfulness but also a strategic device for an optimal collusion.

[18] The on-schedule constraints imply that firms in the announcement stage are truthful along the equilibrium-path (incentive compatible) vector $\mathbf{z}$. Hence, a firm lies at the announcement stage only for a subsequent off-schedule deviation.

[19] See Lemma 2 in their paper.

3. Incentive Compatibility with Contract

In this section, we demonstrate that there is an interaction between contractual form for agents' truthfulness and incentive-compatibility constraints for firms. Among other alternatives, we consider a contract that is enforceable only under Assumption 1: when agent 1 reports low cost and produces q^1, then the agent receives

$$t^1\left(q^1, L\right) = \theta_L q^1 + \triangle \cdot \overline{q}_H^1, \tag{5}$$

and when agent 1 reports high cost and produces q^1, then the agent receives

$$t^1(q^1, H) = \begin{cases} \theta_H q^1 & \text{if } q^1 = q_{Hk}^1 \\ \theta_H q^1 + \alpha\left(\rho - \theta_H\right)\left(\overline{q}_L^1 - \overline{q}_H^1\right) & \text{if } q^1 = q_{Lk}^1, \end{cases} \tag{6}$$

where $0 \leq \alpha \leq 1$. The contract for agent 2 is analogous.

Note that the compensation to agent i is conditional on the actual quantity q^i that agent i produces and other expected quantities, $\overline{q}_H^i$ and $\overline{q}_L^i - \overline{q}_H^i$, that are not conditional on the agent's current report of cost type. Given that market-share allocations are monotone ($\overline{q}_L^i > \overline{q}_H^i$), this contract contains a commitment device that induces the firm's truthfulness: when a high-cost agent produces a large quantity that is assigned for a low-cost agent, the agent receives an extra payment. If an agent reports high cost but the firm lies and announces low cost (understates) to increase its market share, then the firm will have the expected gain $\left(\rho - \theta_H\right)\left(\overline{q}_L^i - \overline{q}_H^i\right)$ but suffer the expected expense $\alpha\left(\rho - \theta_H\right)\left(\overline{q}_L^i - \overline{q}_H^i\right)$. The contractual form for agent i seen in (5) and (6) is hereafter denoted by $x^i(\alpha)$.[20] The level of α is used to represent a contractual parameter that captures how strongly firms are bounded to truthfulness by the contractual form $x^i(\alpha)$. In the previous literature, firms own private information; they can observe and distort private information at no extra costs. In our model, agents hold private information and firms deliver it following the agents' report. Having an incentive to distort the agents' private information, firms use the internal contract and endogenize the level of costs a high-cost firm suffers when it falsifies its agent's report. The level of α reflects the level of costs that a high-cost firm incurs to falsify its agent's report. The contractual arrangement captures the circumstance in which firms deliberately reduce the degree to which they control of private information; their incentive to distort the agents' private information is bounded by the contract with agents.[21]

[20] The qualitative results would be unaffected by a different commitment device, $t^i(q^i; H) = \theta_H q^i + \alpha$, where $\alpha > 0$ if firm i lies, and zero otherwise. Our analysis also does not resort to an immediate solution, where α is a very large number, in order to show that a contract of this nature can be easily modified to a more realistic contract in later analysis.

[21] In relation to the previous literature, our analysis can be extended to the model in which firms hold private information but face some costs to distort it; a higher level of falsification costs may reflect a lower degree of private information. A similar falsification cost is found in the principal-agent model by Maggi and Rodríguez-Clare (1995), where the agent can distort its private information at some costs.

The contract $x^i(\alpha)$ has important features under the monotonicity, $\overline{q}^i_L > \overline{q}^i_H$. First, the contract has a direct commitment effect and yet is enforceable only under Assumption 1. Second, the two-tier revelation problems are closely intertwined: agents' truthful reports are ensured if and only if firms do not lie. If firms are truthful, then agents are truthful under the contract. If high-cost firms do lie (understate the reported cost types), then low-cost agents have an incentive to overstate their cost types. If a low-cost agent i lies and reports high cost and firm i understates it, then the agent receives $t^i(q^i, H) = \theta_H q^i + \alpha(\rho - \theta_H)(\overline{q}^i_L - \overline{q}^i_H)$ as in (6). This agent will then have the expected net gain:

$$(\theta_H - \theta_L)\overline{q}^i_H - \triangle \cdot \overline{q}^i_H + \alpha(\rho - \theta_H)(\overline{q}^i_L - \overline{q}^i_H) > 0.$$

The low-cost agent who lies can get the expected cost saving in production (first term) but lose the information rents (second term) that the agent could earn without overstatement. The first two terms cancel each other out. Third, if firms are truthful, then they can optimally allocate market shares by the criterion of productive efficiency, which enhances contractual efficiency; firms can reduce information rents by mitigating the low-cost agent's incentive to overstate the cost type. Information rents will be realized for two low-cost agents in (L, L), and for one low-cost agent in (H, L) or (L, H). Letting $I^i \equiv \triangle \cdot \overline{q}^i_H$, the overall expected information rents will be the same as in the optimal monopoly contracting:

$$\mu^2(I^1 + I^2) + \mu(1-\mu)(I^1 + I^2) = \mu(1-\mu)\triangle \cdot (q^1_{HH} + q^2_{HH}).$$

At this point, it is worthwhile to present an overview of how the contract acts as a commitment device in later analysis. First, we will establish a Pareto-frontier line segment of APPE values. Construction of the line segment is possible only when the segment is sufficiently long, so that a high-cost firm is persuaded to be truthful today by a sufficiently high continuation value drawn within the segment even if the firm may end up with zero market share. If the contract $x^i(\alpha)$ is selected such that a high-cost firm suffers some falsification costs when it lies, then the high-cost firm can be induced to be honest today by a reduced continuation-value reward, which relaxes the restriction that the segment must be sufficiently long. In this way, the internal contract acts as a commitment device that elicits the firm's truthfulness. Second, we will characterize an optimal SPPE, wherein to prevent the high-cost firm's understatement today, a low continuation value (as future penalty) is given to the firms that report low cost together. If the contract $x^i(\alpha)$ is selected such that a high-cost firm gains nothing by telling a lie today (because α is high enough) today, then it becomes unnecessary to penalize the firms that report low cost together tomorrow. The contract then acts to avoid the equilibrium-path penalization that would otherwise follow a pair of low-cost announcement.

We now fully express incentive-compatibility constraints for a given contract $x^i(\alpha)$. Suppose that prices are fixed at ρ, and market-share schedules are monotone, $\overline{q}^i_L > \overline{q}^i_H$. Both conditions hold in equilibrium. If the internal contract has a commitment device as in (5) and (6), then the current-period profit for firm 1 is

$$\begin{aligned}\Pi^1(\hat{\jmath},j) &= \sum_{k\in\{L,H\}} \mu_k \pi^1\left(\rho, q^1_{\hat{\jmath}k}, j\right) \\ &= \sum_{k\in\{L,H\}} \mu_k \left[\rho q^1_{\hat{\jmath}k} - t^1\left(q^1_{\hat{\jmath}k}, j\right)\right].\end{aligned}$$

To represent the interim-stage profits in a direct form, let $U^i(\hat{\jmath},j) \equiv \Pi^i(\hat{\jmath},j) + \delta\overline{v}^i_{\hat{\jmath}}$.[22] The on-schedule constraints are then given by

$$U^i(H,H) \geq U^i(L,H) \qquad (\text{On-IC}^i_H)$$

$$U^i(L,L) \geq U^i(H,L). \qquad (\text{On-IC}^i_L)$$

Our analysis focuses on the binding downward incentive constraint (On-IC^i_H), based on the following Lemma.

Lemma 2. *Assume that prices are fixed at ρ and that $\overline{q}^i_L > \overline{q}^i_H$. Under a contract $x^i(\alpha)$, if On-IC^i_H is binding, then On-IC^i_L is slack.*

The proof is in the Appendix. The binding On-IC^i_H is assumed, given that a high-cost firm has an incentive to mimic a low-cost type that has a higher market share. The relevant incentive problem is how to dissuade a high-cost firm from mimicking a low-cost type.[23]

We next find the expected profit function when On-IC^i_H is binding. Under a contract $x^i(\alpha)$, if a high-cost firm is truthful, then it earns

$$U^i(H,H) = \Pi^i(H,H) + \delta\overline{v}^i_H,$$

and if a high-cost firm lies, then it earns

$$U^i(L,H) = \Pi^i(H,H) + (1-\alpha)(\rho-\theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right) + \delta\overline{v}^i_L.$$

The binding On-IC^i_H means

$$\delta(\overline{v}^i_H - \overline{v}^i_L) = (1-\alpha)(\rho-\theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right). \qquad (7)$$

The binding On-IC^i_H thus indicates the balance between the expected current-period gain from understatement (RHS) and the expected continuation-value loss that a firm will suffer from telling a lie (LHS). It then follows that the interim profit for a low-cost firm is

$$U^i(L,L) = U^i(H,H) + \triangle \cdot \left(\overline{q}^i_L - \overline{q}^i_H\right) + \alpha(\rho-\theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right).$$

[22] All the interim-stage profits are provided by the proof of Lemma 2 in the Appendix.

[23] As will be seen below, there is a constraint that a Pareto-frontier line segment must be sufficiently long for it to exist. The assumption that On-IC^i_H is binding provides the minimum length of the segment that satisfies this constraint.

It is immediate to derive this equation from $U^i(H,H) = U^i(L,H)$. The second term on the RHS represents information rents, and the last term is the extra costs that a high-cost firm will incur when it lies. Thus, for a given contract $x^i(\alpha)$, if On-IC^i_H is binding, then the expected equilibrium payoff, $u^i(\mathbf{z}(\alpha)) = \sum_j \mu_j U^i(j,j)$, is given by

$$u^i(\mathbf{z}(\alpha)) = (\rho - \theta_H)\,\overline{q}^i_H + \delta\overline{v}^i_H + \mu\triangle\cdot\left(\overline{q}^i_L - \overline{q}^i_H\right) + \alpha\mu(\rho-\theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right). \tag{8}$$

As mentioned above, there are two types of off-schedule deviations: (i) a firm can slightly undercut the price and capture the entire market *after* the communication with the other firm, and (ii) a firm can overstate or understate *at* the communication and then undercut the price. No firms will gain by undertaking the first type of deviation if $\forall j,k,(j,k)$

$$\delta\left(v^1_{jk} - \underline{v}\right) \geq \rho - \theta_j - \left(\rho q^1_{jk} - t^1(q^1_{jk}, j)\right) \qquad (\text{Off-IC}^1_{jk})$$
$$\delta\left(v^2_{jk} - \underline{v}\right) \geq \rho - \theta_k - \left(\rho q^2_{jk} - t^2(q^2_{jk}, k)\right). \qquad (\text{Off-IC}^2_{jk})$$

The Nash-equilibrium value, denoted by $\underline{v}$, is used as the punishment that follows any off-schedule deviation. The RHS represents the current-period gain that firm i can have by undercutting slightly the price ρ in state (j,k), whereas the LHS represents the loss that firm i will suffer in the future. Note that $\rho - \theta_j$ is the highest current-period payoff that firm i of type j can get for any payment scheme $t^i(q^i, j) \geq \theta_j q^i$. We can show that since a misrepresentation has no cost savings in terms of payments to the agent under $x^i(\alpha)$, the constraint for the second deviation (Off-m-IC^i_j) is redundant in a price-fixing collusion; Off-m-IC^i_j holds whenever other constraints are satisfied. For instance, consider Off-m-IC^1_L. Firm 1 of a low-cost type has its interim payoff under $x^i(\alpha)$:

$$U^1(L,L) = \sum_{k\in\{L,H\}} \mu_k\left[(\rho-\theta_L)q^1_{Lk} + \delta v^1_{Lk} - \triangle\cdot\overline{q}^1_H\right] \geq \sum_{k\in\{L,H\}} \mu_k\left[(\rho-\theta_L) + \delta\underline{v}\right].$$

The RHS represents the highest expected payoff that a low-cost firm can get when it misrepresents its cost type for a subsequent deviation. The inequality is given by Off-IC$^1_{Lk}$.

4. Optimal APPE

In this section, we establish the existence of an optimal APPE as follows. In regard to the existence, we show that a contract $x^i(\alpha)$ can be designed such that there exists a value set $V(\alpha)$ generated by incentive compatible vectors $\mathbf{z}(\alpha) \in Z^{IC}(V(\alpha))$ under the contract:

$$V(\alpha) = \left\{(u^1, u^2) : \exists\mathbf{z}(\alpha) \in Z^{IC}(V(\alpha)) \text{ such that } u^i = u^i(\mathbf{z}(\alpha))\ \forall i\right\}.$$

Agents' truthful reports are ensured under $x^i(\alpha)$ by firms' incentive compatibility under $\mathbf{z}(\alpha)$. In regard to the optimality, given that a contract $x^i(\alpha)$ provides a value set $V(\alpha)$, we choose the contract such that values in $V(\alpha)$ achieve the optimal monopoly values: $V(\alpha) \subset V^m$, where $V^m \equiv \{(u^1,u^2) : u^1 + u^2 = \frac{\pi^m}{1-\delta}\}$. Then, we say that there exists a set $V(\alpha) \subset V^m$ such that $V(\alpha) \cup V^n$ is a self-generating set of PPE values. For the base model, we preserve Assumption 1 together with the assumption, $\mu > \frac{1}{2}$. These assumptions will be relaxed.

4.1. Contractual Range

To establish an optimal APPE-value set $V(\alpha)$, we find a contract $x^i(\alpha)$ (a range of α in (6)), wherein for any $(u^1, u^2) \in V(\alpha)$, there exists a vector $\mathbf{z}(\alpha) \in Z^{IC}(V(\alpha))$ such that $u^i = u^i(\mathbf{z}(\alpha))$ $\forall i$. Since the set V^m has slope $\frac{du^2}{du^1} = -1$, any optimal value set $V(\alpha) \subset V^m$ is a line segment with slope -1. The line segment is defined as $V(\alpha) = [(\underline{u}, \overline{u}), (\overline{u}, \underline{u})]$, where $\overline{u} > \underline{u}$ and $\underline{u} + \overline{u} = \frac{\pi^m}{1-\delta}$. As is standard, we first explore only the on-schedule constraints, assuming that firms are sufficiently patient so that off-schedule constraints hold.

Consider first the parameter range $\rho \geq C_H \equiv \theta_H + \frac{\mu}{1-\mu}\triangle$ in which $q^1_{HH} + q^2_{HH} = 1$ in an optimal collusion, as seen in Lemma 1. It follows from the optimality that price is fixed at ρ, and market shares in states (L, H) and (H, L) are fixed at $q^1_{LH} = q^2_{HL} = 1$. Each point in $V(\alpha)$ is therefore established by varying market shares in ties, q^i_{LL} and q^i_{HH}. At an endpoint $(\underline{u}, \overline{u})$ of the line segment $V(\alpha)$, for example, firm 1 receives the smallest value $\underline{u}$ by being assigned to the least favored market shares such that q^1_{LL} and q^1_{HH} are close or equal to zero. Attention is thus on how to elicit a firm's truthfulness at its "disadvantaged" position, where the firm draws high cost and thus may have zero market share. A high continuation value v^1_{HL} (as future reward) is afforded to firm 1, in order to induce its truthfulness today at the least favored endpoint. The level of such future reward is determined by the binding On-IC^i_H in (7):

$$v^1_{HL} - v^1_{LH} = \frac{(1-\alpha)(\rho - \theta_H)}{\delta}. \tag{9}$$

The equation is derived by adding up both sides of the binding On-IC^i_H:[24]

$$\sum_{i=1}^{2} \delta(\overline{v}^i_H - \overline{v}^i_L) = \sum_{i=1}^{2} (1-\alpha)(\rho - \theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right). \tag{10}$$

The equation (9) implies that if a line segment $V(\alpha) \subset V^m$ exists, then its width must be sufficiently long: the value $\overline{u}$ must exceed $\underline{u}$ by at least the RHS of (9).[25] Only then is it feasible to reward a high-cost firm with a high continuation value v^1_{HL} drawn from the segment. Thus, the on-schedule constraints imply that there is an "additional" constraint:

$$\overline{u} - \underline{u} \geq \frac{(1-\alpha)(\rho - \theta_H)}{\delta}. \tag{Add-IC}$$

We now establish that there exists a vector $\mathbf{z}(\alpha)$ that satisfies the binding On-IC^i_H and Add-IC if for any δ,

$$\alpha \geq \alpha^*(\delta) \equiv \frac{1 - \delta + \delta(2\mu - 1)(1 - \mu\gamma)}{1 - \delta + \delta 2\mu^2},$$

where γ represents the ratio $\frac{\triangle}{\rho - \theta_H}$.

To clarify the exposition, consider the case where $\alpha = \alpha^*(\delta)$ for any δ.[26] Define a vector $\mathbf{z}(\alpha)$

[24] The derivation of (9) is included in the proof of Lemma 3 in the Appendix.

[25] The length given by the RHS of (9) satisfies the required length for the segment to exist. If On-IC^i_H is not binding (say, slack), then the width on the RHS is not long enough.

[26] In the Appendix, we show that for *any* $\alpha \geq \alpha^*(\delta)$ for a given δ, there exist $\mathbf{z}(\alpha) \in Z^{IC}(V(\alpha))$ that establishes

such that (i) prices are fixed at ρ, (ii) market shares are allocated by the criterion of productive efficiency with $q^1_{LL} = q^1_{HH} = 0$ and (iii) continuation values are $v^1_{LH} = v^1_{HH} = \underline{u}$,

$$\begin{aligned} v^1_{HL} &= v^1_{LH} + \frac{(1-\alpha)(\rho-\theta_H)}{\delta} = \underline{u} + \frac{(1-\alpha)(\rho-\theta_H)}{\delta} \\ v^1_{LL} &= \underline{u} + \frac{2\mu-1}{\mu} \cdot \frac{(1-\alpha)(\rho-\theta_H)}{\delta}, \end{aligned}$$

together with the condition, $v^1_{ij}+v^2_{ij} = \underline{u}+\overline{u}\ \forall(j,k)$. Given the price and productive efficiencies, this vector $\mathbf{z}(\alpha)$ achieves optimality. To see this, using the equilibrium payoff in (8), let $u^1(\mathbf{z}(\alpha)) = \underline{u}$ and $u^2(\mathbf{z}(\alpha)) = \overline{u}$. The vector $\mathbf{z}(\alpha)$ then obtains the values at the endpoint:

$$\underline{u} = \frac{\left(\mu - \mu^2\alpha\right)(\rho-\theta_H) + \mu(1-\mu)\triangle}{1-\delta} \tag{11}$$

$$\overline{u} = \frac{\left(1 - \mu + \mu^2\alpha\right)(\rho-\theta_H) + \mu^2\triangle}{1-\delta}. \tag{12}$$

Note that $\underline{u}$ decreases in α whereas $\overline{u}$ increases in α, and that $\forall\alpha$

$$\underline{u} + \overline{u} = \frac{\rho - E(\theta)}{1-\delta} = \frac{\pi^m}{1-\delta}.$$

The defined vector indicates that firm 1, at the end point $(\underline{u}, \overline{u})$, receives the least favored market shares in ties, and produces zero output in (H, L). Given the vector, the binding On-IC^i_H in (9) implies that firm 1 receives a high continuation value $v^1_{HL} = \underline{u}+\frac{(1-\alpha)(\rho-\theta_H)}{\delta}$. This continuation-value reward will be delivered when firm 1 takes more favored market shares in ties in the future after the realization of (H, L). We confirm that this vector $\mathbf{z}(\alpha)$ is (on-schedule) incentive compatible. The value v^1_{LL} is chosen to satisfy the binding On-IC^1_H, and the value v^1_{HL} is chosen to satisfy the equation (10), given the other values. Because of the binding On-IC^1_H and the equation (10), On-IC^2_H is also binding. It follows from Lemma 2 that $\forall i$ if On-IC^i_H is binding, then On-IC^i_L is slack. We still need to confirm that Add-IC holds. The level of $\alpha = \alpha^*(\delta)$ is determined to satisfy the binding Add-IC: $\overline{u} - \underline{u} = \frac{(1-\alpha)(\rho-\theta_H)}{\delta}$. Lastly, we verify that the continuations are drawn from the value set $[(\underline{u}, \overline{u}), (\overline{u}, \underline{u})]$. If Add-IC holds, then $\underline{u} < v^1_{HL} \leq \overline{u}$, and if $\mu > \frac{1}{2}$, then $\underline{u} < v^1_{LL} < \overline{u}$.[27] Hence, $v^i_{jk} \in V(\alpha)\ \forall i, (j,k)$.

We next construct the other endpoint $(\overline{u}, \underline{u})$ of $V(\alpha)$. If there exists an incentive compatible vector $\mathbf{z}(\alpha)$ that establishes an endpoint $(\underline{u}, \overline{u})$, then there exists an analogous vector $\mathbf{z}'(\alpha)$ that establishes the other endpoint $(\overline{u}, \underline{u})$. Then, the remainder of the segment can be constructed by a convex combination of two vectors. The reason is that given the fixed price ρ, firms' payoffs and the on-schedule constraints are linear in terms of market shares and continuation values, for a given level of α.

We also emphasize that the contract $x^i(\alpha)$ can be used to *lengthen* the width of $V(\alpha)$. Observe that the gap $\overline{u} - \underline{u}$ increases in α. To investigate how the equilibrium payoff in (8) changes with

the end point $(\underline{u}, \bar{u})$. This result and the arguments that follow are detailed by the proof of Lemma 3 in the Appendix.

[27] Note that the assumption $\mu > \frac{1}{2}$ is necessary for the continuation values to be drawn from the value set $V(\alpha)$.

α, suppose that α rises. Then, the payoff of firm 2 ($\overline{u}$) increases. The last term in (8) $\alpha\mu(\rho - \theta_H)(\overline{q}_L^2 - \overline{q}_H^2)$ rises with α, and this positive effect is maximized, because of the most favored market shares, $q_{LL}^2 = q_{HH}^2 = 1$, given the continuation values, $v_{LH}^2 = v_{HH}^2 = \overline{u}$ under $\mathbf{z}(\alpha)$.[28] When α rises, however, the payoff of firm 1 ($\underline{u}$) falls. The positive effect through the last term is minimized for firm 1, because of the least favored market shares, $q_{LL}^1 = q_{HH}^1 = 0$, and the continuation-value reward $v_{HL}^1 = v_{LH}^1 + \frac{(1-\alpha)(\rho-\theta_H)}{\delta}$ falls, given $v_{LH}^1 = \underline{u}$ under $\mathbf{z}(\alpha)$. To see how the width is lengthened, consider an alternative contract $x^i(\widehat{\alpha})$, where $\widehat{\alpha} = \widehat{\alpha}(\delta) > \alpha^*(\delta)$ for any δ. The associated vector $\mathbf{z}(\widehat{\alpha})$ is defined such that productive efficiency is achieved with $q_{LL}^1 = q_{HH}^1 = 0$, and continuation values are similar to those in $\mathbf{z}(\alpha)$ except that $\widehat{\alpha}$ replaces α in v_{HL}^1 and v_{LL}^1. We can then construct a value set $V(\widehat{\alpha})$ whose width is longer than that of $V(\alpha)$ under $\alpha = \alpha^*(\delta)$. Since the RHS of Add-IC decreases in α, Add-IC is slack under the alternative contract.

Our argument can be summarized as follows. It is highly beneficial that firms allocate market shares by productive efficiency. Variations of continuation values are necessary to induce firms' current-period truthfulness. These continuation-value transfers are delivered in the form of market-share favors in ties (in terms of favored location on the segment). Construction of the optimal APPE-value set is possible only when the frontier of value set is long enough. The frontier can be sufficiently lengthened by the contractual device such that the least favored firm is persuaded to be truthful by a continuation-value reward drawn from the segment. Given the recursive structure of the model, any collusive scheme is designed to elicit firms' truthfulness in each period, regardless of their previous cost reports. Even after a history of 10 consecutive draws of (H, L), for example, firm 1 is induced to be truthful today by the promise of the most favored market shares tomorrow, if it is patient enough to endure asymmetric market-share arrangements in ties.[29]

Lemma 3. *Assume that $\mu > \frac{1}{2}$ and $\rho \geq C_H$, and that firms are sufficiently patient. If $\alpha \geq \alpha^*(\delta)$, there exists a set*

$$V(\alpha) = \left\{(u^1, u^2) : \exists \mathbf{z}(\alpha) \in Z^{IC}(V(\alpha)) \text{ such that } u^i = u^i(\mathbf{z}(\alpha)) \ \forall i\right\} \subset V^m.$$

A detailed proof is provided in the Appendix. An example of the locus $\alpha = \alpha^*(\delta)$ is in Fig. 1. Note that $\alpha^*(\delta)$ is decreasing in δ. If δ rises, the gap $\overline{u} - \underline{u}$ rises but the RHS of Add-IC falls. Thus, if δ is higher, then Add-IC may hold for a lower α. Intuitively, when firms are more patient and thus more willing to wait for future reward than to capture the current-period gain by understating their cost types, they may depend on a lower level of contractual commitment. Note also that the level of $\alpha^*(\delta)$ is decreasing in the ratio $\gamma = \frac{\triangle}{\rho - \theta_H}$. Intuitively, when $\triangle$ is higher, high-cost firms are more willing to wait for the continuation-value reward; when the continuation-value reward is delivered in terms of market-share favors (e.g., $q_{LL}^i = q_{HH}^i = 1$), the payoff will increase more

[28] Letting market shares in ties $q_{LL}^2 = q_{HH}^2 = q_T^2$, the last term in (8) becomes $\alpha\mu(\rho - \theta_H)[(2\mu - 1)q_T^2 + (1 - \mu)]$, which rises in q_T^2 if $\mu > \frac{1}{2}$.

[29] If firms are not sufficiently patient, some inefficiency begins to have an effect on the APPE-value set as is detailed by Athey and Bagwell (2001).

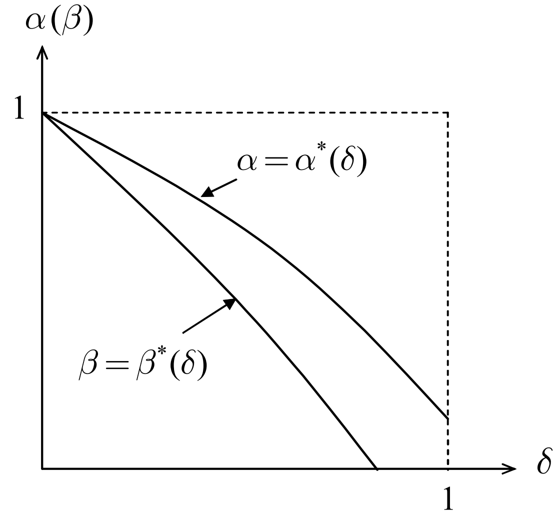

Figure 1: Example of $\alpha = \alpha^*(\delta)$ and $\beta = \beta^*(\delta)$.

significantly for a higher $\triangle$. Firms may then need a lower level of contractual commitment. On the other hand, when the margin $\rho - \theta_H$ is higher, high-cost firms are more tempted to capture the current-period gain, and thus their incentives should be more strongly bounded by the contract.

Consider next the parameter range $\rho < C_H$ in which $q^1_{HH} + q^2_{HH} = 0$ in an optimal collusion. The contract $x^i(\beta)$ that corresponds to (5) and (6) becomes

$$t^i\left(q^i, L\right) = \theta_L q^i \text{ and} \tag{13}$$

$$t^i\left(q^i, H\right) = \theta_H q^i + \beta\left(\rho - \theta_H\right) q^i, \tag{14}$$

where $0 \leq \beta \leq 1$. A high-cost firm is assigned to zero output in an optimal collusion. Any positive production is accomplished by either a low-cost firm or a high-cost firm that surely lies. If a high-cost firm lies and ever produces $q^i > 0$, it has gain $(\rho - \theta_H)\, q^i$ but suffers costs $\beta\,(\rho - \theta_H)\, q^i$. Note that there is no information-rent term in the payment. The binding On-IC^i_H implies that

$$v^1_{HL} - v^1_{LH} = \frac{(1-\beta)(2-\mu)(\rho - \theta_H)}{\delta}. \tag{15}$$

Thus, if a line segment $V(\beta) \subset V^m$ exists, then its width must be sufficiently long:

$$\overline{u} - \underline{u} \geq \frac{(1-\beta)(2-\mu)(\rho - \theta_H)}{\delta}. \tag{Add-IC}$$

There is a vector $\mathbf{z}(\beta)$ that satisfies the binding On-IC^i_H and Add-IC if for any δ,

$$\beta \geq \beta^*(\delta) \equiv \max\left\{\frac{(2-\mu)\left(1-\delta+\mu\delta\right) - \mu^2\delta\gamma}{2-\mu+(3\mu-2)\,\delta},\ 0\right\}.$$

Following the same arguments as above, we can obtain the following result.[30]

[30] The vector $\mathbf{z}(\beta)$ is detailed in the proof of Proposition 1 in the Appendix. Note that the assumption $\mu >$

Lemma 4. *Assume that* $\mu > \frac{3-\sqrt{5}}{2}$ *and* $\rho < C_H$*, and that firms are sufficiently patient. If* $\beta \geq \beta^*(\delta)$*, there exists a set*

$$V(\beta) = \left\{(u^1, u^2) : \exists \mathbf{z}(\beta) \in Z^{IC}(V(\beta)) \text{ such that } u^i = u^i(\mathbf{z}(\beta))\ \forall i\right\} \subset V^m.$$

An example of $\beta = \beta^*(\delta)$ is illustrated in Fig. 1. The level of $\beta^*(\delta)$ is decreasing in δ and $\gamma = \frac{\triangle}{\rho - \theta_H}$; as above, firms needs a lower level of contractual commitment when δ and γ are higher. The inequality $\rho < C_H$ is rewritten as $\gamma > \frac{1-\mu}{\mu}$. If γ is low and close to $\frac{1-\mu}{\mu}$, then $\beta^*(\delta)$ is higher than $\alpha^*(\delta)$ for a given δ. If γ keeps rising, then $\beta^*(\delta)$ shifts down below $\alpha^*(\delta)$. If γ is higher than $\frac{2-\mu}{\mu}$ and if δ is sufficiently high, then $\beta^*(\delta) = 0$. This finding implies that the contractual commitment becomes unnecessary if γ and δ are sufficiently high.

4.2. Optimal APPE with Assumption 1

As of yet, our analysis has been confined to the on-schedule constraints. In this subsection, we identify the critical discount factor δ^* above which the off-schedule constraints also hold. To find δ^*, we use the above vector $\mathbf{z}(\alpha)$. It then suffices to check whether a firm's off-constraints hold at its disadvantaged endpoint, since the firm there is more tempted to undercut the price than at any other point of the segment. Firm 1 is disadvantaged at the endpoint $(\underline{u}, \overline{u})$ and assigned to $q^1_{LL} = q^1_{HH} = 0$ under $\mathbf{z}(\alpha)$. In (L, H), firm 1 would not undercut the price, since it captures the entire market in equilibrium. Comparing (H, L) to (H, H), firm 1 is less tempted to deviate in (H, L) than in (H, H); the continuation value v^1_{HL} is higher than v^1_{HH}, and the RHS remains the same under $q^1_{HL} = q^1_{HH} = 0$ in Off-IC$^1_{HL}$ and Off-IC$^1_{HH}$:

$$\begin{aligned} \delta\left(v^1_{HL} - \underline{v}\right) &\geq \rho - \theta_H - (\rho - \theta_H)\, q^1_{HL} \\ \delta\left(v^1_{HH} - \underline{v}\right) &\geq \rho - \theta_H - (\rho - \theta_H)\, q^1_{HH}. \end{aligned}$$

Given that $q^1_{LL} = q^1_{HH} = 0$ under $\mathbf{z}(\alpha)$, Off-IC$^1_{LL}$ is $\delta(v^1_{LL} - \underline{v}) \geq \rho - \theta_L$. Hence, the off-schedule constraints reduced to Off-IC$^1_{LL}$ and Off-IC$^1_{HH}$. If $\rho \geq C_H$, for example, the two relevant constraints are

$$\delta\left(\underline{u} + \frac{2\mu - 1}{\mu} \cdot \frac{(1-\alpha)(\rho - \theta_H)}{\delta} - \underline{v}\right) \geq \rho - \theta_L \qquad \text{(Off-IC}^1_{LL}\text{)}$$

$$\delta\left(\underline{u} - \underline{v}\right) \geq \rho - \theta_H. \qquad \text{(Off-IC}^1_{HH}\text{)}$$

By plugging $\alpha = \alpha^*(\delta)$ and $\underline{u}$ in (11) and (12) into the constraints, we can find the associated critical discount factors, δ^*_{LL} and δ^*_{HH}. Then, $\delta^* = \max\{\delta^*_{LL}, \delta^*_{HH}\}$.

Example. Suppose that $\rho = 4$, $\theta_H = 2$, $\theta_L = 1$ and $\mu = 0.6$. Consider first the on-schedule constraints. A contract $x^i(\alpha)$ can be designed to establish the existence of a value set $V(\alpha) = [(\underline{u}, \overline{u}), (\overline{u}, \underline{u})] \subset V^m$. If $V(\alpha)$ exists, then the on-schedule constraints imply that the width of $V(\alpha)$

$\frac{1}{2}$ is relaxed to $\mu > \frac{3-\sqrt{5}}{2} \approx 0.382$. As above, this assumption is necessary for v^1_{LL} to be drawn from $V(\beta)$.

must be long enough (Add-IC): $\overline{u}-\underline{u} \geq \frac{2(1-\alpha)}{\delta}$. The endpoint $(\underline{u},\overline{u})$ can be constructed by a vector $\mathbf{z}(\alpha)$ in which productive efficiency is achieved with $q_{LL}^1 = q_{HH}^1 = 0$, and continuation values are assigned to satisfied the binding On-IC$_H^i$: $v_{LH}^1 = v_{HH}^1 = \underline{u}$,

$$v_{HL}^1 = v_{LH}^1 + \frac{2(1-\alpha)}{\delta} \text{ and } v_{LL}^1 = \underline{u} + \frac{2(1-\alpha)}{3\delta}.$$

The vector constructs the values:

$$\underline{u} = \frac{1.44 - 0.72\alpha}{1-\delta} \text{ and } \overline{u} = \frac{1.16 + 0.72\alpha}{1-\delta}.$$

Note that $\underline{u} + \overline{u} = \frac{2.6}{1-\delta} = \frac{\pi^m}{1-\delta}$. All the on-schedule constraints are satisfied: On-IC$_H^i$ is binding, On-IC$_L^i$ is slack and Add-IC is binding if $\alpha = \alpha^*(\delta) = \frac{1-0.86\delta}{1-0.28\delta}$. Consider next the off-schedule constraints for firm 1 at $(\underline{u},\overline{u})$. The relevant off-schedule constraints are Off-IC$_{LL}^1$ and Off-IC$_{HH}^1$:

$$\delta\left(v_{LL}^1 - \underline{v}\right) \geq 3 \text{ and } \delta\left(v_{HH}^1 - \underline{v}\right) \geq 2.$$

Using the values v_{LL}^1 and v_{HH}^1 under $\mathbf{z}(\alpha)$ and $\alpha = \alpha^*(\delta)$, we find that $\delta_{LL}^* \approx 0.729$ and $\delta_{HH}^* \approx 0.678$. Thus, $\delta^* = \max\{\delta_{LL}^*, \delta_{HH}^*\} \approx 0.729$. This example highlights that $V(\alpha) \subset V^m$ exists even if firms are not infinitely patient: if $\delta = 0.8 > \delta^*$, then $\alpha^*(\delta) \approx 0.402$, $\underline{u} \approx 5.753$ and $\overline{u} \approx 7.247$.

As argued above, the contract $x^i(\alpha)$ can be used to lengthen the width of $V(\alpha)$. Under an alternative contract $x^i(\widehat{\alpha})$, if $\widehat{\alpha} = \widehat{\alpha}(\delta) > \alpha^*(\delta)$ for any δ, then there exists a vector $\mathbf{z}(\widehat{\alpha})$ in which productive efficiency is achieved with $q_{LL}^1 = q_{HH}^1 = 0$, and continuation values are assigned as in $\mathbf{z}(\alpha)$ except that $\widehat{\alpha}$ replaces α in continuation values. The width of the associated value set $V(\widehat{\alpha})$ is longer than that of $V(\alpha)$ under $\alpha = \alpha^*(\delta)$. Add-IC becomes slack under the alternative contract. Note, however, that the values, v_{LL}^1 and v_{HH}^1, are lower under $\mathbf{z}(\widehat{\alpha})$ than $\mathbf{z}(\alpha)$, and thus lengthening the value segment may be constrained by the off-schedule deviation of the least favored firm unless δ is sufficiently high.

By contrast, the alternative contract $x^i(\widehat{\alpha})$ may be used to *shorten* the width of the value set. Under the contract $x^i(\widehat{\alpha})$, there exists a vector $\widehat{\mathbf{z}}(\widehat{\alpha}) \neq \mathbf{z}(\widehat{\alpha})$ such that productive efficiency is achieved with $q_{LL}^1 = q_{HH}^1 \in (0, \frac{1}{2})$ while continuation values remain similar to those under $\mathbf{z}(\widehat{\alpha})$. A distinct feature here is that market shares of firm 1 in ties are above zero and Add-IC is binding; the level of $\widehat{\alpha} = \widehat{\alpha}(\delta)$ and market shares in ties are tailored to satisfy the binding Add-IC.[31] In this case, the width of the corresponding set $V(\widehat{\alpha})$ is shorter than that of $V(\alpha)$ under $\alpha = \alpha^*(\delta)$. Note that the RHS of Off-IC$_{LL}^1$ and Off-IC$_{HH}^1$ decreases when q_{LL}^1 and q_{HH}^1 decrease, and that the value $\underline{u}$ in the LHS rises when the segment is shortened. Thus, shortening the segment may relax the off-schedule constraints of the least favored firm in some parameter range. We summarize our findings as follows:

Proposition 1. *Assume that firms are sufficiently patient. (i) If $\rho \geq C_H$ and $\mu > \frac{1}{2}$, then for $\alpha \geq \alpha^*(\delta)$, there exists a set $V(\alpha) \subset V^m$ such that $V(\alpha) \cup V^n$ is a self-generating set of APPE*

[31] It is shown by the proof of Lemma 3 that for any $\alpha \geq \alpha^*(\delta)$ for a given δ, there exists a vector $\mathbf{z}(\alpha)$ in which On-IC$_H^i$ and Add-IC are binding.

values. (ii) If $\rho < C_H$ and $\mu > \frac{3-\sqrt{5}}{2}$, then for $\beta \geq \beta^*(\delta)$, there exists a set $V(\beta) \subset V^m$ such that $V(\beta) \cup V^n$ is a self-generating set of APPE values.

The proof is in the Appendix. The benefit of the interaction between internal contracting and inter-firm collusion is substantial. Even if firms are not infinitely patient, they may be able to duplicate the monopolist's optimal performance. If the information that facilitates collusion is held by other parties (agents), then the contract with agents may act as a commitment device to relax the truth-telling constraint for colluding firms to allocate market shares by the criterion of productive efficiency. Conversely, collusive market-share allocations may act to discipline the agents' overstating incentive and thus reduce their information rents. This finding is, however, based on Assumption 1.

4.3. Optimal APPE without Assumption 1

In this subsection, we relax Assumption 1. In the previous analysis, the continuation-value reward was delivered in the form of market-share favors in ties (or in terms of favored position on the segment), and construction of an optimal APPE-value set was possible, since market shares in ties, q^i_{LL} and q^i_{HH}, varied along the equilibrium-value set without causing any inefficiency. The market-sharing arrangements are, however, feasible only when they are verifiable for agents. Our objective here is to find a payment scheme $t^i(q^i, j)$ that is conditional only on the real quantity q^i that agent i produces after the report of j. A difficulty with finding such a payment scheme is that the information-rent term for a low-cost agent, $\triangle \cdot \overline{q}^i_H$, involves the market-sharing schemes that the agent would face only after the report of high cost. Given this difficulty, we put a restriction on market shares in ties: q^i_{HH} is held constant at $\frac{1}{2}$ and only q^i_{LL} is used for market-share favors. Using the restriction, we can construct a simple and enforceable contract. Assuming that $\rho \geq C_H$, define a contract $x^i(\alpha)$:

$$t^i\left(q^i, L\right) = \theta_L q^i + \frac{(1-\mu)\triangle}{2} \text{ and} \tag{16}$$

$$t^i\left(q^i, H\right) = \begin{cases} \theta_H q^i & \text{if } 0 \leq q^i \leq \frac{1}{2} \\ \theta_H q^i + \alpha\left(\rho - \theta_H\right)\left(q^i - \frac{1}{2}\right) & \text{if } q^i > \frac{1}{2}. \end{cases} \tag{17}$$

Note that payments to agent i are conditional only on the actual output q^i that the agent produces. A low-cost agent receives a fixed information rent, and a high-cost agent receives an extra payment when the agent produces more than a fixed output $\frac{1}{2}$. Given the restriction $q^i_{HH} = \frac{1}{2}$, information rent is fixed at the minimal level, $\triangle \cdot \overline{q}^i_H = \frac{(1-\mu)\triangle}{2}$, and production above $\frac{1}{2}$ is carried out by either a low-cost firm or a high-cost firm that surely lies. If firms are truthful with the commitment device, then agents also are truthful with the minimal information rent. The overall expected information rent for both agents is the same as that in the optimal monopoly contract, $\mu(1-\mu)\triangle$.

Following the previous procedure, we can establish the existence of a value set $V(\alpha) \subset V^m$

under the contract $x^i(\alpha)$ if for any δ,

$$\alpha \geq \alpha^*(\delta) \equiv \frac{1-\delta+\delta\mu(1-\mu\gamma)}{1-\delta+\delta\mu(1+\mu)}. \tag{18}$$

As above, the contract $x^i(\alpha)$ can be used to lengthen (or shorten) the width of $V(\alpha)$. The scope of market-sharing arrangements at (H,H) is reduced from $[0,1]$ to $\frac{1}{2}$. At the same time, however, the rigidity, $q^i_{HH} = \frac{1}{2}$, reduces the market-share disadvantage in ties. The assumption on μ is necessary in order to make the continuation-value reward sufficient for a firm to be truthful at its disadvantaged endpoint. The assumption is now relaxed to $\mu > \frac{1}{3}$, as the rigidity reduces the market-share disadvantage in ties.

Proposition 2. *Assume that firms are sufficiently patient. If $\rho \geq C_H$ and $\mu > \frac{1}{3}$, then for $\alpha \geq \alpha^*(\delta)$ as in (18), there exists a set $V(\alpha) \subset V^m$ such that $V(\alpha) \cup V^n$ is a self-generating set of APPE values.*

The proof is in the Appendix. In the parameter range $\rho < C_H$, the result in Proposition 1 (ii) remains without depending on Assumption 1. Any modification of the contract $x^i(\beta)$ in (13) and (14) is unnecessary, since the payments are conditional only on the real output q^i.

In the previous literature, firms own private information; firms can observe and distort private information at no extra costs. In our model, agents hold private information and firms deliver it; firms deliberately reduce the degree to which they control of private information. A very simple contractual arrangement that reduces the firms' incentive to distort the agents' private information makes it possible to establish a sufficiently long Pareto-frontier segment, so that firms are induced to be truthful by continuation-value transfers. As a result, an interaction between internal contracting and market-sharing collusion is exploited: firms achieve an optimal market allocation, and the benefit of the optimal market allocation is not limited to productive efficiency but expanded to contractual efficiency. This argument becomes apparent in comparison with the no-agent setting as in Athey and Bagwell (2001). To characterize first-best collusion, Athey and Bagwell restrict $\gamma = \frac{\triangle}{\rho-\theta_H}$ to be above a certain level. As argued above, when γ falls, the need for the commitment device grows, and thus $\alpha^*(\delta)$ rises. If this restriction fails in the no-agent model, then first-best profit is approximated only when firms are infinitely patient; as firms becomes more patient, the width of Pareto-frontier segment grows, but not sufficiently for $\delta < 1$. In this paper, however, the width restriction on the equilibrium-value set is relaxed even if firms are moderately patient.

5. Optimal SPPE

In the previous section, continuation-value transfers were used to construct an optimal APPE-value set, and they were delivered in the form of *asymmetric* market shares in ties without sacrificing any efficiency; market shares in ties, q^i_{LL} and q^i_{HH}, are unrestricted (changeable) as in Proposition

1, and q^i_{LL} is unrestricted and q^i_{HH} is restricted to $\frac{1}{2}$ as in Proposition 2. We now impose a stronger restriction on market-shares in ties: *both* q^i_{LL} and q^i_{HH} are restricted to $\frac{1}{2}$. This symmetry restriction makes it impossible to exchange market-share favors in ties. In this section, to allow for the symmetry restriction, we employ *Symmetric Perfect Public Equilibrium* (SPPE) and characterize the features that would not be found in the no-agent model because of its symmetry restriction.

5.1. Symmetry Restriction

In this subsection, we study what symmetry means. Recall that a strategy of firm i in period τ is a mapping from the set of potential public histories to the set of stage-game strategies. A typical strategy of firm i in period τ is $\sigma^i_\tau(h_\tau)$. In SPPE, firms follow the public history and adopt symmetric strategies, $\sigma^i_\tau(h_\tau) = \sigma^j_\tau(h_\tau)\ \forall i, j, \tau, h_\tau$. This means that stage-game strategies are symmetric across firms for all histories. Among the stage-game strategies we have used, only strategies $(\mathbf{p}, \mathbf{q})$ are affected by the symmetry restriction in this section; we have considered symmetric contracts, $x^i = x^j$, and firm announcements are truthful in both APPE and SPPE. Thus, when $\rho \geq C_H$, the optimal symmetric vectors of prices and market shares are

$$\forall i, (j,k),\ p^i_{jk} = \rho,\ q^1_{LH} = q^2_{HL} = 1, \text{ and } q^i_{LL} = q^i_{HH} = \frac{1}{2}. \tag{19}$$

Note that prices and market-share schedules are symmetric across firms.

Again, we use the previous two-tier mechanism design program to establish an SPPE-value set: assuming the agents' truthfulness, we find current-period strategies $(\mathbf{p}, \mathbf{q})$ and continuation-value function $\mathbf{v}$ that satisfy the on- and off-schedule constraints, and select a contract x^i that induces the agent's truthfulness and achieves optimality. Let V^s denote the set of SPPE continuation values. The values in V^s are restricted to the 45-degree line, and the Pareto-frontier value set of SPPE is reduced to a point $(\widehat{u}, \widehat{u})$: $V^s \subset \left\{(u^1, u^2) : u^1 = u^2 \leq \widehat{u}\right\}$.[32] In SPPE, any continuation-value reduction (below the Pareto-frontier value set) is suffered by all firms together. In APPE, by contrast, a continuation-value loss for one firm may imply a continuation-value gain for another. Thus, efficient continuation-value transfers across firms are unavailable in SPPE.

To emphasize how the presence of privately-informed agents affects the SPPE-value set $V^s(\alpha)$, we first consider the no-agent model and argue that symmetry is a real restriction. In no-agent model, On-IC^1_H is

$$\sum_{k \in \{L,H\}} \left[\mu_k \left(p_{Hk} - \theta_H\right) q^1_{Hk}\right] + \delta \overline{v}^1_H \geq \sum_{k \in \{L,H\}} \left[\mu_k (p_{Lk} - \theta_H) q^1_{Lk}\right] + \delta \overline{v}^1_L.$$

On-IC^2_H is similarly given. The constraint shows that (i) if a vector $(\mathbf{p}, \mathbf{q}, \mathbf{v})$ achieves price efficiency $(p_{jk} = \rho)$ and Pareto-efficient continuation values $(v^i_{jk} = \widehat{u})$, then it entails productive inefficiency $(\overline{q}^i_L = \overline{q}^i_H)$, and (ii) if the vector achieves productive efficiency and Pareto-efficient continuation

[32] The value $\widehat{u}$ is the supremum of SPPE continuation values. We show below that the Pareto-frontier set of SPPE values includes this supremum.

values, then it entails price inefficiency ($p_{jk} < \rho$ for some (j,k)). These two cases occur in an SPPE that is *stationary*, wherein firms repeatedly use the same current-period strategies $(\mathbf{p},\mathbf{q})$ to satisfy all the constraints, fixing continuation values at $\widehat{u}$. Any SPPE that is *nonstationary* involves variations of continuation values, which necessarily entails a reduction of some continuation values below $\widehat{u}$ to satisfy On-IC^i_H. Therefore, because of the intrinsic nature of SPPE that continuation-value transfers are wasteful, optimal SPPE values are lower than the optimal monopoly values in the no-agent model.[33]

Lemma 5. *In the no-agent model, the Pareto-frontier SPPE values are lower than the optimal monopoly values.*

5.2. Optimal SPPE Values

The result in Lemma 5 seems fairly straightforward. However, it becomes far different if each firm has a privately-informed agent. In this subsection, we establish a nonstationary SPPE and show that along with a simple contractual arrangement, the SPPE-value set $V^s(\alpha)$ may approximate the point: $V^m = \{(u^1,u^2) : u^1 = u^2 = \frac{\pi^m}{2(1-\delta)}\}$. We construct an SPPE-value set $V^s(\alpha) = [(\underline{u},\underline{u}),(\overline{u},\overline{u})]$, where $\overline{u} > \underline{u}$. This set has the two endpoints of SPPE values, $(\underline{u},\underline{u})$ and $(\overline{u},\overline{u})$, on the 45-degree line. If firms randomize over the two vectors that construct the two endpoints, the SPPE-value set becomes convex and fully characterized. We thus focus on the construction of the two endpoints.

To establish $V^s(\alpha)$, we find a contract $x^i(\alpha)$, wherein for any $(u^1,u^2) \in V^s(\alpha)$, there exists a vector $\mathbf{z}(\alpha) \in Z^{IC}(V^s(\alpha))$ such that $u^i = u^i(\mathbf{z}(\alpha))\ \forall i$. We for now consider only the on-schedule constraints, assuming that the off-schedule constraints hold. Assuming that $\rho \geq C_H$, define a contract $x^i(\alpha)$:[34]

$$t^i\left(q^i, L\right) = \theta_L q^i + \frac{(1-\mu)\triangle}{2} \text{ and} \tag{20}$$

$$t^i\left(q^i, H\right) = \begin{cases} \theta_H q^i & \text{if } 0 \leq q^i \leq \frac{1}{2} \\ \theta_H q^i + \frac{\alpha(\rho-\theta_H)}{2(1-\mu)} & \text{if } q^i > \frac{1}{2}. \end{cases} \tag{21}$$

Note that payments to agent i are conditional only on the real output q^i that the agent produces. A low-cost agent receives a fixed information rent, and a high-cost agent receives a fixed extra payment when the agent produces more than $\frac{1}{2}$. Given the symmetry restriction $q^i_{HH} = \frac{1}{2}$, information rent is fixed at the minimal level, $\triangle \cdot \overline{q}^i_H = \frac{(1-\mu)\triangle}{2}$, and production above $\frac{1}{2}$ is carried out by either a low-cost firm or a high-cost firm that surely lies. The high-cost firm that lies incurs falsification costs with probability of $(1-\mu)$. If market shares follow the optimal symmetric vectors in (19),

[33] See Lemma 5 in Athey and Bagwell (2001) to find how symmetry restricts the Pareto frontier.

[34] There are various forms of contract and associated vector $\mathbf{z}(\alpha)$ that can establish a SPPE-value set $V^s(\alpha)$. The qualitative result, however, would be unaffected: V^m can be approximated by $V^s(\alpha)$ along with a simple contract $x^i(\alpha)$.

the high-cost firm that lies can have the current-period gain:

$$(\rho-\theta_H)\left(\overline{q}^i_L-\overline{q}^i_H\right)-(1-\mu)\frac{\alpha\,(\rho-\theta_H)}{2(1-\mu)}=\frac{(1-\alpha)(\rho-\theta_H)}{2}.$$

Thus, under $x^i(\alpha)$, the binding On-IC^i_H becomes

$$\delta\left(\overline{v}^i_H-\overline{v}^i_L\right)=\frac{(1-\alpha)(\rho-\theta_H)}{2}. \tag{22}$$

We first construct the higher SPPE-values $(\overline{u},\overline{u})$. Define a symmetric vector $\mathbf{z}(\alpha)$ such that current-period strategies $(\mathbf{p},\mathbf{q})$ are the vectors in (19) and continuation-value vector $\mathbf{v}$ is given by $v^1_{jk}=v^2_{jk}=\overline{u}\ \forall(j,k)$, except

$$v^1_{LL}=v^2_{LL}=\overline{u}-\frac{(1-\alpha)(\rho-\theta_H)}{2\delta\mu}. \tag{23}$$

The continuation value v^i_{LL} is lower than other values and is chosen to satisfy the binding On-IC^i_H in (22). In SPPE, continuation-value transfers across firms are wasteful; any variation of continuation values entails some inefficiency and is suffered by all firms together. The symmetric vector $\mathbf{z}(\alpha)$ uses a lower continuation value v^i_{LL} (as future penalty) to prevent the high-cost firm's understatement today. This future penalty will be delivered subsequent to the realization of (L,L). The equation (23) also implies that if an SPPE-value set $V^s(\alpha)$ exists, then the distance between the two endpoints must be sufficiently long (Add-IC): the value $\overline{u}$ must be greater than $\underline{u}$ at least by $\frac{(1-\alpha)(\rho-\theta_H)}{2\delta\mu}$. Only then is it feasible to penalize low-cost firms with a low continuation value v^i_{LL} drawn from $V^s(\alpha)$. Note that the expected payoff of firm i takes the same form as (8), since On-IC^i_H is binding. Letting $u^i(\mathbf{z}(\alpha))=\overline{u}$ and using the vector $\mathbf{z}(\alpha)$, we can find the value:

$$\overline{u}=\frac{\rho-E\left(\theta\right)}{2}+\delta\overline{u}-\frac{(1-\alpha)\mu(\rho-\theta_H)}{2}. \tag{24}$$

The third term on the RHS reflects the (discounted) potential future penalty that follows the realization of (L,L).

We next construct the lower SPPE values $(\underline{u},\underline{u})$. Define a vector $\mathbf{z}'(\alpha)$ such that prices are fixed at a lower level, $p^i_{jk}=\underline{\rho}<\rho\ \forall i,(j,k)$, and market shares and continuation values, $\mathbf{q}$ and $\mathbf{v}$, are the same as in the previous vector $\mathbf{z}(\alpha)$.[35] Following the vector $\mathbf{z}'(\alpha)$, firms deliver the future penalty (a lower continuation value) by setting the lower price $\underline{\rho}$ after the realization of (L,L). With no prior assumption that On-IC^i_H is binding under $\mathbf{z}'(\alpha)$, the expected payoff of firm i becomes

$$\begin{aligned} u^i(\mathbf{z}'(\alpha)) &= \sum_{j\epsilon\{L,H\}}\mu_j\left[\Pi^i\left(j,j\right)+\delta\overline{v}^i_j\right] \\ &= \sum_{j\epsilon\{L,H\}}\mu_j\left[\left(\rho-\theta_j\right)\overline{q}^i_j+\delta\overline{v}^i_j\right]-\frac{\mu(1-\mu)\triangle}{2}. \end{aligned}$$

[35] In order to construct the lower-value point, firms may use a productive inefficiency rather than a price reduction. An advantage of choosing the price reduction is that it reduces the incentive to undercut the price at the lower-value point.

The last term represents the expected information rent under the contract. Letting $u^i(\mathbf{z}'(\alpha)) = \underline{u}$ and using the vector $\mathbf{z}'(\alpha)$, we can find the value:

$$\underline{u} = \frac{\underline{\rho} - E(\theta)}{2} + \delta\overline{u} - \frac{(1-\alpha)\mu(\rho - \theta_H)}{2}. \tag{25}$$

The lower value $\underline{u}$ captures the current-period price reduction (first term) and the switch to the higher value in the following period (second term) together with the potential future penalty (third term). The level of $\underline{\rho}$ is now chosen to satisfy the binding Add-IC: $\overline{u} - \underline{u} = \frac{(1-\alpha)(\rho-\theta_H)}{2\delta\mu}$, or equivalently

$$\underline{\rho} = \rho - \frac{(1-\alpha)(\rho - \theta_H)}{\mu\delta}. \tag{26}$$

Note that $\underline{\rho} \leq \rho$ for any $\alpha \in [0,1]$ with equality for $\alpha = 1$. The binding Add-IC together with (24) and (25) yields values:

$$\overline{u} = \frac{\rho - E(\theta) - (1-\alpha)\mu(\rho - \theta_H)}{2(1-\delta)} \tag{27}$$

$$\underline{u} = \overline{u} - \frac{(1-\alpha)(\rho - \theta_H)}{2\delta\mu}. \tag{28}$$

As in the Appendix, the level of α is next chosen to ensure that the vectors $\mathbf{z}(\alpha)$ and $\mathbf{z}'(\alpha)$ satisfy the on-schedule constraints.[36] A certain level of α is necessary to prevent the low-cost firm from overstating its cost type to avoid the potential future penalty.

We finally consider the off-schedule constraints. Since firms are more tempted to undercut the price ρ than $\underline{\rho}$, we focus on the firm 1's off-schedule incentive at the endpoint $(\overline{u}, \overline{u})$. When firm 1 is a low-cost type, it will not undercut the price in (L,H) since it captures the entire market in that state. When firm 1 is a high-cost type, it will be more tempted to undercut the price in (H,L) than in (H,H), since the current-period market share q^1_{HL} is lower than q^1_{HH} for the same continuation values, $v^1_{HL} = v^1_{HH} = \overline{u}$. Hence, firm 1 will not undertake any off-schedule deviation if

$$\delta\left(v^1_{LL} - \underline{v}\right) \geq (\rho - \theta_L)q^2_{LL} = \frac{\rho - \theta_L}{2} \tag{Off-IC$^1_{LL}$}$$

$$\delta\left(v^1_{HL} - \underline{v}\right) \geq (\rho - \theta_H)q^2_{HL} = \rho - \theta_H. \tag{Off-IC$^1_{HL}$}$$

The off-schedule constraints of firm 2 are symmetrically described. The continuation values on the LHS are $v^1_{LL} = \underline{u}$ and $v^1_{HL} = \overline{u}$, and the RHS represents the current-period gain that firm 1 can make when it undercuts the price. Letting $\delta^* = \max\{\delta^*_{LL}, \delta^*_{HL}\}$, we obtain the result that corresponds to Lemma 3 in APPE: for some range of α and δ, there exist the two vectors that can establish $(\overline{u}, \overline{u})$ and $(\underline{u}, \underline{u})$, respectively, and the remainder of $V^s(\alpha)$ can be constructed by a convex combination of the two vectors.

We now conclude that a simple contractual arrangement may significantly change the SPPE-value set $V^s(\alpha)$. It is evident that if $\alpha \to 1$, then (i) $\underline{\rho} \to \rho$, (ii) $\underline{u} \to \overline{u}$, (iii) $\overline{u} \to \frac{\pi^m}{2(1-\delta)}$ and (iv) δ^*

[36] See the proof of Proposition 3 in the Appendix.

decreases and converges to the lowest level of δ that solves

$$\delta\left(\frac{\pi^m}{2(1-\delta)}-\frac{\pi^n}{1-\delta}\right)\geq \max\left\{\frac{\rho-\theta_L}{2},\rho-\theta_H\right\}.$$

In other words, if $\alpha \to 1$ and $\delta > \delta^*$, then $V^s(\alpha) \to V^m$. We emphasize that the contract $x^i(\alpha)$ can be used to *shorten* the width of $V^s(\alpha)$. When $\alpha \to 1$, wasteful continuation-value transfers become unnecessary and the width of $V^s(\alpha)$ is shortened to a point V^m. Having an incentive to distort the agents' private information, firms use the contractual arrangement to increase the costs a high-cost firm suffers when it falsifies its agent's report. The high-cost firm then gains nothing by telling a lie today, and thus it becomes unnecessary to penalize the firms that report low cost together. The contract thus acts to avoid the equilibrium-path penalization that would otherwise follow a pair of low-cost announcement.

Proposition 3. *Assume that $\rho \geq C_H$ and that firms are sufficiently patient. If $\alpha \to 1$, then an SPPE can approximate the optimal monopoly profit; if $\alpha \to 1$, then there exists a set $V^s(\alpha)$ such that (i) $V^s(\alpha) \cup V^n$ is a self-generating set of SPPE values and (ii) $V^s(\alpha) \to V^m$.*

The result is not based on the assumption $\mu > \frac{1}{2}$, and the result for $\rho < C_H$ is analogous. Our finding shows that a very simple contractual arrangement can shorten the SPPE-value set so that firms are induced to be truthful without depending on wasteful continuation-value transfers. The symmetry restriction in SPPE affects characteristics of an optimal collusion differently between our model and the no-agent model: SPPE suffers a waste of equilibrium values in the no-agent model, but it can approximate the optimal monopoly profit despite the symmetry restriction in our model.

Our model also contrasts with the no-agent model by Athey et al. (2004), where cost types are continuously distributed. They predict that when the distribution of cost types is log-concave, optimal SPPE is characterized by a pooling equilibrium in which market shares are constant regardless of cost types; firms sacrifice productive efficiency and instead save informational costs that would be necessary to deter higher-cost firms from mimicking lower-cost firms. In our two-type model, firms achieve productive efficiency in market allocation, which, in turn, reduces informational costs.

6. Extensions

In this section, we informally present some possible extensions of the model.[37] First, we discuss the role of communication between firms in comparison with other models. Second, we consider internal contracting with agents who have future prospects.

[37] This section is motivated by referees' reports.

6.1. Non-communicative Firms

The role of communication in the model is to achieve state-dependent market-sharing arrangements. Athey and Bagwell (2001) show that if observable past prices act as public history on which subsequent collusion is coordinated, then first-best profit may be achieved without communication. Skrzypacz and Hopenhayn (2004) find, however, that the scope of collusion is constrained without explicit communication when firms have imperfect public monitoring on past actions. A potential benefit of communication is also suggested in a large and growing private-monitoring literature. It would be very complicated to keep track of each player's belief over rival types as private information is accumulated over time. Compte (1998) and Kandori and Matsushima (1998) show that players can generate a public history, benefiting from communication.[38]

Returning to our model, we follow Athey and Bagwell (2001) and establish a non-communicative APPE. Consider a price vector: $p_H^1 = \rho$, $p_L^1 = \rho - 2\varepsilon$, $p_H^2 = \rho - \varepsilon$ and $p_L^2 = \rho - 3\varepsilon$ for arbitrarily small $\varepsilon > 0$, where p_j^i denotes price of firm i when its cost type is j. Note that prices and market shares correspond to the vector that we used to construct the endpoint $(\underline{u}, \overline{u})$; productive efficiency is achieved, and market shares in ties are in favor of firm 2 in the Bertrand model. Since continuation values can be contingent only on prices, communication is unnecessary. In this way, we can directly use arguments by Athey and Bagwell without depending on their restriction on $\gamma = \frac{\triangle}{\rho - \theta_H}$.[39] Furthermore, we can establish a non-communicative SPPE that approximates the optimal monopoly values. Prices can be defined as $p_H^i = \rho$ and $p_L^i = \rho - \varepsilon$ to construct $(\overline{u}, \overline{u})$, and $p_H^i = \underline{\rho}$ and $p_L^i = \underline{\rho} - \varepsilon$ to construct $(\underline{u}, \underline{u})$. Market shares are symmetric and achieve productive efficiency in each endpoint, and continuation values can be contingent only on prices. Communication is then unnecessary. A relative easiness of collusion in our model is still demonstrated in this non-communicative collusion: no restriction on γ is necessary in APPE and no optimal values are wasted in SPPE.

6.2. Non-myopic Agents

It has been assumed so far that the internal contract with agent lasts for only one period. The contracting scheme is stationary; the same contract is repeatedly offered over time. It is beyond the scope of the present paper to analyze a contracting scheme when each agent has future prospects in various multi-period contractual relationships. In the remainder of this section, we briefly present a possibility that colluding firms may benefit when agents have future prospects; firms can use agent's future prospects to make the internal contract more efficient. To this end, we construct a self-generating set of *agents*' values, $V^A = \{(\underline{u}^A, \overline{u}^A), (\overline{u}^A, \underline{u}^A)\}$, where $\overline{u}^A > \underline{u}^A$. When agent i

[38] Recent work shows that the Folk Theorem seems quite robust in games with private monitoring (without communication) within the class of prisoner's dilemma model (e.g., Sekiguchi, 1997; Bhaskar and Obara, 2002; and Ely and Välimäki, 2002).

[39] For detail, see their Proposition 8. Athey and Bagwell (2001) further address the circumstances where non-communication benefits colluding firms.

observes cost type j and reports $\hat{\jmath}$, the agent receives interim-stage payoff,

$$U^{A_i}(\hat{\jmath}, j) = \bar{t}^i_{\hat{\jmath}} - \theta_j \bar{q}^i_{\hat{\jmath}} + \delta \overline{w}^i_{\hat{\jmath}},$$

where $\bar{t}^i_{\hat{\jmath}}$ and $\overline{w}^i_{\hat{\jmath}}$ represent the expected transfers and continuation values under the report $\hat{\jmath}$: for agent 1, for instance,

$$\bar{t}^1_{\hat{\jmath}} = \sum_{k \in \{L,H\}} \mu_k t^1_{\hat{\jmath}k} \text{ and } \overline{w}^1_{\hat{\jmath}} = \sum_{k \in \{L,H\}} \mu_k w^1_{\hat{\jmath}k}.$$

Observing that the agent has an incentive to overstate the cost type, we focus on the binding constraint of a low-cost type:[40]

$$\delta^A \left(\overline{w}^i_L - \overline{w}^i_H\right) = \bar{t}^i_H - \theta_L \bar{q}^i_H - \left(\bar{t}^i_L - \theta_L \bar{q}^i_L\right), \tag{IC-A^i_L}$$

where δ^A represents agents' common discount factor. An overstatement has current gain (RHS) and future loss (LHS). When IC-A^i_L is binding, the expected payoff of agent i is

$$u^{A_i} \equiv \sum_{j \in \{L,H\}} \mu_j U^{A_i}(j, j) = \bar{t}^i_L - \theta_L \bar{q}^i_L + \delta \overline{w}^i_L - (1-\mu)\triangle \cdot \bar{q}^i_H. \tag{29}$$

As in the contract in (16) and (17), with a predetermined output $q^i_{HH} = \widehat{q}^i > 0$ in mind, we define $x^i(\alpha)$:

$$t^i\left(q^i, L\right) = \theta_L q^i + (1-\mu)\triangle \cdot \widehat{q}^i - \psi^i \text{ and} \tag{30}$$

$$t^i\left(q^i, H\right) = \begin{cases} \theta_H q^i & \text{if } 0 \leq q^i \leq \widehat{q}^i \\ \theta_H q^i + \alpha\left(\rho - \theta_H\right)\left(q^i - \widehat{q}^i\right) & \text{if } q^i > \widehat{q}^i. \end{cases} \tag{31}$$

Since the contract becomes more efficient only when agents' information rents exist and become lower, assuming $\rho \geq C_H = \theta_H + \frac{\mu}{1-\mu}\triangle$, we deal with the case where productive efficiency is achieved and productions in (H, H) are positive: $\widehat{q}^1 + \widehat{q}^2 = 1$. In the previous analysis, information rents were given by $\triangle \cdot \bar{q}^i_H = (1-\mu)\triangle \cdot \widehat{q}^i$. A new constant term $\psi^i \geq 0$ represents a reduction of information rents for a low-cost agent i, and will be derived below. We first construct a point $(\overline{u}^A, \underline{u}^A) \in V^A$, where continuation values are drawn from V^A and conditional on current-period productions:

$$\left(w^1_{LL}, w^2_{LL}\right) = \left(w^1_{LH}, w^2_{LH}\right) = (\overline{u}^A, \underline{u}^A) \text{ and } \left(w^1_{HH}, w^2_{HH}\right) = \left(w^1_{HL}, w^2_{HL}\right) = (\underline{u}^A, \overline{u}^A).$$

Note that the point $(\overline{u}^A, \underline{u}^A)$ is in favor of agent 1. Continuation values indicate that if agent 1 reports low cost (high cost) today, then the agent will preserve (lose) the current favored position tomorrow as future reward (penalty). To deliver continuation-value reward (penalty), a non-stationary contracting schedule and market-sharing arrangements in (H, H) are employed as follows. If agent 1 in its favored position reports low cost (high cost) today, the agent will receive a favorable (unfavorable) contract and market share in (H, H) tomorrow. If any agent i (agent j) is given the favored (disadvantaged) position today, then (i) $q^i_{HH} = \widehat{q}^i = 1$ ($q^j_{HH} = \widehat{q}^j = 0$) so that

[40] Constraints for high-cost types, IC-A^1_H and IC-A^2_H, are slack as below.

information rents $\triangle \cdot \overline{q}_H^i$ are maximized (minimized to zero), and (ii) the contract is offered such that the net surplus $(1-\mu)\triangle \cdot \widehat{q}^i - \psi^i$ is positive (zero with $\psi^j = 0$).

At the point $(\overline{u}^A, \underline{u}^A)$, firm 1 has a market-share favor in (H,H), $q_{HH}^1 = 1$. We now greatly simplify our analysis and focus only on the agents' constraints by neutralizing any firm's relative benefit from market-sharing arrangements. To this end, we find a disadvantaged market share for firm 1 in (L,L), $q_{LL}^1 \in (0, \frac{1}{2})$, that exactly offsets firm 1's benefit in (H,H). Under productive efficiency, if μ is not too low, then there always exists $q_{LL}^1 \in (0, \frac{1}{2})$ in which the firm 1's gain (firm 2's loss) in (H,H) is equal to its loss (its gain) in (L,L) in terms of the expected profit.[41] Along with such market shares in ties, we set $\alpha = 1$ in $x^i(\alpha)$. In this case, the expected profits are the same across firms in each period, firms' continuation values are drawn from the same value and their on-schedule constraints are trivially satisfied.

Returning to the above history-dependent contracting schedule and continuation values, we can rewrite the binding IC-A_L^1:

$$\delta(\overline{u}^A - \underline{u}^A) = \theta_L \overline{q}_L^1 + \Delta \cdot \overline{q}_H^1 - \overline{t}_L^1 = \psi^1. \tag{32}$$

The second equality is given by (30). We can easily find that IC-A_L^2 has zero in both sides. Hence, the "additional" constraint (Add-IC for agents' constraints) is simply reduced to IC-A_L^1.[42] Using the expected payoff in (29) and letting $u^{A_1} = \overline{u}^A$ and $u^{A_2} = \underline{u}^A$, we can find the values, $\overline{u}^A$ and $\underline{u}^A$, and the differential:

$$\overline{u}^A - \underline{u}^A = \frac{1}{1-\delta^A \mu}\left[\overline{t}_L^1 - \theta_L \overline{q}_L^1 - \left(\overline{t}_L^2 - \theta_L \overline{q}_L^2\right) + (1-\mu)\Delta \cdot \left(\overline{q}_H^2 - \overline{q}_H^1\right)\right]. \tag{33}$$

Since agent 1 is favored now, let $\overline{t}_L^2 - \theta_L \overline{q}_L^2 = 0$ and $\overline{q}_H^2 = 0$. Plugging (33) into (32), we can get

$$\overline{t}_L^1 = \theta_L \overline{q}_L^1 + \Delta \cdot \overline{q}_H^1 - \frac{\delta^A \mu}{1-\delta^A(1-\mu)}\Delta \cdot \overline{q}_H^1.$$

The last term represents the information-rent reduction; observe that (i) if $\delta^A > 0$, then $\frac{\delta^A \mu}{1-\delta^A(1-\mu)} > 0$ and (ii) if $\delta^A \to 1$, then $\frac{\delta^A \mu}{1-\delta^A(1-\mu)} \to 1$ and so $\overline{t}_L^1 \to \theta_L \overline{q}_L^1$. Given that $q_{HH}^1 = \widehat{q}^1 = 1$ and that ψ^1 is used to denote the information-rent reduction, we can define the last term as:

$$\frac{\delta^A \mu (1-\mu)\Delta}{1-\delta^A(1-\mu)} \equiv \psi^1 > 0 \text{ for } \delta^A > 0.$$

The other point $(\underline{u}^A, \overline{u}^A) \in V^A$ can analogously be constructed. In the first period where history is null, firms may start with any point. This result shows that in some parameter range, (i) if

[41] Without a prior assumption that On-IC_H^1 is binding, we can find

$$q_{LL}^1 = \frac{1}{2}\left(1 - \frac{(1-\mu)^2(\rho-\theta_H) - \mu(1-\mu)\Delta}{\mu^2(\rho-\theta_L)}\right).$$

If μ is not too low, then $q_{LL}^1 \in (0, \frac{1}{2})$.

[42] As for high-cost agents, IC-A_H^1 is given by $\Delta \cdot (\overline{q}_L^1 - \overline{q}_H^1) + \psi^1 \geq \delta(\overline{w}_L^1 - \overline{w}_H^1)$, where the RHS becomes $\delta(\overline{u}^A - \underline{u}^A) = \psi^1$ by (32). IC-A_H^2 is $\Delta \cdot (\overline{q}_L^2 - \overline{q}_H^2) \geq \delta(\overline{w}_L^2 - \overline{w}_H^2)$, where the RHS becomes zero. Hence, both constraints are slack.

agents ever care about the future ($\delta^A > 0$), then sufficiently patient firms can reduce information rents (and thus increase their expected profits) and (ii) if $\delta^A \to 1$, then they may approximate the first-best profit. This striking result is due to the interaction between internal contracting and market-sharing arrangement.

7. Conclusions

In this paper, we investigated a commonly observed collusive behavior, price-fixing with market-share allocation, when private information is held by the agent who engages in production. We established two classes of an optimal collusion and found some features that are not observed in no-agent models. In particular, our findings provided a new perspective on collusive conduct, arguing that colluding firms, facing the contrasting incentives on internal and inter-firm level, may be able to exploit the interaction between internal contracting and market-sharing arrangement. The presence of privately-informed agents may thus provide firms with a strategic opportunity to achieve an optimal APPE in a wider parameter range and save a potential waste of optimal SPPE values. The argument can be extended at a broad level: if each firm has to determine whether it should identify its agent's cost type (with informational costs) before signing the internal contract, then it may deliberately delay getting informed of its agent's cost type to take advantage of the strategic opportunity.

In the literature, firms own private information; firms can observe and distort private information at no extra costs. The contractual arrangement in this paper captures the circumstance in which firms deliberately reduce the degree to which they control of private information; their incentive to distort the agents' private information is bounded by the contract with agents. We showed that a very simple contractual arrangement that reduces the firms' incentive to distort the agents' private information makes it possible to establish a sufficiently long Pareto-frontier segment in APPE and avoid wasteful continuation-value transfers in SPPE. The internal incentive problem seen here is not new, and market-allocation collusion is commonly observed. Despite extensive research, the literature that links the two is rarely found. Our paper raises new challenging questions: What is the degree of asymmetric information within firms? How is the degree to which firms own private information related to inter-firm behaviors?

Appendix

Proof of Lemma 2. Given the contract defined in (5) and (6) and the fixed price ρ, the interim-stage profits for firm i are

$$\begin{aligned} U^i(H,H) &= (\rho-\theta_H)\overline{q}^i_H + \delta\overline{v}^i_H \\ U^i(L,H) &= (\rho-\theta_H)\overline{q}^i_L - \alpha(\rho-\theta_H)(\overline{q}^i_L - \overline{q}^i_H) + \delta\overline{v}^i_L \\ U^i(L,L) &= (\rho-\theta_L)\overline{q}^i_L - \triangle\cdot\overline{q}^i_H + \delta\overline{v}^i_L \\ U^i(H,L) &= (\rho-\theta_L)\overline{q}^i_H - \triangle\cdot\overline{q}^i_H + \delta\overline{v}^i_H. \end{aligned}$$

Before we prove Lemma 2, we first show that a weak monotonicity, $\overline{q}^i_L \geq \overline{q}^i_H$, is a necessary feature of equilibrium, since the on-schedule constraints, $U^i(H,H) \geq U^i(L,H)$ and $U^i(L,L) \geq U^i(H,L)$, imply the weak monotonicity. To see this, note that the on-schedule constraints imply

$$U^i(H,H) - U^i(H,L) \geq U^i(L,H) - U^i(L,L).$$

This inequality is equivalent to

$$\triangle\cdot\left(\overline{q}^i_L - \overline{q}^i_H\right) + \alpha(\rho-\theta_H)(\overline{q}^i_L - \overline{q}^i_H) \geq 0. \tag{A1}$$

Since this inequality must be satisfied in equilibrium, a weak monotonicity, $\overline{q}^i_L \geq \overline{q}^i_H$, is necessarily satisfied in equilibrium.

We next prove Lemma 2. The binding On-IC^i_H implies that

$$U^i(H,H) = U^i(L,H) = (\rho-\theta_H)\overline{q}^i_L - \alpha(\rho-\theta_H)(\overline{q}^i_L - \overline{q}^i_H) + \delta\overline{v}^i_L.$$

It then follows that

$$\begin{aligned} U^i(L,L) - U^i(H,H) &= U^i(L,L) - U^i(L,H) \\ &= \triangle\cdot\left(\overline{q}^i_L - \overline{q}^i_H\right) + \alpha(\rho-\theta_H)(\overline{q}^i_L - \overline{q}^i_H) \geq 0. \end{aligned}$$

The last inequality comes from (A1). Lastly, we show that the term $U^i(H,H)$ equals $U^i(H,L)$:

$$\begin{aligned} U^i(H,L) &= (\rho-\theta_L)\overline{q}^i_H - \triangle\cdot\overline{q}^i_H + \delta\overline{v}^i_H \\ &= (\rho-\theta_H)\overline{q}^i_H + \delta\overline{v}^i_H = U^i(H,H). \end{aligned}$$

Hence, $U^i(L,L) \geq U^i(H,L)\ \forall\alpha$. Further, if $\overline{q}^i_L > \overline{q}^i_H$ holds as in the optimal collusion, then $U^i(L,L) > U^i(H,L)\ \forall\alpha$. ■

Proof of Lemma 3. Before we establish the existence of $V(\alpha) \subset V^m$, we derive the equation (9), which is a necessary feature implied by the the binding On-IC^i_H. Adding the two binding On-IC^i_H yields

$$\delta\sum_{i=1}^{2}(\overline{v}^i_H - \overline{v}^i_L) = (1-\alpha)(\rho-\theta_H)\sum_{i=1}^{2}\left(\overline{q}^i_L - \overline{q}^i_H\right). \tag{A2}$$

The LHS of (A2) is

$$\begin{aligned} &\delta\left[\mu(v^1_{HL} - v^1_{LL}) + (1-\mu)(v^1_{HH} - v^1_{LH}) + \mu(v^2_{LH} - v^2_{LL}) + (1-\mu)(v^2_{HH} - v^2_{HL})\right] \\ &= \delta\left[\mu(v^1_{HL} - v^1_{LL}) + (1-\mu)(v^1_{HH} - v^1_{LH}) + \mu(v^1_{LL} - v^1_{LH}) + (1-\mu)(v^1_{HL} - v^1_{HH})\right] \\ &= \delta\left[v^1_{HL} - v^1_{LH}\right]. \end{aligned}$$

The first equality transforms the continuation values for firm 2 into the ones for firm 1; since continuation values are drawn from $V(\alpha) = [(\underline{u}, \overline{u}), (\overline{u}, \underline{u})]$, we can show that

$$v^2_{LH} - v^2_{LL} = v^1_{LL} - v^1_{LH} \text{ and } v^2_{HH} - v^2_{HL} = v^1_{HL} - v^1_{HH}.$$

A simplification confirms the second equality. The term, $\sum_{i=1}^{2}(\overline{q}^i_L - \overline{q}^i_H)$, on the RHS of (A2) becomes

$$\mu\left(q^1_{LL} + q^2_{LL}\right) + (1-\mu)\left(q^1_{LH} + q^2_{HL}\right) - \mu\left(q^1_{HL} + q^2_{LH}\right) - (1-\mu)\left(q^1_{HH} + q^2_{HH}\right) = 1.$$

Hence, (A2) boils down to

$$v^1_{HL} - v^1_{LH} = \frac{(1-\alpha)(\rho - \theta_H)}{\delta}. \tag{A3}$$

Thus, if $V(\alpha)$ exists, then its width must be at least as long as the RHS:

$$\overline{u} - \underline{u} \geq \frac{(1-\alpha)(\rho - \theta_H)}{\delta}. \tag{Add-IC}$$

We hereafter establish the existence of $V(\alpha)$. To this end, we first construct an endpoint $(\underline{u}, \overline{u})$ of a segment $V(\alpha)$. Consider a vector $\mathbf{z}(\alpha)$:

$$\mathbf{z}(\alpha) = \begin{cases} p^i_{jk} = \rho, \\ q^1_{LL} = q^1_{HH} = q^1_T \ (q^2_{LL} = q^2_{HH} = q^2_T), \\ q^1_{LH} = q^2_{HL} = 1 \ (q^1_{HL} = q^2_{LH} = 0), \\ v^1_{LH} = v^1_{HH} = \underline{u} \ (v^2_{LH} = v^2_{HH} = \overline{u}). \end{cases} \tag{A4}$$

The term q^i_T represents the firm i's market share when two firms tie as in (L, L) and (H, H), and $q^1_T + q^2_T = 1$. Note that the continuation values, v^1_{HL} and v^1_{LL}, are not specified yet, and will be defined below. If On-IC^i_H $\forall i$ is binding, then the firm's expected payoff is

$$u^i(\mathbf{z}(\alpha)) = (\rho - \theta_H)\,\overline{q}^i_H + \mu\triangle\cdot\left(\overline{q}^i_L - \overline{q}^i_H\right) + \alpha\mu(\rho - \theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right) + \delta\overline{v}^i_H. \tag{A5}$$

Using the vector $\mathbf{z}(\alpha)$ in (A4), we find the values, $\underline{u}$ and $\overline{u}$, in the self-generating set. Letting $u^1 = \underline{u}$ in (A5), we can find

$$\begin{aligned} \underline{u} = {} & (\rho - \theta_H)[1 - \mu + \mu\alpha(2\mu - 1)]q^1_T + \triangle\mu(2\mu - 1)q^1_T \\ & + (\rho - \theta_H)\mu(1-\mu)\alpha + \triangle\mu(1-\mu) + \delta\left[\mu v^1_{HL} + (1-\mu)v^1_{HH}\right]. \end{aligned} \tag{A6}$$

We rearrange the last continuation-value terms:

$$\begin{aligned} \delta\left[\mu v^1_{HL} + (1-\mu)v^1_{HH}\right] &= \delta\left[\mu v^1_{LH} + \frac{\mu(1-\alpha)(\rho - \theta_H)}{\delta} + (1-\mu)v^1_{HH}\right] \\ &= \delta\underline{u} + \mu(1-\alpha)(\rho - \theta_H). \end{aligned}$$

The first equality holds because of (A3) and the second equality comes from (A4). Plugging this into (A6), we can get

$$\begin{aligned} (1-\delta)\underline{u} = {} & (\rho - \theta_H)\left[1 - \mu + \mu\alpha(2\mu - 1)\right] q^1_T + \triangle\mu(2\mu - 1)q^1_T \\ & + (\rho - \theta_H)\mu(1 - \mu\alpha) + \triangle\mu(1-\mu). \end{aligned} \tag{A7}$$

Likewise, letting $u^2 = \overline{u}$ in (A5), we can find

$$(1-\delta)\overline{u} = (\rho - \theta_H)\left[1 - \mu + \mu\alpha(2\mu - 1)\right] q_T^2 + \triangle\mu(2\mu-1)q_T^2 \\ + (\rho-\theta_H)\mu(1-\mu)\alpha + \triangle\mu(1-\mu). \tag{A8}$$

Note that a line segment $V(\alpha) = [(\underline{u}, \overline{u}), (\overline{u}, \underline{u})]$ achieves the optimality, $V(\alpha) \subset V^m$:

$$\underline{u} + \overline{u} = \frac{\rho - E(\theta)}{1-\delta} = \frac{\pi^m}{1-\delta} \ \forall \alpha.$$

We next look for the range of α in which the constraint Add-IC is satisfied. We claim that the constraint Add-IC is binding,

$$\overline{u} - \underline{u} = \frac{(1-\alpha)(\rho-\theta_H)}{\delta}, \tag{A9}$$

if and only if

$$q_T^1 = \frac{1}{2} - \frac{(1-\alpha)\left(\mu + \frac{1-\delta}{\delta}\right)}{2(1-\mu) + 2\mu(2\mu-1)(\alpha+\gamma)}. \tag{A10}$$

To prove this, we plug (A7) and (A8) into (A9), recollecting $q_T^2 = 1 - q_T^1$ and $\gamma = \triangle/(\rho - \theta_H)$. If firm 1 is in the least favored position ($q_T^1 = q_{LL}^1 = q_{HH}^1 = 0$), then we can get

$$\alpha^*(\delta) \equiv \frac{1 - \delta + \delta(2\mu-1)(1-\mu\gamma)}{1-\delta+\delta 2\mu^2}. \tag{A11}$$

It follows from (A10) that (i) if $\alpha = \alpha^*(\delta)$, then $q_T^1 = 0$, (ii) if $\alpha^*(\delta) < \alpha < 1$, then $0 < q_T^1 < \frac{1}{2}$ and (iii) if $\alpha = 1$, then $q_T^1 = \frac{1}{2}$. Hence, for any $\alpha \geq \alpha^*(\delta)$ for a given δ, there exists a vector $\mathbf{z}(\alpha)$ in which Add-IC is binding.

We next verify that all the continuation values are drawn from the segment $V(\alpha)$: $v_{ij}^i \in V(\alpha)\ \forall i, (j,k)$. By the vector $\mathbf{z}(\alpha)$, all the continuation values are in $V(\alpha)$ except v_{HL}^1 and v_{LL}^1, which are not specified by $\mathbf{z}(\alpha)$ in (A4). We thus need to prove that given the vector $\mathbf{z}(\alpha)$, v_{HL}^1 and v_{LL}^1 are also drawn from $V(\alpha)$: $\underline{u} \leq v_{HL}^1 \leq \overline{u}$ and $\underline{u} \leq v_{LL}^1 \leq \overline{u}$. Recall that the continuation value v_{HL}^1 is determined to satisfy (A2):

$$\begin{aligned} v_{HL}^1 &= v_{LH}^1 + \frac{(1-\alpha)(\rho-\theta_H)}{\delta} \\ &= \underline{u} + \frac{(1-\alpha)(\rho-\theta_H)}{\delta} \leq \overline{u}. \end{aligned} \tag{A12}$$

The last inequality comes from Add-IC. If q_T^1 is chosen as in (A10), then Add-IC is binding as in (A9) and

$$v_{HL}^1 = \underline{u} + \frac{(1-\alpha)(\rho-\theta_H)}{\delta} = \overline{u}.$$

Thus, if Add-IC is binding, then $v_{HL}^1 = \overline{u}$, and if Add-IC is slack, then $\underline{u} < v_{HL}^1 < \overline{u}$. The continuation value v_{LL}^1 is determined to satisfy the binding On-IC$_H^1$:

$$v_{LL}^1 = \underline{u} + \frac{2\mu-1}{\mu} \cdot \frac{(1-\alpha)(\rho-\theta_H)\left(1-q_T^1\right)}{\delta}. \tag{A13}$$

Given the assumption $\mu > \frac{1}{2}$, $\underline{u} \leq v_{LL}^1 \leq \overline{u}$. Hence, $v_{ij}^i \in V(\alpha)\ \forall i, (j,k)$.

We now prove that all the on-schedule constraints are satisfied. On-IC^1_H is binding, since v^1_{LL} is chosen to satisfy the binding On-IC^1_H. Given that v^1_{HL} is chosen to satisfy (A2), we can confirm that On-IC^2_H also is binding:

$$\begin{aligned}\delta(\overline{v}^2_H - \overline{v}^2_L) &= (1-\alpha)(\rho-\theta_H)\left(\overline{q}^2_L - \overline{q}^2_H\right) \\ &= (1-\alpha)\mu(\rho-\theta_H)\left[(2\mu-1)q^2_T + (1-\mu)\right].\end{aligned}$$

Under $\mathbf{z}(\alpha)$, the RHS of On-IC^2_H can be rewritten as the second equality, and the LHS is

$$\begin{aligned}&\delta\left[\mu v^2_{LH} + (1-\mu)v^2_{HH} - \mu v^2_{LL} - (1-\mu)v^2_{HL}\right] \\ &= \delta\left[\mu\overline{u} + (1-\mu)\overline{u} - \mu\left(\overline{u} - \frac{2\mu-1}{\mu}\cdot\frac{(1-\alpha)(\rho-\theta_H)q^2_T}{\delta}\right) - (1-\mu)\left(\overline{u} - \frac{(1-\alpha)(\rho-\theta_H)}{\delta}\right)\right] \\ &= (1-\alpha)\mu(\rho-\theta_H)\left[(2\mu-1)q^2_T + (1-\mu)\right].\end{aligned}$$

The values, v^2_{LL} and v^2_{HL}, are given by (A12) and (A13) and by the condition, $v^1_{jk} + v^2_{jk} = \underline{u} + \overline{u}$ $\forall(j,k)$. Thus, On-IC^2_H is binding. We then invoke Lemma 2 to show that since On-IC^i_H $\forall i$ is binding, On-IC^i_L $\forall i$ is slack. Hence, all the on-schedule constraints are satisfied.

Until now, we have found that for $\alpha \geq \alpha^*(\delta)$, there exists a vector $\mathbf{z}(\alpha) \in Z^{IC}(V(\alpha))$ that establishes the endpoint $(\underline{u}, \overline{u})$. Letting $\mathbf{z}'(\alpha) \in Z^{IC}(V(\alpha))$ denote an analogous vector that implements the other endpoint, the remainder of the segment can be established with the convex combination of $\mathbf{z}(\alpha)$ and $\mathbf{z}'(\alpha)$. This is possible, since for any α, firms' payoffs and the on-schedule constraints are linear in terms of market shares and continuation values.

Lastly, we show that Add-IC can be *slack* in the range $\{(\delta, \alpha) : \alpha^*(\delta) < \alpha \leq 1\}$. To this end, consider the vector $\mathbf{z}(\alpha)$ in which Add-IC is binding under $\alpha = \alpha^*(\delta)$ for a given δ. Then, this vector $\mathbf{z}(\alpha)$ specifies $q^1_T = 0$ as in (A10). The values in (A7) and (A8) become

$$\begin{aligned}\underline{u} &= \frac{\left(\mu - \mu^2\alpha\right)(\rho-\theta_H) + \mu(1-\mu)\triangle}{1-\delta} \\ \overline{u} &= \frac{\left(1-\mu+\mu^2\alpha\right)(\rho-\theta_H) + \mu^2\triangle}{1-\delta}.\end{aligned}$$

For an alternative contract $x^i(\widehat{\alpha})$, where $\widehat{\alpha} = \widehat{\alpha}(\delta) > \alpha^*(\delta)$ for a given δ, define a vector $\mathbf{z}(\widehat{\alpha})$ in which $q^1_T = 0$ is preserved and continuation values are assigned as in $\mathbf{z}(\alpha)$ except that $\widehat{\alpha}$ replaces α in v^1_{HL} and v^1_{LL}. Then, it follows that the vector $\mathbf{z}(\widehat{\alpha})$ (along with $\mathbf{z}'(\widehat{\alpha})$) can establish the value set $V(\widehat{\alpha})$ whose width is longer than the width of $V(\alpha)$, since $\overline{u}$ is higher and $\underline{u}$ lower under $\widehat{\alpha} = \widehat{\alpha}(\delta)$ than under $\alpha = \alpha^*(\delta)$. On the other hand, the RHS of Add-IC is lower under $\widehat{\alpha} = \widehat{\alpha}(\delta)$ than under $\alpha = \alpha^*(\delta)$:

$$\frac{(1-\widehat{\alpha})(\rho-\theta_H)}{\delta} < \frac{(1-\alpha)(\rho-\theta_H)}{\delta}.$$

Hence, Add-IC is slack. Note that when Add-IC is slack, $v^1_{HL} = \underline{u} + \frac{(1-\widehat{\alpha})(\rho-\theta_H)}{\delta} < \overline{u}$. ■

Proof of Proposition 1. To establish the existence of $V(\alpha)$ and $V(\beta)$, the proof focuses on the two vectors $\mathbf{z}(\alpha)$ and $\mathbf{z}(\beta)$ when $\alpha = \alpha^*(\delta)$ and $\beta = \beta^*(\delta)$, respectively. In these cases, $q^1_T = 0$. Suppose first that $\rho \geq C_H$. To establish an endpoint $(\underline{u}, \overline{u})$ of a segment $V(\alpha)$, define a vector

$\mathbf{z}(\alpha)$:

$$\mathbf{z}(\alpha) = \begin{cases} p^i_{jk} = \rho \\ q^1_{LL} = q^1_{HH} = 0 \; (q^2_{LL} = q^2_{HH} = 1) \\ q^1_{LH} = q^2_{HL} = 1 \; (q^1_{HL} = q^2_{LH} = 0) \\ v^1_{LH} = v^1_{HH} = \underline{u} \; (v^2_{LH} = v^2_{HH} = \overline{u}). \end{cases} \tag{A14}$$

It is shown by the proof of Lemma 3 that (i) all the on-schedule constraints and Add-IC are satisfied and (ii) $v^i_{ij} \in V(\alpha)\ \forall i, (j,k)$. It now suffices to prove that off-schedule constraints are satisfied. We restrict attention to the Off-IC of firm 1 that is the least favored position at the endpoint $(\underline{u}, \overline{u})$. Because $\triangle \cdot \overline{q}^1_H = 0$ under $\mathbf{z}(\alpha)$, the associated off-schedule constraints are

$$\begin{aligned} \delta(v^1_{LL} - \underline{v}) &\geq \rho - \theta_L - (\rho - \theta_L)\, q^1_{LL} && (\text{Off-IC}^1_{LL}) \\ \delta(v^1_{LH} - \underline{v}) &\geq \rho - \theta_L - (\rho - \theta_L)\, q^1_{LH} && (\text{Off-IC}^1_{LH}) \\ \delta(v^1_{HL} - \underline{v}) &\geq \rho - \theta_H - (\rho - \theta_H)\, q^1_{HL} && (\text{Off-IC}^1_{HL}) \\ \delta(v^1_{HH} - \underline{v}) &\geq \rho - \theta_H - (\rho - \theta_H)\, q^1_{HH} && (\text{Off-IC}^1_{HH}) \end{aligned}$$

Because $q^1_{LL} = q^1_{HL} = q^1_{HH} = 0$ and $\overline{q}^1_H = 0$ under $\mathbf{z}(\alpha)$, Off-IC$^1_{LH}$ is slack and Off-IC$^1_{HL}$ is implied by Off-IC$^1_{HH}$. Thus, the off-schedule constraints are reduced to Off-IC$^1_{LL}$ and Off-IC$^1_{HH}$:

$$\delta(v^1_{LL} - \underline{v}) \geq \rho - \theta_L \text{ and } \delta(v^1_{HH} - \underline{v}) \geq \rho - \theta_H.$$

The values v^1_{LL} and v^1_{HH} are

$$\begin{aligned} v^1_{LL} &= \underline{u} + \frac{2\mu - 1}{\mu} \cdot \frac{(1-\alpha)(\rho - \theta_H)}{\delta} \text{ and} \\ v^1_{HH} &= \underline{u} = \frac{\left(\mu - \mu^2 \alpha\right)(\rho - \theta_H) + \mu(1-\mu)\triangle}{1 - \delta}. \end{aligned}$$

Plugging the values with $\alpha = \alpha^*(\delta)$ into two inequalities, we can get δ^*_{LL} and δ^*_{HH}. The critical discount factor is $\delta^* = \max\{\delta^*_{LL}, \delta^*_{HH}\}$.

Suppose next that $\rho < C_H$. For $\beta = \beta^*(\delta)$, define a vector $\mathbf{z}(\beta)$:

$$\mathbf{z}(\beta) = \begin{cases} p^i_{jk} = \rho \\ q^1_{LH} = q^2_{HL} = q^2_{LL} = 1 \\ q^1_{HH} = q^2_{HH} = 0 \\ v^1_{LH} = v^1_{HH} = \underline{u} \; (v^2_{LH} = v^2_{HH} = \overline{u}). \end{cases} \tag{A15}$$

The equation (A2) boils down to

$$v^1_{HL} - v^1_{LH} = \frac{(1-\beta)(2-\mu)(\rho - \theta_H)}{\delta}. \tag{A16}$$

Thus, if a line segment $V(\beta) \subset V^m$ exists, then its width must be sufficiently long:

$$\overline{u} - \underline{u} \geq \frac{(1-\beta)(2-\mu)(\rho - \theta_H)}{\delta}. \tag{Add-IC}$$

The continuation values, v_{HL}^1 and v_{LL}^1, are determined as follows. The value v_{HL}^1 is assigned to satisfy (A16):

$$\begin{aligned} v_{HL}^1 &= v_{LH}^1 + \frac{(1-\beta)(2-\mu)(r-\theta_H)}{\delta} \\ &= \underline{u} + \frac{(1-\beta)(2-\mu)(r-\theta_H)}{\delta}. \end{aligned} \tag{A17}$$

The value v_{LL}^1 is chosen to satisfy the binding On-IC$_H^1$:

$$v_{LL}^1 = \underline{u} + \frac{3\mu - \mu^2 - 1}{\mu} \cdot \frac{(1-\beta)(r-\theta_H)}{\delta}. \tag{A18}$$

If On-IC$_H^i$ $\forall i$ is binding, then the firm's expected payoff takes the same form as (A5). Letting $u^1 = \underline{u}$ and $u^2 = \overline{u}$, the vector $\mathbf{z}(\beta)$ yields

$$\begin{aligned} \underline{u} &= \frac{[\mu(2-\mu) - \mu\beta](\rho - \theta_H) + \mu(1-\mu)\triangle}{1-\delta} \\ \overline{u} &= \frac{\mu\beta(r-\theta_H) + \mu\triangle}{1-\delta}. \end{aligned}$$

Note that $\underline{u}$ decreases in β whereas $\overline{u}$ increases in β, and that

$$\underline{u} + \overline{u} = \frac{[1-(1-\mu)^2](\rho - \theta_L)}{1-\delta} = \frac{\pi^m}{1-\delta} \ \forall \beta.$$

We now confirm that all the constraints hold. If Add-IC holds, then $\underline{u} < v_{HL}^1 \leq \overline{u}$, and if $\mu > \frac{3-\sqrt{5}}{2}$, then $\underline{u} < v_{LL}^1 < \overline{u}$. Hence, $v_{jk}^i \in V(\beta)$ $\forall i, (j,k)$. Because of the binding On-IC$_H^1$ and the equation (A2), On-IC$_H^2$ is binding. It follows from Lemma 2 that if On-IC$_H^i$ is binding, then On-IC$_L^i$ is slack. Lastly, the level of $\beta = \beta^*(\delta)$ is determined to satisfy Add-IC:

$$\beta \geq \beta^*(\delta) \equiv \max\left\{ \frac{(2-\mu)(1-\delta+\mu\delta) - \mu^2\delta\gamma}{2-\mu+(3\mu-2)\delta},\ 0 \right\}.$$

If the first term in $\max\{\cdot,\cdot\}$ is positive, then Add-IC is binding, and if it is negative, then Add-IC is slack. Hence, $\mathbf{z}(\beta)$ satisfies all the on-schedule constraints and constructs the endpoint $(\underline{u}, \overline{u})$. The remainder of $V(\beta)$ is constructed by a convex combination of two analogous vectors. The previous arguments directly hold; when β rises, the width of $V(\beta)$ is lengthened. As above, the relevant off-schedule constraints are Off-IC$_{LL}^1$ and Off-IC$_{HH}^1$:

$$\delta\left(v_{LL}^1 - \underline{v}\right) \geq \rho - \theta_L \text{ and } \delta\left(v_{HH}^1 - \underline{v}\right) \geq \rho - \theta_H.$$

Plugging the continuation values in (A17) and (A18) with $\beta = \beta^*(\delta)$, we can get $\delta^* = \max\{\delta_{LL}^*, \delta_{HH}^*\}$. ■

Proof of Proposition 2. We here focus on the case of $\alpha = \alpha^*(\delta)$ for any δ. The extension to the case of $\alpha > \alpha^*(\delta)$ follows the previous proof. To establish an endpoint $(\underline{u}, \overline{u})$ of a segment

$V(\alpha)$, consider a vector $\mathbf{z}(\alpha)$:

$$\mathbf{z}(\alpha) \equiv \begin{cases} p^i_{jk} = \rho \\ q^1_{LH} = q^2_{HL} = q^2_{LL} = 1 \\ q^1_{HH} = q^2_{HH} = \frac{1}{2} \\ v^1_{LH} = v^1_{HH} = \underline{u} \; (v^2_{LH} = v^2_{HH} = \overline{u}). \end{cases} \tag{A19}$$

Given this vector, the equation (A2) becomes

$$v^1_{HL} - v^1_{LH} = \frac{(1-\alpha)(\rho - \theta_H)}{\delta}. \tag{A20}$$

Thus, if $V(\alpha) \subset V^m$ exists, then its width must be sufficiently long:

$$\overline{u} - \underline{u} \geq \frac{(1-\alpha)\,(\rho - \theta_H)}{\delta}. \tag{Add-IC}$$

The continuation value v^1_{HL} is chosen to satisfy (A20):

$$v^1_{HL} = \underline{u} + \frac{(1-\alpha)\,(\rho - \theta_H)}{\delta} \tag{A21}$$

The continuation value v^1_{LL} is determined to satisfy the binding On-IC^1_H:

$$v^1_{LL} = \underline{u} + \frac{3\mu - 1}{2\mu} \cdot \frac{(1-\alpha)(\rho - \theta_H)}{\delta}. \tag{A22}$$

If On-IC^i_H $\forall i$ is binding, then the firm's expected payoff takes the same form as (A5). Letting $u^1 = \underline{u}$ and $u^2 = \overline{u}$, the vector $\mathbf{z}(\alpha)$ yields

$$\begin{aligned} \underline{u} &= \frac{[1 + \mu - \mu\,(1+\mu)\,\alpha]\,(\rho - \theta_H) + \mu(1-\mu)\triangle}{2(1-\delta)} \\ \overline{u} &= \frac{[1 - \mu + \mu\,(1+\mu)\,\alpha]\,(\rho - \theta_H) + \mu\,(1+\mu)\,\triangle}{2(1-\delta)}. \end{aligned}$$

Note that

$$\underline{u} + \overline{u} = \frac{\rho - E\,(\theta)}{1-\delta} \;\forall \alpha.$$

As above, the value $\overline{u}$ increases in α but $\underline{u}$ decreases in α. We next show that all the constraints are satisfied. If Add-IC holds, then $\underline{u} < v^1_{HL} \leq \overline{u}$, and that if $\mu > \frac{1}{3}$, then $\underline{u} < v^1_{LL} < \overline{u}$. Hence, $v^i_{jk} \in V(\beta)\ \forall i, (j,k)$. Because of the binding On-IC^1_H and the equation (A2), On-IC^2_H is binding. By Lemma 2, On-IC^i_L is slack. The level of $\alpha = \alpha^*(\delta)$ is determined to satisfy the binding Add-IC:

$$\alpha^*\,(\delta) \equiv \frac{1 - \delta + \delta\mu\,(1 - \mu\gamma)}{1 - \delta + \delta\mu\,(1+\mu)}.$$

Hence, if $\alpha = \alpha^*(\delta)$, there exists the vector $\mathbf{z}(\alpha) \in Z^{IC}(V(\alpha))$ that can establish the endpoint $(\underline{u}, \overline{u})$. The remainder of the segment can be established by convex combination of $\mathbf{z}(\alpha)$ and $\mathbf{z}'(\alpha)$. Consider next the off-schedule constraints. The relevant off-schedule constraints are

$$\delta\left(v^1_{LL} - \underline{v}\right) \geq \rho - \theta_L - (\rho - \theta_L)\,q^1_{LL} \tag{Off-IC$^1_{LL}$}$$

$$\delta\left(v^1_{HH} - \underline{v}\right) \geq \rho - \theta_H - (\rho - \theta_H)\,q^1_{HH} \tag{Off-IC$^1_{HH}$}$$

$$\delta\left(v^1_{HL} - \underline{v}\right) \geq \rho - \theta_H - (\rho - \theta_H)\,q^1_{HL} \tag{Off-IC$^1_{HL}$}$$

Given the above continuation values, $q^1_{LL} = q^1_{HL} = 0$ and $q^1_{HH} = \frac{1}{2}$ under $\mathbf{z}(\alpha)$, it is immediate that Off-IC$^1_{LL}$ implies Off-IC$^1_{HL}$ since $v^1_{HL} > v^1_{LL}$ for all $\mu > \frac{1}{3}$. The off-schedule constraints are reduced to Off-IC$^1_{LL}$ and Off-IC$^1_{HH}$. Plugging the continuation values with $\alpha = \alpha^*(\delta)$ into the inequalities, we can get $\delta^* = \max\{\delta^*_{LL}, \delta^*_{HH}\}$. ■

Proof of Proposition 3. We here confirm that the vector $\mathbf{z}(\alpha)$ satisfies the on-schedule constraints. Under $\mathbf{z}(\alpha)$, the continuation value v^i_{LL} is chosen to satisfy the binding On-IC^i_H. If On-IC^i_H is binding, then On-IC^i_L is slack by Lemma 2. Add-IC is satisfied by the choice of $\underline{\rho}$. Because of the binding Add-IC, v^i_{LL} in (23) becomes

$$v^i_{LL} = \overline{u} - \frac{(1-\alpha)(\rho - \theta_H)}{2\delta\mu} = \underline{u}.$$

It follows that all the continuation values are drawn from $V^s(\alpha)$: $v^i_{jk} = \overline{u}$ except $v^i_{LL} = \underline{u}$ $\forall i, (j,k)$. We also can verify that the other vector $\mathbf{z}'(\alpha)$ satisfies the on-schedule constraint. Under $\mathbf{z}'(\alpha)$, On-IC^i_H becomes

$$\delta(\overline{v}^i_H - \overline{v}^i_L) \geq (\underline{\rho} - \theta_H)\left(\overline{q}^i_L - \overline{q}^i_H\right) - (1-\mu)\frac{\alpha\,(\rho - \theta_H)}{2(1-\mu)}.$$

The RHS represents the current-period gain that a high-cost firm can make when it lies. Given the continuation values and (28), the LHS boils down to $\delta\mu(\overline{u} - \underline{u}) = \frac{(1-\alpha)(\rho-\theta_H)}{2}$. A simplification shows that On-IC^i_H becomes $\rho \geq \underline{\rho}$. Thus, as indicated by (26), this constraint is slack (binding) if $0 \leq \alpha < 1$ (if $\alpha = 1$). Under $\mathbf{z}'(\alpha)$, On-IC^i_L becomes

$$\delta(\overline{v}^i_H - \overline{v}^i_L) \leq (\underline{\rho} - \theta_L)\left(\overline{q}^i_L - \overline{q}^i_H\right).$$

This can be rearranged as $(1-\alpha)(\rho - \theta_H) \leq \underline{\rho} - \theta_L$. If α rises, then the LHS falls, but the RHS rises because the price $\underline{\rho}$ in (26) rises. This inequality is thus reduced to

$$\alpha \geq \alpha^*(\delta) \equiv \max\left\{1 - \frac{\mu\delta}{1+\mu\delta}\left(\frac{\rho - \theta_L}{\rho - \theta_H}\right), 0\right\}.$$

Hence, the vectors $\mathbf{z}(\alpha)$ and $\mathbf{z}'(\alpha)$ satisfy the on-schedule constraints if $\alpha \geq \alpha^*(\delta)$. This range of α counters the low-cost firm's incentive to overstate its cost type and avoid the possible future penalty. ■

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