Documentos CEDE
ISSN 1657-7191 edición electrónica

The Difference Indifference Makes in Strategy-Proof Allocation of Objects

Paula Jaramillo
Vikram Manjunath

17
ABRIL DE 2011

Universidad de los Andes
Facultad de Economía

CEDE
Centro de Estudios sobre Desarrollo Económico

Serie Documentos Cede, 2011-17
ISSN 1657-7191

Abril de 2011

Calle 19A No. 1 – 37, Bloque W.
Bogotá, D. C., Colombia
Teléfonos: 3394949- 3394999, extensiones 2400, 2049, 3233
infocede@uniandes.edu.co
http://economia.uniandes.edu.co

Ediciones Uniandes
Carrera 1ª Este No. 19 – 27, edificio Aulas 6, A. A. 4976
Bogotá, D. C., Colombia
Teléfonos: 3394949- 3394999, extensión 2133, Fax: extensión 2158
infeduni@uniandes.edu.co

Edición, diseño de cubierta, preprensa y prensa digital:
Proceditor ltda.
Calle 1ª C No. 27 A – 01
Bogotá, D. C., Colombia
Teléfonos: 2204275, 220 4276, Fax: extensión 102
proceditor@etb.net.co

Impreso en Colombia – *Printed in Colombia*

Universidad de los Andes
Facultad de Economía

CEDE
Centro de Estudios
sobre Desarrollo Económico

The difference indifference makes in strategy-proof allocation of objects*

Paula Jaramillo† and Vikram Manjunath‡

Abstract

We study the problem of allocating objects among people. We consider cases where each object is initially owned by someone, no object is initially owned by anyone, and combinations of the two. The problems we look at are those where each person has a need for exactly one object and initially owns at most one object (also known as "house allocation with existing tenants"). We split with most of the existing literature on this topic by dropping the assumption that people can always strictly rank the objects. We show that, without this assumption, problems in which either some or all of the objects are not initially owned are equivalent to problems where each object is initially owned by someone. Thus, it suffices to study problems of the latter type.

We ask if there are efficient rules that provide incentives for each person not only to participate (rather than stay home with what he owns), but also to state his preferences honestly. Our main contribution is to show that the answer is positive. The intuitive "top trading cycles" algorithm provides the only such rule for environments where people are never indifferent (Ma 1994). We generalize this algorithm in a way that allows for indifference without compromising on efficiency and incentives.

JEL classification: C71, C78, D71, D78

Keywords: strategy-proofness, indivisible goods, indifference, housing market, house allocation, kidney exchange

*We thank William Thomson for supervising our work. We express our gratitude to an associate editor and two anonymous referees, Dolors Berga, Lars Ehlers, Aytek Erdil, Jay Sethuraman, Çağatay Kayı, Bettina Klaus, Fuhito Kojima, Rodrigo Velez, Alexander Westkamp, and seminar/conference participants at Universidad de Los Andes, Universidad del Rosario, the 6th meeting of the Society for Economic Design, the Hausdorff Institute for Mathematics, the 5th Spain-Italy-Netherlands Meetings on Game Theory, and the Indian Statistical Institute, Delhi for their helpful comments and discussions.

†Universidad de Los Andes

‡University of Rochester and Université de Montréal

La diferencia que hacen las indiferencias en la asignación sin manipulación de objetos

Paula Jaramillo* y Vikram Manjunath**

Resumen

Estudiamos problemas de asignación de objetos entre personas. Consideramos casos en los que cada persona es dueña de un objeto, nadie es dueño de un objeto y combinaciones de los dos. Los problemas que estudiamos son aquellos en que cada persona necesita un único objeto e inicialmente es dueño de un objeto (también conocido como "house allocation with existing tenants"). Nosotros nos diferenciamos de la mayoría de la literatura, al relajar el supuesto que cada persona puede ordenar estrictamente los objetos según sus preferencias. Nosotros mostramos que, al considerar indiferencias en las preferencias, problemas en el que algún o ningún objeto pertenece a alguien son equivalentes a problemas en los que cada persona es dueña de un objeto. Entonces, es suficiente trabajar con problemas del último tipo.

Nuestra mayor contribución es mostrar que hay reglas eficientes que provean incentivos a cada persona no solo para participar (en vez de quedarse en su casa con el objeto del que es dueño), si no para reportar sus preferencias honestamente. El intuitivo algoritmo de "ciclos comercio en la cima" provee la única regla cuando las personas nunca son indiferentes (Ma, 1994). Nosotros generalizamos este algoritmo de forma que se permiten indiferencias sin comprometer eficiencia e incentivos.

Clasificación JEL : C71, C78, D71, D78

Palabras claves: no manipulaicón, bienes indivisibles, indiferencias, mercado de casas, asignación de casas, intercambio de riñones

*Universidad de Los Andes

**University of Rochester y Université de Montréal

1 Introduction

Consider a setting where each person in a group has a need for a single object (such as a seminar slot, an on-campus apartment, or an organ for transplant) and may or may not be endowed with such an object. Further, suppose that there are no divisible goods, such as money. Even when every person is endowed with an object, the initial distribution is not necessarily efficient.

When people are never indifferent between objects, there are *strategy-proof, Pareto-efficient,* and *individually rational* rules (Abdulkadiroğlu and Sönmez 1999). In fact, a group of such rules is characterized by these three axioms with the help of consistency and neutrality axioms (Sönmez and Ünver 2010). Moreover, the class of rules satisfying *group strategy-proofness* and *Pareto-efficiency* has been characterized (Pycia and Ünver 2009).

At one extreme of this class of problems are those where nobody is endowed with an object and there is only a social endowment (Hylland and Zeckhauser 1979). At the other extreme are problems where everybody is endowed with an object but there is no social endowment (Shapley and Scarf 1974). For these problems, when people are never indifferent between objects, there are rules with desirable efficiency and incentive properties. The *core* contains a unique allocation which is also the unique *competitive allocation* (Roth and Postlewaite 1977). The rule that maps each problem with its unique core allocation is not only *strategy-proof* (Roth 1982) but also *group strategy-proof* (Bird 1984). Further, it is the only *strategy-proof, Pareto-efficient,* and *individually rational* rule (Ma 1994, Sönmez 1999). It is also *non-bossy* and *anonymous* (Miyagawa 2002).

We argue that there are many real-world situations where people's preferences do exhibit indifference. For instance, if preferences are based on coarse descriptions (say, from a housing brochure), there may be insufficient information to break ties. Alternatively, if preferences are based on checklists of criteria (like blood and tissue types for organ transplant), distinct objects satisfying exactly the same criteria are equivalent.

Appropriate design of rules should take these indifferences into account since breaking ties arbitrarily may lead to inefficiency.

We show that when we drop the assumption that people are never indifferent, all of the problems mentioned above can be thought of as ones where every person is endowed with an object. Thus, we study only such problems. Without strict preferences, many of the results mentioned above no longer hold. Though the *weak core* is not empty, the a *core* allocation is no longer guaranteed to exist (Shapley and Scarf 1974).[1] The set of *competitive allocations* no longer coincides with the *core* (Wako 1991). *Group strategy-proofness* and *Pareto-efficiency* are incompatible (Ehlers 2002).

We show that there may not even be a *competitive allocation* that is *Pareto-efficient*. We provide a direct proof that *strategy-proofness, Pareto-efficiency,* and *individual rationality* are not compatible with *non-bossiness*.[2] Further, we show that, even when we drop *individual rationality*, they are not compatible with *anonymity*.

Our main contribution is a novel algorithm that is associated with a *strategy-proof, Pareto-efficient,* and *individual rational* rule. Moreover, the allocation selected is in the "weak core".

Recently, the assumption of strict preferences has received attention for various matching models. Alcalde-Unzu and Molis (2011) have independently come up with a different class of rules for the model studied in this paper. For the "two-sided matching" model, the *deferred acceptance* algorithm can be adapted to preserve "stability" and *Pareto efficiency* in the presence of indifference (Erdil and Ergin 2006). However, this adaptation does not preserve "one-sided" *strategy-proofness*, which means that an application of this algorithm to our model does not satisfy *strategy-proofness*.

The remainder of the paper is organized as follows. We present the model in Section 2. We describe some desiderata of allocations and rules in Section 3 and define our rules, along with some others, in Section 4. In Section 5 we present our results. We show how the more general problems involving social endowments can be encoded as problems with only private endowments in Section 6.

2 The Model

Let $\boldsymbol{O}$ be a set of distinct objects. Let $\boldsymbol{N}$ be a set of people. There are exactly as many objects as people: $|O| = |N|$. An **endowment** is a bijection, $\omega : N \to O$, that associates an object with each person. For each $i \in N$, i's component of the endowment is $\omega(i)$. Each person has a preference relation over O. Let the set of all preference relations be $\mathcal{R}$. A **preference profile** associates each individual with a preference relation in $\mathcal{R}$. Let $\mathcal{R}^N$ be the set of all preference profiles. Given

[1]Quint and Wako (2004) provide necessary and sufficient conditions on preference profiles for the core to be non-empty.

[2]Bogomolnaia, Deb and Ehlers (2005) show this by characterizing, for problems with no private endowment, classes of *strategy-proof* and *Pareto-efficient* rules satisfying two different forms of *non-bossiness* and some auxillary axioms.

a profile $R \in \mathcal{R}^N$, for each $i \in N$, i's preference relation is R_i. For each pair of alternatives, $a, b \in O$, if i finds a to be at least as good as b, we write $a\ R_i\ b$. If a is better than b, that is, $a\ R_i\ b$ but not $b\ R_i\ a$, we write $a\ P_i\ b$. Similarly, if i is indifferent between a and b, we write $a\ I_i\ b$. Let $\mathcal{P} \subset \mathcal{R}$ be the set of "strict" preference relations. That is, $\mathcal{P} \equiv \{R_0 \in \mathcal{R} :$ for each $a, b \in O, a\ I_0\ b \Leftrightarrow a = b\}$.

We use the notation R_{-i} to denote the preference relations of everyone but i. For each group $S \subseteq N$, we denote the preferences of all the people in S by R_S, and those not in S by R_{-S}. We denote the set of all preferences for people in the group S by $\mathcal{R}^S$.

Let $\boldsymbol{A}$, the set of all bijections from N to O, be the set of all possible allocations. For each $\alpha \in A$, and each $i \in N$, let $\alpha(i)$ denote i's component of α. Similarly, for each $S \subseteq N$, let $\alpha(S)$ be the collective assignment to members of S under α. That is, $\alpha(S) = \bigcup_{i \in S}\{\alpha(i)\}$.

A **problem** consists of a preference profile and an endowment, $(R, \omega) \in \mathcal{R}^N \times A$. A **rule**, $\varphi : \mathcal{R}^N \times A \to A$, selects an allocation for each problem.

3 Properties of allocations and rules

In this section, we list some desiderata of allocations and rules. Let φ be a rule.

The first requirement is that a rule respects each individual's endowment. That is, the allocation selected by the rule should not assign, to any person, an object that he finds worse than his endowment.

For each $(R, \omega) \in \mathcal{R}^N \times A$ and $\alpha \in A$, we say that **$\boldsymbol{\alpha}$ is individually rational at $\boldsymbol{(R, \omega)}$** if for each $i \in N, \alpha(i)\ R_i\ \omega(i)$. Let $\boldsymbol{IR(R, \omega)}$ be the set of all *individually rational allocations at* (R, ω).

Individual Rationality: For each $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega) \in IR(R, \omega)$.

Before we state the next requirement, we define an efficiency relation between allocations. For each $\alpha, \beta \in A$ and $R \in \mathcal{R}^N$, **$\boldsymbol{\alpha}$ Pareto dominates $\boldsymbol{\beta}$ at $\boldsymbol{R}$** if at least one person is better off at α than at β and nobody is worse off. That is, for some $i \in N$, $\alpha(i)\ P_i\ \beta(i)$ and for each $i \in N, \alpha(i)\ R_i\ \beta(i)$.

For each $R \in \mathcal{R}^N$, let the set of allocations that are not *Pareto dominated* by any other allocation be $\boldsymbol{PE(R)}$.

Pareto-efficiency: For each $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega) \in PE(R)$.

The next property says that unilaterally misreporting one's preferences is never beneficial.

Strategy-proofness: For each $(R, \omega) \in \mathcal{R}^N \times A$, there is no $i \in N$ for whom there is $R'_i \in \mathcal{R}$ such that

$$\varphi(\underbrace{R'_i}_{\text{lie}}, R_{-i}, \omega)(i) \underbrace{P_i}_{\text{truth}} \varphi(\underbrace{R_i}_{\text{truth}}, R_{-i}, \omega)(i).$$

The following is the requirement that nobody can affect what the rule assigns to others without affecting his own assignment.

Non-bossiness: For each $(R, \omega) \in \mathcal{R}^N \times$, there are no $i \in N$ and $R'_i \in \mathcal{R}$ such that

$$\varphi(R, \omega)(i) = \varphi(R'_i, R_{-i}, \omega)(i) \text{ and } \varphi(R, \omega) \neq \varphi(R'_i, R_{-i}, \omega).$$

The next desideratum is that the rule is a function of preferences and endowments, but not identities. Let $\pi : N \to N$ be a permutation of N. For each $(R, \omega) \in \mathcal{R}^N \times A$, define the **permutation of $(\boldsymbol{R}, \boldsymbol{\omega})$ with respect to $\boldsymbol{\pi}$, $(\boldsymbol{R^\pi}, \boldsymbol{\omega^\pi})$** $\in \mathcal{R}^N \times A$, such that for each $i \in N$, $R^\pi_{\pi(i)} \equiv R_i$ and $\omega^\pi_{\pi(i)} \equiv \omega_i$.

Anonymity: For each $i, j \in N$, each $R \in \mathcal{R}^N$, $\omega \in A$, each $\pi : N \to N$, and each $i \in N$,

$$\varphi(R^\pi, \omega^\pi)(\pi(i)) = \varphi(R, \omega)(i).$$

The final requirement is that no group of people would rather re-allocate their endowments among themselves than participate in the application of the rule. This can be expressed in two ways. First, for each $\alpha \in A$, $(R, \omega) \in \mathcal{R}^N \times A$, and $S \subseteq N$, we say that **$\boldsymbol{\alpha}$ is blocked by $\boldsymbol{S}$** if members of S can re-allocate their endowments in a way that makes each of them better off than at α. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$ and for each $i \in S, \beta(i)\ P_i\ \alpha(i)$. Second, we say that **$\boldsymbol{\alpha}$ is weakly blocked by $\boldsymbol{S}$** if members of S can re-allocate their endowments in a way that makes at least one of them better off than at α, without making any of the rest worse off than at α. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$, for some $i \in S, \beta(i)\ P_i\ \alpha(i)$, and for each $i \in S, \beta(i)\ R_i\ \alpha(i)$.

The **weak core, $\boldsymbol{C^W(R, \omega)}$**, is the set of allocations that are not *blocked* by any coalition and the **core, $\boldsymbol{C(R, \omega)}$**, is the set of allocations that are not *weakly blocked* by any coalition.

4 Rules

Let $\prec$ be a linear ordering of N.

Sequential priority rules: Let a *tie-breaker* $\theta : \mathbb{P}(A) \setminus \{\varnothing\} \to A$ be such that for each $A' \subseteq A$, $\theta(A') \in A'$.[3] The **sequential priority rule with respect to $\boldsymbol{\prec}$ and $\boldsymbol{\theta}$, $\boldsymbol{SP^{\prec,\theta}}$,** is defined as follows. Suppose $\prec$ is such that $1 \prec 2 \prec \cdots \prec n$.

[3]Given a set S, we denote the *power set of S* by $\mathbb{P}(S)$.

For each $(R,\omega) \in \mathcal{R}^N \times A$, we define a sequence of subsets of A, $\{A_i^{R,\omega}\}_{i=0}^{i=n}$. Let $A_0^{R,\omega} = A$. For each $i = 1, \ldots n$,

$$A_i^{R,\omega} = \{\alpha \in A_{i-1}^{R,\omega} : \text{ for each } \beta \in A_{i-1}^{R,\omega}, \alpha(i)\ R_i\ \beta(i)\}.$$

Finally, define $SP^{\prec}(R,\omega) = \theta(A_n^{R,\omega})$.[4]

Sequential priority rules are *strategy-proof* and *Pareto-efficient* (Svensson 1994). But they are not *individually rational*.

Sequential priority selections from $\boldsymbol{IR}$: Let θ be a *tie-breaker*. The **sequential priority selection from $\boldsymbol{IR}$ with respect to $\prec$ and $\boldsymbol{\theta}$, $\boldsymbol{SP\text{-}IR^{\prec,\theta}}$,** is defined exactly as $SP^{\prec,\theta}$ except that $A_0^{R,\omega} = IR(R,\omega)$.

Sequential priority selections from IR are not *strategy-proof* but are *Pareto-efficient* and, by definition, *individually rational*.[5]

The notion of "most preferred" objects among a subset of O is critical for the definition of our next rule. For each $R \in \mathcal{R}^N$, $O' \subseteq O$, and $i \in N$, let **$\boldsymbol{i}$'s most preferred objects, under $\boldsymbol{R_i}$, among $\boldsymbol{O'}$**, be denoted by $\boldsymbol{\tau(R_i, O')} \equiv \{a \in A :$ for each $b \in O', a\ R_i\ b\}$.

Gale's "top trading cycles" algorithm (Shapley and Scarf 1974) is applicable after breaking ties arbitrarily. The algorithm, defined for preferences in $\mathcal{P}^N$, proceeds by asking each person to point at the person endowed his most preferred object. Since each person points, there is at least one cycle and the members of such cycles exchange their objects accordingly. Once the objects have been exchanged, these people are removed and the algorithm continues among those remaining.

The associated rules are *strategy-proof* and *individually rational* but not *Pareto-efficient*. To see this, consider the following example in which, no matter how ties are broken, the result of the "top trading cycles" algorithm is not *Pareto-efficient*.

Example 1. *Breaking ties.*

Let $N \equiv \{1,2,3\}$, $\omega \equiv (a,b,c)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3
$b\ c$	a	a
a	c	b
	b	c

[4] This definition is taken from Svensson (1994).

[5] It is easy to show that the *sequential priority selections from IR* are *Pareto-efficient* and *individually rational*. To see that they are not *strategy-proof* consider the following example. Let $N = \{1,2,3\}$, $\omega = \{a,b,c\}$, and the following preferences: $P_1 : c\ a\ b$, $P_2 : a\ b\ c$, and $P_3 : a\ b\ c$. Suppose that $1 \prec 2 \prec 3$. Then, for each θ, $SP\text{-}IR^{\prec,\theta}(R,\omega) = (c,a,b)$. If 3 reports $P_3' : a\ c\ b$ instead, for each θ, $SP\text{-}IR^{\prec,\theta}(R_{-3}, R_3', \omega) = (c,b,a)$ and 3 is better off.

There are exactly two ways to break ties: $P^1, P^2 \in \mathcal{P}^N$:

P^1_1	P^1_2	P^1_3
b	a	a
c	c	b
a	b	c

P^2_1	P^2_2	P^2_3
c	a	a
b	c	b
a	b	c

But for (P^1, ω) the reccommendation is (b, a, c) and for (P^2, ω) it is (c, b, a). Neither of these is *Pareto-efficient*.[6]

The next class of rules that we define are based on an adaptation of this algorithm.[7] Two questions need to be answered to adapt the *top-trading cycles* algorithm:

1. When can a person be removed? In Example 1, if 2 gets a, 1 cannot leave with b since that would violate *Pareto-efficiency*. Clearly, it is not enough for a person to leave when he "holds" one of his most preferred objects. With the description of our algorithm, we state a more sophisticated condition that needs to be met for a person (or group of people) to be removed. This aspect of our algorithm is crucial to achieving a *Pareto-efficient* allocation.

2. When a person is indifferent between objects, where does he point? This is a more difficult question. We explain how to deal with this issue in a way that does not compromise *strategy-proofness*. In addition, our condition guarantees that the algorithm terminates.

Corresponding to the two questions that we have just raised, we change the algorithm at each step in two significant ways:

i) Condition for departure: A group of people can leave only if there is no trade with people outside of the group that can make someone outside the group better off without hurting someone inside the group. This condition ensures *Pareto-efficiency* of the final allocation. In Example 1, if 2 gets a, he can leave with it since any further trade would make him worse off. However, 1 cannot leave with b: there is a trade, between 1 and 3 that would make 3 better off without hurting 1. Thus, 1 and 3 trade and 1 leaves with c and 3 with b.

[6]The reason for this is that regardless of how we break ties, the result of the *top-trading cycles* algorithm is a "competitive allocation" (Shapley and Scarf 1974). In fact every *competitive allocation* can be found in this way. However, as evidenced by the above example, for some profiles of preferences, no *competitive allocation* is *Pareto-efficient*.

[7]However, we have the "departure phase" at the beginning of each step rather than at the end. We have done this for expositional simplicity since it rules out people "pointing" at themselves.

ii) Condition for pointing: A natural way to solve the problem is to use a priority order over the people. However, naïvely breaking ties according to a fixed priority order does not work. We illustrate this through the following examples.

(a) Let $N = \{1, 2, 3\}$, $\omega = (a, b, c)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3
$a\ b\ c$	$a\ b$	c
	c	$a\ b$

Let $1 \prec 2 \prec 3$. Suppose each person cannot point at himself when he is indifferent between what he holds and what someone else holds.[8] In this case, 1 and 2 trade at each step between them and the condition of departure is never satisfied. The algorithm does not terminate.

To guarantee that the algorithm terminates, the priority order defined in the algorithm is updated at every stage to give higher priority to people who do not hold one of their most preferred objects than to people who hold one of their most preferred objects.

(b) Let $N = \{1, 2, 3, 4, 5\}$, $\omega = (a, b, c, d, e)$ and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5
$c\ d$	$a\ d\ e$	b	b	a
a	b	a	d	e
		c		

Let $2 \prec 3 \prec 4 \prec 5 \prec 1$. If we use a naïve pointing scheme, in the first step, 1 points at 3 and 2 at 4. Then, 2 and 4 trade and 4 leaves with b. In the second step, since $2 \prec 3$ and $5 \prec 1$, 1 points at 2 and 2 points at 5. Then, 1, 2, and 5 trade. The algorithm terminates and the final allocation assigns c to 3. However, if 3 reports

R'_3
a
c

, 3's assignment is a.

Therefore, 3 is better off reporting the lie R'_3 when his true preference relation is R_3 and others' preferences are R_{-3}.

To guarantee that the rule is *strategy-proof*, at each step, each i points at the same person that he pointed at in the previous step as long as that person holds the same object (that is, he did not trade).

[8] Otherwise, 1 will point to himself at each step of the algorithm. Since the departure condition is not met, the algorithm does not terminate.

We bring these ideas together and define a modification of the *top-trading cycles* algorithm.

Top cycles rules: For each $(R, \omega) \in \mathcal{R}^N \times A$, we define the allocation selected by **top cycles rule with priority $\boldsymbol{\prec}$, $\boldsymbol{TC^{\prec}(R, \omega)}$,** via the following algorithm.

Each step of the algorithm proceeds in three phases: departure, pointing, and trading. The goal of the algorithm is to enlarge, at each step, the set of "satisfied" people: those holding one of their most preferred objects. However, we aim to do this in a way that provides incentives for every person to report his true preferences. To achieve this, the algorithm favors people who have higher priority by connecting more people to them (via direct or indirect pointing) as compared to people with lower priority.

In the first step, the set of remaining people is N and each $i \in N$ holds $\omega(i)$.

1. **Departure:** A group of people is chosen to "depart" if two conditions are met.

   i) What each person in the group holds is among his most preferred objects (among the remaining ones), and

   ii) All of the most preferred objects (among the remaining ones) of the group are held by them.

   Once a group departs, each of them is assigned what he holds and removed from the set of *remaining people*. In addition, their objects are removed from the *remaining objects*. There may be another group that can be chosen to depart. The process continues until there are no more groups that can depart. If the two conditions are not met by any group, then nobody departs.

2. **Pointing:** Each person points at a person holding one of his top objects (among the remaining ones). Since there may be more than one such person, the problem of figuring whom each person points at is a complicated one. We solve it in stages as follows:

Stage 1) For each remaining j such that j holds the same object that he held in the previous step, each i that pointed at j in the previous step points at j in the current step. Of course, this does not apply for the very first step.

Stage 2) Each i with a unique top object (among the remaining ones) points at the person holding it.

Stage 3) Each person who has at least one of his top objects (among the remaining ones) held by an *unsatisfied* person points at whomever has the highest priority among such *unsatisfied* people.

Stage 4) Each person who has at least one of his top objects (among the remaining ones) held by a *satisfied* person who points at an *unsatisfied* person points at whomever points at the *unsatisfied* person with highest priority. If two or more of his *satisfied* "candidates" point at the *unsatisfied* person with highest priority, he points at the *satisfied* candidate with the highest priority.

Stage $\cdots$) And so on.

3. **Trading:** Since each *remaining* person points at someone, there is at least one cycle of remaining people. For each such cycle, people trade according to the way that they point and what they hold for the next step is updated accordingly.

The algorithm terminates when everyone has departed.

To see that the algorithm terminates, note that at each step, since N is finite and there is at least one cycle involving an unsatisfied person, either

1. At least one person departs with his holding, or
2. At least one person's holding is switched to an object that he ranks highest among those remaining. That is at least one person becomes *satisfied*.

Therefore, the algorithm terminates in a finite number of steps.[9] Since the algorithm terminates and provides a unique allocation for every problem, $TC^{\prec}$ is a well-defined rule.

A rigorous formal description of the algorithm is in Appendix A.

To help illustrate the *top cycles rule*, we provide Example 2. First, we state some useful definitions. In the t^{th} step, after the *departure phase*, there is a set of **remaining objects,** $\boldsymbol{O_t \subseteq O}$, and **remaining people,** $\boldsymbol{N_t \subseteq N}$. Each *remaining* person, $i \in N_t$, **holds** the object $\boldsymbol{h_t(i)}$.[10] For each $i \in N_t$, the **person whom** $\boldsymbol{i}$ **points at,** $\boldsymbol{p_t(i)}$. For notational convenience, for each $i, j \in N_t$, we use $\boldsymbol{i \underset{t}{\longrightarrow} j}$ to denote $p_t(i) = j$. If $p_t(p_t(i)) = j$, we write $\boldsymbol{i \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} j}$, and so on. Given $M \subseteq N_t$, if $p_t(i) \in M$, we write $\boldsymbol{i \underset{t}{\longrightarrow} M}$. If $p_t(p_t(i)) \in M$, we write $\boldsymbol{i \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} M}$, and so on.

At each step, the set of remaining people is partitioned into two sets: **satisfied people,** $\boldsymbol{S_t} \equiv \{i \in N_t : h_t(i) \in \tau(R_i, O_t)\}$, who hold an object that they rank

[9] The algorithm is polynomial time: $O(|N|^5)$ where $|N|$ is the number of people in the problem.

[10] Note that for each i, $h_t(i)$ denotes the object that i holds at the beginning of Step t, while $h_{t+1}(i)$ denotes the object i is holding at the end of Step t. If i trades in Step t, the object that he holds in Step t, $h_t(i)$, is different from the one he has at the end of Step t and beginning of Step $t+1$, $h_{t+1}(i)$. If i does not trade $h_t(i) = h_{t+1}(i)$.

at the top of the remaining objects, and the **unsatisfied people** who do not, $\boldsymbol{U_t} \equiv N_t \setminus S_t$.

Example 2. *Top cycles rule.*

Let $O = \{a, b, c, d, e, f, g, h, i, j, k\}$, and $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Consider $(R, \omega) \in \mathcal{R}^N \times A$ such that $\omega = (a, b, c, d, e, f, g, h, i, j, k)$ and $R \in \mathcal{R}^N$ as follows:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
a	a	f	$c\ d\ e\ f$	$d\ e\ g$	$d\ e$	d	$d\ h\ i$	$c\ i$	$a\ b\ j$	$e\ i\ k$
$\vdots$	b	$\vdots$	$\vdots$	$\vdots$	$\vdots$	g	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$					$\vdots$				

Let $\prec$ be such that $1 \prec 2 \prec 3 \prec 4 \prec 8 \prec 5 \prec 6 \prec 7 \prec 9 \prec 10 \prec 11$.

We start with $O_0 = O, N_0 = N$, and $h_1 = \omega$.

Step 1:

Departure phase:

The first group to depart is $\{1\}$. To see this, note that 1's most preferred object in O is the unique object a, his endowment. Given that 1 leaves with a, the second group to departs is $\{2, 10\}$, 2's most preferred object in $O \setminus \{a\}$ is the unique object b and 10's most preferred objects in $O \setminus \{a\}$ are b and j. Now, $TC^{\prec}(R, \omega)(1) = a$, $TC^{\prec}(R, \omega)(2) = b$, and $TC^{\prec}(R, \omega)(10) = j$. Further, the *remaining people* are $N_1 = \{3, 4, 5, 6, 7, 8, 9, 11\}$, and the *remaining objects* are $O_1 = \omega(N_1)$. From this, the *satisfied people* are $S_1 = \{4, 5, 8, 9, 11\}$.

Pointing phase: This is illustrated in Figure 2.

Stage 1) Not applicable to the first step.

Stage 2) Each person with a unique most preferred object in O_1 points at whomever holds that object. In this case, $3 \underset{1}{\longrightarrow} 6$ and $7 \underset{1}{\longrightarrow} 4$.

Stage 3) Each person such that one of their most preferred objects is held by an unsatisfied person points at an unsatisfied person. Such people are 4, 5, and 9. Since 5 and 9 have only one unsatisfied person to point at, they point accordingly. That is, $5 \underset{1}{\longrightarrow} 7$ and $9 \underset{1}{\longrightarrow} 3$. However, 4 is indifferent between the objects held by 3 and 6. In accordance with $\prec$, $4 \underset{1}{\longrightarrow} 3$.

Stage 4) Each person whose most preferred objects are held by satisfied people pointing at an unsatisfied person. 6, 8, and 11 are such people. Since 4 and 5 hold 6's most preferred objects, we consider who they are pointing at. Since $4 \underset{1}{\longrightarrow} 3, 5 \underset{1}{\longrightarrow} 7$, and $3 \prec 7$, we have $6 \underset{1}{\longrightarrow} 4$ rather than $6 \underset{1}{\longrightarrow} 5$. 8's preferred objects (among the remaining ones) are hold by 9 and 4. Note that both point at 3. Since $4 \prec 9$, $8 \underset{1}{\longrightarrow} 4$. Since 5 and 9 hold 11's most preferred objects, $5 \underset{1}{\longrightarrow} 7, 9 \underset{1}{\longrightarrow} 3$, and $3 \prec 7$, we have $11 \underset{1}{\longrightarrow} 9$

Trading phase:

We observe that there is only one cycle and it involves 3, 4, and 6. Thus, $h_2 = (-, -, f, c, e, d, g, h, i, -, k)$.

Step 2:

Departure phase:

The only group satisfying the departure condition is $\{3\}$. From this, $TC^{\prec}(R, \omega)(3) = f$, $N_2 = \{4, 5, 6, 7, 9, 11\}$, $O_2 = \{c, d, e, g, h, i, k\}$, and $S_2 = \{4, 5, 6, 8, 9, 11\}$.

Pointing phase: This is illustrated in Figure 2.

Stage 1) Note that 5 was pointing at 7 in Step 1 and 7 is holding the same object. That is, $5 \underset{1}{\longrightarrow} 7 \in N_2$ and $h_2(7) = h_1(7)$. Then, we have $5 \underset{2}{\longrightarrow} 7$. In addition, since $11 \underset{1}{\longrightarrow} 9 \in N_2$ and $h_2(9) = h_1(9)$, we have $11 \underset{2}{\longrightarrow} 9$.[11]

Stage 2) Since 7's unique most preferred object is d, $7 \underset{2}{\longrightarrow} 6$.

Stage 3) No person, other than 5, most prefers g (7's holding) among O_2.

Stage 4) 4 and 6 point at 5 whose pointing to an unsatisfied person: $4 \underset{2}{\longrightarrow} 5$ and $6 \underset{2}{\longrightarrow} 5$.

Stage 5) 8 and 9 are pointing to someone that is pointing to 5. Thus, $8 \underset{2}{\longrightarrow} 6$ and $9 \underset{2}{\longrightarrow} 4$.

Trading Phase:

At the end of this Step, there is one cycle and it involves 5, 6, and 7. In the *trading phase*, we get $h_3 = (-, -, -, c, g, e, d, h, i, -, k)$.

Step 3:

Departure phase:

We end after the departure phase of Step 3 since all the remaining people, $\{4, 5, 6, 7, 8, 9, 11\}$, satisfy the conditions. Then, $N_3 = \varnothing$ y $O_3 = \varnothing$.

Thus, $TC^{\prec}(R, \omega) = (a, b, f, c, g, e, d, h, i, j, k)$.

[11] Without Stage 1, 11 would point to 5 whose pointing at an unsatisfied person.

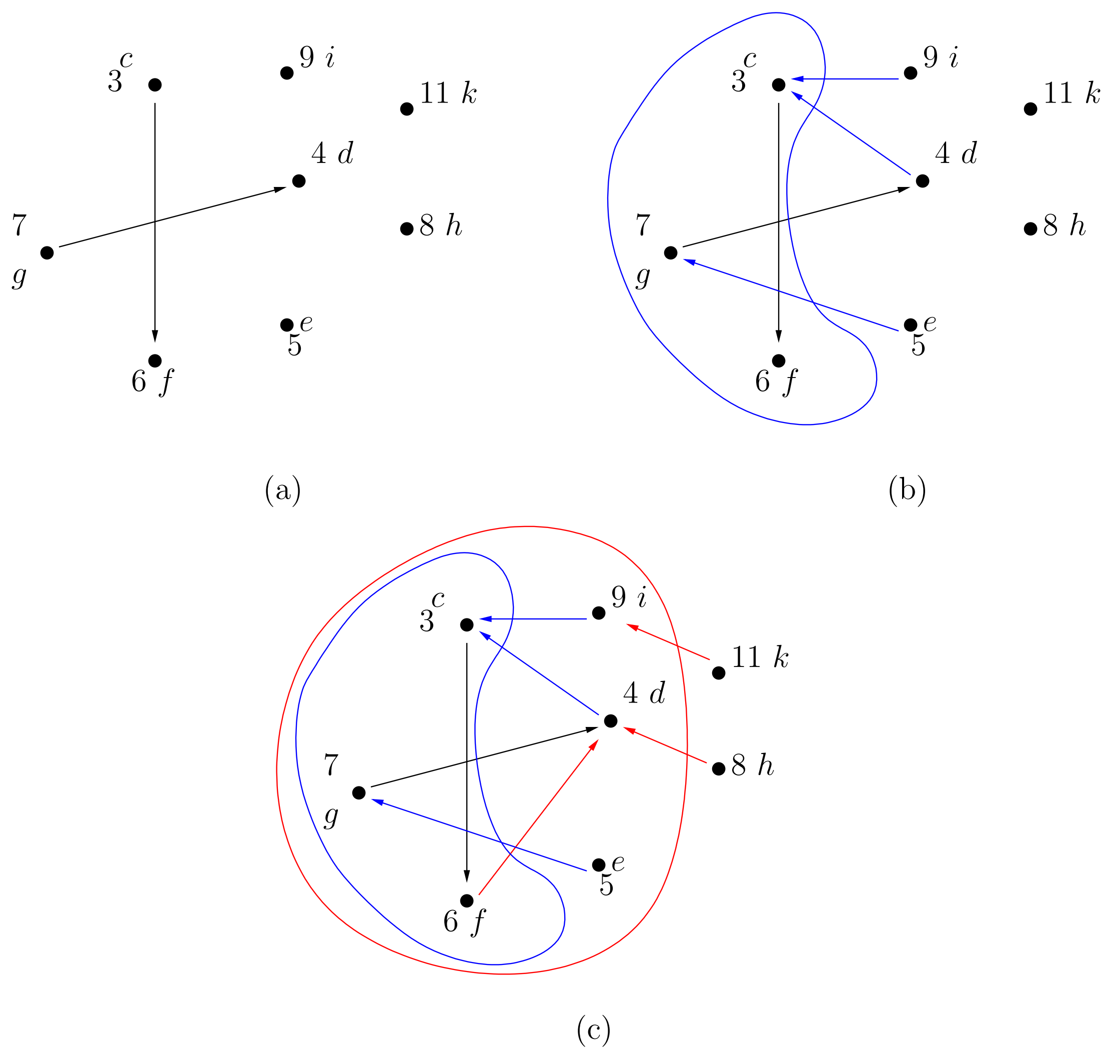


Figure 1: Pointing phase of Step 1: (a) Since 3 and 7 have unique most preferred objects, they point at whoever holds those objects. (b) Next, we consider 4, 9 and 5: those who have a most preferred object that is held by an unsatisfied person in the bubble. (c) Finally, we consider 6, 8, and 11: those who have a most preferred object that is held by a member of the bigger bubble: people who can point at an unsatisfied person.

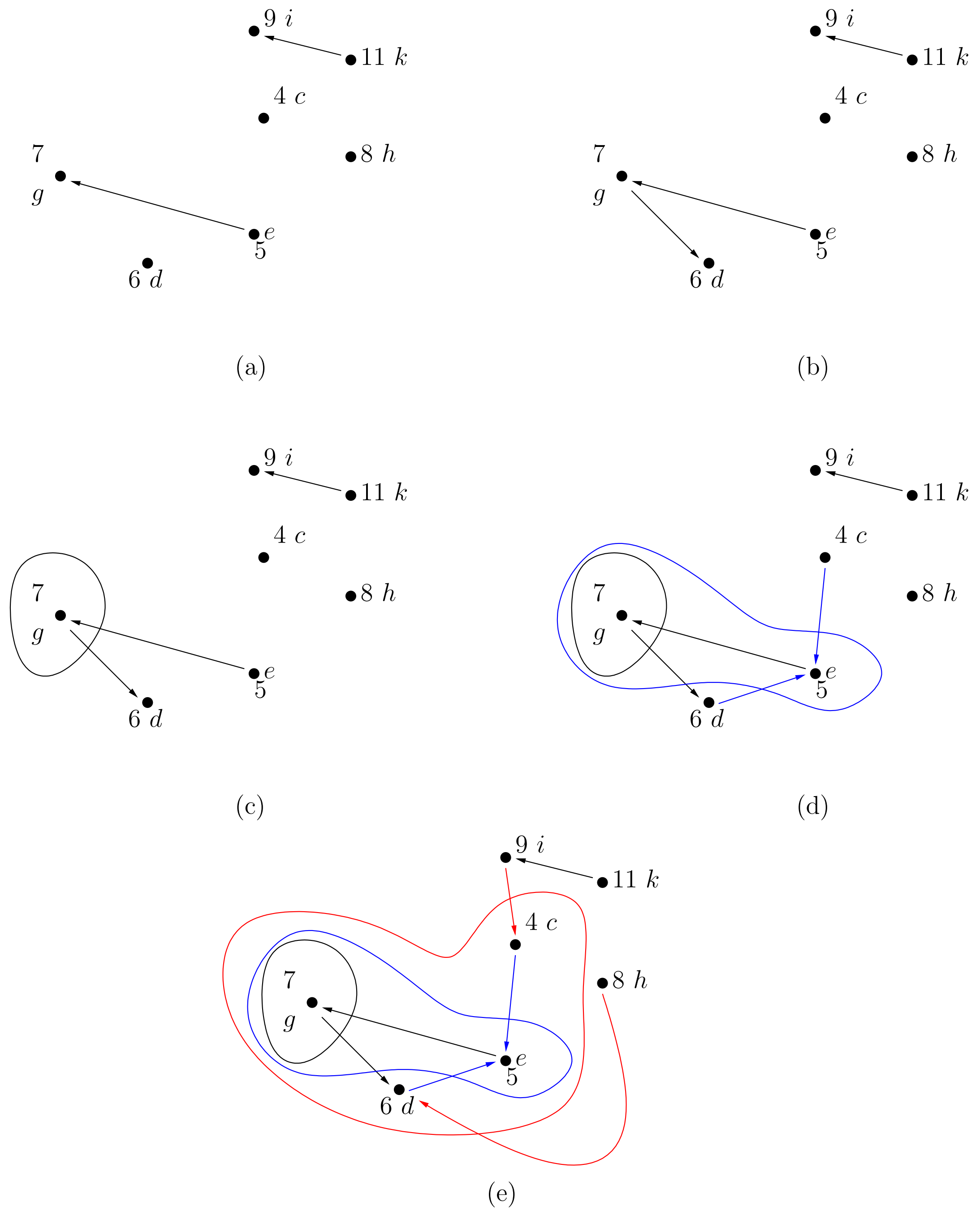


Figure 2: Pointing phase of Step 2: (a) Since 5 pointed at 7 in Step 1 and 7 has not traded, 5 points at 7 in Step 2 as well. Similarly, 11 points at 9 in Step 2 as well. Without Stage 1, 11 would point to 5 in Step 2. (b) 7 is the only person with a unique most preferred object. (c) We now consider people who can point at the only unsatisfied person, 7. However, there is no such person. (d) Next, we consider 4 and 6 who point into the bubble containing 7 and 5. (e) Finally, 8 and 9 point into the biggest bubble.

In the next section, we show that $TC^{\prec}$ is *strategy-proof*, *Pareto-efficient*, and *individually rational*. We also show that $TC^{\prec}$ always picks an allocation from the *weak core*.

When the input preference profile does not involve any indifference, the priority order $\prec$ plays no role in the definition of $TC^{\prec}$ since for each $i \in N$ and each t, $\tau(R_i, O_t)$ is a singleton and $p_t(i)$ is defined in the first two stages of the *pointing phase*. Thus, for each $(P, \omega) \in \mathcal{P}^N \times A$ and each pair of priority orders $\prec$ and $\prec'$, $TC^{\prec}(P, \omega) = TC^{\prec'}(P, \omega)$.

Remark 1. It is natural to ask whether, for each $(R, \omega) \in \mathcal{R}^N \times A$, there is a corresponding problem $(P', \omega) \in \mathcal{P}^N \times A$ such that,

1. For each $i \in N$ and each pair $x, y \in O$, if $x \; P'_i \; y$, then $x \; R_i \; y$, and
2. $TC^{\prec}(R, \omega) = TC^{\prec}(P', \omega)$.

However, this is not the case. Let us go back to Example 1. For each $\prec$ such that $2 \prec 3$, $TC^{\prec}(R, \omega) = (c, a, b)$, and for each $\prec'$ such that $3 \prec' 2$, $TC^{\prec'}(R, \omega) = (b, c, a)$. But $TC^{\prec}(P^1, \omega) = (b, a, c)$ and $TC^{\prec}(P^2, \omega) = (c, b, a)$, neither of which coincides with $TC^{\prec}(R, \omega)$ or $TC^{\prec'}(R, \omega)$. [12] ◦

5 Results

We first show that *strategy-proofness* and *Pareto-efficiency* are incompatible with *anonymity*. We also show that the additional requirement of *individual rationality* leads to an incompatibility with *non-bossiness*. We then show that these incompatibilities are tight by proving that *top cycles* rules satisfy all three of our central axioms. The proofs of these results are in the appendix D.

Proposition 1. *If $N > 2$, no rule is strategy-proof, Pareto-efficient and anonymous.*

Proposition 2. *If $N > 2$, no rule is strategy-proof, Pareto-efficient, individually rational, and non-bossy.*[13]

Next, we show that both Propositions 2 and 1 are tight. When we drop *non-bossiness* or *anonymity* from the list of requirements, the incompatibility does not persist, as evidenced by *top cycles* rules.

By definition *top cycles* rules are not anonymous. To see that they are *bossy*, consider the following example.

---

[12] As evidenced by the above example, $TC^{\prec}(R, \omega)$ need not to be a competitive allocation.

[13] This is a corollary of Theorem 2 in (Bogomolnaia et al. 2005). We provide a direct proof in the appendix.

Example 3. Bossiness of top cycles rules: Let $O = \{a, b, c\}, N = \{1, 2, 3\}, \omega = (a, b, c)$, and $1 \prec 2 \prec 3$. Let $R, R' \in \mathcal{R}^N$ be such that,

R_1	R_2	R_3
$a\ b\ \textcircled{c}$	$\textcircled{a}$	a
	b	$\textcircled{b}$
	c	c

R'_1	R_2	R_3
$\textcircled{c}$	a	$\textcircled{a}$
$a\ b$	$\textcircled{b}$	b
	c	c

.

Then, $TC^{\prec}$ selects the circled allocations above, showing that it is *bossy*. •

Proposition 3. *For each priority order $\prec$, $TC^{\prec}$ is Pareto-efficient and individually rational. That is, for each $(R, \omega) \in \mathcal{R}^N \times A$ and each $\prec$, $TC^{\prec}(R, \omega) \in PE(R) \cap IR(R, \omega)$.*

Proof: By definition of $TC^{\prec}$, it is *individually rational.*

We show that it is *Pareto-efficient* using the conditions of the Departing Phase in the algorithm. Consider the sequence of groups of people who leave at the first step. By condition (i) of the Departure Phase, each member of the first group leaves with one of his most preferred objects. By condition (ii) of the Departure Phase, each of them can be made no better off. By the same reasoning, each member of the second group leaves with one of his most preferred objects after members of the first group have left and can be made no better off without hurting at least one member of the first group. Continuing, each member of a group that in Step 1 leaves with one of his most preferred objects after members of all the previous groups have left and can be made no better off without hurting at least one person who has left.

A similar argument applies to the subsequent steps. Those leaving in later steps can be made no better off without hurting those who have left in prior steps. Thus, $TC^{\prec}$ is *Pareto-efficient.* ■

Proposition 4. *For each priority order $\prec$, $TC^{\prec}$ selects an allocation from the weak core. That is, for each $(R, \omega) \in \mathcal{R}^N \times A$ and each $\prec$, $TC^{\prec}(R, \omega) \in C^W(R, \omega)$.*

Proof: Suppose not. Then, there are $(R, \omega) \in \mathcal{R}^N \times A$ and $S \subseteq N$ *such that* S *blocks* $\alpha \equiv TC^{\prec}(R, \omega)$. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$ and for each $i \in S$, $\beta(i)\ P_i\ \alpha(i)$.

For each t and each $i \in S$, if $\beta(i) \in O_t$, then there is no $j \in N$ such that i points at him in Step t ($i \underset{t}{\not\longrightarrow} j$) and $\beta(i)\ P_i\ h_t(j)$.

Let $\hat{t}$ be the first step at which there is $i \in S$ such that i is part of a trading cycle at the end of step $\hat{t}$. Once i trades, his welfare is determined and he is made neither better nor worse off during the remainder of the algorithm. Thus, he is indifferent between the object that makes his first trade for and the object he ends up with. That is, $h_{\hat{t}+1}(i)\ I_i\ \alpha(i)$. So $\beta(i)\ P_i\ h_{\hat{t}+1}(i)$. This implies that there is $j \in N$ such that i points at j in Step $\hat{t}$ ($i \underset{\hat{t}}{\longrightarrow} j$) and $\beta(i)\ P_i\ h_{\hat{t}}(j) = h_{\hat{t}+1}(i)$. Thus, $\beta(i) \notin O_{\hat{t}}$. However, since $\beta(S) = \omega(S)$, there is $k \in S$ such that $\beta(i) = \omega(k)$ and since $\omega(k) \notin O_{\hat{t}}$, k is part of a trading cycle at some $\tilde{t} < \hat{t}$. This contradicts the definition of $\hat{t}$. ∎

In order to show that for each $\prec$, $TC^{\prec}$ is *strategy-proof*, we make a preliminary remark and state two key lemmas.

For each problem $(R, \omega) \in \mathcal{R}^N \times A$, the "state" of the algorithm at Step t is summarized by the tuple (O_t, N_t, h_{t+1}, p_t). Our remark and lemmas pertain to how these tuples change in response to changes in the input problem. These lemmas provide useful insight into the dynamics of the algorithm.

Remark 2. (Persistence) *If i points at j at Step t, then he points at j as long as j holds the same object. That is, if $i \underset{t}{\longrightarrow} j$, then for every $t' > t$ such that $h_{t'}(j) = h_{t'-1}(j) = h_{t'-2}(j) = ... = h_t(j)$, $i \underset{t'}{\longrightarrow} j$.*

Before we proceed to our first lemma, we introduce some additional notation. Let $(R, \omega) \in \mathcal{R}^N \times A$ and $i \in N$. At the Step t of the algorithm, let the set of people **connected to *i*, $\boldsymbol{CONN(i, R, t)}$**, be those, including i, connected to i via p_t. That is,

$$CONN(i,R,t) = \left\{ j \in N_t : \begin{array}{l} j \equiv i, \text{ or} \\ j \underset{t}{\longrightarrow} i, \text{ or} \\ j \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} i, \text{ or} \\ \dots \end{array} \right\}.$$

Fix $\omega \in A$ and priority order $\prec$. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. Let $R' = (R'_i, R_{-i})$. For each $\hat{t} = 0, 1, \dots$, let $h_{\hat{t}}$ be the *holding vector* at Step $\hat{t}$ of the algorithm for the problem (R, ω). Similarly define $h'_{\hat{t}}$ for the problem (R', ω). We also define, $O_{\hat{t}}, O'_{\hat{t}}$, $N_{\hat{t}}, N'_{\hat{t}}, p_{\hat{t}}, p'_{\hat{t}}, S_{\hat{t}}, S'_{\hat{t}}$, $U_{\hat{t}}$, and $U'_{\hat{t}}$. Finally, for each $i, j \in N$, we indicate $p_{\hat{t}}(i) = j$ by $i \overset{R}{\underset{\hat{t}}{\longrightarrow}} j$ and $p'_{\hat{t}}(i) = j$ by $i \overset{R'}{\underset{\hat{t}}{\longrightarrow}} j$. We also use $i \overset{R}{\underset{\hat{t}}{\not\longrightarrow}} j$ to indicate $p_{\hat{t}}(i) \neq j$.

Let t be the step at which i either leaves or makes his first trade under R. Define t' similarly with respect to R'. Let $\bar{t}$ be the first step at which i is satisfied for exactly one of the two problems if such a number exists, and ∞ otherwise. That is, $i \in S_{\bar{t}}$ and $i \in U'_{\bar{t}}$, $i \in U_{\bar{t}}$ and $i \in S'_{\bar{t}}$, or $\bar{t} = \infty$.[14]

[14] The last case, $\bar{t} = \infty$, only occurs if $t = t'$.

Let $\underline{t} \equiv \min\{t, t', \bar{t}\}$. Then, $\underline{t}$ determines the first period when i is satisfied under R or R'.

Our first lemma states that up to Step $\underline{t}$, there is no difference in the *state* of the algorithm, regardless of whether i reports R_i or R'_i.

Lemma 1. ($\underline{t}$ equality) *At $\underline{t}$, for both R and R', the objects and people remaining, as well as the holding vector and previous step's pointing vector, except for i's component, are the same. That is,*

$$\begin{matrix} O_{\underline{t}} & N_{\underline{t}} & h_{\underline{t}} \\ \shortparallel & \shortparallel & \shortparallel \\ O'_{\underline{t}} & N'_{\underline{t}} & h'_{\underline{t}} \end{matrix} \quad \text{and for each } j, k \in N_{\underline{t}} \text{ such that } j \neq i, \quad \begin{matrix} j \xrightarrow[\underline{t}-1]{R} k \\ \Updownarrow \\ j \xrightarrow[\underline{t}-1]{R'} k. \end{matrix}$$

While the formal proof is in appendix B, Example 4 should help the reader build some intuition with regards to Lemma 1.

Example 4. *Lemma $\underline{t}$-equality.*

Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\omega = (a, b, c, d, e, f, g, h)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
$c\ e$	$a\ b\ c$	e	$b\ d$	$b\ e\ f$	e	$d\ f\ g$	$c\ h$
f		h					

Let $\prec$ be such that $1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 7 \prec 8$. Figure 3. shows the pointing stages of Step 1, 2, and 3. Notice that 1 does not trade until Step 3. Suppose that, instead of R_1, 1 reports R'_1 such that he is unsatisfied. We will show that the departure, pointing, and trading stages remain unaffected until 1 trades under one of the two announcements.

Since 1 is unsatisfied, no one leaves in the departure phase of Step 1 under $R'(\equiv (R'_1, R_{-1}))$. Moreover, each person who points at 1 under R points at him under R'. That is, only 2 points at 1. All but 1 point the same under R and R'. Therefore, all the trading cycles not involving 1 that are realized under R are also realized under R'. If 1 is not part of a cycle (then he does not trade) under R', then the objects, people, and holdings are the same under R and R' at the beginning of Step 2.

If 1 is unsatisfied under R' in Step 2, the departure stage of Step 2 is the same under R and R'. Since 1 is unsatisfied under R and R', people who point at 1 do not change. That is, 2 still points at 1. All but 1 again point in the same way. All the trading cycles not involving 1 are realized. If 1 does not trade under R' then the objects, people, and holdings are the same under R and R' at the beginning of Step 3.

Following the same argument as in the previous steps, we conclude that under R and R' the departure phase is the same. And until 1 trades, the objects, people, and holdings are the same at the beginning of the the next Step. Note that the $\underline{t}$-equality Lemma does not say anything about the pointing phase and trading phase of Step 3 when 1 trades. •

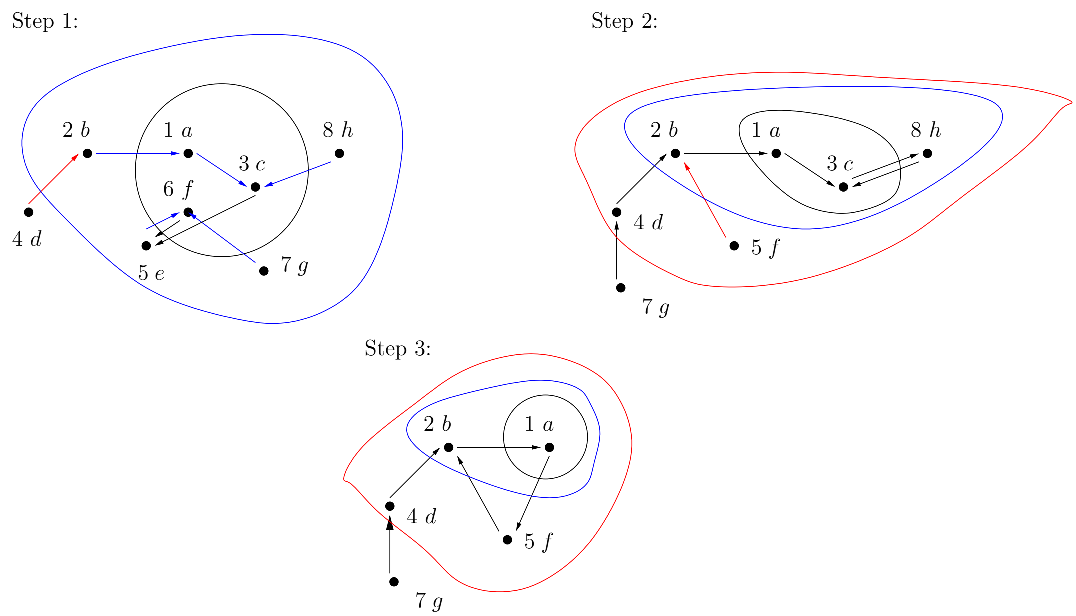


Figure 3: Example of $\underline{t}$-equality Lemma. If 1 reports R'_1 rather than R_1, then each stage in Step 1 and Step 2 remain the same.

For each $R_0 \in \mathcal{R}$, and each $a \in O$, let the **indifference class of $\boldsymbol{a}$ at $\boldsymbol{R_0}$, $\boldsymbol{I(a, R_0)}$**, be

$$I(a, R_0) = \{b \in O \mid b\ I_0\ a\}.$$

Given $R_0 \in \mathcal{R}$, and $a \in O$, let the **local push-up of $\boldsymbol{R_0}$ at $\boldsymbol{a}$, $\boldsymbol{R_0^{a^\uparrow}}$** $\in \mathcal{R}$ be the relation that differs from R_0 *only* in that it ranks a above all objects in $I(a, R_0)$, as shown in Figure 3. That is,

$$R_0|_{O\setminus\{a\}} = R_0^{a^\uparrow}|_{O\setminus\{a\}} \text{ and for each } b \in O \setminus \{a\}, \begin{array}{l} b\ P_0\ a \Rightarrow b\ P_0^{a^\uparrow}\ a \text{ and} \\ a\ R_0\ b \Rightarrow a\ P_0^{a^\uparrow}\ b. \end{array}$$

To prove that $TC^{\prec}$ is *strategy-proof* we will have to consider all possible preference relations that a person can misreport. However, we can split all the available misreports into two categories. The first category includes only preference relations

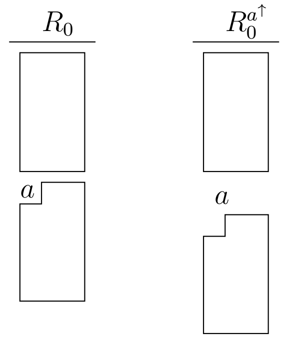


Figure 4: *Local push-up* of a preference relation: Given $R_0 \in \mathcal{R}$ and $a \in O$, the *local push-up of R_0 at a, $R_0^{a\uparrow}$* is as shown above.

under which the person is not indifferent between the object that he is assigned and any other object. The second category consists of all the remaining preference relations. The following lemma implies that for each preference relation in the second category, we can find a preference relation in the first category such that the person is assigned the same object regardless of which of the two preference relations he reports. We use this to prove that $TC^{\prec}$ is *strategy-proof* since it means that we only need to rule out successful misreports from the first category.

Consider the case in which i's preference relation changes from R_i to R'_i. Moreover, R'_i is a *local push-up* of R_i at a. Note that if $\bar{t} < \min\{t, t'\}$, i becomes satisfied in only one of the two problems. Since the only difference between these preference relations is that $I(a, R'_i) \subset I(a, R_i)$, then, by the $\underline{t}$ *equality* lemma, $\tau_i(R', O'_{\bar{t}}) \subset \tau_i(R, O_{\bar{t}})$. Moreover, $i \in S_{\bar{t}}$, $i \in U'_{\bar{t}}$, and $a \in \tau_i(R, O_{\bar{t}}) \cap \tau_i(R', O'_{\bar{t}})$. We use this fact to prove the following lemma. In the proof, we follow the same structure as in the proof of the $\underline{t}$ *equality* lemma. That is, we establish the state of the algorithm between $\bar{t}$ and $\min\{t, t'\}$, and then between t and t'.

Lemma 2. (Invariance) *If the preference relation of a person changes to a local push-up of his original preference at his assignment, then his assignment is unchanged. That is, if $\alpha = TC^{\prec}(R, \omega)$, $R'_i = R_i^{\alpha(i)\uparrow}$, and $\alpha' = TC^{\prec}(R', \omega)$, then $\alpha(i) = \alpha'(i)$.*

Proof: By the $\underline{t}$ *equality* lemma and by the definition of R_i and R'_i, in each step before $\underline{t}$, i points to the same person (holding the same object) under R or R'. Thus, $O_{\underline{t}} = O'_{\underline{t}}$, $N_{\underline{t}} = N'_{\underline{t}}$, $h_{\underline{t}} = h'_{\underline{t}}$, and for each $j \in N_{\underline{t}} \setminus \{i\}$, $p_{\underline{t}-1}(j) = p'_{\underline{t}-1}(j)$. Since R'_i is a *local push-up* of R_i at α_i, $\tau(R'_i, O_{\underline{t}}) = \{\alpha(i)\} \subseteq \tau(R_i, O_{\underline{t}})$.
The $\underline{t}$ *equality* lemma also implies that $CONN(i, R, \underline{t}-1) = CONN(i, R', \underline{t}-1)$.

The rest of the proof proceeds as follows. First we show that at $\min\{t, t'\}$, any person connected to i under R is connected to i under R'. Then, we show that i's

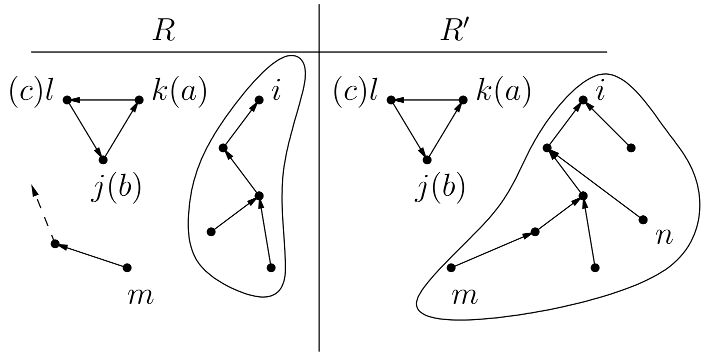


Figure 5: Pre-trade inclusion.

component of the allocation chosen under R is the same as his component of the allocation chosen under allocation under R': $\alpha'(i) = \alpha(i)$.

Claim 1. ***(Pre-trade inclusion)***[15] *For each* $\ddot{t} = \underline{t}, ..., \min\{t, t'\}$,[16]

(i) The objects and people remaining at $\ddot{t}$ *under* R *are a subset of those remaining under* R'*. Further, those remaining under* R' *but not under* R *are connected to* i*. That is,*

$$O_{\ddot{t}} \subseteq O'_{\ddot{t}}, \qquad N_{\ddot{t}} \subseteq N'_{\ddot{t}}$$
$$O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1)), \text{ and } N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1).$$

(ii) Every person who is satisfied at $\ddot{t}$ *under* R' *is satisfied under* R*. Every person who is not satisfied under* R' *but is satisfied under* R *is connected to* i *under* R'*. That is,* $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$.

(iii) Every person not connected to i *at* $\ddot{t}$ *under* R' *points at the same person under* R *as under* R'*. That is, for each* $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

(iv) Every person not connected to i *at* $\ddot{t}$ *under* R' *holds the same object under* R *as under* R'*. That is, for each* $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

(v) The set of people connected to i *under* R *is a subset of the people connected to* i *under* R'*. That is,* $CONN(i, R, \ddot{t}) \subseteq CONN(i, R', \ddot{t})$.

While the formal proof is in appendix C, Example 5 should help the reader build some intuition with regards to Claim 1.

[15] As illustrated in Figure 5.

[16] If $\underline{t} = \min\{t, t'\}$ statements (i) - (v) are implied by the $\underline{t}$ *equality* lemma.

Example 5. *Pre-trade Claim.*

Let $N = \{1,2,3,4,5,6,7,8\}$, $\omega = (a,b,c,d,e,f,g,h)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R'_1
$a\ b\ c$	c	e	e	$a\ d\ e$	$a\ f$	h	$e\ g\ h$	b
		d						$a\ c$

The Pre-trade Claim deals with situations where 1 reports, instead of R_1, a *local push-up* of R_1 at his assignment $TC^{\prec}(R,\omega)(1) = b$, R'_1. The Pre-trade Claim says that at each step before 1 trades under $R'(\equiv (R'_1, R_{-1}))$, the set of people who point at 1 under R is a subset of the set of people who point at 1 under R'. Moreover, trades that do not involve 1 that occur under R occur under R' as well.

In this example, the departure stage of Step 1 is the same under R and R' (figure 6.) In the pointing stage, since 1 is satisfied under R but not under R', more people point at 1 under R' than under R. Under R', 5 points at 1. In addition, note that since 6 points at 1 under R, where 1 is satisfied, he also points at 1 under R', where 1 is unsatisfied. Following this logic we can show that the set of people who point at 1 under R is a subset of the set of people who point at 1 under R'. Moreover, trading cycles that do not involve 1 are the same under R and R'. In this case, 7 points at 8 and 8 points at 7. Therefore, only trades that involve 1 under R might not happen under R'. Since under R' 1 trades in Step 1, the Pre-trade Claim does not say anything about Step 2. •

If $t' \leq t$, by *pre-trade inclusion*, and the $\underline{t}$ *equality* lemma, $O_t \subseteq O'_t$ and $O'_t \subseteq O'_{t'}$. Thus, $\alpha(i) \in O'_{t'}$. Since i is part of a *trading cycle* at Step t' and by definition of R', i points in Step t' at whoever holds $\alpha(i)$ at $\underline{t}$. Then, i is assigned one of his most preferred objects in $O'_{t'}$ which is uniquely $\alpha(i)$. Thus, $\alpha'(i) = \alpha(i)$.

We only need to show that $\alpha(i) = \alpha'(i)$ when $t' > t$. We first state the following claim.

Claim 2. ***(Post-trade inclusion)*** *For each $\ddot{t} \in \{t.., t'\}$,*

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$, $O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i,R',\ddot{t}-1))$, *and* $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i,R',\ddot{t}-1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i,R',\ddot{t}-1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i,R',\ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, *and*

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i,R',\ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

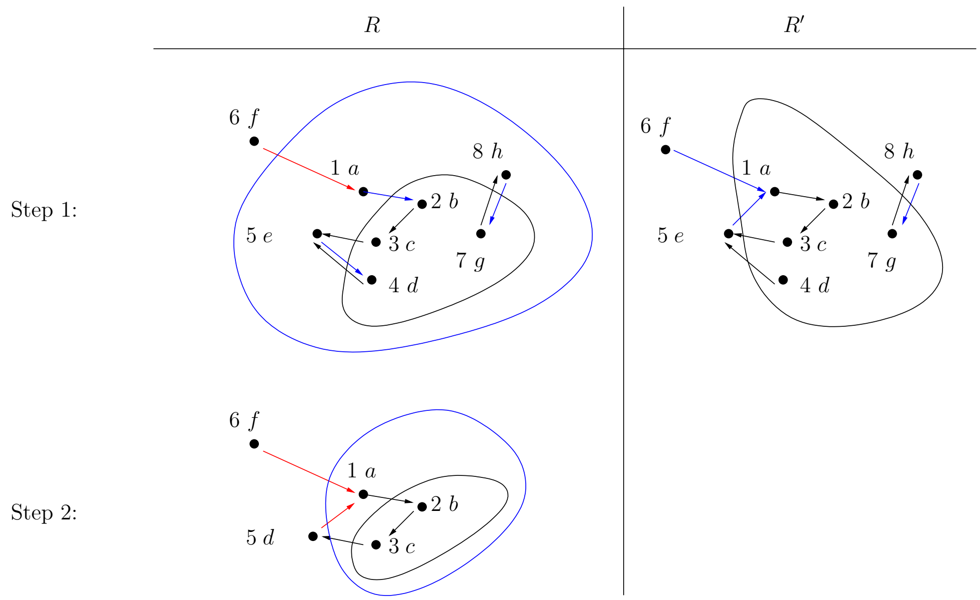


Figure 6: Example of Pre-trade Claim. If 1 reports R'_1 rather than R_1, the set of people who point at 1 has under R' is a superset of those who point at him under R. Moreover, trades that do not involve 1 that are realized under R are also realized under R'.

The proof of this claim is similar to that of *pre-trade inclusion* and is also included in the appendix C, Example 6 should help the reader build some intuition with regards to Claim 2.

Example 6. *Post-trade Claim.*
Let $N = \{1,2,3,4,5,6,7,8\}$, $\omega = (a,b,c,d,e,f,g,h)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R'_1
$b\ c$	$a\ d$	g	$a\ c\ d\ e$	$a\ d\ e\ f$	b	$d\ g\ h$	g	c
		$b\ d$						$a\ b$

Like the Pre-trade Claim, the Post-trade Claim deals with situations where 1 reports R'_1, a *local push-up* of R_1 at his assignment $TC^{\prec}(R,\omega)(1) = c$, rather than R_1 (figure 7.) It provides us with a description of each step between the step at which 1 trades under R and the step that he trades under $R'(\equiv (R'_1, R_{-1}))$. The Post-trade Claim says that before 1 trades under R', the set of people who point at 1 under R is a subset of those who point at 1 under R'. Moreover, trades that do not involve 1 that occur under R also occur under R'. Therefore, the people that trade under R is a superset of those who trade under R'. Finally, since there is less trade under R, the sets of objects and people remaining at a particular step under R are subsets of the objects and people remaining at the same step under R'.

The claim does not say anything about Step 1 because 1 has not traded yet. By the $\underline{t}$-equality lemma, we know that the departure stage of Step 1 is the same under R and R' and in the pointing stage, everyone but 1 points in the same way under R and R'. In this example, 1 trades under R but not under R'. In Step 2, the departure phase is the same. In the pointing stage, since 1 is satisfied under R but not under R', 1 has more people who point at him under R'. Under R', 4 and 5 point at 1. Therefore, the set of people who point at 1 under R is a subset of those who point at 1 under R'. Even though it is not shown in this example, using the same reasoning as in the Pre-trade Claim, trading cycles that do not involve 1 that occur under R also occur under R' and at least as many people trade under R as under R'. Since under R' 1 trades in Step 2, the Post-trade Claim does not say anything about Step 3. •

Suppose $\alpha'(i) \neq \alpha(i)$. Since i is assigned $\alpha(i)$ under R, there is $\tilde{t}$ such that $h_{\tilde{t}+1}(i) = \alpha(i)$. By *post-trade inclusion*, $\tilde{t} < t'$. Since $\tau(R_i', O_{\tilde{t}}) = \{\alpha(i)\}$, we have $i \xrightarrow[\tilde{t}]{R'} j \in N_{\tilde{t}}'$ such that $h_{\tilde{t}}'(j) = \alpha(i)$. Since $\alpha'(i) \neq \alpha(i)$, $h_{\tilde{t}+1}'(i) \neq \alpha(i)$. Thus $j \notin CONN(i, R', \tilde{t}+1)$. Thus by *post-trade inclusion*, $h_{\tilde{t}}(j) = h_{\tilde{t}}'(j) = \alpha(i)$. Since $j \notin CONN(i, R', \tilde{t})$, we have $j \xrightarrow[\tilde{t}]{R'} j_1(\neq i) \xrightarrow[\tilde{t}]{R'} j_2(\neq i) \ldots \xrightarrow[\tilde{t}]{R'} j_r(\neq i)$. Again, by *post-trade inclusion*, $j \xrightarrow[\tilde{t}]{R} j_1(\neq i) \xrightarrow[\tilde{t}]{R} j_2(\neq i) \ldots \xrightarrow[\tilde{t}]{R} j_r(\neq i)$. This contradicts $h_{\tilde{t}}(i) = \alpha(i)$. ♣

We are now ready to show that $TC^{\prec}$ is *strategy-proof.*

Proposition 5. *For priority order $\prec$, $TC^{\prec}(R,\omega)$ is strategy-proof.*

Proof: Suppose that $TC^{\prec}$ is not *strategy-proof.* Then, there is $(R,\omega) \in \mathcal{R}^N \times A$, $i \in N$ and $R_i' \in \mathcal{R}$ such that $TC^{\prec}(R_i', R_{-i}, \omega)(i)\ P_i\ TC^{\prec}(R,\omega)(i)$. Let $\alpha \equiv TC^{\prec}(R,\omega)$ and $\alpha' \equiv TC^{\prec}(R_i', R_{-i}, \omega)$. By the *invariance* lemma, we only need to consider R_i' such that $I(\alpha'(i), R_i) = \{\alpha'(i)\}$. Otherwise, there is $R_i^{\alpha'(i)^{\uparrow}} \in R$ such that $TC^{\prec}(R_i^{\alpha(i)^{\uparrow}}, R_{-i}, \omega)(i) = \alpha'(i)$ and thus, $TC^{\prec}(R_i^{\alpha'(i)^{\uparrow}}, R_{-i}, \omega)(i)\ P_i\ \alpha(i)$.

Define $t, t', \bar{t}$, and $\underline{t}$ as in the proof of the *invariance* lemma. Since $\alpha'(i) \neq \omega(i), \alpha'(i)\ P_i'\ \omega(i)$ and for each $\ddot{t} \leq t'$, $i \in U_{\ddot{t}}'$. We consider the following cases.
Case 1: $\underline{t} = \bar{t} \leq t'$. In this case, $i \in S_{\bar{t}}$. That is, $\omega(i) \in \tau(R_i, O_{\bar{t}})$. By the $\underline{t}$ *equality* lemma, $O_{\bar{t}} = O_{\bar{t}}'$. Since $\alpha'(i) \in O_{\bar{t}}'$, $\alpha'(i) \in O_{\bar{t}}$. Thus, $\omega(i)\ R_i\ \alpha'(i)$ and by *individual rationality*, $\alpha(i)\ R_i\ \alpha'(i)$.

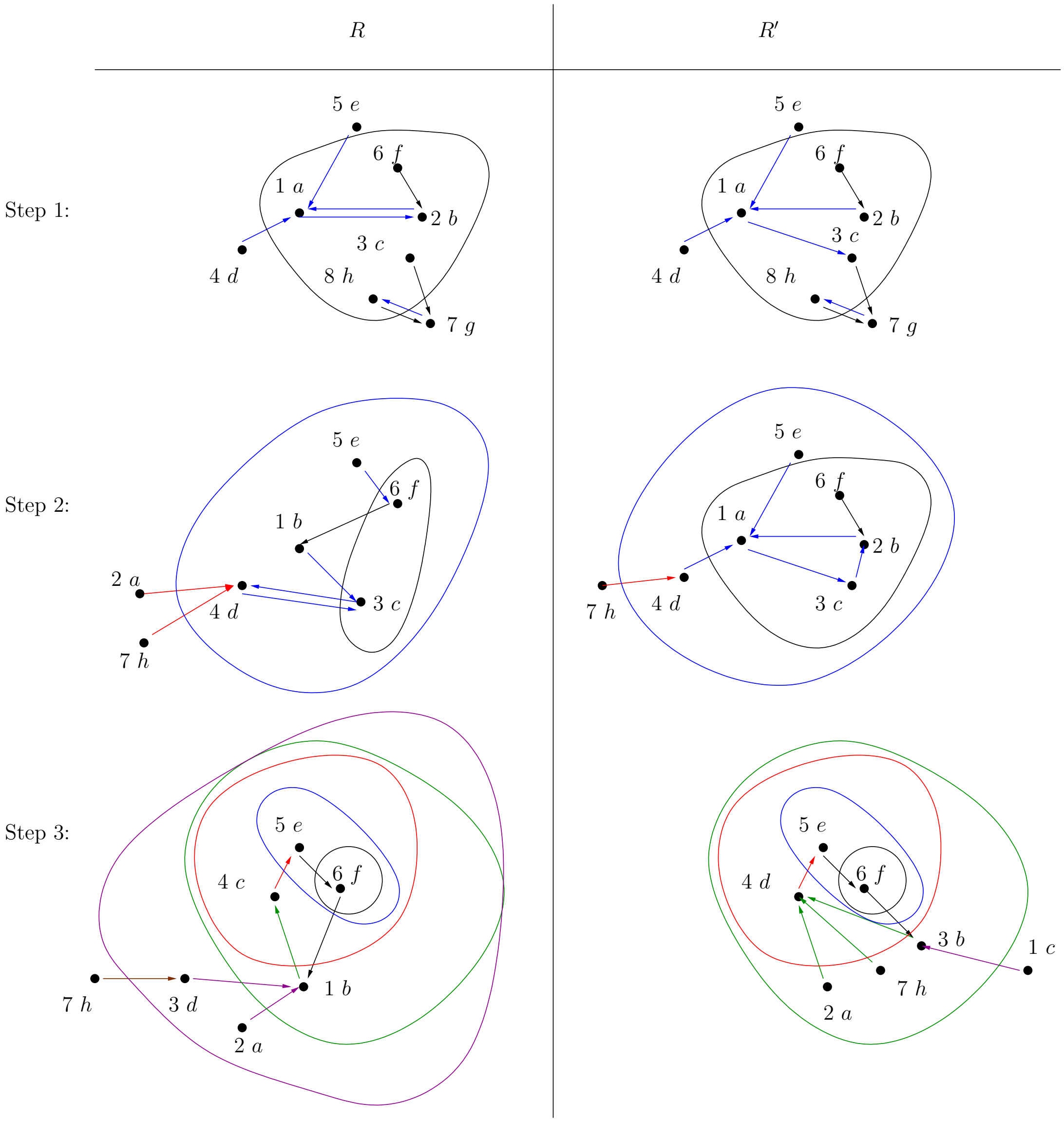


Figure 7: Example of Post-trade Claim. If 1 reports R_1' rather than R_1, the set of people who point at him has under R' includes those who point at him under R. Moreover, trades that do not involve 1 are realized under R are also realized under R'.

Case 2: $\underline{t} = t' < t$. By the $\underline{t}$ *equality* lemma, $O_{t'} = O'_{t'}$, $N_{t'} = N'_{t'}$, and for each $j \in N'_{t'} \setminus \{i\}$, $p_{t'}(j) = p'_{t'}(j)$ and $h_{t'}(j) = h'_{t'}(j)$. Since i trades under R' at t' and by definition of R', $I(\alpha'(i), R_i) = \{\alpha'(i)\}$, then $\{h'_{t'+1}(i)\} = \{\alpha'(i)\} = \tau(R'_i, O'_{t'})$. Then, i leaves with $\alpha'(i)$. Therefore, there is $\{j_1, j_2, \ldots, j_3\} \subseteq N'_{\underline{t}}$ such that $j_1 \xrightarrow[\underline{t}]{R'} j_2 \xrightarrow[\underline{t}]{R'} j_3 \ldots \xrightarrow[\underline{t}]{R'} j_r \xrightarrow[\underline{t}]{R'} i$ and $h'_{\underline{t}}(j_1) = \alpha'(i)$. Then, by the $\underline{t}$ *equality* lemma, $j_1 \xrightarrow[\underline{t}]{R} j_2 \xrightarrow[\underline{t}]{R} j_3 \ldots \xrightarrow[\underline{t}]{R} j_r \xrightarrow[\underline{t}]{R} i$ and $h_{\underline{t}}(j_1) = \alpha'(i)$. By *persistence*, $h_{t+1}(i)\ R_i\ \alpha'(i)$.
Case 3: $\underline{t} = t \leq t'$. Since $h_{t+1}(i) \in \tau(R_i, O_t)$ and $h_{t+1}(i)\ I_i\ \alpha(i)$, $\alpha(i) \in \tau(R_i, O_t)$. Since $\alpha'(i) \in O'_t$ and by the $\underline{t}$ *equality* lemma $O'_t = O_t$ we have $\alpha'(i) \in O_t$. Thus, $\alpha(i)\ R_i\ \alpha'(i)$. ■

6 Generality of our model

In this section, we show that the model that we have studied is general enough to include the problems where there may or may not be a private endowment in addition to a social endowment (Hylland and Zeckhauser 1979, Abdulkadiroğlu and Sönmez 1999).

Let $\tilde{O}$ be a set of objects and $\tilde{N}$ be a set of people. Let $\varnothing \notin \tilde{O}$ be the **null object**. The **private endowment, $\tilde{\omega} : \tilde{N} \to \tilde{O} \cup \{\varnothing\}$**, is such that for each $i, j \in \tilde{N}$, $\tilde{\omega}(i) \neq \tilde{\omega}(j)$ unless $\tilde{\omega}(i) = \varnothing$. Let $\tilde{\mathcal{R}}$ be the set of preference relations over $\tilde{O}$. Let $\tilde{R} \in \tilde{\mathcal{R}}^{\tilde{N}}$. The tuple $(\tilde{O}, \tilde{N}, \tilde{\omega}, \tilde{R})$ defines a problem. We show how this problem can be encoded as a problem in our original model without *social endowments*.

Define (O, N, ω, R) as follows. For each $a \in \tilde{O} \setminus \tilde{\omega}(\tilde{N})$, we introduce i_a, a "dummy person" with degenerate preferences, $R_{i_a} = \overline{I_0}$. For each $i \in \tilde{N}$ such that $\tilde{\omega}(i) = \varnothing$, we introduce d_i, a "dummy object" which every person considers to be worse than any object in $\tilde{O}$. For each person in $\tilde{N}$, his preferences over $\tilde{O}$ are kept the same. That is,

$$\begin{aligned} O &\equiv \tilde{O} \cup \{d_i : \text{for each } i \in \tilde{N} \text{ such that } \tilde{\omega}(i) = \varnothing\}, \\ N &\equiv \tilde{N} \cup \{i_a : \text{for each } a \in \tilde{O} \setminus \tilde{\omega}(\tilde{N})\}, \\ \text{For each } i \in N, \omega(i) &\equiv \begin{cases} \tilde{\omega}(i) & \text{if } i \in \tilde{N} \text{ and } \tilde{\omega}(i) \neq \varnothing \\ d_i & \text{if } i \in \tilde{N} \text{ and } \tilde{\omega}(i) = \varnothing \\ a & \text{if } i = i_a, \text{ and} \end{cases} \end{aligned}$$

$R \in \mathcal{R}^N$ is such that for each $i \in \tilde{N}$, $R_i|_{\tilde{O}} = \tilde{R}_i|_{\tilde{O}}$, and for each $d_j \in O \setminus \tilde{O}$ and each $a \in \tilde{O}, a\ P_i\ d_j$.

When the preferences are strict and the problem includes private endowment, the *top cycles* rule associated with a priority coincides with the *house for turn* rule associated with the same priority.[17] For the same domain of preferences with no public endowment, our family of rules collapses to the *core*. Moreover, when there is no private endowment, the *top cycles* rule associated with a priority coincides with the *serial dictatorship* rule associated with the same priority.

We also point out that the *top cycles rules* described in this paper can be generalized to problems in which each person may be endowed with any number of objects (Pápai 2000).

For school choice problems, the objects that have to be allocated among students are seats at schools. These problems have three important characteristics: First, each seat at a school can be modeled as a copy of the same objects. Second, students are indifferent between seats at the same school, but not between seats at different schools. Finally, schools have weak priorities over students. An adaptation of the *top trading* algorithm can be used in this environment by treating each school's priority as an "inheritance heirarchy"(Pápai 2000, Kesten 2006).

Appendices

A A formal definition of the Top Cycles algorithm

For $t = 0, 1, 2, \ldots$, in Step t, we define the $\boldsymbol{O_t} \subseteq \boldsymbol{O}$, the $\boldsymbol{N_t} \subseteq \boldsymbol{N}$, and $\boldsymbol{h_{t+1} : N_t \to O_t}$. We also define, for each $i \in N_t$, the $\boldsymbol{p_t(i)}$.

Let $O_0 \equiv O$, $N_0 \equiv N$, and $h_1 \equiv \omega$.

At step $t = 1, 2, \ldots$, we get (O_t, N_t, h_{t+1}, p_t) as follows.

Departure phase:

To determine who leaves we use an iterative procedure. Let G_t^1 be the *largest* group in N_{t-1} such that:

i) What each $i \in G_t^1$ holds is among his most preferred objects:

$$h_t(i) \in \tau(R_i, h_t(N_{t-1}))$$

ii) The most preferred objects of the group are hold by them:

$$\tau(R_i, h_t(N_{t-1})) \subseteq h_t(G_t^1)$$

[17] This rule is known in the literature as *You request my house I get your turn* rule (Abdulkadiroğlu and Sönmez 1999).

Among the remaining people, $N_{t-1} \setminus G_t^1$ we determine G_t^2 using the same conditions. We continue until we find $K \in \{1, ..., |N_{t-1}|\}$ such that $G_t^K = \varnothing$ and for each $k < K$, $G_t^k \neq \varnothing$. Then, each $i \in \cup_{k=1}^K G_t^k$, departs with $h_t(i)$. That is, $TC^{\prec}(R, \omega)(i) \equiv h_t(i)$. Further,

$$\begin{array}{rcl} N_t & \equiv & N_{t-1} \setminus \cup_{k=1}^K G_t^k \text{ and} \\ O_t & \equiv & h_t(N_t).^{18} \end{array}$$

Note that the if $K = 1$ the set of people departing is empty, then $N_t = N_{t-1}$ and $O_t = O_{t-1}$.

Pointing phase:

We determine p_t in stages, as follows: For each $i \in N_t$, let i's **candidate pointees,** $\boldsymbol{C_{i,t}} \equiv \{j \in N_t : h_t(j) \in \tau(R_i, O_t)\}$, be the people that hold one of i's most preferred objects.

Stage 1) If $t \neq 1$, we first consider $i \in N_t$ such that i's pointee in Step $t-1$ has not departed and holds the same object as he did at Step $t-1$. Then, i points at the same person in Step t as well. That is, if $t \neq 1$, for each $i \in N_t$ such that $i \underset{t-1}{\longrightarrow} j \in N_t$, and $h_t(j) = h_{t-1}(j)$, we have $i \underset{t}{\longrightarrow} j$.

Stage 2) We consider $i \in N_t$ that has only one candidate pointee. He points at his unique candidate pointee. That is, for each $i \in N_t$ such that $C_{i,t} = \{j\}$, we have $i \underset{t}{\longrightarrow} j$.

Stage 3) We consider $i \in N_t$ with at least one unsatisfied candidate pointee. He points at the unsatisfied candidate pointee with highest priority.[19] That is,

$$p_t(i) \equiv \arg \underset{j \in C_{i,t} \setminus S_t}{\prec\text{-max}}\, j.^{20}$$

Stage 4) We consider $i \in N_t$ with only satisfied candidate pointees, at least one of whom has an unsatisfied pointee, $C_{i,t}^1 \equiv \{j \in C_{i,t} : j \underset{t}{\longrightarrow} U_t\} \subseteq S_t$. Then i points at the satisfied candidate whose unsatisfied pointee has highest priority (breaking ties with respect to $\prec$). That is, $p_t(i) = J_t(i)$, where $J_t(i) \equiv \arg \underset{j \in C_{i,t}^1}{\prec\text{-max}}\, p_t(j)$ and $|J_t(i)| = 1$. In case two or more

[19]The order within a stage is unimportant. In addition, since stages are performed sequentially, if $p_t(i)$ is defined at Stage k, then $p_t(p_t(i))$ is defined at Stage $k' < k$. Further, $p_t(i)$ is independent of $p_t(j)$ if $p_t(j)$ is defined at Stage $k'' \geq k$.

[20]We define $\arg \underset{j \in C_{i,t} \setminus S_t}{\prec\text{-max}}\, j$ as the person that maximizes the priority $\prec$. In general, for each $f : X \rightarrow N$ and each $X' \subseteq X$, we define $\arg \underset{j \in X'}{\prec\text{-max}}\, f(j) \equiv i \in X'$ such that for each $j \in X' \setminus \{i\}, f(i) \prec f(j)$.

of these satisfied candidates points at the unsatisfied pointee with the highest priority ($|J_t(i)| > 1$), i points at the satisfied candidate with the highest priority. That is, $p_t(i) \equiv \arg \underset{j \in J_t(i)}{\prec\text{-max}}\, j$.

Stage 5) We consider $i \in N_t$ whose candidate pointees are all satisfied and have satisfied pointees, at least one of whom has an unsatisfied pointee. He points at the candidate who points at the person who points at the unsatisfied person with highest priority (again, breaking ties with $\prec$). That is,

$$
\begin{aligned}
C^2_{i,t} &\equiv \{j \in C_{i,t} : j \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} U_t\} \subseteq S_t, \\
J_t(i) &\equiv \arg \underset{j \in C^2_{i,t}}{\prec\text{-max}}\, p_t(p_t(j)), \text{ and} \\
p_t(i) &\equiv \arg \underset{j \in J_t(i)}{\prec\text{-max}}\, j.
\end{aligned}
$$

Stage ...) The process is repeated until for each $i \in N_t$, $p_t(i)$ is defined.

By definition of the departure phase, each $i \in N_t$ points, directly or indirectly, at an unsatisfied person.[21] Thus, the pointing phase terminates in a finite number of stages.

Trading phase:

There is at least one cycle $C \equiv \{i_1, i_2, \ldots, i_s\}$ such that $i_1 \underset{t}{\longrightarrow} i_2 \underset{t}{\longrightarrow} \ldots \underset{t}{\longrightarrow} i_s \underset{t}{\longrightarrow} i_1$. Further, each $i \in N_t$ is a member of at most one cycle. We get h_{t+1} by performing the trades prescribed by each cycle. That is, for each cycle, $\{i_1, i_2, \ldots, i_s\}$, and each $k = 1, \ldots s$, $h_{t+1}(i_{k-1}) = h_t(i_k)$. For each $i \in N_t$ who is not in a cycle, $h_{t+1}(i) = h_t(i)$.

The algorithm terminates at Step $\mathring{t}$ such that $N_{\mathring{t}} = \varnothing$.

B Proofs of the $\underline{t}$-equality Lemma

Lemma 1: ($\underline{t}$ equality) *At $\underline{t}$, for both R and R', the objects and people remaining, as well as the holding vector and previous step's pointing vector, except for i's component, are the same. That is,*

$$
\begin{matrix} O_{\underline{t}} & N_{\underline{t}} & h_{\underline{t}} \\ \shortparallel & \shortparallel & \shortparallel \\ O'_{\underline{t}} & N'_{\underline{t}} & h'_{\underline{t}} \end{matrix}
\quad \textit{and for each } j, k \in N_{\underline{t}} \textit{ such that } j \neq i, \quad
\begin{matrix} j \xrightarrow[\underline{t}-1]{R} k \\ \Updownarrow \\ j \xrightarrow[\underline{t}-1]{R'} k. \end{matrix}
$$

[21] This condition implies that each cycle includes at least one unsatisfied person.

Proof: Note that if in Step 1, $i \in S_1$ and $i \in S'_1$, the statement of this Lemma is vacuously satisfied. Then, assume $i \in U_1$ and $i \in U'_1$.

Step 1: Since $i \in U_1$ and $i \in U'_1$, and for each $j \in N \setminus \{i\}$, $R_j = R'_j$ and $h_1(j) = h'_1(j) = \omega(j)$, we have $S_0 = S'_0$. Thus, $O_1 = O'_1$ and $N_1 = N'_1$. Therefore, for each $j \in N_1 \setminus \{i\}$, $p_1(j) = p'_1(j)$.

If $1 < \underline{t}$, i does not trade at Step 1 under either R or R'. Therefore, the cycles formed under p_1 and p'_1 are the same and do not involve i. Then, for each $j \in N_1$, $h_2(j) = h'_2(j)$.

Step 2: Since $i \in U_1$ and $i \in U'_1$, and for each $j \in N_1 \setminus \{i\}, R_j|_{O_1} = R'_j|_{O_1}$ and $h_2(j) = h'_2(j)$, we have $S_1 = S'_1$. Thus, $O_2 = O'_2$ and $N_2 = N'_2$.

As an **induction hypothesis**, suppose that for some $\ddot{t} < \underline{t}$, $O_{\ddot{t}} = O'_{\ddot{t}}$, $N_{\ddot{t}} = N'_{\ddot{t}}$, $h_{\ddot{t}} = h'_{\ddot{t}}$, and for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}-1}(j) = p'_{\ddot{t}-1}(j)$.

Step $\ddot{t}+1$:

We show that for $\ddot{t} < \underline{t}$, $O_{\ddot{t}+1} = O'_{\ddot{t}+1}$, $N_{\ddot{t}+1} = N'_{\ddot{t}+1}$, $h_{\ddot{t}+1} = h'_{\ddot{t}+1}$, and for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

Since $\ddot{t} < \underline{t}$, $i \in U_{\ddot{t}}$ and $i \in U'_{\ddot{t}}$. In addition, by our *induction hypothesis*, $O_{\ddot{t}} = O'_{\ddot{t}}$, $N_{\ddot{t}} = N'_{\ddot{t}}$, $h_{\ddot{t}} = h'_{\ddot{t}}$ and $R_j|_{O_{\ddot{t}}} = R'_j|_{O_{\ddot{t}}}$. Thus, for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

Since $\ddot{t} < \underline{t}$, i does not trade under R or R' at $\ddot{t}$. Therefore, the cycles formed by $p_{\ddot{t}}$ and $p'_{\ddot{t}}$ are the same and do not involve i. Thus, for each $j \in N_{\ddot{t}}$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. Also, $O_{\ddot{t}+1} = O'_{\ddot{t}+1}$ and $N_{\ddot{t}+1} = N'_{\ddot{t}+1}$.

C Proof of the "inclusion" Claims

Claim 1: (Pre-trade inclusion) *For each $\ddot{t} = \underline{t}, ..., \min\{t, t'\}$,*

(i) The objects and people remaining at $\ddot{t}$ under R are a subset of those remaining under R'. Further, those remaining under R' but not under R are connected to i. That is,

$$O_{\ddot{t}} \subseteq O'_{\ddot{t}}, \qquad N_{\ddot{t}} \subseteq N'_{\ddot{t}}$$
$$O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1)), \text{ and } \quad N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1).$$

(ii) Every person who is satisfied at $\ddot{t}$ under R' is satisfied under R. Every person who is not satisfied under R' but is satisfied under R is connected to i under R'. That is, $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$.

(iii) *Every person not connected to i at $\ddot{t}$ under R' points at the same person under R as under R'. That is, for each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.*

(iv) *Every person not connected to i at $\ddot{t}$ under R' holds the same object under R as under R'. That is, for each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.*

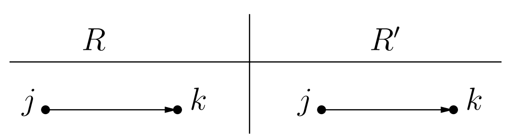
(v) *The set of people connected to i under R is a subset of the people connected to i under R'. That is, $CONN(i, R, \ddot{t}) \subseteq CONN(i, R', \ddot{t})$.*

Proof: Suppose $\underline{t} \neq \min\{t, t'\}$. Then $\underline{t} = \bar{t}$. Since $\tau(R'_i, O_{\bar{t}}) = \{\alpha(i)\}$, $\underline{t} < t'$, and by definition of $\bar{t}$, $i \in U'_{\bar{t}}$ and $i \in S_{\bar{t}}$.

Let $\ddot{t} = \bar{t}$. Statements (i) and (ii), for $\bar{t}$, are implied by the $\underline{t}$ *equality* lemma. Further, $S_{\bar{t}} = S'_{\bar{t}} \cup \{i\}$.

We now prove statement (iii), for $\bar{t}$, by following the progression of the pointing phase.[22] By the $\underline{t}$ *equality* lemma, each $j \in N_{\bar{t}} \setminus \{i\}$ pointed at the same person under R as he did under R' at step $\bar{t} - 1$.

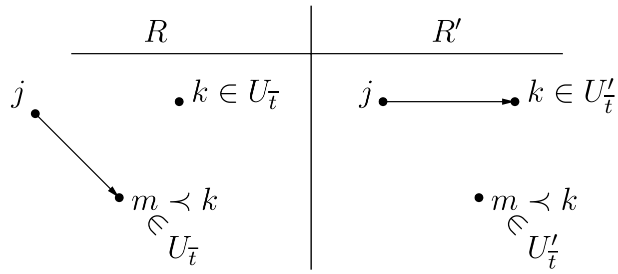
Stage 1) At the beginning of the pointing phase we consider people who were pointing at someone who remains in $N_{\bar{t}}$ and holds the same object. In particular, we consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}-1]{R'} k \in N'_{\bar{t}}$ and $h'_{\bar{t}}(k) = h'_{\bar{t}-1}(k)$. Then, $j \xrightarrow[\bar{t}]{R'} k$. By the $\underline{t}$ *equality* lemma, $j \xrightarrow[\bar{t}-1]{R} k$ and $h_{\bar{t}}(k) = h_{\bar{t}-1}(k) = h'_{\bar{t}-1}(k)$. Thus $j \xrightarrow[\bar{t}]{R} k$.



Stage 2) Now we consider people who have a unique most preferred object. They point at the same person under R as under R'.

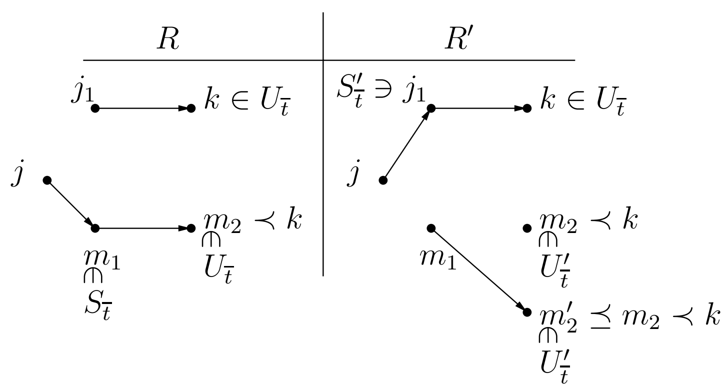
Stage 3) Next, we consider the people who point at unsatisfied people under R'. In particular, $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Since $j \notin CONN(i, R', \bar{t})$, $k \notin CONN(i, R', \bar{t})$. Since $k \in U'_{\bar{t}}$ and $S_{\bar{t}} = S'_{\bar{t}} \cup \{i\}$, $k \in U_{\bar{t}}$. Further, $h_{\bar{t}}(k) = h'_{\bar{t}}(k) = \omega(k)$. Suppose $j \xrightarrow[\bar{t}]{R} m \neq k$. Then, $m \in U_{\bar{t}} \subseteq U'_{\bar{t}}$ and so $h_{\bar{t}}(m) = h'_{\bar{t}}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\bar{t}]{R'} k$.

[22] We provide a graphical illustration of the argument following each stage of the pointing phase.



Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}]{R'} j_1 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Then, by (ii), $j_1 \in S_{\bar{t}}$.

By the preceding arguments, $j_1 \xrightarrow[\bar{t}]{R} k$ and $k \in U_{\bar{t}}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. If $m_1 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1) = \omega(m_1)$ and $m_1 \in U'_{\bar{t}}$. But this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_1 \in S_{\bar{t}}$ and $m_1 \xrightarrow[\bar{t}]{R} m_2$ such that $m_2 \in U_{\bar{t}}$ and $m_2 \prec k$. Then, $m_2 \in U'_{\bar{t}}$ and thus $m_1 \xrightarrow[\bar{t}]{R'} m'_2 \in U'_{\bar{t}}$ such that $m'_2 \preceq m_2 \prec k$. [23] By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1)$. This contradicts $j \xrightarrow[\bar{t}]{R'} j_1$.

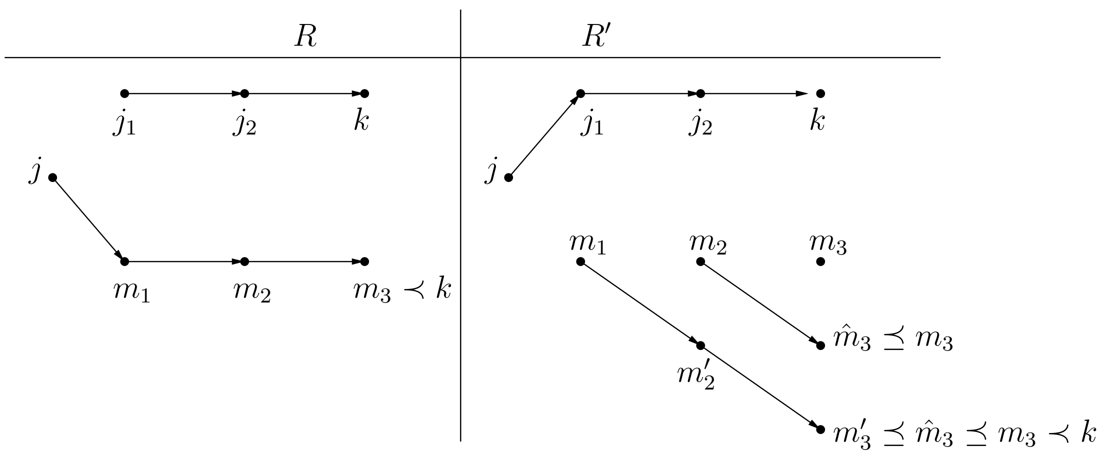


Stage 5) Now we consider the people who point at satisfied people with satisfied pointees whose pointees are unsatisfied, under R'. Particularly, consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ be such that $j \xrightarrow[\bar{t}]{R'} j_1 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} j_2 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Then, $j_1, j_2 \in S_{\bar{t}}$.

By the preceding arguments, $j_1 \xrightarrow[\bar{t}]{R} j_2 \xrightarrow[\bar{t}]{R} k \in U_{\bar{t}}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. If $m_1 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1) = \omega(m_1)$ and $m_1 \in U'_{\bar{t}}$. But this contra-

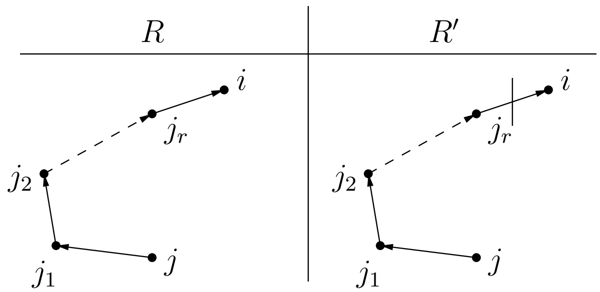
[23] We use the notation $i \preceq j$ to indicate $i \prec j$ or $i = j$.

dicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_1 \in S_{\bar{t}}$. By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1)$. Let $m_1 \xrightarrow[\bar{t}]{R} m_2$. If $m_2 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_2) = h'_{\bar{t}}(m_2) = \omega(m_2)$ and $m_2 \in U'_{\bar{t}}$. So $m_1 \xrightarrow[\bar{t}]{R'} U'_{\bar{t}}$. But this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_2 \in S_{\bar{t}}$. By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_2) = h'_{\bar{t}}(m_2)$. Since $k \in U_{\bar{t}}$, $m_2 \xrightarrow[\bar{t}]{R} m_3 \in U_{\bar{t}}$ and $m_3 \prec k$. Then, $m_3 \in U'_{\bar{t}}$ and so $m_2 \xrightarrow[\bar{t}]{R'} \hat{m}_3 \in U'_{\bar{t}}$ such that $\hat{m}_3 \preceq m_3 \prec k$. If $m_1 \xrightarrow[\bar{t}]{R'} m_2$, this contradicts $j \xrightarrow[\bar{t}]{R'} j_1$. Then $m_1 \xrightarrow[\bar{t}]{R'} m'_2 \neq m_2$ and $m'_2 \xrightarrow[\bar{t}]{R'} m'_3$. Note that $m'_2 \in S'_{\bar{t}}$, otherwise this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. In addition, since $m_1 \not\xrightarrow[\bar{t}]{R'} m_2$ and $m_1 \xrightarrow[\bar{t}]{R'} m'_2$, we have $m'_3 \in U'_{\bar{t}}$ and $m'_3 \prec \hat{m}_3$. Then, $m_3 \prec k$, which contradicts $j \xrightarrow[\bar{t}]{R'} j_1$.



Stage ...) Repeating this argument for the rest of the pointing phase, we show (iii).

We show that (v) $CONN(i, R, \bar{t}) \subseteq CONN(i, R', \bar{t})$ is a consequence of (iii). To see this, suppose $j \in CONN(i, R, \bar{t}) \setminus CONN(i, R', \bar{t})$. Then, there is a sequence $\{j_1, j_2, ..., j_r, i\} \subset N_{\bar{t}}$, such that $j \xrightarrow[\bar{t}]{R} j_1 \xrightarrow[\bar{t}]{R} j_2 \xrightarrow[\bar{t}]{R} ... \xrightarrow[\bar{t}]{R} j_r \xrightarrow[\bar{t}]{R} i$. Since $j \notin CONN(i, R', \bar{t})$, then by (iii), $j \xrightarrow[\bar{t}]{R'} j_1$. Then, $j_1 \notin CONN(i, R', \bar{t})$. Again, by (iii), $j_1 \xrightarrow[\bar{t}]{R'} j_2$ and $j_2 \notin CONN(i, R', \bar{t})$. Repeating the argument r times, $j_r \notin CONN(i, R', \bar{t})$. By (iii), $j_r \xrightarrow[\bar{t}]{R'} i$, and this contradicts $j \notin CONN(i, R', \bar{t})$.



Finally, we prove (iv) for Step $\bar{t}$. We show that for each $j \in N'_{\bar{t}} \backslash CONN(i, R', \bar{t})$, $h_{\bar{t}+1}(j) = h'_{\bar{t}+1}(j)$. Note that since at $\bar{t} < t'$, i does not trade. Then, no *trading cycle* under R' involves i. So, no *trading cycle* involves any member of $CONN(i, R', \bar{t})$. That is, for each *trading cycle* $C' \subset N'_{\bar{t}}$, $CONN(i, R', \bar{t}) \cap C' = \varnothing$. By (iii) and since $N_{\bar{t}} = N'_{t''}$, we have $C' \subset N_{\bar{t}}$ is also a *trading cycle* under R. Therefore, for each $j \in N'_{\bar{t}} \backslash CONN(i, R', \bar{t})$, $h'_{\bar{t}+1}(j) = h_{\bar{t}+1}(j)$. Moreover, for each $j \in CONN(i, R', \bar{t})$, $h'_{\bar{t}+1}(j) = h'_{\bar{t}}(j)$.

As an **induction hypothesis**, suppose that for some $\ddot{t} \in \{\underline{t}, ..., \min\{t, t'\} - 1\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$
$O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t} - 1))$, and $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t} - 1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t} - 1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$,

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$, and

(v) $CONN(i, R, \ddot{t}) \subseteq CONN(i, R', \ddot{t})$.

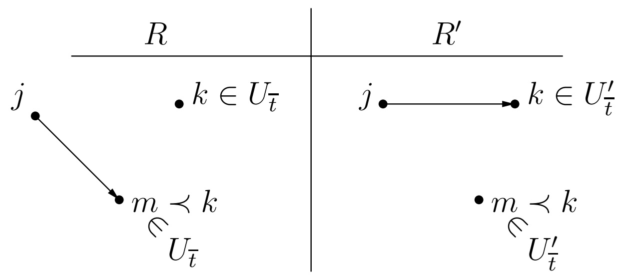
We prove that these statements are true of $\ddot{t}+1$. To prove (i) and (ii) for $\ddot{t}+1$, note that by (iv) and (v) of the *induction hypothesis*, if $C \in N_{\ddot{t}}$ is a *trading cycle* under R and is not a *trading cycle* under R', then $C \subseteq CONN(i, R', \ddot{t})$. Thus, at Step $\ddot{t}+1$, we have statements (i) and (ii).

We now prove (iii), for $\ddot{t}+1$, by following the progression of the pointing phase just as in the case of $\bar{t}$.

Stage 1) We consider people whose pointee at $\ddot{t}$ remains at $\ddot{t}+1$ and holds the same object under R as R'. In particular, we consider $j \in N'_{\ddot{t}+1} \backslash CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}]{R'} k \in N'_{\ddot{t}}$ and $h'_{\ddot{t}+1}(k) = h'_{\ddot{t}}(k)$. Then, $j \xrightarrow[\ddot{t}+1]{R'} k$. By the *induction hypothesis*, $j \xrightarrow[\ddot{t}]{R} k$ and $h_{\ddot{t}+1}(k) = h_{\ddot{t}}(k) = h'_{\ddot{t}}(k)$. Thus $j \xrightarrow[\ddot{t}+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, if $\tau(R_j, O'_{\ddot{t}+1}) = \{a\}$, then by the *induction hypothesis*, $h^{-1}_{\ddot{t}+1}(a) = h'^{-1}_{\ddot{t}+1}(a) \notin CONN(i, R', \ddot{t}+1)$. Thus, $a \in O_{\ddot{t}+1}$ and so $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}}$. Since $j \notin CONN(i, R', \ddot{t}+1)$, $k \notin CONN(i, R', \ddot{t}+1)$. Since $k \in U'_{\ddot{t}+1}$ and $S_{\ddot{t}+1} \setminus S'_{\ddot{t}+1} \subseteq CONN(i, R', \ddot{t}+1)$, $k \in U_{\ddot{t}+1}$. Further, $h_{\ddot{t}+1}(k) = h'_{\ddot{t}+1}(k) = \omega(k)$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m \neq k$. Then, $m \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and so $h_{\ddot{t}+1}(m) = h'_{\ddot{t}+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} k$.



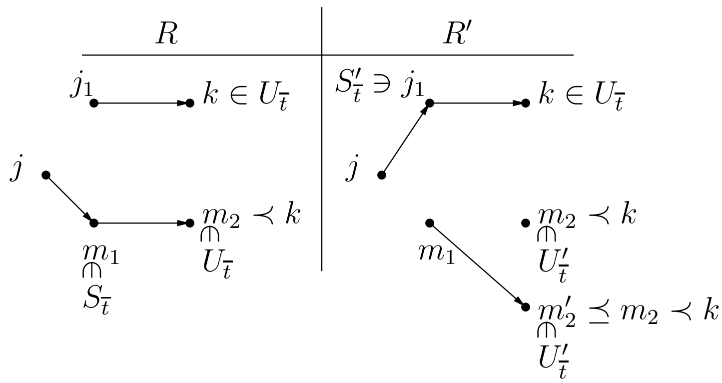
Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, by (ii), $j_1 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} k$ and $k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$. We consider the following two cases.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$ and $k \in U_{\ddot{t}+1}$, then $m_2 \in U_{\ddot{t}+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{\ddot{t}+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and $m_1 \prec j_1]$. Since $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have $m_1 \not\xrightarrow[\ddot{t}+1]{R'} m_2$. Let $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$. Since

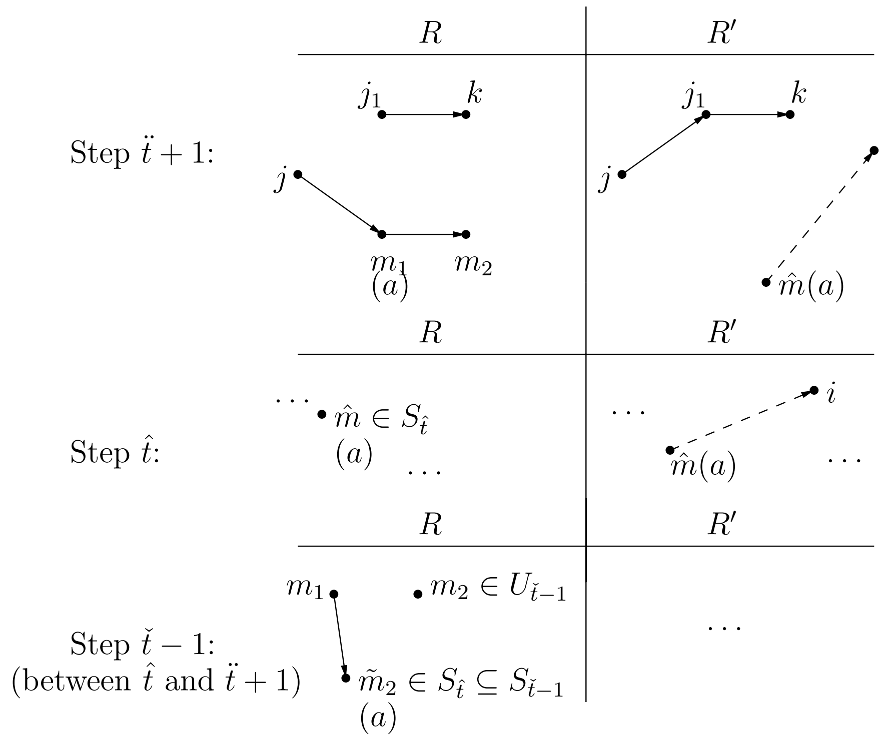
$m_2 \in U'_{\ddot{t}+1}$, we have $m'_2 \in U'_{\ddot{t}+1}$ and $m'_2 \prec m_2$. Then, $m'_2 \prec k$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.



$\boldsymbol{h'_{\ddot{t}+1}(m_1) \not\asymp h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in CONN(i, R', \ddot{t})$. Thus, $m_1 \in CONN(i, R', \ddot{t}+1)$. Further, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$, we have $m_2 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[\ddot{t}+1]{R'} \hat{m}$, $\hat{m} \in S'_{\ddot{t}+1}$. Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Then, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.



Stage 5) Next we consider the people who point at satisfied people whose pointees satisfied and have unsatisfied pointees, under R'. Particularly, consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ be such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} j_2 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, $j_1, j_2 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. Let $m_1 \xrightarrow[\ddot{t}+1]{R} m_2 \xrightarrow[\ddot{t}+1]{R} m_3$. We consider the following cases.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

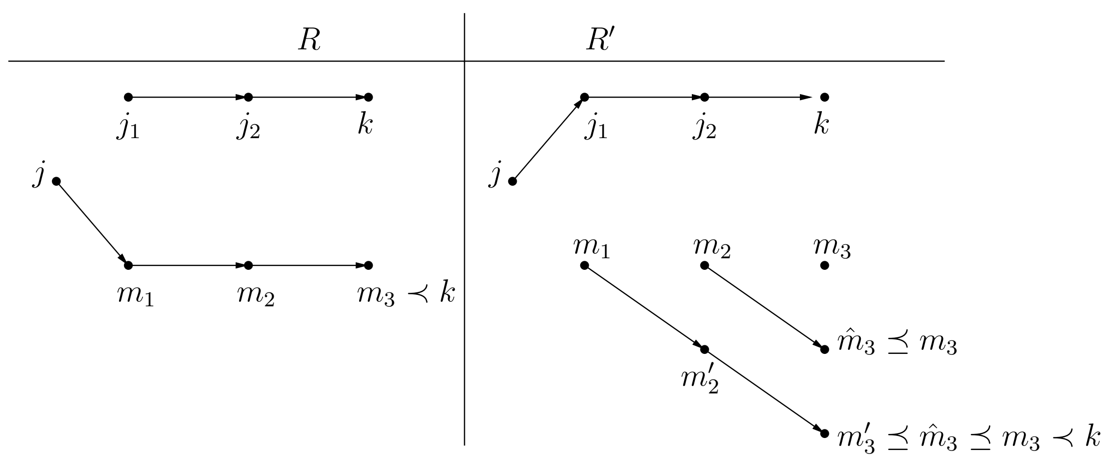
Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Two sub-cases are as follows:

$\boldsymbol{h'_{\ddot{t}+1}(m_2) = h_{\ddot{t}+1}(m_2)}$: If $m_2 \in U_{\ddot{t}+1}$, then $m_2 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_2) = h'_{\ddot{t}+1}(m_2) = \omega(m_2)$. Then, $m_1 \xrightarrow[\ddot{t}+1]{R} U'_{\ddot{t}+1}$ and $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_2 \in S_{\ddot{t}+1}$.

Since $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$, $m_3 \in U_{\ddot{t}+1}$. Further, $m_3 \in U'_{\ddot{t}+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, then either,

(a) $m_1 \xrightarrow[\ddot{t}+1]{R'} m_2 \xrightarrow[\ddot{t}+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{\ddot{t}+1}$, $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

(b) $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[\ddot{t}+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{\ddot{t}+1}$. Suppose $m_2 \xrightarrow[\ddot{t}+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[\ddot{t}+1]{R'} m'_3$. Since $m_3 \in U'_{\ddot{t}+1}$, $\hat{m}_3 \in U'_{\ddot{t}+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{\ddot{t}+1}$, we have $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \preceq \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.



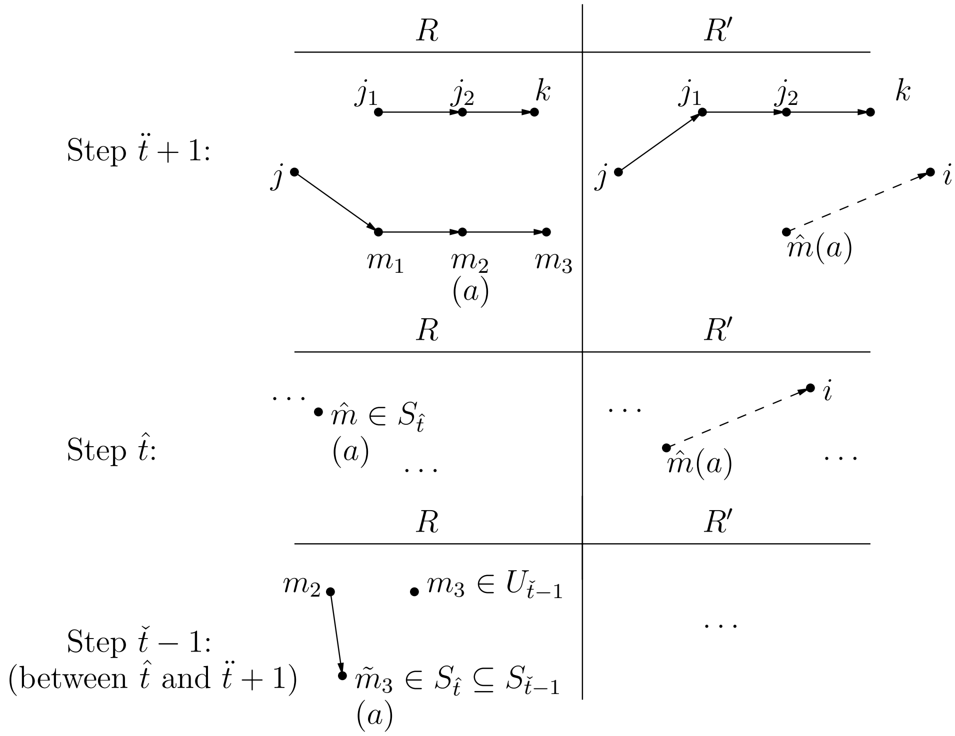
$\boldsymbol{h'_{\ddot{t}+1}(m_2) \neq h_{\ddot{t}+1}(m_2)}$: Let $a \equiv h_{\ddot{t}+1}(m_2)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_2) \neq a$, we have $m_2 \in S_{\ddot{t}+1}$. Since $j \xrightarrow[\ddot{t}]{R} m_1$, $m_3 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$.

Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$ and by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+ 1)$.

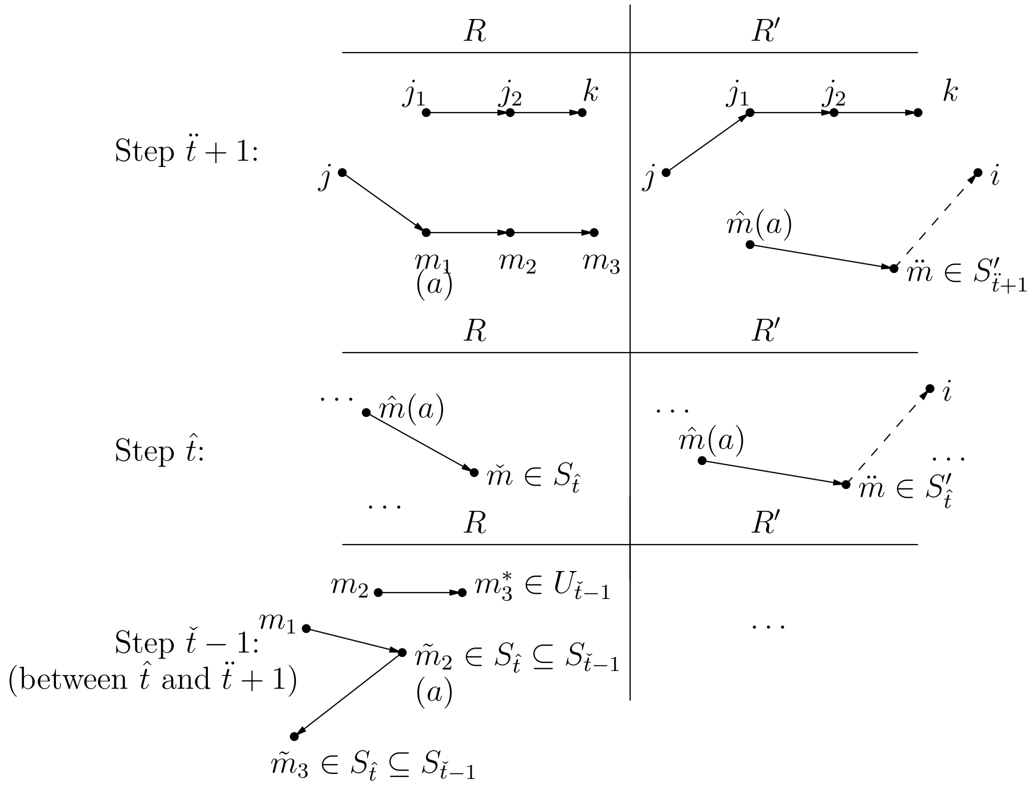
Since $a \; I_{m_1} \; h_{\ddot{t}+1}(m_1)$, and $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have that $m_1 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$, and $\hat{m} \in S'_{\ddot{t}+1}$.

Since $h_{\ddot{t}+1}(\hat{m}) \neq a$ and since there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now consider the first $\check t$, which is between $\hat t$ and $\ddot t+1$, such that $h_{\check t}(m_2) = a$. Then, $m_2 \xrightarrow[\check t-1]{R} S_{\check t-1}$ which contradicts $m_3 \in U_{\ddot t+1} \subseteq U_{\check t-1}$.



$\boldsymbol{h'_{\ddot t+1}(m_1) \not\le h_{\ddot t+1}(m_1)}$: Let $a \equiv h_{\ddot t+1}(m_1)$. Since $h'_{\ddot t+1}(m_1) \neq a$, $m_1 \in S_{\ddot t+1}$. Since $O_{\ddot t+1} \subseteq O'_{\ddot t+1}$, there is $\hat m \in N'_{\ddot t+1}$ such that $h'_{\ddot t+1}(\hat m) = a$. Since $j \xrightarrow[\ddot t+1]{R'} j_1$, we have that $\hat m \in S'_{\ddot t+1}$ and $\hat m \xrightarrow[\ddot t+1]{R'} S'_{\ddot t+1}$. Since $h_{\ddot t+1}(\hat m) \neq a$, by the *induction hypothesis*, $\hat m \in CONN(i, R', \ddot t+1)$ and there is a first $\hat t$ such that $\hat m \in CONN(i, R', \hat t)$. Since $\hat m \in S'_{\ddot t+1}$ and $p'_{\ddot t+1}(\hat m) = p'_{\hat t}(\hat m)$, we have that $\hat m \in S'_{\hat t}$. This implies that $\hat m \in S_{\hat t}$ and $h_{\hat t}(\hat m) = a$. Since $\hat m \xrightarrow[\hat t]{R'} S'_{\hat t}$, then $\hat m \xrightarrow[\hat t]{R} S_{\hat t}$. And for each $\hat{\hat t} > \hat t$, we have $\hat m \xrightarrow[\hat{\hat t}]{R} S_{\hat{\hat t}}$. Now consider the first $\check t$, which is between $\hat t$ and $\ddot t+1$, such that $h_{\check t}(m_1) = a$. Then, $m_1 \xrightarrow[\check t-1]{R} S_{\check t-1} \xrightarrow[\check t-1]{R} S_{\check t-1}$. However, if $m_2 \in U_{\ddot t+1} \subseteq U_{\check t}$, then $m_1 \xrightarrow[\ddot t+1]{R} U_{\ddot t+1} \subseteq U_{\check t}$ and if $m_2 \in S_{\ddot t+1}$, then $m_3 \in U_{\ddot t+1}$ and $m_1 \xrightarrow[\ddot t+1]{R} S_{\ddot t+1} \xrightarrow[\ddot t+1]{R} U_{\ddot t+1}$. In either case, we have reached a contradiction.



Stage ...) Repeating this argument for the rest of the pointing phase we show (iii).

Now, we prove (v) for $\ddot{t}+1$. Suppose $j \in CONN(i, R, \ddot{t}+1) \setminus CONN(i, R', \ddot{t}+1)$. Then, there is $\{j_1, j_2, ..., j_r, i\} \subset N_{\ddot{t}+1} \subseteq N'_{\ddot{t}+1}$, such that $j \xrightarrow[\ddot{t}+1]{R} j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} ... \xrightarrow[\ddot{t}+1]{R} j_r \xrightarrow[\ddot{t}+1]{R} i$. Since $j \notin CONN(i, R', \ddot{t}+1)$, by (iii), $j \xrightarrow[\ddot{t}+1]{R'} j_1$. Then, $j_1 \notin CONN(i, R', \ddot{t}+1)$. Again, by (iii), $j_1 \xrightarrow[\ddot{t}+1]{R'} j_2$ and $j_2 \notin CONN(i, R', \ddot{t}+1)$. Repeating the argument r times, $j_r \notin CONN(i, R', \ddot{t}+1)$. By (iii), $j_r \xrightarrow[\ddot{t}+1]{R'} i$, and this contradicts $j \notin CONN(i, R', \ddot{t}+1)$.

Finally, we prove (iv) for Step $\ddot{t}+1$. We show that for each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. By (iii) each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h_{\ddot{t}+2}(j)$. Moreover, for each $j \in CONN(i, R', \ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h'_{\ddot{t}+1}(j)$. $\diamond$

Claim 2: (Post-trade inclusion)

For each $\ddot{t} \in \{t.., t'\}$,

(i) $\begin{array}{ll} O_{\ddot{t}} \subseteq O'_{\ddot{t}}, & N_{\ddot{t}} \subseteq N'_{\ddot{t}}, \\ O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1)),\ and & N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1) \end{array}$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(iii) *For each* $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, *and*

(iv) *For each* $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

Proof: Let $\ddot{t} = t + 1$. First, we prove statements (i) and (ii) for $t + 1$. At t, i is a member of a *trading cycle* under R, but not under R'. By *pre-trade inclusion*, each trading cycle that does not involve people connected to i under R' is also a trading cycle under R. In addition, for each $j \in N'_t \setminus CONN(i, R', t)$, $h_{t+1}(j) = h'_{t+1}(j)$. Thus, if $C \in N_t$ is a trading cycle under R but not under R', then $C \subset CONN(i, R', t)$. Therefore, at Step $t+1$, $O_{t+1} \subset O'_{t+1}$, $O'_{t+1} \setminus O_{t+1} \subseteq h_{t+1}(CONN(i, R', t))$, $N_{t+1} \subset N'_{t+1}$, and $N'_{t+1} \setminus N_{t+1} \subseteq CONN(i, R', t)$. Further, $S'_{t+1} \subset S_{t+1}$ and $S_{t+1} \setminus S'_{t+1} \subseteq CONN(i, R', t)$.

We now prove (iii) for $t+1$, by following the progression of the pointing phase.

Stage 1) We first consider people whos pointee in Step t remains in N'_{t+1} and holds the same object. In particular, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t]{R'} k \in N'_{t+1}$ and $h'_{t+1}(k) = h'_t(k)$. Then, $j \xrightarrow[t+1]{R'} k$. By *pre-trade inclusion*, $j \xrightarrow[t]{R} k$ and $h_{t+1}(k) = h_t(k) = h'_t(k)$. Thus, by (ii), $j \xrightarrow[t+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{t+1} \setminus CONN(i, R', t+1)$, if $\tau(R_j, O'_{t+1}) = \{a\}$, then by *pre-trade inclusion*, $h^{-1}_{t+1}(a) = h'^{-1}_{t+1}(a) \notin CONN(i, R', t+1)$. Thus, $a \in O_{t+1}$ and so $p_{t+1}(j) = p'_{t+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t+1]{R'} k \in U'_{t+1}$. Since $j \notin CONN(i, R', t+1)$, $k \notin CONN(i, R', t+1)$. Since $k \in U'_{t+1}$ and $S_{t+1} \setminus S'_{t+1} \subset CONN(i, R', t+1)$, $k \in U_{t+1}$. Further, $h_{t+1}(k) = h'_{t+1}(k) = \omega(k)$. Suppose $j \xrightarrow[t+1]{R} m \neq k$. Then, $m \in U_{t+1} \subset U'_{t+1}$ and so $h_{t+1}(m) = h'_{t+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[t+1]{R'} k$.

Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1} \xrightarrow[t+1]{R'} k \in U'_{t+1}$. Then, by (ii), $j_1 \in S_{t+1}$.

By the preceding arguments, $j_1 \xrightarrow[t+1]{R} k$ and $k \in U_{t+1}$. Suppose $j \xrightarrow[t+1]{R} m_1 \neq j_1$. We consider the following cases.

$\boldsymbol{h'_{t+1}(m_1)} = \boldsymbol{h_{t+1}(m_1)}$: If $m_1 \in U_{t+1}$, then $m_1 \in U'_{t+1}$ and $h_{t+1}(m_1) = h'_{t+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[t+1]{R'} m_1$, which contradicts $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1}$. Thus, $m_1 \in S_{t+1}$. Suppose $m_1 \xrightarrow[t+1]{R} m_2$. Since $j \xrightarrow[t+1]{R} m_1$ and $k \in U_{t+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{t+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and $m_1 \prec j_1]$. Since $j \not\xrightarrow[t+1]{R'} m_1$, we have $m_1 \not\xrightarrow[t+1]{R'} m_2$. Let $m_1 \xrightarrow[t+1]{R'} m'_2$. Since $m_2 \in U'_{t+1}$, we have $m'_2 \notin S'_{t+1}$ and $m'_2 \prec m_2$. Then, $m'_2 \prec k$, which contradicts $j \xrightarrow[t+1]{R'} j_1$.

$\boldsymbol{h'_{t+1}(m_1)} \neq \boldsymbol{h_{t+1}(m_1)}$: Let $a \equiv h_{t+1}(m_1)$. By *pre-trade inclusion*, since $h'_{t+1}(m_1) \neq a$, $m_1 \in CONN(i, R', t)$. Thus, $m_1 \in CONN(i, R', t+1)$. Further, $m_1 \in S_{t+1}$. Since $O_{t+1} \subseteq O'_{t+1}$, there is $\hat{m} \in N'_{t+1}$ such that $h'_{t+1}(\hat{m}) = a$. Suppose $m_1 \xrightarrow[t+1]{R} m_2$. Since $j \xrightarrow[t+1]{R} m_1$, we have $m_2 \in U_{t+1} \subseteq U'_{t+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[t+1]{R'} \hat{m}$, $\hat{m} \in S'_{t+1}$.

Since $h_{t+1}(\hat{m}) \neq a$, by *pre-trade inclusion*, $\hat{m} \in CONN(i, R', t+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $t+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$, which contradicts $m_2 \in U_{t+1} \subseteq U_{\check{t}-1}$.

Stage 5) Now we consider the people who point at satisfied people whose pointees are satisfied people with unsatisfied pointees under R'. Particularly, we consider $j \in N'_{t+1} \setminus CONN(i, R', t+1)$ such that $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1} \xrightarrow[t+1]{R'} j_2 \in S'_{t+1} \xrightarrow[t+1]{R'} k \in U'_{t+1}$. Then, by (ii), $j_1, j_2 \in S_{t+1}$.

By the preceding arguments, $j_1 \xrightarrow[t+1]{R} j_2 \xrightarrow[t+1]{R} k \in U_{t+1}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. Let $m_1 \xrightarrow[t+1]{R} m_2 \xrightarrow[t+1]{R} m_3$. We consider the following cases.

$h'_{t+1}(m_1) = h_{t+1}(m_1)$: If $m_1 \in U_{t+1}$, then $m_1 \in U'_{t+1}$ and $h_{t+1}(m_1) = h'_{t+1}(m_1) = \omega(m_1)$. Then, $j \xrightarrow[t+1]{R'} m_1$, which contradicts $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1}$. Thus, $m_1 \in S_{t+1}$. Two sub-cases are as follows:

$h'_{t+1}(m_2) = h_{t+1}(m_2)$: If $m_2 \in U_{t+1}$, then $m_2 \in U'_{t+1}$ and $h_{t+1}(m_2) = h'_{t+1}(m_2) = \omega(m_2)$. Then, $m_a \xrightarrow[t+1]{R'} U'_{t+1}$ and $j \xrightarrow[t+1]{R'} m_1$, which contradicts $j \xrightarrow[t+1]{R'} j_1 \in S'_{t+1}$. Thus, $m_2 \in S_{t+1}$.

Since $j \xrightarrow[t+1]{R} m_1 \neq j_1$, $m_3 \in U_{t+1}$. Further, $m_3 \in U'_{t+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \not\xrightarrow[t+1]{R'} m_1$, then either,

(a) $m_1 \xrightarrow[t+1]{R'} m_2 \xrightarrow[t+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{t+1}$, $m'_3 \in U'_{t+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[t+1]{R'} j_1$.

(b) $m_1 \xrightarrow[t+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[t+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{t+1}$. Suppose $m_2 \xrightarrow[t+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[t+1]{R'} m'_3$. Since $m_3 \in U'_{t+1}$, $\hat{m}_3 \in U'_{t+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{t+1}$, $m_1 \xrightarrow[t+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{t+1}$, we have $m'_3 \in U'_{t+1}$ and $m'_3 \prec \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[t+1]{R'} j_1$.

$h'_{t+1}(m_2) \neq h_{t+1}(m_2)$: Let $a \equiv h_{t+1}(m_2)$. By *pre-trade inclusion*, since $h'_{t+1}(m_2) \neq a$, we have $m_2 \in S_{t+1}$. Since $j \xrightarrow[\ddot{t}]{R} m_1$, $m_3 \in U_{t+1} \subseteq U'_{t+1}$.

Since $O_{t+1} \subseteq O'_{t+1}$, there is $\hat{m} \in N'_{t+1}$ such that $h'_{t+1}(\hat{m}) = a$ and by *pre-trade inclusion*, $\hat{m} \in CONN(i, R', t+1)$.

Since $a\ I_{m_1}\ h_{t+1}(m_1)$, and $j \not\xrightarrow[t+1]{R'} m_1$, we have that $m_1 \in S'_{t+1}$, $m_1 \xrightarrow[t+1]{R'} S'_{t+1}$, and $\hat{m} \in S'_{t+1}$.

Since $h_{t+1}(\hat{m}) \neq a$ and since there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now consider the first $\check{t}$, which is between $\hat{t}$ and $t+1$, such that $h_{\check{t}}(m_2) = a$. Then, $m_2 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_3 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.

$\boldsymbol{h'_{t+1}(m_1)} \not\mathrel{R} \boldsymbol{h_{t+1}(m_1)}$: Let $a \equiv h_{t+1}(m_1)$. Subce $h'_{t+1}(m_1) \neq a$, $m_1 \in S_{t+1}$. Since $O_{t+1} \subseteq O'_{t+1}$, there is $\hat{m} \in N'_{t+1}$ such that $h'_{t+1}(\hat{m}) = a$. Since $j \xrightarrow[t+1]{R'} j_1$, we have that $\hat{m} \in S'_{t+1}$ and $\hat{m} \xrightarrow[t+1]{R'} S'_{t+1}$. Since $h_{t+1}(\hat{m}) \neq a$, by *pre-trade inclusion*, $\hat{m} \in CONN(i, R', t+1)$ and there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Since $\hat{m} \in S'_{t+1}$ and $p'_{\hat{t}}(\hat{m}) = p'_{t+1}(\hat{m})$, we have that $\hat{m} \in S'_{\hat{t}}$. This implies that $\hat{m} \in S_{\hat{t}}$ and $h_{\hat{t}}(\hat{m}) = a$. Since $\hat{m} \xrightarrow[\hat{t}]{R'} S'_{\hat{t}}$, then $\hat{m} \xrightarrow[\hat{t}]{R} S_{\hat{t}}$. And for each $\hat{\hat{t}} > \hat{t}$, we have $\hat{m} \xrightarrow[\hat{\hat{t}}]{R} S_{\hat{\hat{t}}}$. Now consider the first $\check{t}$, which is between $\hat{t}$ and $t+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1} \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$. However, if $m_2 \in U_{t+1} \subseteq U_{\check{t}}$, then $m_1 \xrightarrow[t+1]{R} U_{t+1} \subseteq U_{\check{t}}$ and if $m_2 \in S_{t+1}$, then $m_3 \in U_{t+1}$ and $m_1 \xrightarrow[t+1]{R} S_{t+1} \xrightarrow[t+1]{R} U_{t+1}$. In either case, we have reached a contradiction.

Stage ...) Repeating this argument for the rest of the pointing phase we show (iii).

Now, we prove (iv) for $t+1$. That is, we show that for each $j \in N'_{t+1} \setminus CONN(i, R', t+1)$, $h_{t+1}(j) = h'_{t+1}(j)$. Note that since at $t+1 < t'$, i is not part of any *trading cycle* under R'. Thus is, no *trading cycle* under R' involves people connected to i under R'. That is for each trading cycle $C' \subset N'_{t+1}$, $CONN(i, R', t+1) \cap C' = \varnothing$. By (iii), each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{t+1} \setminus CONN(i, R', t+1)$, $h'_{t+2}(j) = h_{t+2}(j)$. Moreover, for each $j \in CONN(i, R', t+1)$, $h'_{t+2}(j) = h'_{t+1}(j)$.

As an **induction hypothesis**, suppose that for some $\ddot{t} \in \{t, ..., t'-1\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t}-1))$, and $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$, $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t}-1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$, and

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}}(j) = h'_{\ddot{t}}(j)$, and

Now we prove that these statements are true of $\ddot{t}+1$. To prove (i) and (ii) for $\ddot{t}+1$ note that by (iv) and (v) of the *induction hypothesis*, if $C \in N_{\ddot{t}}$ is a *trading cycle* under R and is not a *trading cycle* under R', then $C \subseteq CONN(i, R', \ddot{t})$. Thus, at Step $\ddot{t}+1$, we have statements (i) and (ii).

We now prove (iii), for $\ddot{t}+1$, by following the progression of the pointing phase just as in the case of $t+1$.

Stage 1) At the beginning of the pointing phase we consider people who were pointing at someone who remains in $N'_{\ddot{t}+1}$ and holds the same object. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}]{R'} k \in N'_{\ddot{t}}$ and $h'_{\ddot{t}+1}(k) = h'_{\ddot{t}}(k)$. Then, $j \xrightarrow[\ddot{t}+1]{R'} k$. By the *induction hypothesis*, $j \xrightarrow[\ddot{t}]{R} k$ and $h_{\ddot{t}+1}(k) = h_{\ddot{t}}(k) = h'_{\ddot{t}}(k)$. Thus $j \xrightarrow[\ddot{t}+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, if $\tau(R_j, O'_{\ddot{t}+1}) = \{a\}$, then by the *induction hypothesis*, $h^{-1}_{\ddot{t}+1}(a) = h'^{-1}_{\ddot{t}+1}(a) \notin CONN(i, R', \ddot{t}+1)$. Thus, $a \in O_{\ddot{t}+1}$ and so $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}}$. Since $j \notin CONN(i, R', \ddot{t}+1)$, $k \notin CONN(i, R', \ddot{t}+1)$. Since $k \in U'_{\ddot{t}+1}$ and $S_{\ddot{t}+1} \setminus S'_{\ddot{t}+1} \subseteq CONN(i, R', \ddot{t}+1)$, $k \in U_{\ddot{t}+1}$. Further, $h_{\ddot{t}+1}(k) = h'_{\ddot{t}+1}(k) = \omega(k)$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m \neq k$. Then, $m \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and so $h_{\ddot{t}+1}(m) = h'_{\ddot{t}+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} k$.

Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, by (ii), $j_1 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} k$ and $k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$. We consider the following two cases.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$ and $k \in U_{\ddot{t}+1}$, then $m_2 \in U_{\ddot{t}+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{\ddot{t}+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and

$m_1 \prec j_1]$. Since $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have $m_1 \not\xrightarrow[\ddot{t}+1]{R'} m_2$. Let $m_1 \xrightarrow[\ddot{t}+1]{R'} m_2'$. Since $m_2 \in U'_{\ddot{t}+1}$, we have $m_2' \in U'_{\ddot{t}+1}$ and $m_2' \prec m_2$. Then, $m_2' \prec k$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) \neq h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in CONN(i, R', \ddot{t})$. Thus, $m_1 \in CONN(i, R', \ddot{t}+1)$. Further, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$, we have $m_2 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[\ddot{t}+1]{R'} \hat{m}$, $\hat{m} \in S'_{\ddot{t}+1}$.

Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Then, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = \ a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.

Stage 5) Next we consider the people who point at satisfied people whose pointees satisfied and have unsatisfied pointees, under R'. Particularly, consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ be such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} j_2 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, $j_1, j_2 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\bar{t}]{R} \ m_1 \ \neq \ j_1$. Let $m_1 \xrightarrow[\ddot{t}+1]{R} m_2 \xrightarrow[\ddot{t}+1]{R} m_3$. We consider the following cases.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)}$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Two sub-cases are as follows:

$\boldsymbol{h'_{\ddot{t}+1}(m_2) = h_{\ddot{t}+1}(m_2)}$: If $m_2 \in U_{\ddot{t}+1}$, then $m_2 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_2) = h'_{\ddot{t}+1}(m_2) = \omega(m_2)$. Then, $m_1 \xrightarrow[\ddot{t}+1]{R} U'_{\ddot{t}+1}$ and $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_2 \in S_{\ddot{t}+1}$.

Since $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$, $m_3 \in U_{\ddot{t}+1}$. Further, $m_3 \in U'_{\ddot{t}+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, then either,

(a) $m_1 \xrightarrow[\ddot{t}+1]{R'} m_2 \xrightarrow[\ddot{t}+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{\ddot{t}+1}$, $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

(b) $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[\ddot{t}+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{\ddot{t}+1}$. Suppose $m_2 \xrightarrow[\ddot{t}+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[\ddot{t}+1]{R'} m'_3$. Since $m_3 \in U'_{\ddot{t}+1}$, $\hat{m}_3 \in U'_{\ddot{t}+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{\ddot{t}+1}$, we have $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \preceq \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

$\boldsymbol{h'_{\ddot{t}+1}(m_2) \neq h_{\ddot{t}+1}(m_2)}$: Let $a \equiv h_{\ddot{t}+1}(m_2)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_2) \neq a$, we have $m_2 \in S_{\ddot{t}+1}$. Since $j \xrightarrow[\ddot{t}]{R} m_1$, $m_3 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$.
Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$ and by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t} + 1)$.
Since $a\ I_{m_1}\ h_{\ddot{t}+1}(m_1)$, and $j \not\xrightarrow[\ddot{t}+1]{R'} m_1$, we have that $m_1 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$, and $\hat{m} \in S'_{\ddot{t}+1}$.
Since $h_{\ddot{t}+1}(\hat{m}) \neq a$ and since there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.
Now consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_2) = a$. Then, $m_2 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_3 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.

$\boldsymbol{h'_{\ddot{t}+1}(m_1) \neq h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. Since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Since $j \xrightarrow[\ddot{t}+1]{R'} j_1$, we have that $\hat{m} \in S'_{\ddot{t}+1}$ and $\hat{m} \xrightarrow[\ddot{t}+1]{R'} S'_{\ddot{t}+1}$. Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t} + 1)$ and there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Since $\hat{m} \in S'_{\ddot{t}+1}$ and $p'_{\ddot{t}+1}(\hat{m}) = p'_{\hat{t}}(\hat{m})$, we have that $\hat{m} \in S'_{\hat{t}}$. This implies that $\hat{m} \in S_{\hat{t}}$ and $h_{\hat{t}}(\hat{m}) = a$. Since $\hat{m} \xrightarrow[\hat{t}]{R'} S'_{\hat{t}}$, then $\hat{m} \xrightarrow[\hat{t}]{R} S_{\hat{t}}$. And for each $\hat{\hat{t}} > \hat{t}$, we have $\hat{m} \xrightarrow[\hat{\hat{t}}]{R} S_{\hat{\hat{t}}}$. Now consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1} \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$. However, if $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}}$, then $m_1 \xrightarrow[\ddot{t}+1]{R} U_{\ddot{t}+1} \subseteq U_{\check{t}}$ and if $m_2 \in S_{\ddot{t}+1}$, then $m_3 \in U_{\ddot{t}+1}$ and $m_1 \xrightarrow[\ddot{t}+1]{R} S_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R} U_{\ddot{t}+1}$. In either case, we have reached a contradiction.

Stage ...) Repeating this argument for the rest of the pointing phase we show (iii).

Finally, we prove (iv) for Step $\ddot{t}+1$. That is, we show that for each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. By (iii) each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. Moreover, for each $j \in CONN(i, R', \ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h'_{\ddot{t}+1}(j)$. ◇

D Proof of Propositions 1 and 2

Proposition 1: *If $N > 2$, no rule is strategy-proof, Pareto-efficient and anonymous.*

Proof: Let φ by a rule satisfying the axioms. We prove this for the case of $N = 3$.

Suppose φ is a *strategy-proof* and *Pareto-efficient*. Let $O = \{a, b, c\}$, $N = \{1, 2, 3\}$, and let $\omega = (a, b, c)$. Consider the following preference profile:

R_1	R_2	R_3
$a\ b\ c$	a	ⓐ
	$b\ c$	$b\ c$

By *efficiency*, $\varphi(R, \omega)(1) \neq a$. Thus, either $\varphi(R, \omega)(2) = a$ or $\varphi(R, \omega)(3) = a$. Suppose $\varphi(R, \omega)(3) = a$.

Claim *(Limited favoritism)*: *If 1 is indifferent between all three objects, and if 3's unique most preferred object is a, it is assigned to him. That is, for each $R' \in \mathcal{R}^N$,*[24]

$$\left.\begin{array}{l} R'_1 = \overline{I_0}, \text{ and} \\ \tau(R'_3, O) = \{a\} \end{array}\right\} \Rightarrow \varphi(R', \omega)(3) = a.$$

[24] $\overline{I_0}$ is indifference between all objects.

Proof: By *strategy-proofness*, for each $R'_3 \in \mathcal{R} \setminus \{R_3\}$ such that $\tau(R'_3, O) = \{a\}$, $\varphi(R'_3, R_{-3}, \omega)(3) = a$. Otherwise,

$$\varphi(\underbrace{R_3}_{\text{lie}}, R_{-3}, \omega)(3) \underbrace{P'_3}_{\text{truth}} \varphi(\underbrace{R'_3}_{\text{truth}}, R_{-3}, \omega)(3).$$

Also by *strategy-proofness*, there is no $R'_2 \in \mathcal{R}$, such that $\varphi(R'_2, R_{-2}, \omega)(2) = a$. Otherwise,

$$\varphi(\underbrace{R'_2}_{\text{lie}}, R_{-2}, \omega)(2) \underbrace{P_2}_{\text{truth}} \varphi(\underbrace{R_2}_{\text{truth}}, R_{-2}, \omega)(2).$$

Thus, for any $R' \in \mathcal{R}^N$ such that $R'_1 = \overline{I_0}$ and $\tau(R'_3, O) = \{a\}$, $\varphi(R', \omega)(3) = a$.$\diamond$

Since φ exhibits *limited favoritism*, it cannot be *anonymous*. ■

Proposition 2: *If $N > 2$, no rule is strategy-proof, Pareto-efficient, individually rational, and non-bossy.*

Proof: Suppose φ is *strategy-proof, Pareto-efficient, individually rational*, and *non-bossy.* We begin by noting that it satisfies *limited favoritism* as in the proof of the previous proposition.

Claim *(General favoritism)*: *If a is not assigned to 1, then 3 finds his assignment to be at least as good as a. That is, for each $R \in \mathcal{R}^N$,*

$$\varphi(R, \omega)(1) \neq a \Rightarrow \varphi(R, \omega)(3)\ R_3\ a.$$

Proof: Suppose not. Then, there is $R \in \mathcal{R}^N$ such that $\varphi(R, \omega)(1) \neq a$ and $a\ P_3\ \ \varphi(R, \omega)(3)$. Let $\alpha \equiv \varphi(R, \omega)$. Since $\alpha(1) \neq a$ and $\alpha(3) \neq a$, we have $\alpha(2) = a$.

Case $b\ P_3\ a$: Since $\alpha(1) \neq a$, by *individual rationality*, there is $x \in \{b, c\}$ such that $x\ R_1\ a$. Since $b\ P_3\ a$, $\alpha(3) \neq b$. Thus, by *Pareto-efficiency*, $\alpha(1) = b$. Further, by *Pareto-efficiency*, $b\ P_1\ c$ and by *individual rationality*, $b\ R_1\ a$. There are four possible configurations for the preference profile:

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ \hline \textcircled{b} & \textcircled{a} & b \\ \vdots & \vdots & a \\ & & \textcircled{c} \end{array}, \quad \begin{array}{ccc} R_1 & R_2 & R_3 \\ \hline \textcircled{b} & \textcircled{a}\ b & b \\ \vdots & c & a \\ & & \textcircled{c} \end{array},$$

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ \hline \textcircled{b}\ a & \textcircled{a} & b \\ c & \vdots & a \\ & & \textcircled{c} \end{array} \quad \text{or} \quad \begin{array}{ccc} R_1 & R_2 & R_3 \\ \hline \textcircled{b}\ a & \textcircled{a}\ b & b \\ c & c & a \\ & & \textcircled{c} \end{array}.$$

The circled allocation in each of the above is α. By *strategy-proofness*, if α is chosen at any one of the four configurations, it is chosen at the first. Thus, it is suffices to show that α cannot be chosen for the first configuration.

Consider the following preference profile:

R_1'	R_2	R_3
$\textcircled{b}$	$\textcircled{a}$	b
$a\ c$	$\vdots$	a
		$\textcircled{c}$

By *strategy-proofness*, b is assigned to 1. By *non-bossiness*, a is assigned to 2 and c is assigned to 3.

Now, consider another preference profile:

R_1'	R_2	R_3'
$\textcircled{b}$	$\textcircled{a}$	a
$a\ c$	$\vdots$	$b\ \textcircled{c}$

At (R_1', R_2, R_3'), by *strategy-proofness*, c is assigned to 3 and by *Pareto-efficiency*, b is assigned to 1 and a is assigned to 2. By *strategy-proofness* and *non-bossiness* the allocation is unchanged for the following profile.

R_1'	R_2'	R_3'
$\textcircled{b}$	$\textcircled{a}$	a
$a\ c$	c	$b\ \textcircled{c}$
	b	

Now suppose 1 reports $\overline{I_0}$,

$\bar{R}_1$	R_2'	R_3'
$a\ \textcircled{b}\ c$	a	$\textcircled{a}$
	$\textcircled{c}$	$b\ c$
	b	

At $(\bar{R}_1, R_2', R_3')$, by *limited favoritism*, a is assigned to 3 and by *Pareto-efficiency* c is assigned to 2, leaving b for 1. But by *strategy-proofness*, b is assigned to 1 at (R_1', R_2', R_3'). By *non-bossiness*, the circled allocation cannot be chosen.

Case $a\ P_3\ b$: This case is similar. $\diamond$

Now, we show that *general favoritism* is incompatible with *individual rationality* and *Pareto-efficiency*. Consider the following profile.

$\tilde{R}_1$	$\tilde{R}_2$	$\tilde{R}_3$
b	a	a
a	b	$b\ c$
c	c	

By *Pareto-efficiency* and *individual rationality*, b is assigned to 1 and a is assigned to 2. This violates *general favoritism*.

E List of abbreviations

Abbrev.	Description
O	Set of distinct objects.
N	Set of people.
$\omega : N \to O$	List of endowments.
$\omega(i)$	i's component of the endowment.
R_i	i's weak preference relation over O.
$\mathcal{R}$	Set of all preference relations over O.
R	A preference profile.
$\mathcal{R}^N$	Set of all preference profiles.
$\mathcal{P}$	Set of strict preference relations.
R_{-i}	Preference relation of everyone but i.
R_S	Preference profile of people in $S \subseteq N$.
R_{-S}	Preference profile of people not in $S \subseteq N$.
A	Set of allocations.
$\alpha(i)$	i's component of $\alpha \in A$.
$\alpha(S) = \bigcup_{i \in S}\{\alpha(i)\}$	Collective assignment to members of $S \in N$ under α.
$\varphi : \mathcal{R}^N \times A \to A$	A rule that selects an allocation for each problem.
$O_t \subseteq O$	Remaining objects determined after the *departure phase* in Step t of the *top cycles* algorithm.
$N_t \subseteq N$	Remaining preople determined after the *departure phase* in Step t of the *top cycles* algorithm.
$h_{t+1} : \; N_t \to \; O_t$	Holding vector determined after the *trading phase* in Step t of the *top cycles* algorithm.
$p_t(i)$	Person whom i points at.
$i \underset{t}{\longrightarrow} j$	i points at j in Step t of the *top cycles* algorithm.
$i \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} j$	i points at someone who is pointing at j in Step t of the *top cycles* algorithm.
$i \underset{t}{\longrightarrow} M,\ M \subset N_t$	i points at someone in $M \in N_t$ in Step t of the *top cycles* algorithm.
S_t	Set of satisfied people who hold one of their most preferred objects among O_t
$U_t = N_t \setminus S_t$	Set of unsatisfied people.

References

Abdulkadiroğlu, Atila and Tayfun Sönmez (1999) "House Allocation with Existing Tenants," *Journal of Economic Theory*, Vol. 88, No. 2, pp. 233–260, October.

Alcalde-Unzu, Jorge and Elena Molis (2011) "Exhange of indivisible goods and indifferences: The top trading absorving sets mechanisms," *Games and Economic Behavior, forthcoming.*

Bird, Charles G. (1984) "Group incentive compatibility in a market with indivisible goods," *Economics Letters*, Vol. 14, No. 4, pp. 309–313.

Bogomolnaia, Anna, Rajat Deb, and Lars Ehlers (2005) "Strategy-proof assignment on the full preference domain," *Journal of Economic Theory*, Vol. 123, No. 2, pp. 161–186, August.

Ehlers, Lars (2002) "Coalitional Strategy-Proof House Allocation," *Journal of Economic Theory*, Vol. 105, No. 2, pp. 298–317, August.

Erdil, Aytek and Haluk Ergin (2006) "Two-sided matching with indifferences," *unpublished.*

Hylland, Aanund and Richard Zeckhauser (1979) "The Efficient Allocation of Individuals to Positions," *Journal of Political Economy*, Vol. 87, No. 2, pp. 293–314, April.

Kesten, Onur (2006) "On two competing mechanisms for priority-based allocation problems," *Journal of Economic Theory*, Vol. 127, pp. 155–171.

Ma, Jinpeng (1994) "Strategy-Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory*, Vol. 23, No. 1, pp. 75–83.

Miyagawa, Eiichi (2002) "Strategy-Proofness and the Core in House Allocation Problems," *Games and Economic Behavior*, Vol. 38, No. 2, pp. 347–361, February.

Pápai, Szilvia (2000) "Strategyproof Assignment by Hierarchical Exchange," *Econometrica*, Vol. 68, No. 6, pp. 1403–1434, November.

Pycia, Marek and Utku Ünver (2009) "A Theory of House Allocation and Exchange Mechanisms," *MIMEO.*

Quint, Thomas and Jun Wako (2004) "On houseswapping, the strict core, segmentation, and linear programming.," *Mathematics of Operations Research*, Vol. 29, pp. 861–877.

Roth, Alvin E. (1982) "Incentive compatibility in a market with indivisible goods," *Economics Letters*, Vol. 9, No. 2, pp. 127–132.

Roth, Alvin E. and Andrew Postlewaite (1977) "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economics*, Vol. 4, No. 2, pp. 131–137, August.

Shapley, Lloyd and Herbert Scarf (1974) "On cores and indivisibility," *Journal of Mathematical Economics*, Vol. 1, pp. 23–37.

Sönmez, Tayfun (1999) "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica*, Vol. 67, No. 3, pp. 677–690, May.

Sönmez, Tayfun and M. Utku Ünver (2010) "House Allocation with Existing Tenants: A Characterization," *Games and Economic Behavior*, Vol. 69, No. 2, pp. 425–445.

Svensson, L-G (1994) "Queue allocation of indivisible goods," *Social Choice and Welfare*, Vol. 11, pp. 323–330.

Wako, Jun (1991) "Some properties of weak domination in an exchange market with indivisible goods," *Exonomic Studies Quarterly*, Vol. 42, pp. 303–314.